Weak instabilities in many-dimensional Hamiltonian systems

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The problem of long-time predictions in many-dimensional Hamiltonian systems is examined. Some geometrical methods of nonlinear dynamics are reviewed, and applied to the study of a class of instabilities that are peculiar of systems with more than two degrees of freedom. These are called "weak instabilities," since they manifest themselves only after a long time. A qualitative analysis of the weak instability induced by slow parametric modulation (the modulational diffusion) is developed. The relevance of these phenomena to the problem of stability of charged particles in accelerators and storage rings is discussed.

CONTENTS

I. INTRODUCTION

Recent developments in the theory of nonlinear dynamical systems have rekindled interest in the so-called stability problem of classical mechanics, whose original formulation was related to the age-old question of the stability of the solar system (Poincaré, 1892; Moser, 1973). Similar problems of stability in many degrees of freedom conservative systems are now found to be relevant to various branches of physics and engineering. They all share the same kind of mathematical difficulties, which arise from the lack of global dynamical invariants. At the same time the analytical and numerical approach to such problems is guided by the immense historical tradition of classical mechanics, which provides mathematical tools and methods of investigation, as well as a large amount of rigorous results.

This paper is intended as a practical guide for those interested in some qualitative methods of nonlinear dynamics that are relevant to stability problems. Emphasis will be placed on the geometrical aspects of the phenomena we shall consider, which will be illustrated by means of simple models. The latter have been chosen from among the ones most frequently encountered in the local analysis of nonlinear systems and in particular of time-dependent ones.

In order to motivate the introduction of some essential ingredients (such as nonlinearity and multidimensionality) we will consider an example from the physics of charged particles accelerators (Courant and Snyder, 1958), namely, the stability of heavy particles in intersecting storage rings (Tennyson, 1982). In a storage ring, bunches of particles rotating in opposite directions are made to collide periodically, yielding not only a desired elementary particle interaction but also an undesired electromagnetic interaction (beam-beam interaction). Particles' motions are characterized by unavoidable transverse oscillations about the ideal "circular" path (betatron oscillations), whose amplitude becomes time dependent due to the beam-beam interaction. As a result, even if the strength of the perturbation is typically very small, the amplitude of the betatron oscillations may become intolerably large, yielding the disintegration of the beam or at least a decrease in its intensity. The problem is then to predict the time evolution of these motions as a function of the initial conditions and of the machine parameters, and to recognize dynamical regimes which optimize the stability of the beam.

The analytical and numerical modeling of the beam-beam interaction requires some drastic simplifications, since the dimensionality of the original system is too large. A commonly used approximation, the so-called weak-strong approximation, consists of ignoring the influence of one beam (the weak one) upon the other (the strong one); with this assumption the motion of a test particle in the average electromagnetic field of the strong beam is studied. In addition, in the case of heavy particles (e.g., protons and antiprotons), the dissipative effect of electromagnetic radiation can be neglected, and the resulting dynamical system is Hamiltonian.

One then has to deal with two oscillators (radial and azimuthal betatron motions) subject to a periodic perturbation (the beam-beam interaction) which introduces nonlinearity, coupling, and explicit dependence on the time. Furthermore, one also often needs to include a slow periodic variation of some system parameters (parametric modulation) which derives from the existence of low-frequency longitudinal oscillations (synchrotron oscillations).

After the introduction of a natural canonical coordinate system [action-angle variables (Arnol'd, 1978, p. 279)], the phase space associated with a typical model of the beam-

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beam interaction becomes the direct product of two cylinders, which, once the explicit time dependence is taken into account, yields a dynamical system with "2.5 degrees of freedom" (five-dimensional extended phase space).

The problem here is to investigate stability for infinite time, as opposed to stability for long times. Indeed, during the truly "cosmic" time scales that characterize a typical experiment (corresponding to some 10^{11} revolutions), some delicate and subtle phenomena (weak instabilities) peculiar to Hamiltonian systems of more than two degrees of freedom can occur.

The stability problem to be studied is formulated in Sec. II. Section III illustrates the main features of integrable systems on a cylinder, while the appearance of irregular motion is discussed in Sec. IV. Section V is devoted to the analysis of the structure induced in the phase space by parametric modulations. Section VI describes the source of weak instabilities in many-dimensional systems. Finally Secs. VII and VIII provide a qualitative description of the weak instability induced by parametric modulation (modulational diffusion).

An exhaustive introduction to the field of nonlinear dynamics can be found in Ford (1975) or Berry (1978) and a more advanced exposition in Moser (1973), Chirikov (1979), or Lichtenberg and Lieberman (1983). A great deal of useful information concerning the dynamics of the beam-beam interaction is clearly presented in Tennyson (1982), to which this paper may be regarded as an introduction and complement.

II. THE STABILITY PROBLEM

From the theoretical point of view, one is interested in studying the motions generated by the Hamiltonian

$$H(\mathbf{I},\boldsymbol{\theta},t)=H_0(\mathbf{I})+\varepsilon H_1(\mathbf{I},\boldsymbol{\theta},t) , \tag{2.1}$$

where $\mathbf{I}=(I_1,\dots,I_N)$ and $\boldsymbol{\theta}=(\theta_1,\dots,\theta_N)$ are action-angle variables, and H_1 is assumed to be a periodic function of the variables $\theta_1,\dots,\theta_N,t$.

A system described by Hamiltonian (2.1) is said to be a near-integrable system, since the Hamiltonian H_0 is itself integrable and εH_1 is a small perturbation. Here we shall regard H_0 as describing some bounded oscillations. When $\varepsilon=0$, we note that N analytic single-valued constants of the motions, the action functions $I_i=I_i(\mathbf{p},\mathbf{q})$, are defined on the phase space. In addition, these functions are in involution [i.e., all mutual Poisson brackets vanish identically: $(I_i,I_j)=0$], and their gradients $\mathbf{dI}_i=(\partial I_i/\partial p_1,\dots,\partial I_i,/\partial q_N)$ are linearly independent vectors. Under these assumptions, the Liouville-Arnol'd theorem guarantees (Arnol'd, 1978, p. 271) that the system H_0 is completely integrable (by quadratures). Specifically, it is proved that the N-dimensional invariant surface defined by the N equations $I_i=I_i(0)$ is topologically equivalent to an N torus T^N, that is, to the direct product of N circles: $T^N=S\times S\times\cdots\times S$. Motion on a given torus decomposes into the product of N uniform rotations: $\theta_i(t)=\theta_i(0)+\omega_i t$, with frequencies $\omega_i(\mathbf{I})=\partial H_0/\partial I_i$. The invariant tori fiber the whole energy surface.

The main problem is now to determine the evolution of this motion under the effect of a perturbation. Specifically, are there solutions which will eventually become unbounded? If such orbits exist, are they exceptional or do they form a set of positive measure in phase space? The first step is to decide whether toroidal invariants exist in the perturbed system as well. The long-sought answer to this question can be found in the celebrated Kolmogorov-Arnol'd-Moser (KAM) theory (Arnol'd, 1963,1978, p. 399), which guarantees that, when ε is small, a large fraction of those tori of H_0 bearing quasiperiodic motions (corresponding to rationally independent frequencies) are merely distorted, rather than destroyed, by the perturbation. This in turn implies that the majority of initial conditions still yield bounded solutions ["metric" stability (Arnol'd, 1963)]. However, the preserved tori do not form an open set, being separated from each other by a dense set of gaps, the so-called "stochastic layers" (Chirikov, 1979). Orbital motion within the layers is irregular and unpredictable, and so far it has remained unreachable by the tools of a rigorous analysis. For this reason, the stability problem has been attacked with topological and geometrical methods. For systems with two degrees of freedom, criteria for global stability have been formulated, based on the observation that the nondense two-dimensional tori nonetheless disconnect the three-dimensional energy surface—that is, they divide it into invariant disjoint regions. In other words, an orbit initiated in a gap between two tori will remain forever confined in it. When the number of degrees of freedom exceeds two, the N-dimensional tori no longer disconnect the $(2N-1)$-dimensional energy surface. The situation is analogous to that of a three-dimensional space which can be disconnected by a plane (one dimension less) but not by a line (two dimensions less). When this is the case, all layers merge together to form a unique connected dense set of positive measure. Trajectories exist on this set which link any two open domains in the phase space (the Arnol'd diffusion) as illustrated by Arnol'd in a simple example (Arnol'd, 1964). Arnol'd also conjectured that this phenomenon is generic for many-dimensional systems, a fact which is now widely believed, but which still awaits a rigorous proof. This somewhat unpleasant result indicates that a generic Hamiltonian system with more than two degrees of freedom is topologically unstable, in the sense that an arbitrarily small change in the initial conditions may completely alter the character of the motion over a long time, and in particular (if the system is explicitly time dependent) convert a bounded solution into an unbounded one.

Arnol'd diffusion can be regarded as belonging to a larger class of analogous phenomena deriving from the merging of various types of stochastic layers in many-dimensional systems. We shall call them "weak instabilities," since their effect is typically exponentially small in some perturbation parameter.

In order to obtain an accurate picture of the practical applications of weak instabilities, we must note that the full measure of the stochastic domain in phase space is usually small when ε is small and that stochastic diffusion on this domain is a slow process which becomes noticeable only after a long time. Yet phenomena have been observed for which these instabilities constitute a possible, even plausible, explanation. Besides the loss of particles in colliding beams facilities, I should like to mention one example from celestial mechanics, namely, the loss of asteroids in the asteroid belt, which resulted in the formation of the Kirkwood gaps (Chirikov, 1971).

III. INTEGRABLE SYSTEMS

Integrable Hamiltonian systems are extremely useful but quite exceptional (Marcus and Meyer, 1974). The onset of irregular motions in a near-integrable system is most easily understood as a perturbation of the associated integrable case. Here we consider a few simple examples.

We begin the analysis of the system (2.1) by expanding the Hamiltonian in "double" Fourier series

$$H(\mathbf{I},\boldsymbol{\theta},t)=H_0(\mathbf{I})+\varepsilon\sum_{\mathbf{m}}\sum_{n=-\infty}^{\infty}V_{\mathbf{m}}(\mathbf{I})e^{i(\mathbf{m}\cdot\boldsymbol{\theta}+n\Omega t)}, \tag{3.1}$$

where the first sum is extended to all integer combinations $\mathbf{m}=(m_1,\ldots,m_N)$ and Ω is a given constant (the external driving frequency). The Hamiltonian vector field $\mathbf{V}=(\dot I_1,\ldots,\dot\theta_N)$ is defined via the Hamilton's equations $\mathbf{V}=J\,\mathbf{dH}$, which assign one tangent vector at each point of the phase space. Here $\mathbf{dH}$ is the gradient of the Hamiltonian function and

$$J=\begin{bmatrix}0 & -E\\ E & 0\end{bmatrix},$$

E being the $N\times N$ identity matrix. $\mathbf{V}$ generates the phase flow H^t that is a one-parameter group of canonical maps of the phase space onto itself: $H^t(x)=x(t)$, where x denotes a phase point.

The simplest class of integrable systems of the form (3.1) is obtained by letting $N=1$ and $\varepsilon=0$. Then $H_0(I)$ describes a monodimensional nonlinear oscillator, and the action-angle variables (I,θ) are "polar" coordinates in the phase plane (p,q). Without loss of generality we may take the origin $(p=0,\ q=0)\to I=0$ as the equilibrium position of the oscillator (the circular orbit in the accelerator), where the transformation $(p,q)\leftrightarrow(I,\theta)$ is singular. After the removal of this point, the phase space M becomes a cylinder—that is, the direct product of the real line R and the circle S: $M=C\equiv R\times S=\{(I,\theta)\}$. Expanding H_0 about some $I=I_0$ and writing again I for $I-I_0$, we find

$$H_0(I)=a_1I+a_2I^2+O(I^3)\,, \tag{3.2a}$$

$$\mathbf{V}=(0,a_1+2a_2I)\,. \tag{3.2b}$$

Since $\mathbf{V}$ has no axial component, all invariant tori $I=\text{const}$ encircle the cylinder [Fig. 1(a)]. In addition $\mathbf{V}$ depends on I (nonlinearity), and it vanishes identically on

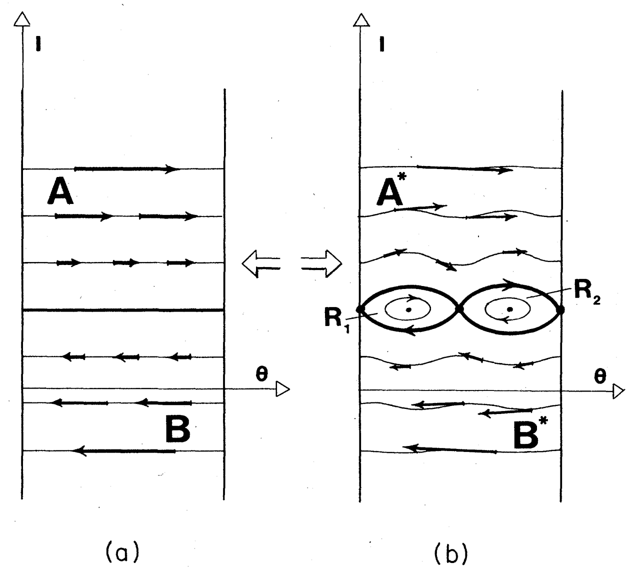

FIG. 1. Integrable systems on the cylinder: identification of phase curves. (a) $\varepsilon=0$. (b) $\varepsilon=0$ and $m=2$.

the torus $I_r=-a_1/2a_2$, which will be called a *resonant torus.* Thus a resonant torus is entirely made of fixed points. This property is a quite exceptional one, and we expect it to fail under the effect of a perturbation. In preparation for future use, we denote by A (B) the region of the cylinder above (below) the resonant torus.

We now add a single angle-dependent term to the Hamiltonian

$$H(I,\theta)=a_1I+a_2I^2-\varepsilon V_0\cos(m\theta)+O(I^3)\,, \tag{3.3a}$$

$$\mathbf{V}=(-\varepsilon mV_0\sin(m\theta),a_1+2a_2I)\,. \tag{3.3b}$$

H is still an invariant function—i.e., the system (3.3) is integrable—but now $\mathbf{V}$ has a nonvanishing axial component, causing the unperturbed variables to oscillate periodically with time (phase oscillations).

Analogies and differences between the flows H_0^t and H^t become evident as one seeks to identify phase curves of the two systems via a diffeomorphism.[1] To this end, we note that the level lines $H=\text{const}$ are still diffeomorphic to (families) of circles (Arnol'd, 1978, p. 285) with one exception, given by the equation

$$H(I,\theta)=\varepsilon V_0-\frac{a_1^2}{4a_2}\,. \tag{3.4}$$

These critical level lines are called *separatrices.* They divide the cylinder into $2+m$ open regions: $A^*,B^*,R_1,\ldots,R_m$ [Fig. 1(b)]. Each of these regions can be mapped into a cylinder by constructing local action-angle coordinates, which reduce the motions to uniform rotations. A natural choice is to identify the phase curves of A (B) with those of A^* (B^*) having the same frequen-

[1]A diffeomorphism is a one-to-one differentiable map with differentiable inverse.

cy, because this map reduces to the identity as $\varepsilon \to 0$. In other words, a nonresonant torus is merely distorted by the perturbation. However, we cannot extend this transformation to the whole phase space, since it is *singular* on the resonant torus (the latter cannot be mapped diffeomorphically onto the R_i's). Therefore, we must conclude that all the information brought by the perturbation is stored within the R_i's themselves. They form a *nonlinear resonance* of order m. Tori within a resonance do not encircle the cylinder and hence describe motion of a different nature (librations). Of the infinity of stationary points which constituted the resonant torus only $2m$ survive, m of which are Lyapounov stable and m unstable (Arnol'd, 1973, p. 155). These are consequences of the Poincaré-Birkhoff fixed-point theorem (Birkhoff, 1927, p. 165). A stable (unstable) fixed point is also referred to as elliptic (hyperbolic), for the invariant curves in a small neighborhood of it are ellipses (hyperbolas) (Berry, 1978).

The geometrical properties of a nonlinear resonance can be easily controlled through the available parameters. Specifically, its location $I_r = -a_1/2a_2$ coincides with that of the unperturbed resonant torus. Then we note that (3.3) can be transformed into the pendulum Hamiltonian

$$H(p,\varphi)=\frac{p^2}{2M}-U_0\cos\varphi \ , \tag{3.5a}$$

$$M=\left[m^2 \frac{\partial^2 H}{\partial I^2}\bigg|_{I=I_r}\right]^{-1}, \quad U_0=\varepsilon V_0 \ , \tag{3.5b}$$

via the canonical transformation generated by

$$F(p,\theta)=(pm+I_r)\theta \ ; \tag{3.6}$$

then standard theory gives (Chirikov, 1979)

$$\Delta I = 2m(\varepsilon M V_0)^{1/2} \ , \tag{3.7}$$

$$\omega_0=\left[\frac{\varepsilon V_0}{M}\right]^{1/2}, \tag{3.8}$$

where ΔI is the resonance halfwidth and ω_0 is the frequency of small librations (Fig. 2). We also note that the term linear in I in (3.3) can be eliminated via the transformation

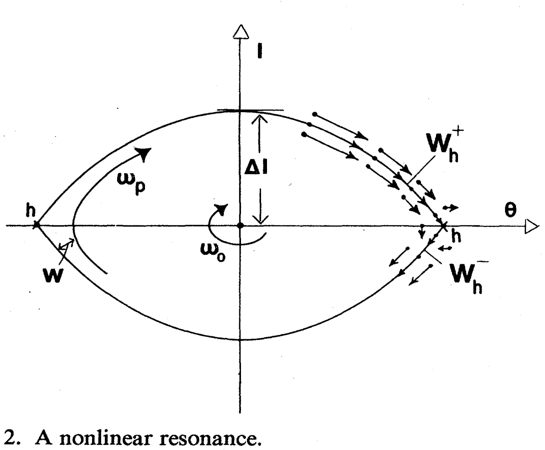

FIG. 2. A nonlinear resonance.

$$F(p,\theta,t)=p(\theta-a_1 t) \ , \tag{3.9}$$

which introduces an explicit time dependence (and vice versa)

$$H' = H + \frac{\partial F}{\partial t} = a_2 p^2 - U_0 \cos[m(\varphi + a_1 t)] \ . \tag{3.10}$$

From (3.10) we see that the location of the resonance can also be determined by requiring the phase $\Psi = \varphi + a_1 t$ to be stationary:

$$\frac{d\Psi}{dt} = 2a_2 p + a_1 = 0 \ . \tag{3.11}$$

The generating functions (3.6) and (3.10), often combined, are commonly employed to reduce a resonant expansion of a time-dependent Hamiltonian into pendulum form (Chirikov, 1979) (see Sec. IV).

The separatrices deserve much attention. We now recall some important facts.

(1) The solutions of (3.3b) depend *discontinuously* on the initial conditions on the separatrices (in the limit $t \to \mp\infty$). This dependence stems from the lack of a global diffeomorphism linking unperturbed and perturbed phase curves.

(2) All points on the separatrices are *homoclinic*—i.e., they are doubly asymptotic ($t \to \mp\infty$) to unstable equilibria. This circumstance makes the separatrix particularly fragile under perturbation. Specifically, from the pendulum Hamiltonian (3.5) and the definition (3.4) the equation of the separatrix is obtained explicitly as

$$p_s = \mp 2\omega_0 \cos(\varphi/2) \ , \tag{3.12}$$

which can be integrated to give (Arnol'd, 1964)

$$\varphi_s(t) = 4\arctan(e^{\omega_0 t}) - \pi \ , \tag{3.13}$$

whence

$$\lim_{t\to\mp\infty}(p_s(t),\varphi_s(t)) = (0,\mp\pi) \ ,$$

which is the location of the hyperbolic fixed point. A point on the separatrix can either approach the fixed point or escape from it, depending on which branch it belongs to. Accordingly, we shall speak of the stable (W_h^+) and the unstable (W_h^-) manifolds (associated with the hyperbolic fixed point h), respectively. Since all points on the separatrix are homoclinic, the roles of W_h^+ and W_h^- are interchanged under time reversal—that is, W_h^+ and W_h^- are joined to each other smoothly (Fig. 2).

(3) Motion in the vicinity of a pendulum separatrix is more conveniently described via the local coordinate (Chirikov, 1979)

$$w = \frac{H-U_0}{U_0} \ , \tag{3.14}$$

which measures the distance from the separatrix in terms of the relative energy (Fig. 2). Since w is itself invariant under the phase flow, we can characterize tori near the separatrix by their frequency $\omega = \omega(w)$. The latter is evaluated by constructing the local action function

(Arnol'd, 1978, p. 280)

$$I'(h)=\frac{1}{2\pi}\oint p\,d\varphi\ , \tag{3.15}$$

where the integral is extended to a phase curve $H=h$. For rotations ($h>U_0$) we find

$$I'(h)=\frac{4\omega_0}{\pi}\frac{E(\kappa)}{\kappa}\ , \tag{3.16}$$

E being the complete elliptic integral of the second kind, with modulus $\kappa=[2U_0/(h+U_0)]^{1/2}=[2/(2+w)]^{1/2}$. ω is then given by

$$\omega=\frac{\partial H}{\partial I'}=\frac{\partial H}{\partial \kappa}\frac{\partial \kappa}{\partial I'}=\frac{\pi\omega_0}{\kappa K(\kappa)}\approx\frac{2\pi\omega_0}{\ln(32/w)}\ , \tag{3.17}$$

where we have used the fact that near separatrix we have $\kappa\approx 1$ and $K(\kappa)\approx\frac{1}{2}\ln(32/w)$ (K is the complete elliptic integral of the first kind). One can easily verify that the expression (3.17) holds also when $h<U_0$ [in this case $\omega(w)$ is twice the frequency of librations near separatrix]. We see that the period of motion grows indefinitely as one approaches the separatrix. It must be noted that in the unperturbed coordinates (I,θ) the rotations described by (3.17) are far from being uniform. In fact, the phase velocity varies considerably along and near the separatrix [see Eq. (3.13)], to become arbitrarily small in a neighborhood of the unstable equilibria where phase points spend most of their time evolution.

Before concluding this section, let me note that the validity of the analysis presented here relies on the smallness of the terms neglected in (3.3a). In addition, $V_0=V_0(I)$ in general, so that (3.7) must be regarded as the first term in the expansion of ΔI about $I=I_0$. This is a legitimate approximation, as long as ε is small.

IV. NEAR-INTEGRABLE SYSTEMS

Most Hamiltonian systems are nonintegrable (Marcus and Meyer, 1974). We shall be concerned with systems close to integrable ones, which exhibit a mixture of regular and irregular motions. Even a short survey of such a vast field is quite impossible here. I shall merely recall some qualitative aspects of this class of phenomena and present a few useful formulas.

Let us now modify Hamiltonian (3.3) to accommodate more resonances (Chirikov, 1979):

$$\begin{aligned} H&=H_0+\varepsilon H_1,\\ H_0&=\frac{I^2}{2}-\omega_0^2\cos\theta\ ,\\ H_1&=\omega_0^2\cos\theta\cos\Omega t=\frac{\omega_0^2}{2}[\cos(\theta-\Omega t)+\cos(\theta+\Omega t)]\ , \end{aligned} \tag{4.1}$$

which describes a pendulum subject to a periodic parametric perturbation. The inclusion of two additional resonances (rather than one) is not essential. It serves the sole purpose of simplifying the calculations by making the system more symmetric.

The time-dependent system (4.1) can be naturally embedded in an autonomous one, with two degrees of freedom. The latter is defined by the Hamiltonian

$$\begin{aligned} K=&\frac{I^2}{2}+I_\zeta\Omega-\omega_0^2\cos\theta\\ &+\frac{\varepsilon\omega_0^2}{2}[\cos(\theta-\zeta)+\cos(\theta+\zeta)]\ , \end{aligned} \tag{4.2}$$

where the new angle variable $\zeta=\Omega t \bmod(2\pi)$ is canonically conjugated to the action $I_\zeta=I_\zeta(I(t),\theta(t),t)=-H/\Omega$. The Hamiltonian (4.2) [and hence also (4.1)] is nonintegrable, because it possesses only one global invariant, namely, K itself. The lack of additional invariants is actually far from being obvious, and it can be demonstrated only by means of a special technique [see Eqs. (4.7) and (4.8) and the following discussion]. The phase space M is now three dimensional, for the auxiliary action variable I_ζ is actually a function of the remaining coordinates, via (4.1) (note, however, that K does not depend on I, θ, or ζ through I_ζ). More precisely, M is the direct product of a cylinder and a circle $M=C\times S=\{(I,\theta,\zeta)\}$. M is called the extended phase space of the original system (4.1).

The analysis of motion in three-dimensional spaces can be considerably simplified via the following construction (Poincaré, 1892, Chap. 27) (Fig. 3). We consider the surface C^* defined by the equation $\zeta=\text{const}$ (surface of section). Then C^* is again a cylinder. Under the action of the phase flow

$$K^t(I,\theta,\zeta)=(I(t),\theta(t),\zeta+\Omega t) \tag{4.3}$$

all phase points cross C^* repeatedly, the time interval Δ

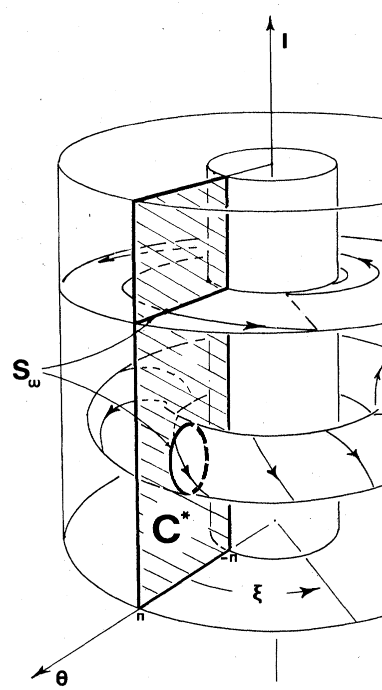

FIG. 3. The extended phase space of a time-dependent system. Construction of a surface of section.

between two successive crossings being the period of the driving perturbation, or $\Delta=2\pi/\Omega$. If we think of K^t as acting on the whole C^*, then for every integer n we have

$$K^{n\Delta}(C^*)=C^* , \tag{4.4}$$

that is, C^* is invariant under the discrete group of canonical maps $K^{n\Delta}$ parametrized by n. K^Δ is called the "period-advance" map, for obvious reasons (Arnol'd, 1973, p. 57). In this way we have again reduced our problem to the study of transformations of the cylinder onto itself.

Let us examine this point more closely. When $\varepsilon=0$, the whole extended phase space foliates into invariant two tori $T^2=S_\omega\times S_\Omega$, which are labeled by the two frequencies ω and Ω. Then the intersection of T^2 and C^* is the one torus (circle) S_ω, which may or may not encircle the cylinder (Fig. 3). The action of $K^{n\Delta}$ on T^2 leaves all points of S_Ω invariant, while it rotates those of S_ω by an angle $n\Delta\omega$. If a point of S_ω comes back on itself, for some n, then the whole torus is made of fixed points, and therefore it is resonant. This will be the case provided that there exist integers n,m such that

$$n\omega+m\Omega=0 . \tag{4.5}$$

Indeed, the action of the phase flow on T^2 during the time interval $n\Delta$ causes each phase point on it to complete m revolutions on S_ω and n revolutions on S_Ω. Since ω varies from torus to torus, due to nonlinearity, one can adjust the value of the frequency (i.e., move on a nearby torus) in such a way as to satisfy the resonance condition (4.5), with some suitable choice of m and n. Moreover, since rational numbers form a dense set of zero measure on the real line, so do the resonant tori in the phase space.

The stability of maps on a cylinder is related in an essential way to the existence of encircling invariant tori, for any region bounded by two such tori is itself invariant. For perturbed systems this fact is in turn established by the KAM theorem, provided that ε is small enough. Indeed, the KAM theory provides a map which identifies some phase curves of the unperturbed system with phase curves of the perturbed one having the same frequency (Fig. 4). This map (which is obtained as the sum of a convergent series of canonical transformations) is a smooth function of ε, so that it reduces to the identity as $\varepsilon\to 0$. On the other hand, it is characterized by a *dense set of singularities* in phase space, indicating that the perturbation has dramatically altered the topological structure of the phase space. The new solutions are confined within the dense set of gaps (stochastic layers) between the preserved tori. The singularities of the KAM map are related not only to the simultaneous disintegration of all resonant tori but also to the progressive destruction of nonresonant ones, as the perturbation strength increases. The description of this phenomenon is one of the central problems in the theory of Hamiltonian systems. It involves formidable mathematical difficulties, because the full resonant structure of a nonintegrable system is immensely more complicated than that which appears explicitly in the Hamiltonian (primary resonances). Indeed, secondary resonances replace each destroyed resonant torus. They in turn have their own secondary resonances and so forth ad infinitum, to form a hierarchic structure which repeats itself on every small scale (Fig. 5).

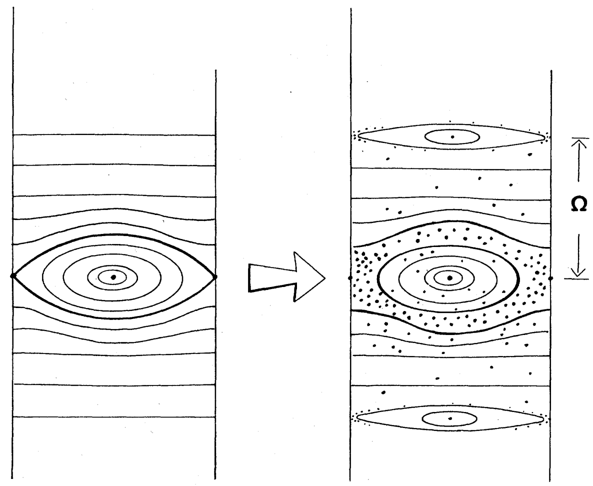

FIG. 4. The KAM theorem.

We begin our analysis by considering primary resonances only. For this approximation to make sense we need to assume that the primary resonances are geometrically well separated from each other (or $\Omega/\omega_0\gg 1$; see below), so that the amplitude of the higher-order terms is small. The relevance of secondary resonances as well as the relation between resonances at different levels in the hierarchy will be discussed at the end of this section.

We shall first be concerned with the evolution of the separatrix itself, which I wish to describe in terms of the unperturbed variable w, now no longer invariant [see (3.14) and (4.1)]. To this end we note that motion in the vicinity of the hyperbolic points is still regular, being governed by a local invariant (Moser, 1956), which we may take as w itself to a first approximation. Then we are forced to conclude that the variation of w takes place during the relatively short time interval corresponding to the transit between neighborhoods of hyperbolic points (Fig. 6). Let us then consider an initial point x located on the unstable manifold W_h^- near the hyperbolic fixed point h ($w=0$). Due to the smallness of the perturbation, for

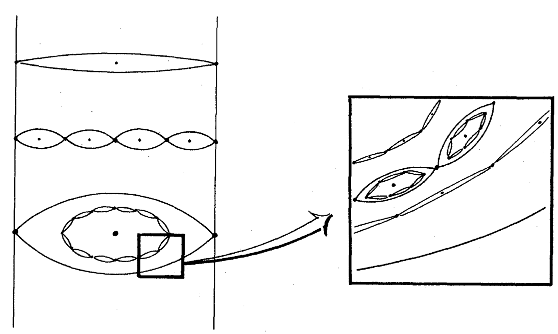

FIG. 5. Hierarchy of resonances.

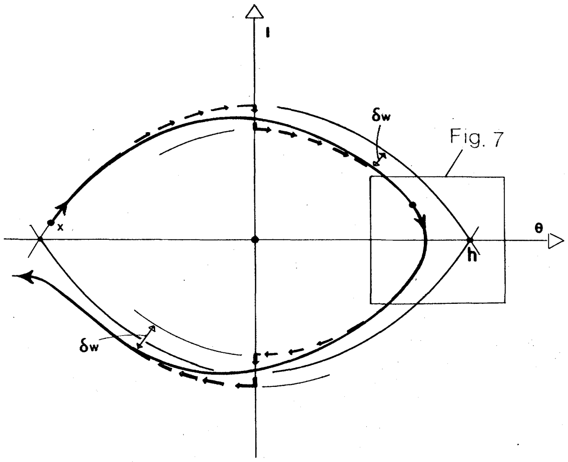

FIG. 6. Motion near a separatrix. The dashed curve represents the whisker map approximation.

some suitably large values of n we expect to find the image $K^{n\Delta}(x)$ of x again in a neighborhood of h near the stable manifold W_h^+ (Figs. 6 and 7). The unperturbed homoclinic trajectory ($w=0$) will be converted into a rotation ($w>0$) or a libration ($w<0$), depending on the sign of the increment δw of w during the phase oscillation. The latter can be approximately determined by integrating the equation [cf. (3.14)]

$$\frac{dw}{dt}=\frac{1}{U_0}\frac{dH_0}{dt}=\frac{1}{U_0}(H,H_0)$$
$$=\frac{\varepsilon}{U_0}I(\theta)\frac{\partial H_1(\theta)}{\partial\theta}, \tag{4.6}$$

where we must substitute for $\theta=\theta(t)$ the phase $\varphi_s(t)$ corresponding to the motion on the unperturbed separatrix [cf. (3.13)] and then extend the limits of integration to infinity:

$$\delta w \approx \varepsilon\omega_0\int_{-\infty}^{\infty}dt[\sin(\varphi_s-\zeta)+\sin(\varphi_s+\zeta)]\cos(\varphi_s/2)\ . \tag{4.7}$$

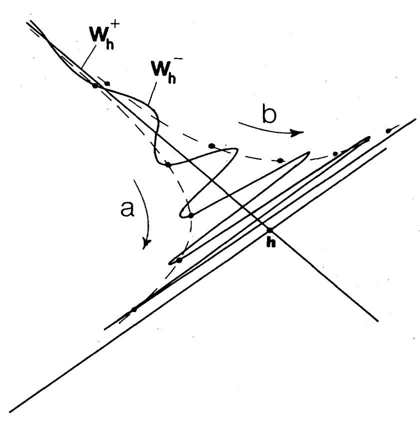

FIG. 7. Homoclinic oscillations of separatrices. (a) Motion on a separatrix. (b) Motion near a separatrix. The dashed curves are level lines $w=$const.

This expression yields the so-called Melnikov-Arnol'd integral (Chirikov, 1979). The first term on the right-hand side (rhs) of (4.7) is dominant on the upper branch of the separatrix, whereas the second term is dominant on the lower one. Due to the symmetry of the perturbation their contribution to the integral is identical in both cases. In the limit $\lambda\equiv\Omega/\omega_0\gg 1$ we obtain

$$\delta w \approx -4\pi\varepsilon\lambda^2\exp\left[-\frac{\pi\lambda}{2}\right]\sin\zeta_0\ . \tag{4.8}$$

Here ζ_0 denotes the phase Ωt_0 of the driving perturbation at the instant of time in which $\theta=0$. The oscillations described by (4.8) are referred to as "homoclinic oscillations," whose amplitude is an exponentially small function of λ. They cause the stable and the unstable manifolds to intersect each other transversally (Fig. 7), the points of intersection being the only homoclinic points left on the separatrices (cf. Sec. III). The presence of the homoclinic intersections [or, equivalently, the nonzero value of the Melnikov-Arnol'd integral (4.7)], is of great significance, since it implies the nonexistence of an additional invariant of the motion. Indeed, if such an invariant existed, the phase portrait near separatrix would be topologically equivalent to that of the unperturbed case—i.e., there would be no homoclinic intersections. The continuity of the period-advance map K^Δ imposes several constraints on the topological structure of the perturbed separatrices. Indeed, one homoclinic intersection implies infinitely many others, and in addition neither W^+ nor W^- must ever intersect itself. For this reason and from the fact that K^Δ is canonical (i.e., it is area preserving) the stable and the unstable manifolds accumulate about each homoclinic point to form an exceedingly intricate web (Moser, 1973, p. 99). If one considers that the solutions of Hamilton's equations depend discontinuously on the initial conditions (in the limit $t\to\mp\infty$) over this set of intersecting separatrices, one can easily understand the failure of the efforts to treat this problem analytically. All this was already known to Poincaré almost a century ago (Poincaré, 1892, Chap. 33).

If we retain the law of motion $\varphi_s(t)$ on the separatrix as an approximation also for motions near the separatrix, then the evolution of w can be reconstructed by iterations of (4.8) $w_n=w_0+\delta w_0+\delta w_1+\cdots+\delta w_{n-1}$, combined with a prescription for computing the corresponding sequence of ζ_i's. The latter is directly derived from (3.17) by assuming that the motion between two consecutive $\theta=0$ crossings takes place entirely on an unperturbed torus $w=$const (Fig. 6). In this way we obtain the "whisker map" (Chirikov, 1979), which again acts on a cylinder $C^+=\{(w,\zeta)\}$

$$w_{n+1}=w_n+\mu\sin\zeta_n\ ,$$
$$\zeta_{n+1}=\zeta_n+\lambda\ln(32/|w_{n+1}|)\ , \tag{4.9}$$
$$\lambda=\frac{\Omega}{\omega_0},\quad \mu=-4\pi\varepsilon\lambda^2\exp\left[-\frac{\pi\lambda}{2}\right].$$

By inspecting (4.9) we note that ζ depends sensitively (in fact, singularly) on w at $w=0$. Due to this fact successive values of ζ tend to become uncorrelated (random phase), resulting in an irregular sequence of rotations and librations. For this reason trajectories exist which are characterized by a truly random evolution (Moser, 1973; Ford, 1983) and which densely fill a two-dimensional region of the phase space. They are believed, although not proved, to constitute a set of positive measure. The stochastic domains generated in this way near a separatrix are sometimes referred to as "thin layers."

When primary resonances approach each other (either because their amplitude increases or because their relative spacing decreases, or both), the tori which are trapped between them are first distorted and then destroyed. Given a pair of resonances, a fundamental problem is to determine the critical value of the perturbation parameters which signals the disappearance of the last encircling torus between them. Indeed, this results in a sudden change of the topological structure of the phase space, because adjacent layers merge together to form a much broader stochastic component (thick layer). A simple (yet quite efficient) estimate of critical parameters is derived by regarding the two resonances as noninteracting, and then computing the condition for the unperturbed separatrices to touch each other [Chirikov resonance overlap criterion (Chirikov, 1979)]. According to this criterion, two resonances will overlap, provided that the sum of their respective halfwidths is larger than their relative spacing (Fig. 8), or

$$S=\frac{\text{sum of halfwidths}}{\text{spacings}}\geq 1\ , \tag{4.10}$$

where the critical parameter corresponds to $S=1$. This estimate is reliable provided that the two resonances have comparable amplitude. When this is not the case, (4.10) should be replaced by the stronger condition $S\gg 1$. The overlap criterion can be readily applied to the Hamiltonian (4.1) to estimate the onset of large-scale stochasticity. We shall consider the resonant terms of (4.1) one at a time, and then follow the analysis developed in Sec. III. The unperturbed resonance is located in the origin ($I=0$), since in the unperturbed Hamiltonian H_0 there is no term linear in I (cf. Sec. III). The resonance halfwidth $\Delta I=2\omega_0$ is then readily computed from (3.7) by letting $m=\varepsilon=M=1$ and $V_0=\omega_0^2$ [compare (3.3a) and (4.1)]. The terms which comprise the perturbation depend explicitly on time. By changing the coordinates via the canonical transformation generated by $F=I'(\theta\mp\Omega t)$ [cf. (3.9)], we shall then obtain the linear term $\mp\Omega I'$. Thus these resonant regions are centered about $I=\mp\Omega$—i.e., the resonance spacing is just Ω (see Fig. 4). Alternatively, one could obtain this result from the stationarity of the phase $\Psi=\theta\mp\Omega t$ [see (3.11)]. By means of (3.7) we compute again the resonance halfwidth, which is found to be equal to $\omega_0\sqrt{2\varepsilon}$. Putting everything together, we rewrite (4.10) as

$$S=\frac{2\omega_0}{\Omega}(1+\sqrt{\varepsilon/2})\geq 1\ . \tag{4.11}$$

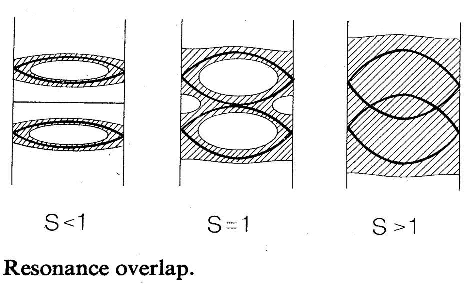

FIG. 8. Resonance overlap.

The overlap criterion has the remarkable feature of being easily applicable even to fairly complicated systems. On the other hand, it is nonrigorous, and it provides only a schematic picture of the interaction between resonances. For instance, both the finite width of all stochastic layers near separatrices (see below) and the presence of secondary resonances between the primary ones (see Fig. 5) affect the overlap process. Additionally, in some particular cases, this procedure shows even more serious defects. One such example will be presented in Sec. V. A more detailed analysis of this problem can be found in Chirikov (1979) or in Lichtenberg and Lieberman (1983, Chap. 4).

We now apply the analysis previously developed to the study of the geometrical structure of the whisker map (4.9). For this purpose we regard the cylinder C^+ as a surface of section in the extended phase space $M=C^+\times S=\{(w,\zeta,2\pi t)\}$, where we define the time-dependent Hamiltonian

$$\begin{aligned}H&=H_0(w)+\mu H_1(\zeta,t)\\&=w\lambda\left[\ln\frac{32}{|w|}+1\right]+\mu\cos(\zeta)\delta_1(t)\ ,\end{aligned} \tag{4.12}$$

$\delta_1(t)$ being the periodic delta function with unit period

$$\delta_1(t)=\sum_{n=-\infty}^{\infty}\cos(2\pi nt)\ . \tag{4.13}$$

One can easily verify by direct calculation that the difference equations (4.9) are the explicit expression of the period-advance map generated by the Hamiltonian (4.12). The relation between the system (4.12) and the original one (4.1) needs to be clarified [Fig. 9(a)]. The phase flow

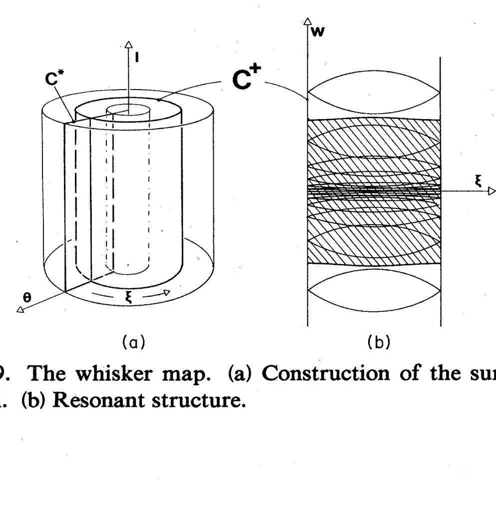

FIG. 9. The whisker map. (a) Construction of the surface of section. (b) Resonant structure.

of (4.1) near separatrix is first converted into a flow in the space $C\times S=\{(w,\theta,\zeta)\}$ via the local transformation $I\to w(H(I))$ [see Eq. (3.14)]. Then we replace the local surface of section $\zeta=\text{const}$ by the surface $\theta=0$ in the same space. In so doing we have merely interchanged the roles of ζ and θ. By regarding θ as a periodic variable with unit period $\theta=2\pi t$ we can naturally identify the latter surface with the cylinder C^+ introduced above, and then approximate the associated period-advance map with the whisker map, i.e., with the Hamiltonian (4.12).

Inserting (4.13) into (4.12), we find

$$H_1=\sum_{n=-\infty}^{\infty}\cos(\zeta-2\pi nt)\ , \tag{4.14}$$

and therefore the whisker map displays an infinite array of primary resonances of order 1 [Fig. 9(b)]. They correspond to the secondary resonances (of order n) which develop near the separatrix of the original system (4.1) in the place of those unperturbed tori whose frequency is an integer submultiple of the driving frequency. Note that the order of the resonances has been renormalized, as we can see by letting $\theta=2\pi t$ in (4.14) and then regard ζ as the independent variable. The location of these resonant domains on C^+ is found from Eq. (3.11)

$$\dot{\zeta}-2\pi n=\frac{\partial H_0}{\partial w}-2\pi n=0 \tag{4.15}$$

or

$$w(n)=32\exp\left[-\frac{2\pi n}{\lambda}\right], \tag{4.16}$$

showing that the resonant values $w=w(n)$ accumulate at $w=0$ (unperturbed separatrix). The relative spacing $\Delta w(n)$ between the nth and the $(n+1)$th resonances is then given by

$$\Delta w(n)=32\exp\left[-\frac{2\pi n}{\lambda}\right](1-e^{-2\pi/\lambda})\ . \tag{4.17}$$

In order to estimate the resonance width, we reduce the Hamiltonian (4.12) to pendulum form. After retaining only one term from the perturbation (4.14), we first introduce new canonical coordinates (I,θ) via the generating function [cf. (3.6) and (3.9)]

$$F(I,\zeta,t)=[I+w(n)](\zeta-2\pi nt) \tag{4.18}$$

and then we expand the new Hamiltonian $K=H+\partial F/\partial t$ about $I=w(n)$, up to second order

$$K=-\frac{\lambda}{2w(n)}I^2+\mu\cos\theta+O(I^2)\ . \tag{4.19}$$

The halfwidth of the nth resonance is now readily computed from (3.7) as

$$\Delta I(n)=2\left[\frac{\mu w(n)}{\lambda}\right]^{1/2}e^{-\pi n/\lambda}\ , \tag{4.20}$$

so that the resonance width decreases more slowly than the spacing. From (4.17) and (4.20) we obtain the overlap parameter $S(n)$ associated to the pair $n,n+1$ [see (4.10)]:

$$S(n)=\sqrt{\mu/8\lambda}\,e^{\pi n/\lambda}(1-e^{-\pi/\lambda})^{-1}\ . \tag{4.21}$$

Therefore, for any values of the perturbation parameters there always exists a neighborhood of the unperturbed separatrix $|w|<|w(n)|$ in which the overlap does take place $[S(n)>1]$ and a stochastic domain appears [Fig. 9(b)]. In other words, thin layers always exist in near-integrable systems.

Having reinterpreted the appearance of stochastic motions in terms of the unavoidable overlap of second-order resonances, we will also attribute the presence of thin layers about the separatrices of resonances at the nth level of the hierarchy to the overlap of those of the $(n+1)$th level. By the same token, the homoclinic intersection of the corresponding stable and unstable manifolds will exist at any scale, and in particular in any neighborhood of any stable fixed point in phase space (Moser, 1973, p. 105).

The width Δw of a thin layer is determined by the location of the encircling torus on C^+ nearest to the separatrix. It will be found between the pair of resonances corresponding to the smallest value of n for which $S(n)\lesssim 1$ [Fig. 9(b)]. Letting $S(n)=1$ in (4.21) and using (4.16), we obtain

$$\Delta w\approx\frac{4}{\pi^2}\mu\lambda=\frac{16}{\pi}\varepsilon\lambda^3\exp\left[-\frac{\pi\lambda}{2}\right], \tag{4.22}$$

where the numerical coefficient should be merely regarded as a crude estimate, for many approximations were involved in its derivation [see also Chirikov (1979) and Chirikov and Shepelyansky (1981)]. The width of the stochastic layer is an exponentially small function of λ, and it is approximately $\lambda 4/\pi^2$ times larger than the amplitude of the homoclinic oscillations [cf. (4.8)]. As a result, thin layers are sometimes very difficult to detect.

Recently, rigorous and elegant techniques have been devised to describe the breakup of KAM surfaces via renormalization-group methods. A discussion of this subject can be found, for instance, in MacKay (1982).

V. MODULATIONAL LAYERS

In this section we investigate the nature of stochastic domains generated by a slow parametric modulation (Tennyson, 1979,1982).

We consider the following time-dependent Hamiltonian:

$$H(I,\theta,t)=\frac{I^2}{2}-\varepsilon\cos(\theta+\lambda\sin\Omega t)\ , \tag{5.1}$$

which describes a pendulum with modulated phase. Here λ is the amplitude and Ω the frequency of the modulation. In order to explore the resonant structure of the system we expand (5.1) as in (3.1) to obtain

$$H(I,\theta,t)=\frac{I^2}{2}-\varepsilon\sum_{n=-\infty}^{\infty}J_n(\lambda)\cos(\theta+n\Omega t)\ , \tag{5.2}$$

where the coefficients J_n are Bessel functions of the first kind. Thus (5.2) is characterized by an infinite sequence of primary resonances of the first order which are evenly spaced along the cylinder (multiplet): $I_n = n\Omega$ (Fig. 10). After selecting one perturbation term from (5.2) we can again reduce the original Hamiltonian into pendulum form with the generating function [cf. (3.9)]

$$F(J,\theta,t) = (J - n\Omega)(\theta + n\Omega t) . \tag{5.3}$$

The corresponding halfwidth $\Delta I(n)$ is then readily obtained from (3.7) as

$$\Delta I(n) = 2[\varepsilon J_n(\lambda)]^{1/2} , \tag{5.4}$$

yielding the overlap parameter $S(n)$ for the pair $n, n+1$ [cf. (4.10)]

$$S(n) = \frac{\Delta I(n) + \Delta I(n+1)}{\Omega} = \frac{2\sqrt{\varepsilon}}{\Omega}\{[J_n(\lambda)]^{1/2} + [J_{n+1}(\lambda)]^{1/2}\} . \tag{5.5}$$

For large enough n, $S(n)$ falls off exponentially: $S(n) \propto (e/n)^{n/2}$ due to the asymptotic form of the Bessel functions. This fact ensures the existence of encircling invariant tori, and therefore the system (5.1) is always globally stable. On the other hand, for sufficiently small n, one can adjust the system parameters to induce the simultaneous overlap of several adjacent multiplet elements, which results in the formation of a quite broad and homogeneous stochastic domain (modulational layer). When $\lambda \gg 1$, the layer width is approximately equal to $\lambda\Omega$, because the amplitude of the resonant terms drops

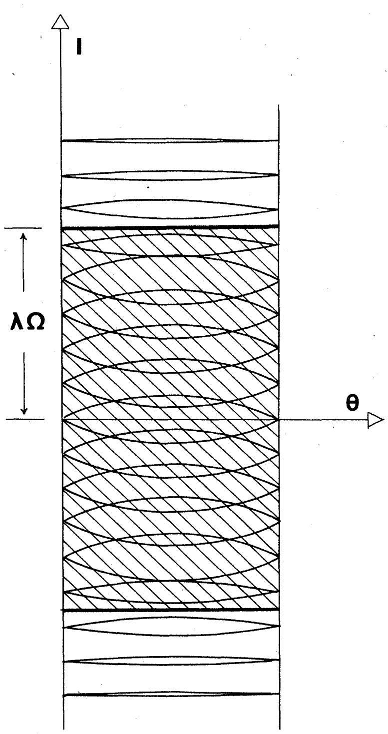

FIG. 10. A modulational layer.

off quite abruptly for $n > \lambda$. At the same time, for $n < \lambda$ one can replace $J_n(\lambda)$ by their average values $|J_n(\lambda)| \approx (\pi\lambda)^{-1/2}$, so that the condition for resonance overlap takes on the simplified form

$$S \approx \frac{4}{\Omega}\left[\frac{\varepsilon}{\sqrt{\pi\lambda}}\right]^{1/2} \geq 1 . \tag{5.6}$$

Below this threshold all stochastic orbits are confined within thin layers.

Equations (5.5) and (5.6) must be interpreted with some care, being singular at $\Omega = 0$, when the system becomes again integrable. The strong overlap regime ($S \gg 1$) which would develop for very small Ω conflicts with the predictions of the theory of adiabatic invariants, guaranteeing a smooth evolution of the action function as the system parameters change very slowly (Arnol'd, 1978, p. 297). This apparent contradiction is resolved by noting that as $\Omega \to 0$ the geometrical arguments on which the overlap criterion is based are no longer appropriate, since several resonances with comparable amplitude collapse into the origin. The resulting picture of the motion is most easily illustrated by introducing an oscillating coordinate frame, via the generating function

$$F(J,\theta,t) = (J - \lambda\Omega\cos\Omega t)(\theta + \lambda\sin\Omega t) + \frac{\lambda^2\Omega^2}{4}\left[t + \frac{\sin(2\Omega t)}{2\Omega}\right] , \tag{5.7}$$

where the new Hamiltonian $K = H + \partial F/\partial t$

$$K = \frac{J^2}{2} - \varepsilon\cos\varphi - \varphi\lambda\Omega^2\sin\mu \tag{5.8}$$

now depends on the "slow time" variable $\mu = \Omega t$. By drawing the level set $K = \text{const}$, for any fixed μ, we note that unless

$$\frac{\lambda\Omega^2}{\omega_0^2} > 1, \quad \omega_0^2 = \varepsilon \tag{5.9}$$

there will always be a resonant domain centered about the stable equilibrium $J_0 = 0$, $\varphi_0 = -\arcsin[(\lambda\Omega^2/\varepsilon)\cos\mu]$ and bounded by a separatrix [Fig. 11(a)]. In other words, when Ω is sufficiently small, a phase point located near (J_0, φ_0) sees a stable local environment, although slowly changing with time. Under these assumptions, the local action function $J' = J'(\mu)$ can be shown to be a perpetual adiabatic invariant (Arnol'd, 1963), in the sense that its departure from the unperturbed value $J'(0)$ is forever bounded. This result is a direct consequence of the existence of KAM tori surrounding the elliptic fixed point. In the original coordinate system there is a moving resonance which forces the orbit located near its center to follow its slow excursion in the phase space, and a periodically oscillating stable region appears within the stochastic domain [Fig. 11(b)]. As $\Omega \to 0$, its relative phase area increases, while the frequency and amplitude of the oscillations decrease, yielding a smooth transition to the integrable limit $\Omega \to 0$. This phenomenon is known as "trapping" (Tennyson, 1979; Chirikov and Shepelyansky,

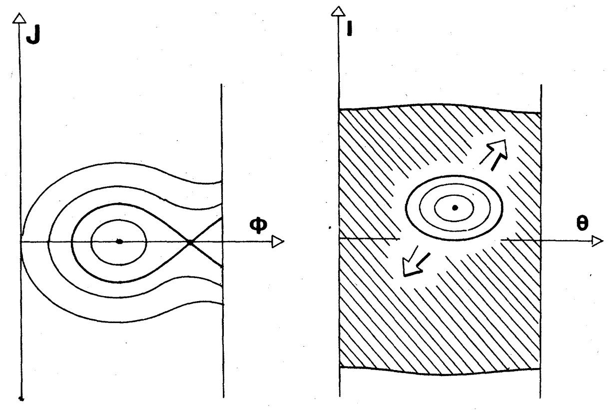

FIG. 11. Trapping. (a) Oscillating coordinate system: instantaneous level lines of Hamiltonian (5.8). (b) Fixed coordinate system.

1980). The conditions (5.6) and (5.9) must be simultaneously satisfied for a uniform modulational layer to exist.

VI. WEAK INSTABILITIES

In this section we shall see how a new class of dynamical instabilities naturally occurs in near-integrable systems whenever the dimension of the phase space exceeds four. Let us consider the periodically time-dependent Hamiltonian

$$\begin{aligned} H &= H_1 + H_2 + \mu H_c \ , \\ H_1 &= \frac{I_1^2}{2} - \varepsilon \cos(\theta_1 + \lambda \sin\Omega t), \\ H_2 &= \frac{I_2^2}{2}, \\ H_c &= \cos(\theta_1 - \theta_2) \ . \end{aligned} \tag{6.1}$$

Here the extended phase space M is five dimensional, being the direct product of two cylinders and a circle $M = C_1 \times C_2 \times S = \{(I_1, \theta_1, I_2, \theta_2, \zeta = \Omega t)\}$. We will first examine an intuitive explanation of the sources of a weak instability; the necessary details will be supplied later. The system under consideration has two essential ingredients:

(1) The Hamiltonian H_1 is nonintegrable, but at the same time globally stable (cf. Sec. V).

(2) A coupling term (H_c) allows energy exchange between the two systems H_1 and H_2.

We shall also require that $\varepsilon \gg \mu$. In so doing, we can neglect the effects of the coupling term on the H_1 motions and clarify the role played by each of the three terms appearing in (6.1).

When $\mu = 0$ our system is globally stable, being the direct product of two globally stable subsystems. In particular, we have three tori embedded in the five-dimensional extended phase space. For $\mu \neq 0$ and sufficiently small, a large fraction of these three tori still exist, but the phase space is no longer a direct product (the energy can flow from one subspace to the other), so that three tori do not disconnect it. This topological change will mostly affect the orbits in the subsystem H_2 (since $\varepsilon \gg \mu$), which we shall now regard as explicitly depending on the time by writing

$$H_2(I_2, \theta_2, t) = \frac{I_2^2}{2} + \mu \cos[f(t) - \theta_2] \ . \tag{6.2}$$

Note that the driving perturbation $f(t) = \theta_1(t)$ differs from the ones previously considered by its lack of periodicity. A trajectory confined on a three torus (f quasiperiodic) can induce only bounded oscillations on C_2, because boundness in the full phase space naturally implies boundness in reduced spaces. The converse statement is generally untrue. Indeed, from (6.2) we see that a bounded stochastic orbit originated in a layer on C_1 results in a *random* external perturbation acting on C_2, whose effect is to destroy all unperturbed encircling tori $I_2 = \text{const}$. Consequently, the motion on C_2 will not only be irregular, but it may become unbounded. In other words, the effect of the coupling term is to transfer stochasticity from one reduced space to another, a mechanism quite pictorially denoted as "stochastic pumping" (Tennyson *et al.*, 1979). The detailed features of this phenomenon depend on the specific nature of the random driving perturbation, that is, on the geometrical structure of the layer in which the random phase originates. Accordingly, the weak instabilities can be regrouped into two classes, which correspond to the case of thin and thick layers, respectively. Thin-layer diffusion is known as the Arnol'd diffusion (Arnol'd, 1964; Chirikov, 1979), which always exists in many-dimensional systems, since it is related to the unavoidable presence of small stochastic domains near separatrices (see Sec. IV). On the other hand, thick-layer diffusion is activated only within those parameter ranges which provide the overlap of some (generally primary) resonances. However, the latter class of phenomena is more relevant to practical applications, because the flow of stochasticity between reduced spaces is enhanced.

To proceed more rigorously, we embed the system (6.1) in an autonomous one, with Hamiltonian

$$\begin{aligned} K &= \frac{I_1^2}{2} + \frac{I_2^2}{2} + \Omega I_\zeta \\ &\quad + \varepsilon \sum_n J_n(\lambda) \cos[\mathbf{m}(n) \cdot \boldsymbol{\theta}] + \mu \cos(\mathbf{m}_c \cdot \boldsymbol{\theta}) \ , \\ I_\zeta &= -H/\Omega, \quad \theta_\zeta = \Omega t \ , \end{aligned} \tag{6.3}$$

where we have introduced a more compact notation:

$$\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_\zeta),$$
$$\mathbf{m}(n)=(1,0,n)\,, \tag{6.4}$$
$$\mathbf{m}_c=(1,-1,0)\,.$$

Our analysis will follow that developed in Tennyson *et al.* (1979) for the conservative case; here we shall emphasize the greater symmetry of time-dependent systems. Many-degrees-of-freedom systems are usually studied in some suitably reduced spaces, typically the action space, $A=R^N=\{(I_1,\ldots,I_N)\}$ or the frequency space $F=R^N=\{(\omega_1,\ldots,\omega_N)\}$. We shall carry out the analysis both in the frequency and in the action representations. However, in the particular example we have considered, these turn out to be quite similar, since from (6.5) and (6.7) (see below) we have $\omega_i=I_i$, $i=1,2$ (quadratic nonlinearity). In the case of time-dependent systems, motion in the reduced spaces A and F is restricted to an invariant surface. In the action space the latter is just the unperturbed energy surface

$$K_0=\frac{I_1^2}{2}+\frac{I_2^2}{2}+\Omega I_\zeta=\text{const}\,. \tag{6.5}$$

The invariance of K_0 is a direct consequence of the way K and I_ζ were constructed. Indeed, from (6.1) and (6.3) it is easy to see that the Poisson bracket of K_0 and K vanishes identically:

$$(K_0,K)\equiv 0\,. \tag{6.6}$$

Note that the unperturbed energy surface K_0 is unbounded, being a paraboloid of revolution about the I_ζ axis [Fig. 12(a)]. The transformation from actions to frequencies

$$\omega_i=\Pi_i(\mathbf{I})=\frac{\partial K_0}{\partial I_i} \tag{6.7}$$

is a Legendre transformation (Arnol'd, 1973, p. 61) which carries the embedding Hamiltonian into the corresponding Lagrangian. Since K_0 is linear in I_ζ and straight lines are taken into points by the Legendre transformation, the energy surface becomes the plane $\omega_3=\Omega$ in F, which we denote by L_0 [Fig. 12(b)]. Besides this simplification, the frequency space is also best suited to the analysis of the resonant structure, which is always linearized on it. In fact, resonant conditions are expressed by equations that are linear in the frequencies but in general not in the actions [see (4.5) above and (6.8)—(6.11) below].

Now let $\mu=0$. The primary resonances in (6.3) are defined by the equations [cf. (3.11) and (4.5)]

$$\mathbf{m}(n)\cdot\boldsymbol{\omega}=0\,, \tag{6.8}$$

which in turn define an infinite sequence of planes in F (resonant surfaces) passing through the origin and orthogonal to the vectors $\mathbf{m}(n)$ (resonance vectors). The location of a resonant domain in the frequency space coincides with the intersection of the corresponding resonance surface with L_0, which is called a resonance line (Fig. 13). Here the multiplet is represented by a set of straight lines parallel to the ω_2 axis: $\omega_1=n\Omega$. If we include also secondary resonances (see Sec. IV), we obtain a dense set of nonintersecting resonance lines: $m\omega_1=n\Omega$.

The component $\mathbf{V}_I$ of the Hamiltonian vector field

$$\dot{\mathbf{I}}=\mathbf{V}_I=\sum_n \mathbf{m}(n)J_n(\lambda)\sin[\mathbf{m}(n)\cdot\boldsymbol{\theta}] \tag{6.9}$$

is a field on the action space, which induces motions of action points. Since it depends on the angle variables, here it should be regarded as explicitly time dependent.

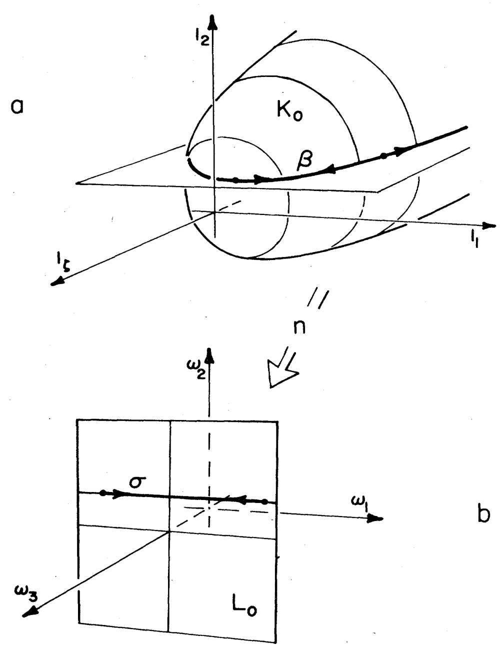

FIG. 12. (a) The action space. (b) The frequency space.

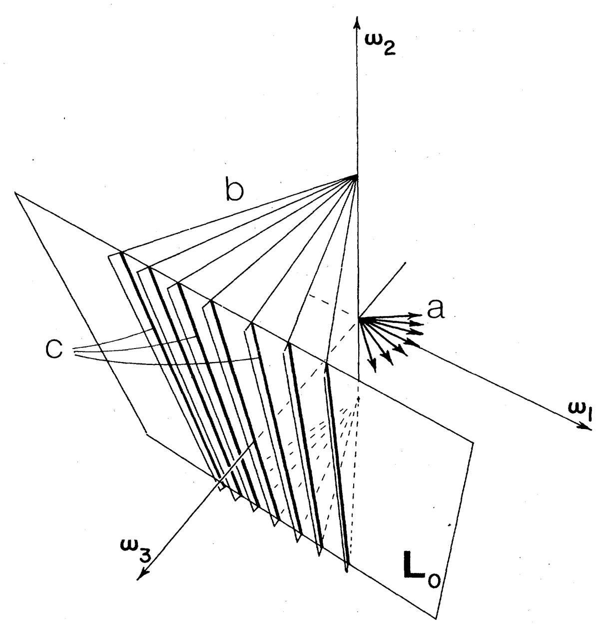

FIG. 13. A multiplet in the frequency space. (a) Resonance vectors. (b) Resonance surfaces. (c) Resonance lines.

From (6.9) we see that an infinite number of terms contribute to $\mathbf{V}_I$, each of which is directed along a resonance vector $\mathbf{m}(n)$. Since all the resonance vectors in the multiplet are coplanar and perpendicular to the I_2 axis, $\mathbf{V}_I$ must also lie in the same plane $I_2=\text{const}$, which therefore is invariant. For this reason motion in the action space can take place only on the intersection β of this plane with K_0 [Fig. 12(a)]. Note that the existence of an invariant plane merely reflects the existence of the global invariant I_2 when $\mu=0$. The action field $\mathbf{V}_I$ is carried into the frequency field $\mathbf{V}_\omega$ in F by the linear transformation [tangent map (Arnol'd, 1973, p. 247]

$$\dot{\omega}_i=(\mathbf{V}_\omega)_i=d\Pi_i(\mathbf{V}_I) \ , \tag{6.10}$$

where $d\Pi_i$ is the differential of the function Π_i [see Eq. (6.7)]. Again, motion in the frequency space can take place only on the set $\sigma=\Pi(\beta)$ in L_0, which in our case is a line $\omega_2=\text{const}$ [Fig. 12(b)]. Since our system is nonintegrable, the evolution of the frequency points on σ will be irregular on and near the resonant lines, due to the presence of stochastic layers. If a multiplet is generated, phase points will perform larger excursions on L_0, even though they never become unbounded, because the uncoupled ($\mu=0$) system has been assumed globally stable. In addition, σ is transverse to all the resonance lines, so that there is no displacement at all along them.

In summary, if we perturb an integrable system with a set of coplanar resonant terms, we still retain an integral of the motion (the plane itself) and action (frequency) vectors cannot explore the whole unperturbed energy surface. This is always the case if the perturbation consists of two terms only.

Now let $\mu\neq0$. The additional primary resonance

$$\mathbf{m}_c\cdot\boldsymbol{\omega}=0 \tag{6.11}$$

defines the resonance surface $\omega_1=\omega_2$, which is transverse to all the resonance surfaces of the multiplet (Fig. 14).

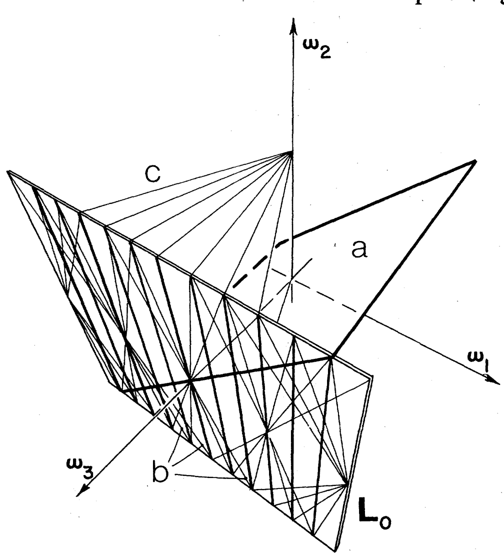

FIG. 14. Noncoplanar perturbation: the Arnol'd web. (a) Primary coupling resonance. (b) Secondary coupling resonances. (c) Multiplet resonances.

The mutual interaction of the primary resonances (6.8) and (6.11) generates an infinite set of secondary ones. The corresponding resonance planes are of the type $\omega_2+m_1\omega_1+m_2\Omega=0$, which define an intricate set of intersecting lines on L_0. This set of secondary resonances will be analyzed in the next section.

We see that a perturbation consisting of a set of noncoplanar resonances causes the appearance on the energy surface of a dense set of intersecting resonance lines (the Arnol'd web). Since there is a layer associated with every resonance line, we also have a dense and connected set of layers. The field $\mathbf{V}_\omega$ now has a nonvanishing component along the ω_2 axis, which allows motions in any direction on L_0. Thus a phase point located inside the network of intersecting layers can come arbitrarily close to any point of L_0 by proceeding along resonance lines, and in particular it can disperse to infinity.

Not all resonances contribute significantly to this process, and in practical applications it is sufficient to take into account only a few selected primary and secondary resonances. The point is that the diffusive motion along resonance lines is significantly rapid only provided that the associated layer is large enough, and this is usually the case for primary resonances only. These resonances are called guiding resonances. The possibility of converting the irregular bounded motion *across* a guiding resonance line [the H_1 motion in our example (6.1)] into motion *along* it (the H_2 motion) arises whenever a guiding resonance intersects some other resonance which we call a driving resonance. In other words, driving resonances are responsible for the mechanism of stochastic pumping discussed above. As we shall see in the next section, the long-range diffusion along guiding resonances is supported mainly by secondary driving resonances, whose amplitude is typically very small. This fact justifies the appellative "weak" we have used to characterize these instabilities. I should like to remark that the distinction between guiding and driving resonances is somewhat arbitrary, and in particular that their roles can be interchanged by rearranging the initial conditions.

VII. THE MODULATION DIFFUSION

In this section we shall analyze the weak instability induced by modulational layers which is referred to as modulational diffusion. It was first considered by Chirikov (1980), and subsequently studied by several authors (Chirikov, Izrailev *et al.*, 1981; Chirikov, Ford *et al.*, 1981; Lichtenberg and Lieberman, 1983, p. 335; Chirikov *et al.*, 1984). As a model for modulational diffusion we retain the Hamiltonian (6.1) [or (6.3)] and we assume that conditions (5.6) and (5.9) are satisfied, so that a modulational layer is constructed in phase space. We shall develop the analysis on the plane L_0 in Ω, where the field

$$\dot{\omega}_1=V_{\omega_1}=\varepsilon\sum_n J_n(\lambda)\sin(\theta_1+n\Omega t)+\mu\sin(\theta_1-\theta_2) \ , \tag{7.1a}$$

$$\dot{\omega}_2=V_{\omega_2}=-\mu\sin(\theta_1-\theta_2) \ , \tag{7.1b}$$

is defined. The guiding resonances for modulational diffusion are the multiplet resonances, and motion along their resonance lines is driven by the component V_{ω_2} of the field, which will be called longitudinal field. The field V_{ω_1} (transverse field) generates only bounded stochastic motion across resonance lines. Under the assumption that $\varepsilon \gg \mu$ we can neglect the second term on the rhs of (7.1a) and regard the transverse field as independent on the two variables. The motion across the modulational layer then becomes identical to that of the uncoupled case ($\mu=0$), and in particular invariant under translation along the ω_2 axis. Clearly this is not the case for the longitudinal component of the field (7.1b), whose dependence on ω_2 is a distinctive feature of the phenomenon under consideration. The main theoretical problem is the evaluation of the rate $D_{\omega_2}=D_{\omega_2}(\omega_2)$, at which the variable ω_2 (or any function of it) diffuses due to longitudinal motion on the modulational layer. We shall describe a procedure yielding an approximate expression for D_{ω_2} similar to that developed in Lichtenberg and Lieberman (1983, p. 335). For a more exhaustive treatment of the problem the reader is referred to Chirikov *et al.* (1984).

The diffusion rate is defined as follows:

$$D_{\omega_2} \equiv \lim_{T\to\infty} \frac{\langle (\Delta\omega_2)_T^2 \rangle}{2T} , \tag{7.2a}$$

$$(\Delta\omega_2)_T = \int_{-T}^{T} dt\, \dot{\omega}_2(t) , \tag{7.2b}$$

where $\langle\ \rangle$ denotes phase averaging. The time evolution of ω_2 is determined by the component V_{ω_2} of the field

$$\dot{\omega}_2 = -\mu \sin[q\theta_1(t) - \theta_2(t)] , \tag{7.3}$$

where we have generalized the expression (7.1b) by introducing the additional parameter q, which is relevant to the case of higher-order driving resonances. The evaluation of the diffusion coefficient (7.2) will be carried out in two steps. We will first derive an explicit expression for the rhs of (7.3) as a function of time, and then perform the integral (7.2b) as well as the average (7.2a).

We begin by noting that since both $\theta_1(t)$ and $\theta_2(t)$ are random functions of time, they cannot be computed exactly, and we must resort to some approximation scheme. In the unperturbed system both θ_1 and θ_2 take on the simple form

$$\theta_1(t) = \theta_1(0) + \omega_1 t , \tag{7.4a}$$

$$\theta_2(t) = \theta_2(0) + \omega_2 t . \tag{7.4b}$$

Equation (7.4b) is already a satisfactory approximation, since we are interested only in the local behavior (in ω_2). On the other hand, the function θ_1 must incorporate at least the effect of the modulation to first order in ε, since $\varepsilon \gg \mu$. This amounts to assuming that the properties of the longitudinal field are primarily determined by the first-order interaction between the coupling resonance (7.1b) and the multiplet resonances (7.1a), that is, by secondary resonances. From Hamilton's equations we have $\dot{\theta}_i = \partial H/\partial I_i = \omega_i$, and thus from (7.1a) $\dot{\omega}_1 = V_{\omega_1} = \ddot{\theta}_1$. Inserting (7.4a) into the rhs of (7.1a), with $\mu=0$, and integrating twice, we obtain

$$\theta_1(t) = \varepsilon \sum_{n=-\infty}^{\infty} \frac{J_n(\lambda)}{(\omega_1+n\Omega)^2} \sin[(\omega_1+n\Omega)t + \theta_1(0)] + \omega_1 t , \tag{7.5}$$

where we have made use of the expansion (5.2). The formula (7.5) is still difficult to manipulate, even if the infinite sum is truncated to include only the resonances within the layer domain, i.e., $|n| \le \Delta\omega/\Omega \equiv N$, where the layer halfwidth $\Delta\omega$ is approximately equal to $\lambda\Omega$ (see Sec. V). Our basic approximation is to retain just one term (to be specified later), while introducing a fitting parameter R to describe the "effective amplitude" of the resonances considered. Substituting (7.5) in (7.3) and expanding again, we have

$$V_{\omega_2} = \sum_{j=-\infty}^{\infty} A_j(\omega_1) \cos\{[(j+q)\omega_1 + jn\Omega - \omega_2]t + \theta_1(0)\} , \tag{7.6a}$$

$$A_j(\omega_1) = J_j \left[\frac{q\varepsilon R J_n(\lambda)}{(\omega_1+n\Omega)^2} \right] . \tag{7.6b}$$

Thus the one term retained still generates an infinite family of secondary resonances of order $j+q$. The equation of the corresponding resonance lines on L_0 reads

$$\omega_2 = (j+q)\omega_1 + jn\Omega . \tag{7.7}$$

We note that, even though we have retained only one term in the first-order expansion (7.5), by varying n and j in (7.7) we can still construct all secondary resonance lines. Their structure is found to be rather complicated (see Fig. 15).

The diffusion coefficient can now be derived by substituting (7.6) into (7.2), integrating over time, and performing suitable phase averages. To this end we observe that the expression (7.6a) is local in ω_1. The time evolution causes ω_1 to explore the whole interval $|\omega_1| \le \Delta\omega$ uniformly, as a consequence of stochastic motion across the modulational layer. For this reason we will average (7.6a) over that domain. The diffusion rate (7.2a) becomes

$$D_{\omega_2} = \lim_{T\to\infty} \frac{1}{2T} \frac{\mu^2}{2\Delta\omega} \left\langle \int_{-\Delta\omega}^{\Delta\omega} d\omega_1 \int_{-T}^{T} dt' V_{\omega_2}(\omega_2,t') \int_{-T}^{T} dt'' V_{\omega_2}(\omega_2,t'') \right\rangle_{\theta_1(0)} . \tag{7.8}$$

The integration in t' introduces a δ function

$$\int dt' V(t') = \sum_{j=-\infty}^{\infty} A_j(\omega_1) \cos\theta_1(0) \frac{2\pi}{j+q} \delta\left[\omega_1 + \frac{jn\Omega - \omega_2}{j+q}\right] , \tag{7.9}$$

which, after integration in ω_1, yields

$$D_{\omega_2}=\lim_{T\to\infty}\frac{\mu^2}{2\Delta\omega}\sum_{j=-\infty}^{\infty}\frac{2\pi}{j+q}A_j^2\left[\frac{\omega_2-jn\Omega}{j+q}\right]\langle\cos^2\theta_1(0)\rangle_{\theta_1(0)}\frac{1}{2T}\int_{-T}^{T}dt'' , \tag{7.10}$$

and the amplitudes (7.6b) become

$$A_j\left[\frac{\omega_2-jn\Omega}{j+q}\right]=J_j\left[\frac{q\varepsilon RJ_n(\lambda)(j+q)^2}{(\omega_2+qn\Omega)^2}\right] . \tag{7.11}$$

From Eq. (7.9) we obtain the condition

$$j=\frac{\omega_2-q\omega_1}{\omega_1+n\Omega} . \tag{7.12}$$

Equation (7.12) represents a family of resonance lines $j=j(n)$, of the type (7.7), passing through the point (ω_1,ω_2). Among these, the one having maximal amplitude gives the dominant contribution to the sum (7.10). We shall now exploit this fact in order to obtain a more compact expression for the diffusion rate. For fixed ω_2 the dominant term in (7.10) is the one for which $j=j(n,\omega_1)$ is the smallest, since the amplitudes (7.11) of the resonances which drive the motion along the layer are Bessel functions of small arguments $(\varepsilon \ll 1)$ with limit behavior

$$J_j(x)\approx\frac{1}{j!}\left[\frac{x}{2}\right]^j, \quad x\ll 1 . \tag{7.13}$$

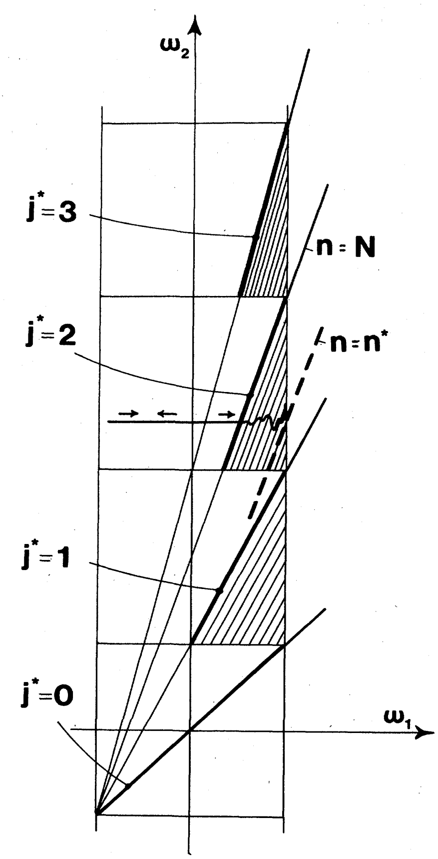

FIG. 15. Resonant structure of a thick layer.

From (7.12) we obtain $(n\Omega=\omega_1=\Delta\omega)$

$$j_{\min}\equiv j^*=\left[\frac{1}{2}\left[\frac{\omega_2}{\Delta\omega}+q\right]\right]_I=[\tfrac{1}{2}(\alpha+q)]_I , \tag{7.14}$$

where $[\ ]_I$ denotes the integer part of its argument. The new normalized variable $\alpha=\omega_2/\Delta\omega$ has also been introduced, measuring the displacement along the layer in units of the layer halfwidth. Furthermore, for fixed α and $j=j^*$, ω_1 and n in (7.12) are functionally dependent, and we can still use the arbitrariness of n to maximize (7.11). Solving (7.12) for n, with $\omega_1=\Delta\omega$ we obtain the optimal (smallest) $n^*=n^*(\alpha)$:

$$n^*(\alpha)=\frac{N}{j^*}[\alpha-(j^*+q)]_I . \tag{7.15}$$

Thus for fixed α the dominant contribution to the diffusion coefficient is given by the resonance (see Fig. 16)

$$\omega_2=[j^*(\alpha)+q]\omega_1+j^*(\alpha)n^*(\alpha)\Omega . \tag{7.16}$$

With this choice for j and n the normalized diffusion rate $D_R=D/(\mu^2/\Delta\omega)$ becomes

$$D_R=\frac{\pi}{2(j^*+q)}J_{j^*}^2\left[\frac{q\varepsilon RJ_{n^*}(\lambda)(j^*)^2}{(\Delta\omega)^2(\alpha-q)^2}\right] , \tag{7.17}$$

where we have averaged over the initial phase $\theta_1(0)$. The latter is a random variable, for the phase average is equivalent to the time average of the random function $\theta_1(t)$ (recall that the motion in the layer is ergodic). To obtain a more explicit dependence of D_R on the system parameters we use (7.13) and the approximate expression $J_n(\lambda)\approx 1/\sqrt{\pi\lambda}$. Equation (7.17) becomes

$$D_R(\alpha)=D_0(j^*)(\alpha-q)^{-4j^*} , \tag{7.18a}$$

$$D_0(j^*)=\frac{\pi}{2(j^*+q)(j^*!)^2}\left[\frac{q\varepsilon R(j^*)^2}{\pi\lambda(\Delta\omega)^2}\right]^{2j^*} . \tag{7.18b}$$

VIII. DISCUSSION

Even though the derivation of the diffusion coefficient (7.18) involved quite drastic simplifications, it nonetheless illustrates many general features of motions along thick layers, which are not restricted to the particular model we have chosen.

In Fig. 16 we see a comparison of the estimate (7.18) with the results of a numerical computation of the diffusion coefficient [for details on the numerical techniques see Chirikov, Ford *et al.* (1981)], with parameters $q=1$, $\lambda=10$, $\Omega=10^2$, $\varepsilon=5\times10^{-4}$. In order to simplify the calculations we have decoupled the system H_1 from H_2—i.e., we have solved numerically the differential equation

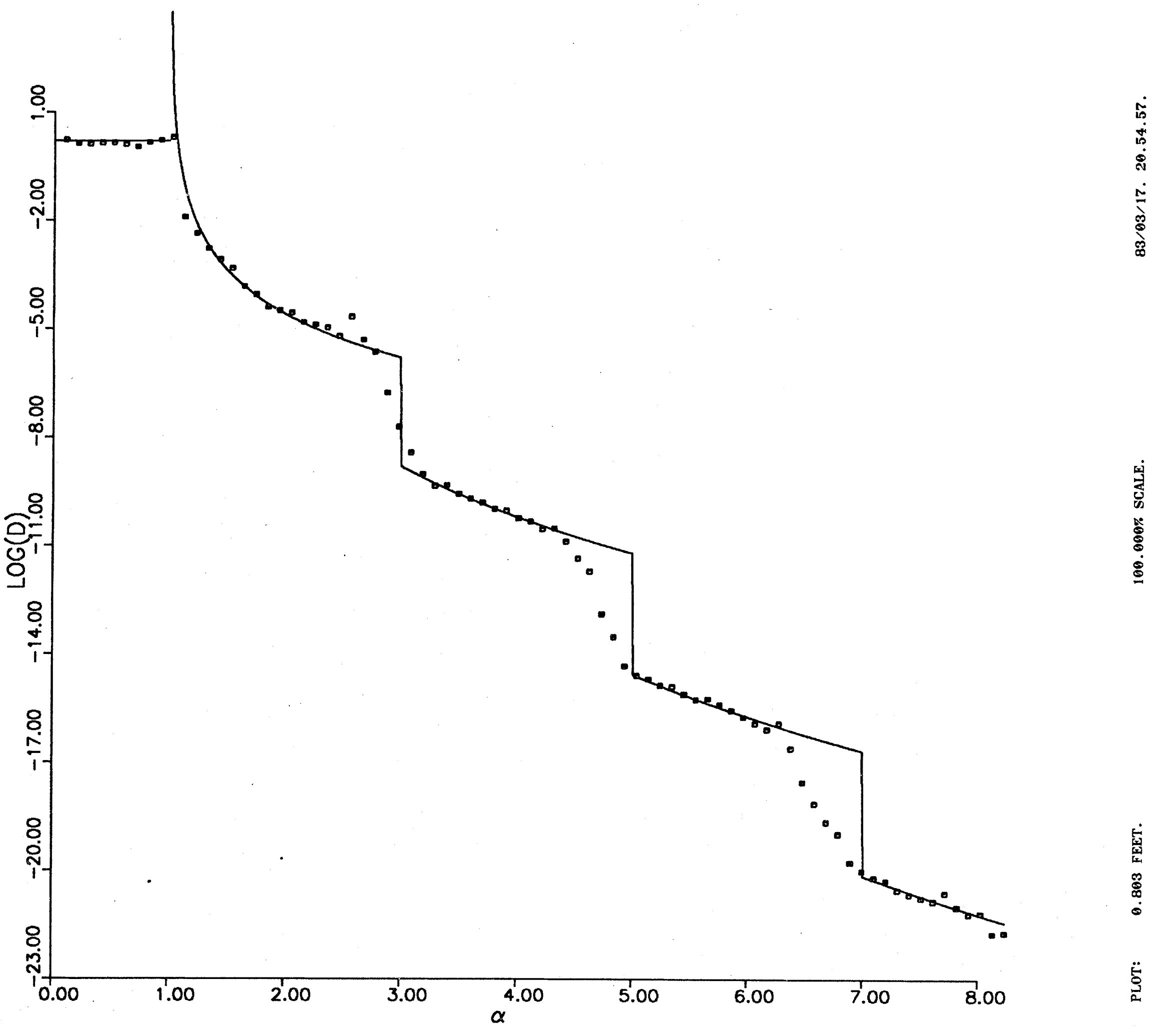

FIG. 16. Diffusion coefficient. The solid curve denotes expression (7.18).

(7.3) with $\mu=1$ and θ_1 evolving according to Hamiltonian (5.1). The diffusion rate depends discontinuously on α at the critical values $\alpha_n=2n+1$, $n=0,1,2,\ldots$, where the integer function $j^*=j^*(\alpha)$ is itself discontinuous [see (7.14)]. The resulting picture is a sequence of descending steps called "plateaus," the nth plateau corresponding to the interval $\alpha=[2n-1,2n+1]$ where j^* is constant. The fitting parameter R (see Sec. VII) has been adjusted to assume different values on each plateau: $R(j^*)=1$, 2.5,4.6,6.2,7.7 giving an overall exponential decay rate. The expression (7.18) is invalid at and near $\alpha=1$, since there the argument of the Bessel function becomes infinite and the approximation (7.13) is no longer appropriate.

The qualitative aspects of this phenomenon can be illustrated by examining the resonant structure on L_0, as depicted in Fig. 15. Here the modulational layer is represented as an unbounded strip. We recall that the one motion is bounded, while the two motion (modulational diffusion) can proceed arbitrarily far in the vertical direction. The multiplet resonance lines (guiding resonances—see Sec. VI) have been omitted: they are parallel to the ω_2 axis (see Figs. 13 and 14). Stochastic motion across the modulational layer causes phase points to cross either primary ($j^*=0$) or secondary ($j^*>0$) resonance lines repeatedly. At exact resonance the point suffers a quite sudden displacement along the ω_2 axis, which in turn gives the main contribution to the integral (7.2b). The magnitude and direction of the displacement depend on both the amplitude of the resonance crossed and the phase of the oscillation at the moment of crossing. Away from resonance the diffusion along the layer is negligible, and the trajectory moves approximately on an invariant surface $\omega_2=$const. On the primary plateau the primary coupling resonance (of order 1) is the dominant one. Since its amplitude is constant, the primary plateau is leveled. For $\alpha>1$ the primary resonance line no longer intersects the layer domain and the motion along it is driven by secondary resonances. On the jth plateau the dominant terms are of order j^*+q—that is,

$$\omega_2=(j^*+q)\omega_1+n\Omega\ , \tag{8.1a}$$

$$n^*\leq n\leq N\ , \tag{8.1b}$$

so that many resonances actually contribute to the diffusion process. The main features of $D_R(\alpha)$ become clear from the following considerations.

(1) The upper bound of the interval (8.1b) is determined by the fact that for $n > N$ the resonances (8.1a) have negligible amplitude, even though they still intersect the layer. Therefore, on each plateau the resonance line corresponding to $n = N$ defines the boundary of a triangular region (the interaction region) where the diffusion along the layer originated.

(2) The lower bound of (8.1b) depends on α, since for $n < n^*(\alpha)$ no interaction can take place. For this reason the number of resonances crossed is maximal at the beginning of each plateau ($2N$) and minimal at the end (1). Therefore, at the end of the plateaus the actual diffusion coefficient is smaller than that predicted by (7.18), which has been derived by assuming a constant "effective" number of resonances over the whole plateau.

(3) The amplitude of the secondary resonances decreases with α, so that the secondary plateaus are not leveled. Moreover, for fixed α such amplitude reaches a maximum for $n = n^*$, which corresponds to the resonance selected in Sec. VII [see Eq. (7.16)]. This implies that the diffusive motion along the layer is enhanced whenever the phase point is located near the right boundary of the interaction region, that is, at one edge of the layer.

The geometrical property of the phase space which most strongly characterizes this class of phenomena is the width of the layer which supports the diffusion, since it determines the size of the plateaus and ultimately the domain of phase space over which the diffusion is rapid enough to be of interest. For this reason if we replace the multiplet with an arbitrary array of resonances, we conjecture that no major change in the qualitative features of the diffusion rate will occur, provided that the layer is sufficiently homogeneous. A precise evaluation of the layer width thus becomes necessary. In our example, using the expression (5.5) of the local overlap parameter $S(n)$ one obtains $\Delta\omega = 1.4\lambda\Omega$, in good agreement with numerical results. This value exceeds the cruder estimate $\Delta\omega = \lambda\Omega$ (cf. Sec. V) mainly because λ is not large enough.

The diffusion coefficient depends sensitively on the overlap parameter S whenever the latter is decreased to reach the critical values $S = 1$, which corresponds to the appearance of the first encircling torus within the layer domain. The layer is then disconnected into two sublayers and the diffusion rate changes discontinuously. This process repeats itself until no adjacent primary resonances overlap and only thin layers are left. Therefore, the transition from a thick layer diffusion to the Arnol'd diffusion takes place through a sequence of intermediate steps.

The method illustrated in Sec. VII is based on the possibility of decomposing the expansion of the Hamiltonian function near resonance into two weakly coupled terms describing transversal and longitudinal motions on the layer, respectively. In practical application this decomposition is quite often possible. In Fig. 17 a typical disposition of driving and coupling resonances in a model for the beam-beam interaction is illustrated. Due to parametric modulation many resonance lines develop multiplets and thick layers appear. The resulting structure is a network of guiding resonances with associated primary and secondary plateaus. Long-range diffusion along layers will be observable whenever the primary plateaus are sufficiently close to each other. If we assume that the local environment near primary plateaus can be described by a Hamiltonian of the type (7.1) with analogous parameter values and a coupling coefficient, say, ten times smaller than the perturbation strength, the diffusion over a primary plateau will take approximately 5×10^6 interactions between beams, that is, a few seconds. Diffusion over secondary plateaus will instead take an interval of time several orders of magnitude larger in accordance with (7.18). However, we remark that in realistic models the driving perturbation is itself modulated. In such a circumstance it can be shown that the diffusion rate over secondary plateaus is considerably enhanced (Chirikov *et al.*, 1984).

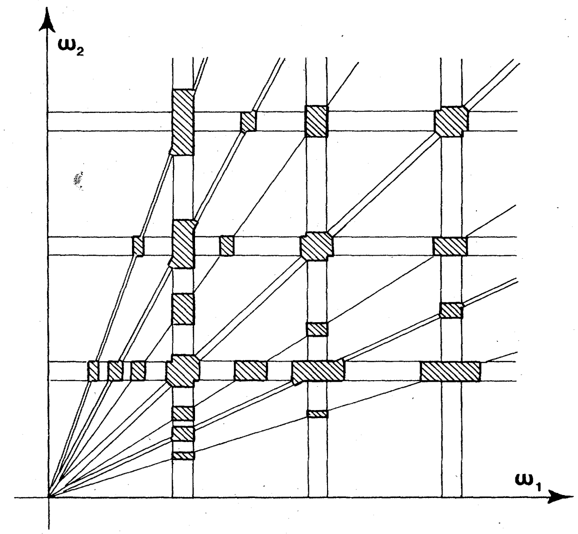

FIG. 17. Typical disposition of resonance lines and associated layers. The shaded areas correspond to primary plateaus.

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