The two-dimensionality of slow motion in a rotating fluid

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The Proudman–Taylor theorem states that, in a fluid rotating steadily about a vertical axis, if a flow pattern is induced —for example by the slow steady motion of a solid body such as a squat cylinder at one level— the same flow pattern is induced at all levels by Coriolis forces. The steady equilibrium situation is simply understood in terms of a balance of the horizontal Coriolis forces at all levels. The mechanism providing the accelerations whereby the equilibrium is established is shown in various examples to involve vertical transport of momentum by flow around loops linking the various levels. The limited role of this mechanism in atmospheric circulation, in the dynamo action in the earth's fluid core responsible for terrestrial magnetism, and in the liquid-drop model of nuclear rotation is briefly discussed.

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I. INTRODUCTION

The slow motion of fluids relative to a rotating coordinate system is encountered both in the meteorology of the earth's atmosphere and in the circulation within the earth's fluid core postulated to explain the earth's magnetism. In both cases Coriolis forces due to the earth's rotation are important and both situations are so complicated that adequate treatment by application of the equations of fluid dynamics has not been achieved. Certain approximate concepts associated with the equations of motion, such as the idea of geostrophic flow, have been important aids to intuition in reaching understanding of some aspects of the motion. Here we seek an intuitive understanding of one very fascinating feature of the slow motion of a fluid in a gravitational field under the influence of strong Coriolis forces.

It was first pointed out by Proudman (1916) and soon afterward demonstrated experimentally by G. I. Taylor (1923) that slow motions in a horizontal slab of fluid rotating about a vertical axis, if they come to a steady equilibrium at all, must be essentially two dimensional. That is, there is a vertical stiffness such that motions induced in one layer of the rotating fluid are reproduced almost exactly in layers above and below it. Taylor's dramatic demonstration consisted of moving a rather short cylinder, standing on end, slowly across the bottom of a rotating slab of water and showing by coloring matter appropriately introduced into the water that the stream lines flow discreetly around the column of water directly above the cylinder—apparently almost as if by magic. (Fig. 1)

A textbook treatment of the equilibrium motion, once established, starts with the equation of motion in the rotating coordinate system

$$\rho\dot{\mathbf{v}} = -\nabla p + \rho\mathbf{g} + 2\rho\mathbf{v}\times\mathbf{\Omega} + \eta\nabla^2\mathbf{v}, \tag{1}$$

where

$$\dot{\mathbf{v}} = (\partial\mathbf{v}/\partial t) + (\mathbf{v}\cdot\nabla)\mathbf{v};$$

the second term being associated with a volume element moving to a region of a different velocity as in a converging stream, and

$$\mathbf{g} = \mathbf{g}_0 + \tfrac{1}{2}[\mathbf{\Omega}\times\mathbf{r}]\cdot[\mathbf{\Omega}\times\mathbf{r}],$$

the "effective gravity" $\mathbf{g}$ being the sum of the actual gravitational term $\mathbf{g}_0$ and the smaller centrifugal force term. For motion that is steady and slow, so that $\dot{\mathbf{v}}\approx 0$, and inviscid, $\eta = 0$, the remaining equation is simply

$$\nabla p = \rho\mathbf{g} + 2\rho\mathbf{v}\times\mathbf{\Omega} \tag{2}$$

and its curl is

$$\nabla\times[2\rho\mathbf{v}\times\mathbf{\Omega}] = 2\rho[(\mathbf{\Omega}\cdot\nabla)\mathbf{v} - (\nabla\cdot\mathbf{v})\mathbf{\Omega}] = 0$$

or, for a homogeneous fluid with $\nabla\cdot\mathbf{v} = 0$,

$$(\mathbf{\Omega}\cdot\nabla)\mathbf{v} = 0. \tag{3}$$

This is an elegantly simple statement of the two dimensionality of the motion, the Proudman–Taylor theorem.

This analysis provides an adequate basis for understanding the equilibrium motion of the Taylor column marching across the fluid above the solid cylinder in Fig. 1. Once the motion is established, this assures that it can continue. But the even more interesting question then arises, how does the motion become established? When a force is applied to the solid cylinder to start its motion, what is the mechanism by which momentum is transferred upward to set the upper parts of the column in motion? This question is addressed here.

It is not commonly addressed in the books on hydrodynamics. Many closely related and very interesting phenomena are treated for example in Greenspan's book on rotating fluids (1968), and the approach to equilibrium of a very different phenomenon that also bears the name of a "Taylor column" is treated there under an index listing "transient development of the Taylor–Proudman column". This phenomenon is a twirling vertical motion of a column above and below a horizontal disc induced by vertical motion of the disc along the axis of rotation of an infinite

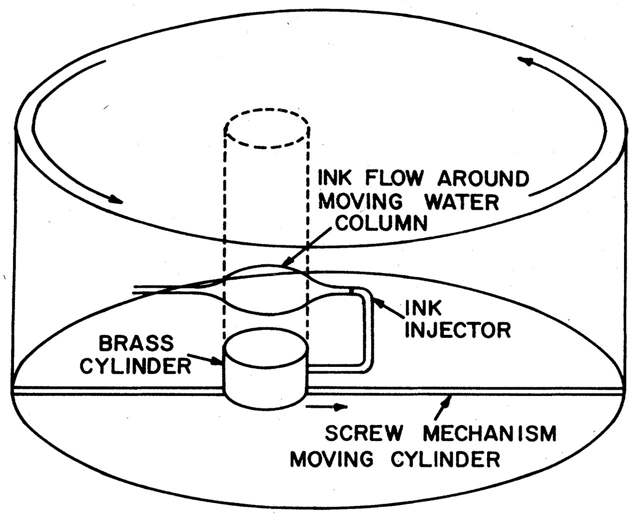

FIG. 1. Taylor's "column" experiment.

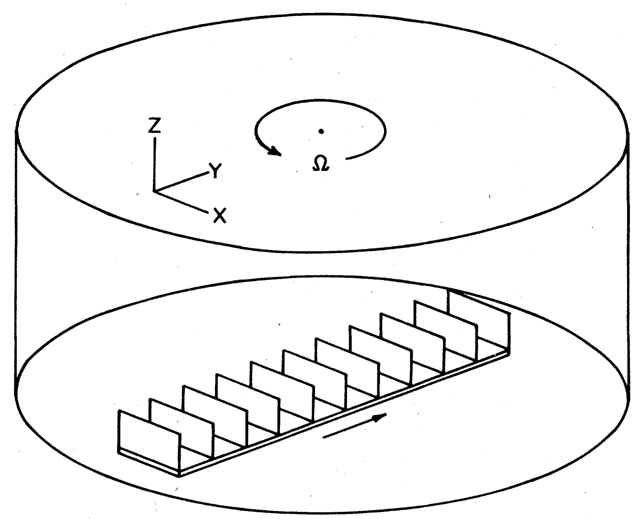

FIG. 2. A chain of paddles to induce slab-shaped columnar motion.

rotating fluid. It is selected for treatment there as simpler to explain than the horizontally moving column, but even so the analytic development in terms of wave propagation is not simple.

For the sake of a more graphic or *anshaulich* understanding of the situation simplified enough that it may rather easily be extended to encompass the transient preceeding equilibrium, we here take a more nearly "handwaving" approach, focusing attention on accelerations within specific closed streamline loops. Integration around a loop is of course related by Stokes theorem to taking the curl locally, as was done to obtain Eq. (3), and it is sometimes easier to visualize the whole than the sum of the parts, or the line integral as compared with the surface integral of the curl.

II. MOVING SLAB OF FLUID ABOVE A CHAIN OF PADDLES

The simplest case to understand first is an approximately one-dimensional motion induced in a layer of rotating liquid. Rather than consider first the effect of a single paddle, we consider a long row of many short paddles in a chain dragged across the bottom of a rotating vessel, each paddle normal to the length of the chain, as in Fig. 2. (They are somewhat wider than the distance between them.)

It is useful to consider the pressures and velocities around the typical loop *abcd* in Fig. 3. An equilibrium situation may very simply be achieved if the slab of water above the chain of paddles moves with the paddles, giving the pressure distribution indicated in Fig. 3(a). Here the paddles induce, at their level, a flow with velocity (in the rotating system) v and this causes in the rotating system a Coriolis force (per unit volume) as in Eq. (1)

$$\mathbf{F}_c = 2\rho\mathbf{v} \times \mathbf{\Omega} \qquad (4)$$

toward the right parallel to the paddles. In equilibrium this must be compensated by a pressure gradient toward the right, or a greater pressure at b than at a. All Coriolis forces being horizontal, the vertical pressure gradient is due to gravity, and in a homogeneous liquid the pressure variation must be the same above a as above b. This then requires the same horizontal pressure gradient along dc as along ab. This in turn means that the velocity of the fluid v (parallel to the chain) must be the same at the level of d and c as between a and b, so as to make the same Coriolis force to compensate the pressure gradient. Thus equilibrium is attained by having the slab of liquid above the chain of paddles move rigidly with the chain. This is clearly a case of simple "geostrophic" flow, the pressure gradient being normal to the velocity at all levels.

It is of interest to inquire "How is this equilibrium motion achieved?" Starting from rest in the rotating system, how is the force that is applied to the paddles transmitted to the fluid in the upper layers of the slab?

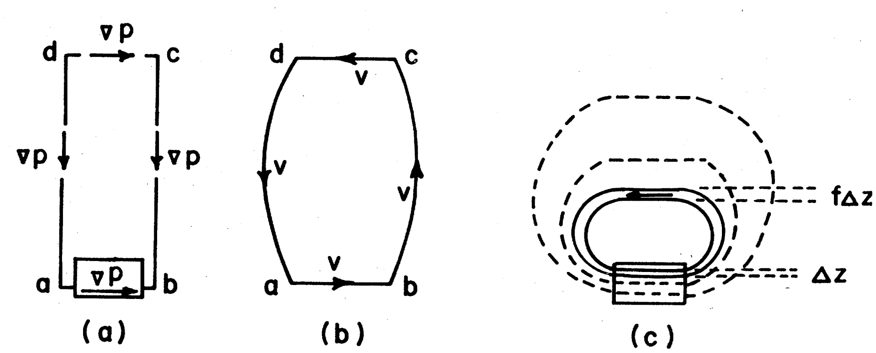

FIG. 3. Integration paths illustrating upward transport of momentum.

Shortly after starting from rest, the pressure at b is not enough greater than at a to balance the Coriolis force, and there is a velocity v between the paddles towards the right, as indicated at the bottom of Fig. 3(b). Continuity requires that the material so transported find its way back towards a, among others by the circuit shown in Fig. 3(b), driven by pressure gradients around the circuit. The Coriolis acceleration due to the flow from c to d is in the direction of the motion of the chain, v, accelerating the fluid in the same direction it would be moved if the paddles were to extend up to this level. The motion v from a to b makes a Coriolis force against the paddles, requiring the paddles to do work, and the opposite motion higher up makes an an opposite Coriolis force in the direction of motion of the paddles, expending some of the work to accelerate the fluid up there. Thus one can expect that there will be such a flow, to the right between the paddles and to the left higher up, gradually diminishing until equilibrium is established.[1] The paddles impart momentum to the bottom of the fluid and the flow around the circuit represents an upward transport of the component of momentum parallel to the chain.

Even in this simple case the dynamics is complicated, but a simplified plausible treatment gives some idea of the time scale of the process whereby equilibrium is established. Let the chain of paddles be moving in the rotating system, in the y direction with velocity v_y^0, the angular velocity Ω being in the z direction. Consider the hypothetical stream tube sketched in Fig. 3(c). It has a thickness dz passing between the paddles (of length l) and has the option of being thicker, fdz with a "fatness factor" f, on the return passage at height z directly above the paddles. We consider the Coriolis factor in these two sections only, neglecting it in the connecting links at the sides, about the shape of which we say little. We take their length to be roughly z, though it is actually somewhat greater because they may bulge and because the upper passage is not necessarily directly above the lower one. The flow around the loop between its stream-line boundaries is such that the mass transport along the loop is the same in the thick upper section as in the higher-speed thin low section, or

$$v_x^{(z)} = -(1/f)v_x^0. \tag{5}$$

When the circulation around the loop has reached a steady value and with constant ρ the local acceleration $\dot{\mathbf{v}}$ makes equal and opposite contributions in the ascending and descending tapered sections (associated with Bernouli changes of pressure) and is zero elsewhere so the integral around the loop vanishes, $\oint \rho\dot{\mathbf{v}}\cdot d\mathbf{s} = 0$. When the hydrodynamic equation (1) is integrated around the loop, the left side thus vanishes as does the first term on the right, the pressure p being single valued. With constant density the integral of the gravitational term, the second term on the right, also vanishes, having equal and opposite contributions from the two vertical legs, and no contribution from the horizontal legs. In the nonviscous case, with the coefficient of viscosity $\eta = 0$, we are left with only the integral of the Coriolis term, the third term on the right, which must thus also vanish. This means that $\mathbf{v}\times\mathbf{\Omega}\cdot d\mathbf{s}$ must be equal and opposite in the upper and lower legs, and $d\mathbf{s}$ has opposite direction in the two so the relevant component v_y must be the same, $v_y^{(z)} = v_y^0$. If there is acceleration of the circulation around the loop (or viscosity or both), the Coriolis force must be different at the two levels. During the initial acceleration from rest this difference arises from the initial lag of the velocity component parallel to the paddles, $v_y^{(z)}$, at the upper level behind that at the lower level between the paddles, v_y^0. Expressed in terms of the acceleration in the lower section, the result is then roughly

$$(1+1/f)(l+z)\dot{v}_x^0 = 2l(v_y^0 - v_y^{(z)})\Omega. \tag{6}$$

A constant density factor ρ has been omitted. This we may call the equation for inertia in the loop. A corresponding equation for inertia in the slab is obtained by equating accelerations within the upper sections:

$$\dot{v}_y^{(z)} = -2\Omega v_x^{(z)} = (2\Omega/f)v_x^{(0)}. \tag{7}$$

A combination of Eqs. (6) and (7) leads to an oscillatory solution, as we shall see, that does not approach equilibrium velocities without a dissipative term. Without attempting a proper treatment of viscosity in so complicated a configuration, we may instead introduce a simple dissipative term in Eq. (1) or (7) or both. It is plausible that viscous drag in the smaller-scale motions of the vertical loops, particularly where they pass between the paddles, is greater than in the slab motion. This we represent by adding a simple dissipative term to Eq. (6)

$$\dot{v}_x^0 = [2l\Omega/(1+1/f)(l+z)](v_y^0 - v_y^{(z)}) - \mu v_x^0, \tag{8}$$

where μ is somehow related to η. Combining Eqs. (7) and (8) and $v_x^0 = -fv_x^{(z)}$, we then have

$$\ddot{v}_y^{(z)} = \mathcal{L}(v_y^0 - v_y^{(z)}) - \mu\dot{v}_y^{(z)}$$
$$\mathcal{L} = l4\Omega^2/(1+f)(l+z). \tag{9}$$

A solution of the form

$$v_y^{(z)} - v_y^{(0)} = A\exp(-gt) \tag{10}$$

yields

$$g^2 + \mu g + \mathcal{L} = 0, \quad g = -\mu/2 \pm [(\mu/2)^2 - \mathcal{L}]^{\frac{1}{2}}. \tag{11}$$

In the limit for small μ the velocity higher up oscillates about that enforced lower down by the paddles with an amplitude depending on initial conditions. In the critically damped case, with μ just great enough to avoid overshooting equilibrium,

$$(\mu/2)^2 - \mathcal{L} = 0,$$
$$v_y^{(z)} = v_y^{(0)}\{1 - \exp[-(l/(1+f)(l+z))^{\frac{1}{2}}2\Omega t]\}, \tag{12}$$

if $v_y^{(z)}$ starts from rest. Since the coefficient $[l/(1+f)(l+z)]^{\frac{1}{2}}$ is of the order of $\frac{1}{2}$, the velocity higher in the slab approaches that at the bottom with a $1/e$ time of the order of the period of rotation of the whole body of the fluid. For smaller μ, $v_y^{(z)}$ will first reach v_y^0 more quickly, but oscillate and close in on it more slowly. For larger μ, the approach is slower. [One takes the physical choice $\pm = +$ in Eq. (11)].

Equilibrium is approached more rapidly for small f and small z, which suggests that within the limitations of laminar flow there will be a tendency for shorter and thinner loops with small z and small f to form and to accelerate more rapidly at first and approach equilibrium sooner than larger ones. Thus the lower parts of the slab, just above the

[1] Whether the approach to equilibrium would be monotonically assymptotic or oscillatory, after the velocity higher up overshoots that below, is expected to depend on the viscosity and initial acceleration.

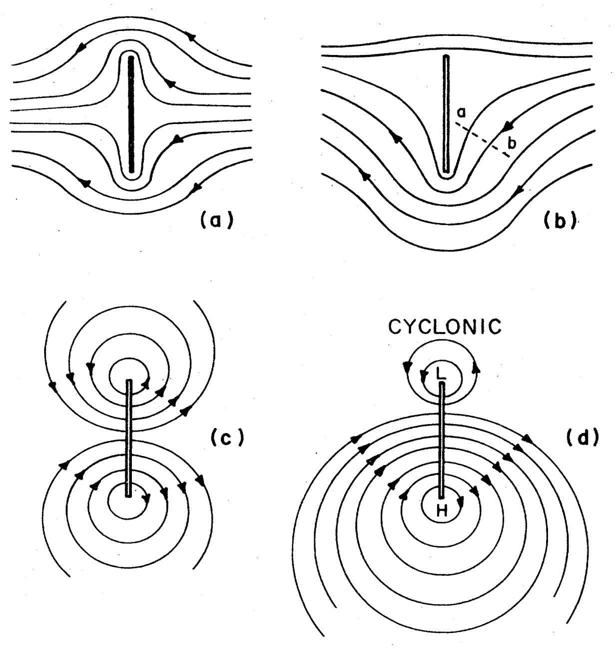

FIG. 4. Flow patterns around a single paddle: (a) and (b) in coordinate system of remove fluid; (c) and (d) in coordinate system of the paddle; (a) and (c) in stationary fluid; (b) and (d) in rotating fluid.

paddles, will approach the equilibrium velocity v_y^0 comparatively quickly, with the upper parts lagging behind, and the deeper the fluid, the longer it will take for the entire slab to approach v_y^0. As the return flow through the lower parts is nearly halted, the flow between the paddles should be increasingly devoted to accelerating the upper parts where there is still a velocity difference $(v_y^{(z)} - v_y^0)$ to drive the fluid around the loop.

A qualitative remark may be made about the upward and downward connecting links without getting into too much analysis. As the flow through the paddles v_x^0, emerges into the space beyond the paddles, it has the same Coriolis force in the $-y$ direction as that which pushes against the paddles, and beyond the paddles the fluid is thus strongly deflected, reducing its v_y below v_y^0 and bending the streamlines backward in the paddles' frame of reference. Now the Coriolis force on this backward flow pushes the first part of the outside leg of the loop to the left, in the $-x$ direction, which tends to make the loop bend in against the sides of the paddles (and the space above them). In the limit when the stream tube bends sharply back, there is no x component of the flow to cause the force to bend it back but still a tendency for it to rise and seek a return path. It seems plausible then, that the compromise between these tendencies as equilibrium is approached will be a rather sharp and short bend backwards and upwards at the exit from the paddles, so that the stream tubes will tend to hug close to the slab being accelerated. Well before equilibrium the upward loop may be expected to extend out beyond the slab and lag backward as it slopes upward, but very close to equilibrium it seems plausible that there should be instead a thin layer of fluid creeping almost directly up along the right wall of the accelerated slab.

The behavior in the descending leg at the left is related to this but there it is mainly the last part of the flow, just before it enters the regimented channel between the paddles, that provides the forces tending to make the motion a nearly vertical one. The initial part of the descent is influenced by the nature of the flow across the top section of the loop. This passage is not directly in the $-x$ direction, but rather curved forward with a considerable $+y$ component due to the accumulated forward acceleration arising from the $-x$ motion. Thus the leftward leg at its left end slopes forward and the upper part of the downward leg bends further forward, with no sharp boundary between them when far from equilibrium The forward motion in the downward leg deflects it in towards the region above the paddles and these forces practically disappear only when the downward motion becomes almost vertical, while the flow in the upper leg approaches the $-x$ direction and the downward bend presumably becomes sharp as equibrium is approached and the circulation becomes slow.

Consider now a different experiment, a single paddle in the $x-z$ plane moving along the bottom of the rotating vessel slowly along the y direction. Let it first be a tall paddle extending the entire depth of the fluid. As seen in the coordinate system of the paddle with the fluid flowing past it, without rotation the fluid would part symmetrically around the right and left sides of the paddle, with stream lines joining again symmetrically behind it, as in Fig. 4(a). In a rotating fluid the parting and joining as influenced by the Coriolis force would be unsymmetrical, as in Fig. 4(b). This is better explained in the coordinate system of the fluid, in which the remote parts of the fluid are at rest, and the paddle moves with velocity v_y. Without rotation the laminar flow is around approximately circular stream lines originating on the front side of the paddle and terminating on the back side, as in Fig. 4(c), of course symmetric around the right and left edges. At constant v_y, without viscosity the propagation of this pattern requires no work. It has a momentum related to the angular momentum around the stream lines, just as in the case of a vortex ring such as a familiar smoke ring. When this velocity field is added vectorially to $-v_y$, the result is a flow pattern like Fig. 4(a) around the stationary paddle. The curvature of the flow lines in Fig. 4(c) is caused by a transverse pressure gradient, which means that there is a low pressure region around each edge of the paddle.

With rotation, all the flow lines of Fig. 4(c) are modified, being bent to the right by the Coriolis force. The clockwise circles at the bottom are bent into tighter circles while some of the outer counter-clockwise paths at the top are bent into clockwise paths around those already at the bottom, leading to the pattern of Fig. 4(d). The tendency to expand the inner counter-clockwise paths at the top of the figure accentuates the low-pressure region there, marked L, requiring a greater pressure gradient to produce the curvature against the Coriolis force, while the squeezing of the clockwise circles by the Coriolis force creates a relatively high-pressure marked H at the edge near the bottom of the figure. In meteorological terms, the smaller circulation at the top of the figure is cyclonic with a low at the center, and the other pattern is anticyclonic with a high at the center. The radial variation of velocity is, however, not typical of natural vortices, being constrained by the paddle.

This pattern added to $-v_y$ yields the flow pattern of Fig. 4(b) relative to the paddle.

So much for the tall paddle: consider now a squat paddle extending only a short distance above the floor of the rotating fluid as it moves slowly across it. Near the bottom in a plane intersecting the paddle, the flow will tend to become like that of Fig. 4(d). One sees that here, too, an equilibrium condition is attained when the same pattern extends up through the fluid, just as if the paddle were the tall one. For here again, as in Fig. 3(a), we can draw a vertical rectangle *abcd* pierced by the flow lines. The integrated pressure gradient around the rectangle *abcd* must vanish in the equilibrium flow pattern and again the equality of the contributions along the vertical legs from the strictly vertical gravitational force requires that the contributions from the horizontal legs, due to the strictly horizontal Coriolis force, must be equal if there are no other forces or accelerations, and this requires the same horizontal flow pattern at the higher level as at the lower.

Thus if we postulate an essentially two-dimensional motion at the level of the paddle, we deduce that the entire flow in equilibrium is two dimensional, the same at all levels. It is expected, however, that the initial motion will have at least a slight vertical component as some of the fluid wells up over the top of the paddle. Imagine, for example, that the inner stream line in Fig. 4(b) rises up over the paddle, out of the plane of the figure, rather than to pass around its lower edge. If R is the radius of the curvature of the projection of such a stream line on the vertical yz plane, this implies a centripetal acceleration making a contribution to the integral of $\dot{\mathbf{v}} \cdot d\mathbf{s}$ from a vertical leg of one of the vertical loops we have just been considering such as the loop of which the lower leg is *ab* in Fig. 4(b). This would upset the balance on which the proof of two-dimensional equilibrium flow was based.[2] This objection vanishes as we go to the limit of slow fluid flow or small v, however, because this effect is proportional to v^2/R and thus becomes small compared with the Coriolis force $\mathbf{v} \times \mathbf{\Omega}$ that is linear in v.

The process of achieving equilibrium at the higher levels is again gradual and for low viscosity oscillatory, requiring an upward transport of momentum by flow around largely vertical loops or spirals. Consider a vertical rectangular loop *abcd* with the lower leg for instance just against the front edge of the paddle, as indicated by the line *ab* in Fig. 4(b), or more generally anywhere at that level at right angles to the flow lines of Fig. 4(b) that will to some degree be approximated by the initial flow. This is analogous to Fig. 3(a) and is not a stream line or tube as in Fig. 3(c), but it samples many such flow tubes and it will suffice to consider line integrals around it. There is a velocity normal to the lower leg *ab*, induced by the paddle motion at that level, initially not matched by the velocity normal to the upper leg *cd* so there exists a line integral of the Coriolis force around the loop that induces a line integral of the component of fluid velocity around the loop (in the rotating coordinate system of the remote fluid). This component of the flow exerts a Coriolis force at the lower level backward and ultimately (by way of pressure gradients) against the paddle, at the upper level forward in the direction of the low-level horizontal flow pattern and accelerating the fluid at the high level so as to tend to establish the same flow pattern there.

Just as in the case of the slab of moving fluid above the chain of paddles, the amount of higher-level fluid to be accelerated by this mechanism is related to the amount of lower-level fluid by the ratio (depth of fluid above paddles)/(height of paddle), and it is plausible that the rate of reaching equilibrium should be about the same in this case as in that, Eqs. (6) to (7) applying approximately also here.

Consider next the case of a long rectangular stick moving along the bottom parallel to its length. This can be compared with the chain of paddles from which it differs only by having the space between the paddles filled in so the fluid cannot flow between them. If the stick is a tall slab extending to the top of the fluid, the flow pattern around it in the coordinate system of the stick is as sketched in Fig. 5(a), a close analog of Fig. 4(d) for a tall paddle, but with longer return paths from the front side around to the back side. The same pattern applies for a squat stick, with vertical dimension considerably less than the depth of the fluid after equilibrium has been attained, but the process of reaching equilibrium is quite different from either the case of the single paddle or the case of the chain of paddles, different in being slower. Consider first a vertical rectangle having as its lower leg the line *ab* in Fig. 5(a) just in front of the front end of the stick (or in nautical terms, across the bow

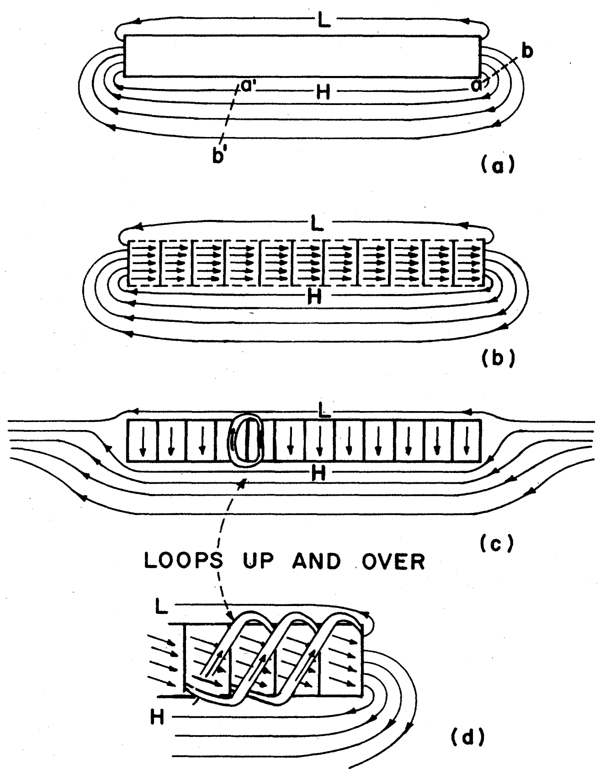

FIG. 5. Flow patterns induced by a chain of paddles.

[2] An analysis by Ingersoll (1969), valid near the limit of low viscosity, shows that such welling up of the fluid over a very low cylindrical obstacle causes curvatures in the horizontal component of the flow which pass around a nascent Taylo. column if the obstacle is made slightly higher.

of the stick) and a similar vertical rectangle at the stern. The circulation induced around these loops by the coriolis force at the lower level induces a forward coriolis force in the upper part of the loops and thus transports momentum upward. But the amount of momentum transported upward is in proportion only to the relatively small area at bow and stern before the lines of flow shown in Fig. 5(a) bend off to the side. The correspondingly rather small forward forces at a higher level has to accelerate the inert mass in all the slab-shaped volume above the stick, so the acceleration is much smaller than in the case of the single paddle where there is no such inert mass, or in the case of the multiple paddles of the chain wherein the flow between the paddles helps accelerate the fluid in the slab above them. The vertical rectangles built on lines such as $a'b'$ in Fig. 5(a), crossing the backward-flowing fluid, alongside the stick produce a backward acceleration at higher levels to tend to establish this part of the flow pattern higher up. It is possible that this may have an indirect effect on the acceleration of the inert matter above the stick by influencing the curvature near the ends and perhaps increasing the area that is effective in providing forward momentum.

A chain of paddles having the same external dimensions as the stick would, after reaching equilibrium, have the same external flow pattern as the stick, while the fluid between the paddles will move along with them simulating the solid matter of the stick, as in Fig. 5(b). The approach to equilibrium, however, is very different, as indicated in Figs. 5(c) and 5(d), wherein the flow between the paddles is part of a circuit extending up and over the paddles to accelerate the slab above. Figure 5(c) is in the coordinate system of the paddles, and 5(d) in the coordinate system of the remote fluid.

Returning now to the case of Sir Geoffry Taylor's demonstration experiment with a squat cylinder moving slowly across the bottom of a rotating fluid (Fig. 1), the mechanism is very similar to that in the case of the single paddle but there is now an inert cylindrical column of fluid to be accelerated by the forces that set up the outside circulation. The rapidity of approach to equilibrium is thus expected to be intermediate between the case of the long stick, where there is even more inert fluid, and the cases of either the single or multiple paddles, where there is none. As an experimental object, the cylinder is presumably superior to a paddle because the discussion assumes laminar flow and this may be achieved at more convenient speeds for the cylinder than for the rather sharp-edged paddle.

III. RELEVANCE TO METEOROLOGY

One would like to find applications of the Proudman–Taylor theorem in the meteorology of the earth's atmosphere (and other planetary atmospheres), in the circulation of ocean currents and in geophysics in the motions postulated in the earth's fluid core. However, the realization of a purely two-dimensional equilibrium motion may be confined to the ideal conditions of the laboratory. Perhaps the only distinctly recognized realization in the chaos of nature is the red spot of Jupiter which, it has been suggested (Hide, 1968), may be analogous to the top of the column in Taylor's demonstration, Fig. 1, with a mountain or depression in the slowly circulating Jovian atmosphere replacing the slowly moving cylinder. Nevertheless, the ideas encountered in discussing the approach to equilibrium in the idealized case do have some relevance to the more complicated situations in meteorology. There are so many competing effects that it is difficult to isolate the influence of any one of them.

The two dimensionality in Taylor's demonstration is strikingly evident because his pot of fluid is deep: the vertical dimension is even greater than the horizontal scale of the flow around the inert column. The atmosphere and the ocean by contrast are shallow in comparison with the scale of the horizontal motions within them. The vertical variation of horizontal shear and the associated viscosity, be it laminar or turbulent, has potentially greater influence in competing with Coriolis effects. There is a wide variation of temperature (and also of density in the atmosphere) between top and bottom of the thin layer of fluid, leading to stratification of flow patterns and possibly the development of waves in the boundary. There is a deep layer of arctic cold water beneath the northward-flowing gulf stream near Florida, for example. The gulf stream itself is many times as wide as it is deep and the vertical distribution of velocity within it may be influenced as much by viscosity as by the Proudman–Taylor phenomenon. In such applications in which the horizontal component of Coriolis force is presumed to be the important one, only the locally vertical component of the earth's angular momentum Ω is relevant, in the approximation we have discussed, as it is for a Foucault pendulum. This is about $\frac{1}{2}\Omega$ at the latitude of Florida and averages somewhat more than that in considerations of temperate-zone meteorology. In the temperate zone of the atmosphere, the circulations in the stratosphere and in the troposphere below it are quite different and to some extend independent. The upper troposphere has its jet stream, sometimes with a snakelike pattern between arctic and tropic boundaries. Its depth is a considerable fraction of that of the atmosphere, but its interaction with the earth's surface is indirect, by way of the cold and warm fronts that characterize lower-troposphere weather. There is a tendency for both patterns to drift eastward together, the advance of azimuthal phase sometimes being delayed by interactions with continental structure.

The cyclonic and anticyclonic eddies in the troposphere each involve circulating streams of air 100 km or more wide and only about 5 km deep, much wider than thick, as in the case of the gulf stream and also the jet stream. Here the Proudman–Taylor aspect of Coriolis force does not seem to be very successful in determining the vertical distribution of velocity. The drag of the contact with ground makes the velocity of the lower atmosphere smaller near the bottom, clearly a viscous effect. (It pays to build windmills high). Temperature differences on the two sides of the flow, such as the warm air before a cold front, make drastic deviations from the idealized discussion. Even with these large deviations from the ideal two-dimensional flow, the Proudman–Taylor mechanism may be helping to stiffen the flow and minimize the vertical variation of velocity within the strata.

IV. LIMITED RELEVANCE TO GEOPHYSICS AND NUCLEAR PHYSICS

In the problem of convective flow within the earth's fluid core to conduct the heat from the radioactively heated internal solid core to the surrounding mantle, one encounters the difficulty of fitting the implicitly cylindrical geometry of the Proudman–Taylor theorem into the spherical geometry of the fluid core, a sort of "square peg in a round hole" difficulty. The situation is perhaps most simply illustrated by thinking of the rotating body of water in Fig. 1 as being terminated at the top by a spherical dome rather than a horizontal plane surface. Imagine the cylindrical column of water above the moving solid cylinder coming up against the sloping side of this dome. In such a two-dimensional motion, the possibility of flow around a loop like *abcd* in Fig. 3 linking *any* two levels, not just a level near the bottom level with a higher one, provides a vertical stiffness to the configuration. When the top of a column encounters the sloping upper boundary, the situation is like that of a man walking upright toward the edge of an attic. When he bumps his head this inhibits the progress of his feet. When the upper leg *cd* of the loop is next to where the top of the column encounters the boundary, it can no longer move sideways matching the motion of the leg *ab* below it and can no longer provide the "back coriolis force" (analogous to "back electromotive force" in an electric motor) to prevent circulation around the loop *abcd*. This circulation then provides a sideways force on *ab* impeding the motion of the column at the lower level.

In the earth's fluid core the influence of gravity on thermal convection must be much greater than that of Coriolis or centrifugal forces. Yet the fact that the magnetic pole is near the rotational axis leads one to suppose that coriolis forces do have some influence on the convective motions that presumably generate the dipole field. If one considers that the turbulent convection carrying heat from the solid inner core outward to the mantle is mainly in a thick slab including and parallel to the equatorial plane, it is attractive to think of large-scale turbulence, involving cylindrical motions with diameters comparable with the thickness of the fluid layer. Because of the difficulty with the spherical boundary, such cylinders would not be long ones of the Proudman–Taylor sort, even if the Coriolis force were strong. The flows patterns on such a large scale, postulated in the earth's core and treated analytically by Bullard (1949) and co-workers and later discussed qualitatively by the writer (Inglis, 1965), were not purely two dimensional in nature and their dynamo mechanism depends critically on deviations from two-dimensional flow. The small-scale turbulence postulated by Parker (1955) likewise depends on three-dimensional motion and in any case would not be so much inhibited by the sloping surface if it should approximately conform to the theorem.

In nuclear physics, classical fluid dynamics has been employed as the basis for an introductory understanding of the moments of inertia of rotating ellipsoidal nuclei. It is perhaps of interest to note that here the Proudman–Taylor axial rigidity does not stand in the way. As Aage Bohr (1952) has pointed out, in the rotating coordinate system the flow, induced in the fluid starting from rest by rotation of the ellipsoidal boundary, follows stream lines that are ellipses. In the equatorial plane (an $x - y$ plane), for example, these have the same shape as a cut made of the ellipsoid by another $x - y$ plane at a different value of z. Thus a two-dimensional motion conforming to the Proudman–Taylor theorem would encounter no obstacle where it contacts the boundaries.[3] The flow is retrogade: the rotation of the fluid lags behind that of the boundary so that the angular momentum and consequent moment of inertia are less than they would be for a rigid motion within the rotating boundary. The observed moments of inertia are intermediate between the rigid and liquid-drop values, a fact that has been extensively interpreted in terms of more explicit treatment of nucleon dynamics.

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[3] Fluid motion within rotating ellipsoidal boundaries is treated by Greenspan (1968), especially Figs. 2 and 4.