from which we derive

$$\partial_\mu j^\mu = m_0^2 \partial_\mu \phi^\mu = ieq(j_e^\mu + m_0^2 A_\mu)\phi^\mu .$$

It will be noted how the presence of the m_0 term is required to obtain the physically necessary equations of nonconservation. The corresponding integral relations are

$$\partial_0 f\mathbf{T} = \int (d\mathbf{x}) ieq(j_e^\mu + m_0^2 A_\mu)\phi^\mu ,$$

which asserts that

$$\partial_0 {}^*\mathbf{T}_{1,2} = \pm(e/f)\gamma^{-1} \int (d\mathbf{x})(j_e^\mu + m_0^2 A_\mu)\phi_{2,1}^\mu$$
$$= -i[{}^*\mathbf{T}_{1,2}, P^0] .$$

We do not try to discuss here whether this mechanism implies reasonable magnitudes for the T_3 dependence of mass multiplets.

We make only one comment about the physical mass problem. For an Abelian gauge field coupled to a conserved current, there are sum rules[5] that require the existence of unit spin excitations with mass m less than m_0. The sum rules are different, however, if the current is not conserved,[6] or if the field is non-Abelian. Then, no simple conclusions about the spectrum in relation to m_0 can be drawn.

[5] K. Johnson, Nucl. Phys. **25**, 435 (1961).

[6] D. G. Boulware and W. Gilbert, Phys. Rev. **126**, 1563 (1962).

Generalized Radial Integrals With Hydrogenic Functions

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The calculation of radial quantum hydrogenic integrals is an important part of wave mechanics. The direct procedure of multiplying, term by term, leads to long and involved series. These series can be summed, but the methods employed are too complicated for student application. Textbooks and reference books have studiously avoided the calculation except for small specific values of n and l. I have discovered a simple method for dealing with this summation problem. The procedure does more than bring the method within the scope of a student whose mathematical skill does not go far beyond that of elementary calculus; it leads to the reduction of generalized radial quantum integrals.

The one-electron wave equation,

$$H\psi = -(h^2/8\pi^2 m)\nabla^2\psi + U\psi = E\psi , \qquad (1)$$

for a Coulomb field,

$$U = -Z\epsilon^2/r , \qquad (2)$$

has the well-known solution[1]

$$\psi = R\Theta\Phi , \qquad (3)$$

a product function.

$$\Phi = (2\pi)^{-\frac{1}{2}} e^{im\varphi} , \qquad (4)$$

a function of the longitude parameter ϕ.

$$\Theta(l,m) = \frac{(|m| + m)}{2}\left[\frac{2l+1}{2}\,\frac{(l-|m|)!}{(l+|m|)!}\right]^{\frac{1}{2}} \times \sin^{|m|}\theta \frac{d^{|m|}}{(d\cos\theta)^{|m|}} P_l(\cos\theta) , \qquad (5)$$

[1] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (The Macmillan Company, New York, 1935), pp. 52 and 117.

where θ is the angle measured from the pole along a meridian. Thus

$$\Theta(l,m) = (-1)^m \Theta(l,-m) . \qquad (6)$$

This choice of signs (or phase) is important for many purposes. $P_l(\cos\theta)$ is the Legendre polynomial. The radial equation is a function of the radius r,

$$R(n,l) = -\left[\frac{Z(n+l)!}{n^2 \mathrm{a}(n-l-1)!}\right]^{\frac{1}{2}} \frac{e^{-Zr/n\mathrm{a}}}{(2l+1)!}\left(\frac{2Zr}{n\mathrm{a}}\right)^{l+1} \times {}_1F_1(-n+l+1, 2l+2, 2Zr/n\mathrm{a}) . \qquad (7)$$

${}_1F_1$ is the confluent hypergeometric function,

$${}_1F_1(-\alpha,\gamma,x) = 1 - \frac{\alpha}{1!\gamma}x + \frac{\alpha(\alpha-1)}{2!\gamma(\gamma+1)}x^2 - \cdots$$
$$= \sum_{j=0}^{\alpha} (-1)^j \frac{\alpha!}{(\alpha-j)!}\,\frac{(\gamma-1)!}{(\gamma-1+j)!}x^j . \qquad (8)$$

These functions are normalized to give

$$\int_0^{2\pi} \Phi(m')\Phi(m)\, d\varphi = \delta(m',m) , \qquad (9)$$

$$\int_0^{\pi} \Theta(l',m)\Theta(l,m) \sin\theta\, d\theta = \delta(l',l) , \qquad (10)$$

$$\int_0^{\infty} R(n',l)R(n,l)\, dr = \delta(n',n) , \qquad (11)$$

where δ is the Kronecker δ, equal to unity if $m' = m$ and equal to zero if $m' \neq m$, and likewise for l and n.

In these equations, ϵ and m are the electronic charge and mass, Z is the nuclear charge, and h is Planck's constant. a is the radius of the first Bohr orbit. n is the total, l the azimuthal, and m the magnetic quantum number, subject to the relations:

$$-l \leqslant m \leqslant l\,, \quad \text{and} \quad n-1 \geqslant l \geqslant 0\,. \tag{12}$$

These wave functions carry the implicit assumption that the Coulomb field extends to infinity. In an actual physical situation, however, neighboring atoms and electrons tend to restrict the field. A more realistic potential function assumes the form:

$$U = -(Z\epsilon^2/r)e^{-r/\eta \mathrm{a}}\,. \tag{13}$$

Near the nucleus, where $r \ll \eta\mathrm{a}$, this function is indistinguishable from that of (2). At large distances it decays exponentially. One may interpret the distance $\eta\mathrm{a}$ as analogous to the "Debye length." It is a function of the density and likewise related to the "excluded volume."

The difference between the two forms of Eq. (1), with the respective potentials of (2) and (13), is

$$U' = (1 - e^{-r/\eta\mathrm{a}})/r\,. \tag{14}$$

We must evaluate matrix components of the form

$$\iiint \psi^*(n',l',m')U'\psi(n,l,m)\,dr \sin\theta\, d\theta\, d\varphi = \langle n'l'm'|U'|nlm\rangle\,. \tag{15}$$

Since U' depends only on r, this integral is diagonal in l and m. However, for the moment, let us consider the more general type of radial function, of the form

$$\int_0^\infty R(n'l')r^k e^{-r/\eta\mathrm{a}}R(n,l)\,dr\,. \tag{16}$$

For still greater generality, we assume that the effective nuclear charge may not be identical for the two radial functions. Designate these effective charges by Z and Z'. Thus the general integral takes the form

$$\begin{aligned}\mathcal{I} = NN'\int_0^\infty r^{l'+l+k+2}\exp\left[-\left(\frac{Z}{n}+\frac{Z'}{n'}+\frac{1}{\eta}\right)\frac{r}{\mathrm{a}}\right] \\ \times {}_1F_1\left(-n'+l'+1, 2l'+2, \frac{2Z'r}{n'\mathrm{a}}\right) \\ \times {}_1F_1\left(-n+l+1, 2l+2, \frac{2Zr}{n\mathrm{a}}\right)dr\,.\end{aligned} \tag{17}$$

N and N^1 are normalizing factors, given by (7).
Let

$$\left(\frac{Z}{n}+\frac{Z'}{n'}+\frac{1}{\eta}\right)\frac{r}{\mathrm{a}} = \rho\,, \tag{18}$$

then

$$\mathcal{I} = ANN'\int_0^\infty \rho^{l'+l+k+2}e^{-\rho} \times {}_1F_1(-\alpha,\gamma,a\rho)\,{}_1F_1(-\beta,\gamma',a\rho)\,d\rho\,, \tag{19}$$

where

$$A = \left[\mathrm{a}\left(\frac{Z}{n}+\frac{Z'}{n'}+\frac{1}{\eta}\right)^{-1}\right]^{l'+l+k+3} \tag{20}$$

and

$$a = \frac{2Z}{n}\left(\frac{Z}{n}+\frac{Z'}{n'}+\frac{1}{\eta}\right)^{-1},$$
$$a' = \frac{2Z'}{n'}\left(\frac{Z}{n}+\frac{Z'}{n'}+\frac{1}{\eta}\right)^{-1}, \tag{21}$$

$$\alpha = n-l-1\,, \quad \beta = n'-l'-1\,,$$
$$\gamma = 2l+2\,, \qquad \gamma' = 2l'+2\,. \tag{22}$$

Let

$$I = \mathcal{I}/ANN'\,, \tag{23}$$

the integral of (19). Then note the following identity

$$I = \int_0^\infty \lim_{x\to 1}(-1)^{l-l'+k+1}\left(\frac{\partial}{\partial x}\right)^{l-l'+k+1} e^{-\rho x}\rho^{2l'+1} \times {}_1F_1(-\alpha,\gamma,a\rho)\,{}_1F_1(-\beta,\gamma',a'\rho)\,d\rho\,. \tag{24}$$

The differentiation merely introduces a factor $\rho^{l-l'+k+1}$ to make the algebraic factors of (24) and (19) agree.

Now set

$$l' - l = \delta\,, \tag{25}$$

choosing the indices l' and l so that δ is positive. Reverse the order of integration and differentiation in (24). Then

$$\begin{aligned}I &= \lim_{x\to 1}(-1)^{-\delta+k+1}\left(\frac{\partial}{\partial x}\right)^{-\delta+k+1}\int_0^\infty e^{-\rho x}\rho^{\gamma'-1} \\ &\quad\times {}_1F_1(-\alpha,\gamma,a\rho)\,{}_1F_1(-\beta,\gamma',a'\rho)\,d\rho \\ &= \lim_{x\to 1}(-1)^{-\delta+k+1}\left(\frac{\partial}{\partial x}\right)^{-\delta+k+1}P\,,\end{aligned} \tag{26}$$

an equation that defines the integral P.

We next compute and simplify the expression for P. Multiply the two series together and integrate term by term [see equation (8)]. The result takes the form:

$$\begin{aligned}P &= \sum_j \frac{\beta!}{j!(\beta-j)!}\frac{(\gamma'-1)!}{(\gamma'-1+j)!}(-a')^j \\ &\quad\times \sum_i \frac{\alpha!}{i!(\alpha-i)!}\frac{(\gamma-1)!}{(\gamma-1+i)!}(-a)^i \\ &\quad\times \int_0^\infty e^{-\rho x}\rho^{\gamma'-1+j+i}\,d\rho = \frac{(\gamma'-1)!}{x^{\gamma'}}\sum_j\frac{\beta!}{j!(\beta-j)!} \\ &\quad\times\left(-\frac{a'}{x}\right)^j \sum_j \frac{\alpha!}{i!(\alpha-i)!}\frac{(\gamma'-1+j+i)!}{(\gamma'-1+j)!} \\ &\quad\times\frac{(\gamma-1)!}{(\gamma-1+i)!}\left(-\frac{a}{x}\right)^i\end{aligned} \tag{27}$$

since

$$\int_0^\infty e^{-\rho x}\rho^r\,d\rho = r!/x^{r+1}. \tag{28}$$

But the ordinary hypergeometric function of four variables is

$$F(-\alpha,\gamma',\gamma,z) = 1 - \frac{\alpha\gamma'}{1!\gamma}z + \frac{\alpha(\alpha-1)\gamma'(\gamma'+1)}{2!\gamma(\gamma+1)}z^2 - \cdots = \sum_{i=0}^{\alpha}\frac{\alpha!}{i!(\alpha-i)!}\frac{(\gamma'-1+i)!}{(\gamma'-1)!}\frac{(\gamma-1)!}{(\gamma-1+i)!}(-z)^i. \tag{29}$$

Hence,

$$P = \frac{(\gamma'-1)!}{x^{\gamma'}}\sum_{j=0}^{\beta}\frac{\beta!}{j!(\beta-j)!}\left(-\frac{a'}{x}\right)^j \times F\left(-\alpha,\gamma'+j,\gamma,\frac{a}{x}\right). \tag{30}$$

Here apply the well-known transformation of the hypergeometric function,[2]

$$F(-\alpha,\gamma'+j,\gamma,z) = (1-z)^\alpha \times F(-\alpha,\gamma-\gamma'-j,\gamma,z/[z-1]). \tag{31}$$

From (22) and (25), we have

$$\gamma - \gamma' = -2\delta. \tag{32}$$

Therefore

$$P = (\gamma'-1)!\frac{(x-a)^\alpha}{x^{\gamma'+\alpha}}\sum_{j=0}^{\beta}\frac{\beta!}{j!(\beta-j)!}\left(-\frac{a'}{x}\right)^j \times F\left(-\alpha,-2\delta-j,\gamma,\frac{a}{a-x}\right). \tag{33}$$

The first "trick" in the present transformation depended on the introduction of the partial derivative in (24), which enabled us to change the exponent of ρ from its original value of $l'+l+k+2$ [Eq. (19)] to γ'. The choice resulted from the necessity of forming the combination $(\gamma'-1+j+i)!/(\gamma'-1+j)!$, which appeared in Eq. (27). The next trick involves simplification of the summation in (33).

Without special reference to the foregoing, consider the expression

$$Q = \lim_{y\to 1}\left(\frac{d}{dy}\right)^\alpha (y-b)^{\gamma+\alpha-1}[y^{2\delta}(y-c)^\beta]. \tag{34}$$

Differentiate this expression as a product, by the Leibnitz formula

$$\left(\frac{d}{dy}\right)^\alpha Y_1Y_2 = \sum_{j=0}^{\alpha}\frac{\alpha!}{j!(\alpha-j)!}\left(\frac{d}{dy}\right)^{\alpha-j}Y_1\left(\frac{d}{dy}\right)^jY_2, \tag{35}$$

$$\left(\frac{d}{dy}\right)^{\alpha-j}(y-b)^{\gamma+\alpha-1} = \frac{(\gamma+\alpha-1)!}{(\gamma-1+j)!}(y-b)^{\gamma-1+j}, \tag{36}$$

$$\left(\frac{d}{dy}\right)^j y^{2\delta}(y-c)^\beta = \frac{\beta!}{(\beta-j)!}(y-c)^{\beta-j}y^{2\delta} \times\left[1+\frac{j(2\delta)}{1!(\beta-j+1)!}\frac{y-c}{y} + \frac{j(j-1)(2\delta)(2\delta-1)}{2!(\beta-j+1)(\beta-j+2)}\left(\frac{y-c}{y}\right)^2+\cdots\right]. \tag{37}$$

Since 2δ is, by definition, a positive integer, this last series (in brackets) will terminate after $2\delta+1$ terms. The bracket is a hypergeometric function, as defined in (29). Setting $y=1$, we get:

$$Q = \frac{(\gamma+\alpha-1)!}{(\gamma-1)!}(1-b)^{\gamma-1}(1-c)^\beta \times\sum_j\frac{\alpha!}{j!(\alpha-j)!}\frac{\beta!}{(\beta-j)!}\frac{(\gamma-1)!}{(\gamma-1+j)!}\left(\frac{1-b}{1-c}\right)^j \times F(-2\delta,-j,\beta-j+1,1-c). \tag{38}$$

Applying the transformation (31) to the hypergeometric function, we finally obtain

$$Q = \frac{(\gamma+\alpha-1)!}{(\gamma-1)!}(1-b)^{\gamma-1}(1-c)^\beta \times\sum_j\left[\frac{\alpha!}{j!(\alpha-j)!}\frac{\beta!}{(\beta-j)!}\frac{(\gamma-1)!}{(\gamma-1+j)!} \times\left(\frac{1-b}{1-c}\right)^j c^{2\delta}F\left(-2\delta,\beta+1,\beta-j+1,\frac{c-1}{c}\right)\right]. \tag{39}$$

The sum ranges from $j=0$ to $j=\alpha$ or β, whichever is smaller.

Now expand the hypergeometric function in (39).

$$F = 1 - \frac{2\delta(\beta+1)}{1!(\beta-j+1)}\frac{c-1}{c} + \frac{2\delta(2\delta-1)(\beta+1)(\beta+2)}{2!(\beta-j+1)(\beta-j+2)}\left(\frac{c-1}{c}\right)^2 - \cdots. \tag{40}$$

Multiply the bracketed term in (39) by 1, the first term of (40), and sum over j. The result is

$$F[-\alpha_1-\beta,\gamma,(1-b)/(1-c)]$$

Now multiply the bracketed term by the second term of (40), and sum again, getting

$$-(2\delta/1!)[(c-1)/c] \times F[-\alpha,-\beta-1,\gamma,(1-b)/(1-c)],$$

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[2] A. R. Forsyth, *A Treatise on Differential Equations* (The Macmillan Company, New York, 1929), p. 193, Eq. XVII.

Continue until all the terms of (40) have been included and collect the results. Hence (39) becomes

$$Q = \frac{(\gamma+\alpha-1)!}{(\gamma-1)!}(1-b)^{\gamma-1}(1-c)^{\beta}c^{2\delta} \times \sum_{j=0}^{2\delta}\frac{(2\delta)!(-1)^j}{j!(2\delta-j)!}\left(\frac{c-1}{c}\right)^j \times F[-\alpha_1-\beta-j,\gamma,(1-b)/(1-c)]\,. \quad (41)$$

Now return to (34), the original expression for Q, and redetermine the result by a different process. First expand the factor $(y-c)^\beta$ by the binomial theorem. We find that

$$Q = \lim_{y\to1}\left(\frac{\partial}{\partial y}\right)^\alpha \times \sum_{j=0}^{\beta}\frac{\beta!}{j!(\beta-j)!}(-c)^{\beta-j}(y-b)^{\gamma+\alpha-1}y^{2\delta+\gamma}\,. \quad (42)$$

Performing the product differentiation as before, we get

$$Q = \frac{(\gamma+\alpha-1)!}{(\gamma-1)!}\sum_{j=0}^{\beta}\frac{\beta!}{j!(\beta-j)!}(-c)^{\beta-j}(1-b)^{\gamma-1} \times F(-\alpha,-2\delta-j,\gamma,1-b)\,. \quad (43)$$

Equate this expression to its equivalent in (41), and we obtain the desired transformation:

$$\sum_{j=0}^{\beta}\frac{\beta!}{j!(\beta-j)!}\left(-\frac{1}{c}\right)^j F(-\alpha,-2\delta-j,\gamma,1-b) = \left(\frac{c-1}{c}\right)^\beta c^{2\delta}\sum_{j=0}^{2\delta}\frac{(2\delta)!(-1)^j}{j!(2\delta-j)!}\left(\frac{c-1}{c}\right)^j \times F\left(-\alpha,-\beta-j,\gamma,\frac{1-b}{1-c}\right)\,. \quad (44)$$

Great simplification results because the number of terms in the summation on the right-hand side of (44) is greatly reduced. Not only is δ constant for a given type of problem, but its magnitude is small for cases of physical importance.

Note that the summation on the left-hand side of (44) becomes identical with that of (33) when we set

$$c = x/a' \quad \text{and} \quad (1-b) = a/(a-x)\,, \quad \text{subject to} \quad \gamma'-\gamma = 2\delta\,. \quad (45)$$

Therefore

$$P = (\gamma'-1)!\frac{(x-a)^\alpha(x-a')^\beta}{x^{\alpha+\beta+\gamma}a'^{2\delta}}\sum_0^{2\delta}\frac{(2\delta)!(-1)^j}{j!(2\delta-j)!} \times\left(\frac{x-a'}{x}\right)^j F\left[-\alpha,-\beta-j,\gamma,\frac{aa'}{(x-a)(x-a')}\right]\,. \quad (46)$$

We may further reduce this summation, if we desire, to not more than two hypergeometric functions, by successive application of the recurrence formula (see Ref. 2, p. 240)

$$[\gamma+2\beta+(\alpha-\beta)z] \times F - \beta(1-z)F_{\beta+} + (\gamma+\beta)F_{\beta-} = 0\,, \quad (47)$$

where

$$F = F(-\alpha,-\beta,\gamma,z)\,,\quad F_{\beta+} = F(-\alpha,-\beta+1,\gamma,z)\,,\quad F_{\beta-} = F(-\alpha,-\beta-1,\gamma,z) \quad (48)$$

To recapitulate, we have reduced our original general integral (17) to the form

$$\mathcal{J} = ANN'\lim_{x\to1}(-1)^{-\delta+k+1}(\partial/\partial x)^{-\delta+k+1}P\,, \quad (49)$$

with P given by (46) and other parameters defined by (20), (21), (22), and (25). The two most important cases involve the zeroth or the first derivative, where

$$\delta = l'-l = k+1 \quad \text{or} \quad k\,. \quad (50)$$

To evaluate the derivative, write P in the expanded form:

$$P = \frac{(\gamma'-1)!}{a'^{2\delta}}\sum_{j=0}^{2\delta}\frac{(2\delta)!(-1)^j}{j!(2\delta-j)!} \times \sum\frac{\alpha!}{i!(\alpha-i)!}\,\frac{(\beta+j)!}{(\beta+j-i)!}\,\frac{(\gamma-1)!}{(\gamma-1+i)!} \times (aa')^i\frac{(x-a)^{\alpha-i}(x-a')^{\beta+j-i}}{x^{\alpha+\beta+\gamma+j}}\,. \quad (51)$$

Differentiate once as a triple product and collect terms as follows:

$$\lim_{x\to1}\left(\frac{\partial P}{\partial x}\right) = \frac{(\gamma'-1)!}{a'^{2\delta}}(1-a)^\alpha(1-a')^\beta \times\sum_0^{2\delta}\frac{(2\delta)!(-1)^j(1-a')^j}{j!(2\delta-j)!} \times\left\{\frac{\alpha}{1-a}F\left[-\alpha+1,-\beta-j,\gamma,\frac{aa'}{(1-a)(1-a')}\right] + \frac{\beta+j}{1-a'}F\left[-\alpha,-\beta-j+1,\gamma,\frac{aa'}{(1-a)(1-a')}\right] - (\alpha+\beta+\gamma+j) \times F\left[-\alpha,-\beta-j,\gamma,\frac{aa'}{(1-a)(1-a')}\right]\right\}\,. \quad (52)$$

Now consider the integral for $n = n'$, with $l'-l = \delta$, $\beta = \alpha-\delta$, and $\eta\to\infty$. For this case $a = a' = 1$ and the expression becomes indeterminate. Expand each of the hypergeometric functions in the braces and keep only the last term of each. The others vanish because of the factor $(1-a)^{2\alpha-\delta+j}$ in the numerator. The first hypergeometric function vanishes except when $j = \delta-1$, the second except when $j = \delta + 1$, and the third except when $j = \delta$. The result becomes

$$\mathcal{J} = \left[\frac{(n+l+\delta)!(n-l-1)!}{(n+l)!(n-l-\delta-1)!}\right]^{\frac{1}{2}} \times \frac{(2\delta+1)!}{\delta!(\delta+1)!}\left(\frac{n\mathrm{a}}{2Z}\right)^{\delta}. \tag{53}$$

When $\delta = 0$,

$$\mathcal{J} = 1\,, \tag{54}$$

as required for normalization. Similarly, for $\delta = 1$, $k = 1$,

$$\mathcal{J} = [n^2 - (l+1)^2]^{\frac{1}{2}}\,\tfrac{3}{2}\,(n\mathrm{a}/Z)\,, \tag{55}$$

in agreement with the value given by Condon and Shortley (Ref. 1, p. 132).

Returning to Eq. (52), we employ the relation from Forsyth,

$$(\alpha - \beta)F = \alpha F_{\alpha+} - \beta F_{\beta+}\,, \tag{56}$$

in the notation of Eq. (48), to eliminate F_{α^+}. Thus,

$$\frac{\partial P}{\partial x} = \frac{(\gamma'-1)!(x-a)^{\alpha-1}(x-a')^{\beta}}{x^{\alpha+\beta+\gamma}a'^{2\delta}} \times \sum_{0}^{2\delta}\frac{(2\delta)!(-1)^j}{j!(2\delta-j)!}\left(\frac{x-a'}{x}\right)^j \times\left\{\frac{2(\beta+j)(x-1)}{(x-a')} \times F\left[-\alpha,-\beta-j+1,\gamma,\frac{aa'}{(x-a)(x-a')}\right] + \frac{1}{x}\left[2(-n'+\delta-j)(x-1) + (na-n'a') + a'(\delta-j)\right] \times F\left[-\alpha,-\beta-j,\gamma,\frac{aa'}{(x-a)(x-a')}\right]\right\}. \tag{57}$$

When we set $x = 1$ in this expression, we get, if a or $a' \neq 1$, for the radial quantum integral

$$\mathcal{J} = \mathrm{a}^{\delta}\left[\frac{ZZ'(n+l)!(n'+l+\delta)}{(n-l+1)(n'-l-\delta-1)!}\right]^{\frac{1}{2}} \times\frac{(-1)^{n'-l-\delta}}{(2l+1)!}\,\frac{(4ZZ'nn')^{l+1}(Z'n-Zn')^{n+n'-2l-\delta-3}}{(Z'n+Zn')^{n+n'-\delta+1}} \times 2n\left(\frac{n'}{2Z'}\right)^{\delta}\sum_{0}^{2\delta}\frac{2\delta!}{j!(2\delta-j)!}\left(\frac{Z'n-Zn'}{Z'n+Zn'}\right)^j \times[n'(Z-Z')+nZ'(\delta-j)] \times F\left[-n+l+1,-n'+l+\delta+1-j,2l+2,-\frac{4ZZ'nn'}{(Z'n-Zn')^2}\right]. \tag{58}$$

This formula holds for $n \to n'$, $l \to l'$, and $l \to l + \delta$. For $\delta = k = 0$ with $Z = Z'$ but $n \neq n'$, the factor $[n'(Z - Z') + nZ'(\delta - j)] = 0$. This demonstrates the orthogonality of the functions. Note that the functions are not orthogonal unless $Z = Z'$.

For $k = \delta = 1$, $n \neq n'$, $l' = l + 1$, $Z = Z'$ we recover the formula for the electric dipole moment, as given by Gordon.[3]

$$\mathcal{J} = \left(\frac{\mathrm{a}}{Z}\right)\frac{(-1)^{n'-l-1}}{4(2l+1)!}\left[\frac{(n+l)!(n'+l+1)!}{(n-l-1)!(n'-l-2)!}\right]^{\frac{1}{2}} \times(4nn')^{l+2}\frac{(n-n')^{n+n'-2l-4}}{(n+n')^{n+n'}} \times\left\{F\left[-n+l+1,-n'+l+2,2l+2,-\frac{4nn'}{(n-n')^2}\right] - \left(\frac{n-n'}{n+n'}\right)^2 F\left[-n+l+1,-n'+l,2l+2,-\frac{4nn'}{(n-n')^2}\right]\right\}. \tag{59}$$

The derivation given here is, however, far shorter and simpler than that employed by Gordon.

When $Z \neq Z'$, we employ (58) and (47), if desired, to eliminate one of the three hypergeometric functions that appear in the summation.

Now turn to the important case of $\delta = 0$, $k = -1$, for which the zero derivative of P applies [see Eq. (49)]. In this development we shall use a and a' as given in (21), with η finite.

$$\mathcal{J} = \frac{1}{\mathrm{a}}\left[\frac{Z}{n}+\frac{Z'}{n'}+\frac{1}{\eta}\right]^{-2l-2} \times\left[\frac{ZZ\,(n+l)!(n'+l')!}{n^2n'^2(n-l-1)!(n'-l-1)!}\right]^{\frac{1}{2}} \times\frac{1}{(2l+1)!}\left(\frac{4ZZ'}{nn'}\right)^{l+1}(1-a)^{n-l-1}(1-a')^{n'-l-1} \times F\left[-n+l+1,-n'+l+1,2l+2,\frac{aa'}{(1-a)(1-a')}\right]. \tag{60}$$

When $n = n'$, $Z = Z'$, and $\eta \to \infty$ we obtain

$$\mathcal{J} = Z/n^2\mathrm{a}\,. \tag{61}$$

When η is not infinite, the diagonal value approaches

$$\mathrm{a} \backsim (Z/n^2\mathrm{a})a^{2n-2l-2}\,, \tag{62}$$

where (for diagonal values only)

$$a = (1 + n/2Z\eta)^{-1}\,. \tag{63}$$

For off-diagonal values, employ (21) in (60).

[3] W. Gordon, Ann. Physik **2**, 1031 (1929).