Theory of Low Energy Nuclear Reactions

ERICH VOGT
Atomic Energy of Canada Limited, Chalk River, Ontario

I. INTRODUCTION

THROUGHOUT the evolution of nuclear reaction theory—from the emergence of the fundamental ideas in 1936, through the development of the basic theory and the subsequent adaptation of the theory to new ideas of nuclear structure—a large fraction of the important steps have been contributed by Wigner and his collaborators.[1] In this paper we shall describe the basic ideas of the theory, derive those results which have recently been found to be most suitable for the description of low energy nuclear reactions, and show how the basic theory of resonance reactions is adapted in a natural way to accommodate the ideas of nuclear structure which have been recently learned from empirical models such as the optical model.

The modern study of nuclear reactions began with the experimental discovery of large neutron cross sections by Fermi[2] and his collaborators and the explanation of the observed resonances by Bohr[3] and simultaneously by Breit and Wigner.[4] The physical picture underlying this explanation is called the compound nucleus.[5] It described the narrow closely-spaced resonances observed in the bombardment of various target nuclei in terms of the rapid sharing of the energy of the incident particle among the many target nucleons.

At the same time that they proposed the compound nucleus picture Breit and Wigner also gave a cross-section formula, the Breit-Wigner one-level formula, which was very successful in fitting a large number of experimental cross section data. However the cross section formula did not initially have a sound basis: It was based on the analogy of the resonance radiation of light by atoms for which case the interaction is weak, whereas the nuclear resonances arise from the strong nuclear reactions underlying the compound nucleus picture.

The formal theory of resonance reactions developed by Wigner and his collaborators,[6–12] Breit,[13] Bethe and Placzek,[14] and Kapur and Peierls[15] soon gave a firm foundation to the Breit-Wigner formula and allowed the calculation of compound nucleus processes more complicated than those connected with a single resonance line.

The various nuclear reaction theories[16] relied heavily on the compound nucleus picture. Because of the short range of the nuclear forces the compound nucleus (in which all the particles interact strongly) was assumed to lie entirely within a nuclear surface in the configuration space of all the particles. The various reaction alternatives or channels (in which two or more groups of particles move without mutual nuclear interactions) project from the nuclear surface in configuration space. The basic ideas of the resonance theories are quite simple. The formality or formidability of these theories rests mainly in the complicated nature of the configuration space, that is, in the large number of possible reaction alternatives which can customarily occur in nuclear reactions. In the next section, we present the basic ideas of the reaction theories with the example of the potential scattering of an s-wave neutron—a problem for which the configuration space is one dimensional. Section III presents a visualizable example of the configuration space for nuclear problems, an example which illustrates all the important points connected with the configuration space. With this preparation, the derivation of the principal results of low energy

[1] Because of the rapid publication schedule of this journal, Appendix B of the present article is probably the most recent contribution by Wigner in this field.
[2] E. Fermi *et al.* Ric. Sci. **5**, 282 (1934).
[3] N. Bohr, Nature **137**, 344 (1936).
[4] G. Breit and E. P. Wigner, Phys. Rev. **49**, 519 (1936).
[5] Surveys of the ideas which led to the compound nucleus picture have been given by E. P. Wigner, Am. J. Phys. **17**, 99 (1949); **23**, 371 (1955); and by V. F. Weisskopf and F. L. Friedmann in *Niels Bohr and the Development of Physics* (McGraw-Hill Book Company, Inc., New York, 1955).
[6] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **27**, 281 (1941).
[7] E. P. Wigner, Phys. Rev. **70**, 15 (1946).
[8] E. P. Wigner, Phys. Rev. **70**, 606 (1946).
[9] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[10] T. Teichmann, Phys. Rev. **77**, 506 (1950).
[11] T. Teichmann, Phys. Rev. **83**, 141 (1952).
[12] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).
[13] G. Breit, Phys. Rev. **58**, 1068 (1940); **69**, 472 (1946).
[14] H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).
[15] P. L. Kapur and R. E. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938).
[16] They are, more properly, mathematical frameworks which lead easily to reasonable nuclear reaction theories.

nuclear reaction theory, in Sec. IV, is relatively straightforward.

The calculation of nuclear cross sections is usually divided into two steps. In the first step the cross section is connected to the asymptotic parts of the wave function (far removed from the compound nucleus) by means of the collision matrix. The connection is a general one; that is, it has nothing to do with the nuclear problem in particular—it deals merely with the geometry of particle beams and detection apparatus. Partly because of the increasing importance of complicated experimental geometries as a tool in nuclear physics, the very elegant quantum theory of angular momentum has been developed by Wigner,[17] Racah,[18] and others,[19] to deal with the connection between cross sections and the components of the collision matrix. This first part of the cross-section calculation can readily be found in standard references[19] and will not be discussed here. The second step of the cross section calculation connects the collision matrix to the parameters of nuclear resonant states.

The various formal resonance theories differ in the details of the connection between the collision matrix and nuclear parameters. Almost all of the analyses of experimental data have employed the Wigner-Eisenbud[6,9] theory, and for the main part we shall discuss this particular theory in this paper. The central point of the Wigner-Eisenbud theory is that the connection between the collision matrix and the nuclear parameters is made through an intermediate quantity called the $\mathfrak{R}$ matrix.[20] As we shall see, the $\mathfrak{R}$ matrix is closely connected to the reciprocal of the logarithmic derivative of the wave function at the general nuclear surface in configuration space. The advantage of using the $\mathfrak{R}$-matrix theory rests partly in the simple manner in which it displays various fundamental conservation theorems. This fact is particularly important for cross sections near thresholds[21] or for elementary particle reactions in high energy physics which usually involve only few reaction channels.[22] However, for low-energy nuclear physics the number of possible reaction channels is usually quite large—commonly greater than the number of resonant states with an appreciable effect on the cross section at a particular energy. Then the $\mathfrak{R}$-matrix theory runs into difficulty because it requires the inversion of a matrix whose rows and columns refer to channels. It was first shown by Thomas[23] that the matrix inversion problem can be avoided by rewriting the collision matrix of Wigner and Eisenbud in a form which no longer contains the $\mathfrak{R}$ matrix explicitly. It is this latter form of the collision matrix which appears to be most useful for low-energy nuclear reactions. We shall give a simple derivation of this result, in Sec. IV, using the $\mathfrak{R}$-matrix theory in the intermediate steps. The same result has also been derived by means of formal scattering theory (without explicit use of the $\mathfrak{R}$ matrix) by Bloch.[24] For a considerable part of the theory of low-energy nuclear reactions the $\mathfrak{R}$ matrix appears to have lost its central role.

The various formal resonance theories were developed at a time when little was known about nuclear forces (except that they were strong and short range) or nuclear structure (except that very complicated compound states existed). More recently both the nuclear interaction and nuclear structure have been found to possess a partial simplicity. The interaction of a nucleon with the nucleus can at least be partly represented by the complex potential well of the optical model; correspondingly, many states at low energy appear to have a structure at least partially described by the nuclear shell model. The simplicity of the nuclear interaction has manifested itself further, during the past decade, in the discovery of a large number of new reaction mechanisms, called direct interactions,[25] in which the compound nucleus does not appear to play a role. Instead, the energy of the incident particle is shared with only one target nucleon, or one group of target nucleons, and the cross section can be calculated by Born approximation. These new ideas of reaction mechanisms and nuclear structure have appeared to clash in several ways with the resonance theories.

The difficulties presented to the resonance theories by the direct interactions (we use this term also for the phenomena described by the optical model) have been of two types. The first type concerns the description of both compound nucleus processes and direct interactions in the same language as any proper

[17] E. P. Wigner, *Group Theory*, translated from the German by J. J. Griffin (Academic Press Inc., New York, 1959).
[18] G. Racah, Phys. Rev. **62**, 438 (1942); **63** 367 (1943).
[19] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York 1959); M. E. Rose, *Angular Momentum* (John Wiley & Sons, Inc., New York, 1957); A. R. Edmunds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).
[20] A thorough review of all features of $\mathfrak{R}$-matrix theory has been given by A. M. Lane and R. G. Thomas, Revs. Modern Phys. **30**, 257 (1958).
[21] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).
[22] R. H. Dalitz, Revs. Modern Phys. **33**, 471 (1961).
[23] R. G. Thomas, Phys. Rev. **97**, 224 (1955).
[24] C. Bloch, Nuclear Phys. **4**, 503 (1957).
[25] An excellent survey of direct interactions has been given by N. Austern in *Fast Neutron Physics*, edited by J. B. Marion and J. L. Fowler (Interscience Publishers, Inc., New York, 1960).

theory of nuclear reactions must do. A fundamental justification of the optical model in terms of resonance parameters was given by Lane, Thomas and Wigner[26] very soon after the first thorough application of the optical model to average low energy cross sections by Feshbach, Porter, and Weisskopf.[27] Although the description of other direct interactions (e.g., deuteron stripping) in the language of resonance theory is far from complete, considerable progress on this problem has been made in recent years by Bloch,[24] Brown and de Dominicis,[28] and others. The second type of difficulty concerns the resonance reactions themselves and is connected with the rather artificial way in which the nuclear surface enters into the simplest results of the resonance theories. For example, in the Breit-Wigner one-level formula the potential scattering in between resonances is that of a hard sphere rather than that of a diffuse-edge potential well as we now feel it should be. Also the sharp division of configuration space into a compound nucleus region and a channel region leads to unphysical reflection of incident particles.[29] Our new knowledge of the nucleus tells us that the division of configuration space is not as sharp as the resonance theories originally supposed it to be.

Because of the above difficulties, several workers,[30-32] have recently attempted to develop new reaction theories which do not employ the division of configuration space of the older theories. However, these newer theories do not presently appear to be valid for many of the situations described adequately by the older theories. We shall show, in Sec. V, how the results of the Wigner-Eisenbud theory may be rewritten so that the surface does not appear so explicitly in the results and so that the basic ideas conform, for example, to the optical model.

The inclusion of these new ideas yields, in fact, a "natural" interpretation of quantities in the theory like the boundary condition numbers and the shift functions which formerly seemed somewhat arbitrary. With this new adaptibility there appears to be no reason why the formal resonance theory should not prove to be useful in low-energy nuclear reactions for many more years.

II. THE SCATTERING OF s-WAVE NEUTRONS BY A SQUARE WELL

The simple problem of the s-wave scattering of neutrons by a square potential well is useful both for the derivation of the main results of nuclear resonance theory and for its direct use in the theory of average cross sections. The application of the resonance theory to this problem will suffice to illustrate almost all the important points of the theory. Furthermore, although the interaction of a nucleon with a nucleus is not that of a simple real potential well, the recent success of the optical model has shown that the results of this simple picture have some applicability to nuclear problems. The results we shall derive here will be of direct use in the discussion of the optical model below. The difference between the present problem and the general nuclear problem is largely one of the geometry of configuration space as discussed in the next section. We choose the potential well to be square because, in doing so, we build into the example many of the ills customarily afflicting the theory of nuclear resonances. In Sec. V, we show how the scattering of s-wave neutrons by a diffuse-edge potential differs from the results of the present section. The comparison will serve to show how the artificial features of the nuclear surface may be removed from the nuclear resonance theory.

The radial part of the Schrödinger equation for the present problem may be written

$$-(\hbar/2m)d^2\phi/dr^2 + V\phi = E\phi \qquad (1)$$

where
$$V = -V_1 \qquad r \leqslant R_1$$
$$= 0 \qquad r > R_1$$

and where m is the neutron mass. The total wave function has to be regular at the origin so that inside the well we have

$$\phi = A \sin Kr \qquad r < R_1 , \qquad (2)$$

in which A is an arbitrary coefficient and K is the wave number inside the well ($E + V_1 = \hbar^2K^2/2m$). Outside the well, the wave function is a linear combination of incoming waves, I [$\equiv (4\pi v)^{-1/2}e^{-ikr}$] and outgoing waves O [$\equiv (4\pi v)^{-1/2}e^{ikr}$],

$$\phi = I - e^{2i\delta}O \qquad r > R_1 \qquad (3)$$

where δ is the phase shift. In the definition of I and O, k is the wave number ($E = \hbar^2k^2/2m$), v is the neutron velocity and the normalizing factor $(4\pi v)^{-1/2}$

[26] A. M. Lane, R. G. Thomas, and E. P. Wigner, Phys. Rev. **98**, 693 (1955).
[27] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).
[28] G. E. Brown and C. T. de Dominicis, Proc. Phys. Soc. (London) **A70**, 668 (1957); **A70**, 686 (1957); **A72**, 70 (1958); and G. E. Brown, C. T. de Dominicis, and J. S. Langer, Ann. Phys. (N. Y.) **6**, 209 (1959).
[29] E. Vogt, Physics Letters **1**, 84, (1962). Some similar results are reported in an earlier paper by D. C. Peaslee, Nuclear Phys. **3**, 255, (1957).
[30] H. Feshbach, Ann. Phys. (N. Y.) **5**, 357 (1959).
[31] R. E. Peierls, Proc. Roy. Soc. (London) **A253**, 16 (1959).
[32] J. Humblet and L. Rosenfeld, Nuclear Phys. **26**, 529, (1961).

is associated with both I and O so that each corresponds to unit flux. By matching the slope and value of (3) and (2) at $r = R_1$, we find the square-well phase shift

$$\delta = \tan^{-1}[(k/K)\tan KR_1] - kR_1 . \quad (4)$$

The scattering cross section σ is related to the phase shift by

$$\sigma = (4\pi/k^2)\sin^2\delta , \quad (5)$$

which together with (4) completely solves the scattering problem of the square well.

In the language of the resonance theories, the solution just obtained would be written in terms of the collision function U, related to the phase shift by

$$U \equiv e^{2i\delta} \quad (6)$$

and, in turn, the cross section (5) then is

$$\sigma = \pi/k^2|1 - U|^2 . \quad (7)$$

It is the generalization of (6) and (7) to the nuclear problem which forms the two main steps of the calculation of nuclear cross sections in the formal theory of resonances. The collision function becomes a collision matrix and its connection to the various reaction cross sections, the analog of (7), is a question of geometry which will not be discussed here. The physics of the problem is in the connection of the collision matrix to nuclear parameters, analogous in the present problem to the connection between U and the properties of the well, as in (6) and (4).

It is evident from the result (5) that the square well scattering cross section has a maximum, or resonance, whenever δ is equal to $(n + \frac{1}{2})\pi$ where n is an integer. For low energies (kR_1 small compared to unity) such a maximum will occur for values of K or R_1 such that $KR_1 \approx (n + \frac{1}{2})\pi$. [An illustration of the $n = 4$ square-well resonance is given below, Fig. 3, in the application of resonance theory to the square well problem]. The resonance condition on KR_1 is a natural consequence of the fact that K is much larger than k. As Fig. 1 shows, the wave function inside the potential well is large only when KR_1 achieves the above condition, that is when the wave function has zero slope at R_1. The nonresonant wave function, with nonzero slope at R_1, rises outside the well to a much larger amplitude than inside.

The neutron wave function, Fig. 1, at the resonance of the scattering cross section may be thought of as almost a standing wave. The reflection which occurs at the edge of the well keeps the neutron in the well for a long time before it finally leaks out. Because the neutron eventually does leak out of the well, the wave function is not exactly a standing

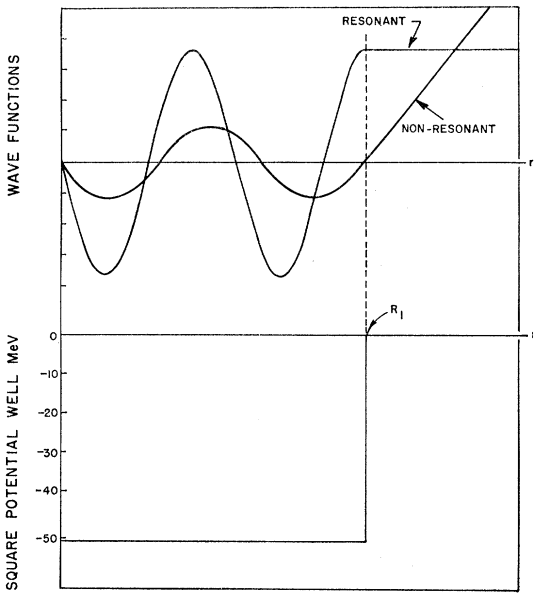

FIG. 1. The square potential well of Eq. (1) and the resonant and nonresonant wave functions at low energies. R_1 is the square-well radius. For low energies, the wave function turns over very slowly outside the well and then the amplitude is large inside only for zero derivative at R_1, as shown.

wave. However, we can construct a complete set of standing waves of which one, in particular, resembles the actual wave function of the low energy resonance very closely. When this is so, then we can show that in the "Fourier" expansion of the actual wave function in terms of the standing waves one term dominates the expansion and this term corresponds to the one standing wave identified with the low-energy resonance. In the nuclear problem the standing waves are the resonance levels of the compound nucleus and the retention of only one term in the Fourier expansion leads to the Breit-Wigner formula.

To construct the standing waves, X_λ, for the square-well problem we use the Schrödinger equation

$$-(\hbar^2/2m)d^2X_\lambda/dr^2 + VX_\lambda = E_\lambda X_\lambda \quad (8)$$

together with a boundary condition at the square well radius, R_1

$$rdX_\lambda/dr = bX_\lambda|_{r=R_1} \quad (9)$$

where b is a real number, so far arbitrary. As we shall see below, and as Fig. 1 suggests, the natural value of b for the present problem is $b = 0$. Then the normalized standing waves are

$$X_\lambda = (2/R_1)^{1/2}\sin K_\lambda r \qquad (b = 0) \quad (10)$$

where

$$K_\lambda = (\lambda + \tfrac{1}{2})\pi R_1^{-1} . \quad (11)$$

Figure 2 shows the lowest few standing waves and

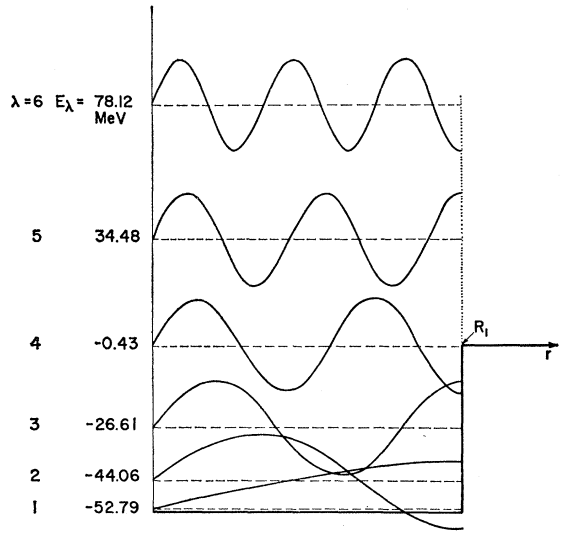

FIG. 2. The first six standing waves of the square well, constructed with a boundary condition number ($b = 0$) appropriate to low-energy scattering.

their characteristic energies for a square potential well with a depth and radius roughly appropriate for the shell model (or real part of the optical model well) of a nucleus with atomic weight $A = 155$. In addition to the state near zero energy there are a few bound states as well as a discrete infinity of states at positive energy. The actual physical problem of a square well of the size shown on Fig. 2 has proper bound states which are *not* those shown. The difference arises because of the boundary condition (9) which we have added to our problem. The physical problem is Hermitian in all space ($0 \leqslant r \leqslant \infty$) and the proper bound levels decay exponentially for $r > R_1$. By means of the boundary condition (9) we make the problem Hermitian in the finite volume ($0 \leqslant r \leqslant R_1$). We do this to construct standing waves which can be used to expand the actual wave function within the square well. If the boundary condition number is properly chosen then the unbound standing waves will bear a close physical resemblance to the wave function at $E = E_\lambda$. However, the bound standing waves should not be connected in a detailed manner with the bound states of the nuclear shell model.

Because the X_λ form a complete orthonormal set of functions, the actual wave function ϕ for $r < R_1$, may be expanded in terms of them

$$\phi = \sum_\lambda C_\lambda X_\lambda \tag{12}$$

with the expansion coefficients

$$C_\lambda = \int_0^{R_1} X_\lambda^* \phi dr \,. \tag{13}$$

Multiplying (1) by X_λ^*, and the complex conjugate of (8) by ϕ, subtracting and integrating we obtain

$$(\hbar^2/2m)[\phi dX_\lambda^*/dr - X_\lambda^* d\phi/dr]_{r=R_1} = (E - E_\lambda) \times \int_0^{R_1} \phi X_\lambda^* dr \,, \tag{14}$$

from which it follows that

$$C_\lambda = (E_\lambda - E)^{-1}(\hbar^2/2mR_1)X_\lambda^*(R_1) \times [\phi'(R_1) - b\phi(R_1)] \,, \tag{15}$$

where ϕ' means $r\, d\phi/dr$. Using this value of C_λ in the Fourier series (12), evaluated at R_1, we obtain

$$\phi(R_1) = \mathfrak{R}[\phi'(R_1) - b\phi(R_1)] \tag{16}$$

where

$$\mathfrak{R} \equiv \sum_\lambda \gamma_\lambda^2/(E_\lambda - E) \tag{17}$$

and

$$\gamma_\lambda^2 \equiv (\hbar^2/2mR_1)|X_\lambda(R_1)|^2 \,. \tag{18}$$

The quantity $\mathfrak{R}$ is the $\mathfrak{R}$ function of the square well problem and the γ_λ^* are the reduced widths,[33] the nuclear parameters of interest in resonance reactions. For $b = 0$, it is clear from (10) that all the reduced widths of the square well problem are equal to $\hbar^2/mR_1^2$.

The $\mathfrak{R}$ function, as defined by (17), is simply the Fourier expansion of that part of the phase shift which refers to the properties of the square well. Thus, we can write the phase shift (4) in the following way:

$$\delta = \tan^{-1}[kR_1\mathfrak{R}/(1 + b\mathfrak{R})] - kR_1 \,. \tag{19}$$

We could then at once write the collision function U in terms of $\mathfrak{R}$ by means of (6). Instead, we shall make the connection by a means more easily adapted to the nuclear problem.

According to (16), the logarithmic derivative ϕ'/ϕ at $r = R$ may be written in terms of $\mathfrak{R}$ in the following way

$$\phi'(R_1)/\phi(R_1) = (1 + b\mathfrak{R})/\mathfrak{R} \,. \tag{20}$$

We next rewrite (3) as

$$\phi = I - UO \tag{21}$$

[33] It may be noted that the reduced widths of (18) and of the subsequent parts of the present paper differ by a factor of R_1^{-1} from the reduced widths employed by Wigner and his collaborators in the early papers on resonance theory. The present definition, which is also that employed by Lane and Thomas,[20] has the advantage that all reduced widths have the dimension of energy and all penetrabilities are dimensionless. On the other hand, the earlier definition had the advantage that for s-wave neutrons the penetrabilities were independent of the nuclear radius.

and then matching the logarithmic derivative of (21), at $r = R_1$, with (20) we obtain

$$\begin{aligned} U &= (O + b\mathfrak{R}O - \mathfrak{R}O')^{-1}(I + b\mathfrak{R}I - \mathfrak{R}I') \\ &= O^{-1}(1 - \mathfrak{R}L)^{-1}(1 - \mathfrak{R}L^*)I \\ &= e^{-2ikR_1}\frac{1 + b\mathfrak{R} + i\mathfrak{R}kR_1}{1 + b\mathfrak{R} - i\mathfrak{R}kR_1}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} L &\equiv O'O^{-1} - b \\ &= -b + ikR_1 . \end{aligned} \tag{23}$$

We can exhibit the resonances in the cross section by assuming that the energy E is close to one particular E_λ and by then approximating $\mathfrak{R}$ by one term

$$\mathfrak{R} \approx \gamma_\lambda^2/(E_\lambda - E) . \tag{24}$$

Using this value of $\mathfrak{R}$ in (22) and then using the resulting collision function in the total scattering cross section (7), yields the Breit-Wigner formula for the present problem:

$$\sigma = \frac{\pi}{k^2}\left|2 \sin kR_1 e^{ikR_1} - \frac{\Gamma_\lambda}{(E_\lambda - E + \Delta_\lambda) - \frac{1}{2}i\Gamma_\lambda}\right|^2, \tag{25}$$

where

$$\begin{aligned} \Gamma_\lambda &\equiv 2kR_1\gamma_\lambda^2 \\ \Delta &\equiv b\gamma_\lambda^2 \end{aligned} \tag{26}$$

are, respectively, the level width and the level shift of the square-well problem. The term $2 \sin kR_1 e^{ikR_1}$, in (25) represents potential scattering; it is, in fact, the scattering due to a hard sphere of radius R_1.

A comparison of the exact square well cross section with the Breit-Wigner formula (for various values of b) is shown on Fig. 3. Instead of varying the energy for a fixed radius the figure gives cross sections at fixed energies for varying R_1: the consequent display shows resonant structure quite reminiscent of the ones commonly observed in nuclear physics. The square-well parameters have been chosen similar to those of Fig. 2 and of the further figures below so that the resonance shown corresponds to the well known $4s$ neutron giant resonance found in the rare-earth region of the atomic table. The figure shows that the one-level Breit-Wigner formula, for the square-well problem, is a good approximation if and only if b is approximately zero. For this value of b, also, the level shift vanishes identically. These facts, together with the discussion of Fig. 1 above, establish $b = 0$ as a natural boundary condition for the present problem.

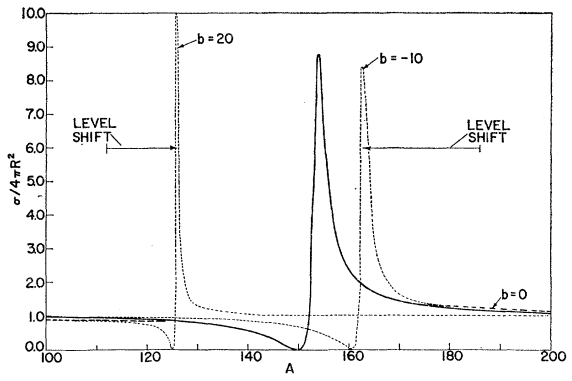

FIG. 3. The s-wave scattering cross sections for a nucleon by a square well of depth 51 MeV and a width $R_1 = 1.25\ A^{1/3}$ F, where A is the atomic weight. The solid line gives the exact cross section (divided by $4\pi a^2$) at 50 keV as a function of A. The broken lines give the one-level approximation to the cross section near the $4s$ resonance for various values of the boundary condition number b. The heavy broken line, corresponding to $b = 0$, merges with the solid line near the resonance.

Much of the content of resonance theory is already contained in the above example. Most of the remainder of the theory has to do with the complicated geometry of nuclear configuration space. In the square-well problem, the nuclear surface on Fig. 1 is the point $r = R_1$, whereas the general nuclear surface is multi-dimensional and all the channels, c, project from this surface. In each channel the wave function is a product of a radial wave function ϕ_c and a channel wave function ψ_c containing the remaining variables. As we shall see the channel wave functions form a complete orthogonal set and may be regarded as unit vectors identifying each channel. The above derivation for the collision matrix still holds if we regard each quantity in (16), (21), (22), and (23) as a matrix with components in the space of the ψ_c. Then the result for the collision matrix is exactly given by the second step of (22). Before going through these steps for the nuclear problem we discuss, in the next section, the properties of the nuclear configuration space and of the channel functions ψ_c.

Although the final result of the nuclear problem has the same form as that found in the present section the exact solution of the Schrödinger equation for the nucleus is not usually possible. In analyzing nuclear reactions the reduced widths and characteristic energies are treated as phenomenological parameters. The usefulness of the resonance theory rests in the easy way in which it allows the extraction of nuclear parameters.

III. CONFIGURATION SPACE FOR NUCLEAR REACTIONS

Because of the short-range of nuclear forces, the compound nucleus occupies only a small region $\mathcal{V}$ of the $3A$-dimensional configuration space of its A nucleons. In the configuration space the various re-

action channels, or reaction alternatives, emerge from the surface $\mathcal{S}$ which encloses $\mathcal{V}$. In this section, we want to make explicit the concept of a channel and the wave function, ψ_c, which labels each channel; further we want to describe, with a simple example, the important properties of the configuration space. When this is done, the derivation of the results of nuclear reaction theory is an almost trivial generalization of the steps carried out in the preceding section.

Roughly speaking, we choose the compound nucleus surface $\mathcal{S}$ so that all the nuclear interactions take place only within $\mathcal{V}$. Outside $\mathcal{S}$ the nucleons are split into two or more groups with only well known interactions between the groups. For low energy nuclear reactions three-body breakup is usually not allowed energetically—and even when it is allowed the process may often be described in terms of two successive two-body breakups—so that we shall consider only two-body breakup of the compound nucleus.[34]

In each channel the two groups of particles have a wave function describing their relative motion, their internal motions and their intrinsic spins. We use the term channel, denoted by c, to represent the collection of quantum numbers describing these motions. In the channel region of configuration space we split up the total wave function ψ, in the following way:

$$\Psi = \sum_c \psi_c \phi_c , \tag{27}$$

where ϕ_c is a solution of the radial equation in each channel and ψ_c is the wave function appropriate to the other quantum numbers. In particular, if r_c is the radius variable in each channel, we write

$$\psi_c = r_c^{-1} \Phi_\alpha \sum_{m_l} \sum_{m_s} (lsm_l m_s | Jm_J) i^l Y_{lm_l} X_{sm_s} \tag{28}$$

where the set of channel quantum numbers is subdivided, $c \equiv (\alpha, l, s, J, m_J)$. Φ_α represents the state of internal motion of the two particles in the channel, $i^l Y_{lm_l}$ is a spherical harmonic describing their relative orbital angular momentum l, and its z component, m_l; χ_{sm_s} is the wave function of the coupled intrinsic spins, $\mathbf{I}$ and $\mathbf{i}$ of the two particles, $(\mathbf{s} = \mathbf{I} + \mathbf{i})$; $(lsm_l m_s | Jm_J)$ is a Clebsch-Gordan coefficient; J and m_J are, respectively, the total angular momentum of the whole system and its z component, both of which are conserved in nuclear reactions.

In splitting up the total wave function as in (27), the ψ_c may be regarded as a set of unit vectors, labeling the channels and the physical content of the theory is contained in the radial wave functions giving the amplitude in each channel. In keeping with this idea we define the channel surface, $\mathcal{S}_c$, by a radius, R_c, in each channel. The whole surface $\mathcal{S}$ enclosing the compound nucleus is composed of the sum of the $\mathcal{S}_c$ as will become clear shortly. The choice of the channel radius is arbitrary, for the moment, although we have assumed it to be sufficiently large so that no strong nuclear interactions take place between the two groups of particles in the channel.

The above decomposition of the total wave function in the channel region of configuration space will be useful for the theory of nuclear reactions if and only if the channel wave functions ψ_c satisfy the following orthogonality condition on the surface $\mathcal{S}$:

$$\int_{\mathcal{S}} \psi_c \psi_{c'}^* d\mathcal{S} = \delta_{cc'} . \tag{29}$$

Equation (29) is only approximate and will be seen (below) to hold if the channel radii are chosen large enough. It is the orthogonality condition (29) which ensures that the ψ_c are the set of unit vectors of which we have been speaking. If (29) is satisfied on $\mathcal{S}$ and on any larger surface, then the channels are independent arms projecting from the compound nucleus.

To give a more explicit picture of the nuclear configuration space and to show how and why the orthogonality condition (29) holds, we consider a simple example in which the details of the configuration space can be drawn. In the example we take four nucleons,[35] each moving in a world of one dimension instead of three. As the top part of Fig. 4 shows, the particles are chosen to be proton (spin-up), proton (spin-down), neutron (spin-up), and neutron (spin-down) and their position along the line which constitutes their world are, respectively, x_1, x_2, x_3, and x_4. The four particles form the compound nucleus He^4. Because the center of mass of the whole system is chosen to be at rest, at the origin

$$x_1 + x_2 + x_3 + x_4 = 0 \tag{30}$$

the entire configuration space of the system has only

[34] The three-body breakup problem in nuclear reaction theory has been considered by Wigner in collaboration with some of his students. See the Princeton Ph.D. dissertations of G. A. Snow (1950) and A. Sauer (1958).

[35] If we chose to consider three nucleons instead of four, the configuration space of the example would be only two dimensional, and simpler than the four-particle example which is discussed. However, for three particles, unlike for four, the center of mass cannot be removed in a symmetrical way and the beautiful symmetries of the figure would disappear.

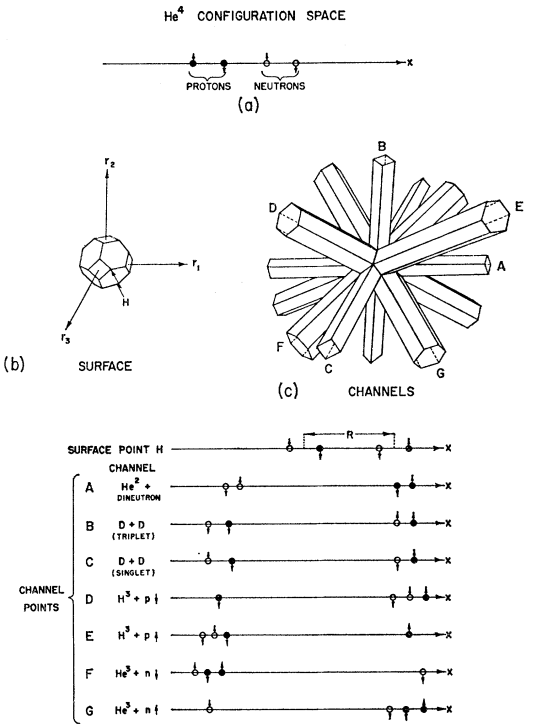

FIG. 4. The configuration space for four nucleons restricted to move on a line as in (a). The nucleons are, as shown, a proton with spin up, a proton with spin down, a neutron with spin up, and a neutron with spin down. The compound system of the four may be thought of as He^4. (b) gives the actual surface of the compound nucleus and (c) the channels emerging from the compound nucleus. The intersection of the channels forms the surface. The bottom part of the figure gives the relative positions of the particles at the surface point H and at a point in each of the channels.

three dimensions. We choose the three coordinates which map out the space to be

$$\begin{aligned} r_1 &= \tfrac{1}{2}(x_1 + x_2 - x_3 - x_4) \\ r_2 &= \tfrac{1}{2}(x_1 - x_2 + x_3 - x_4) \\ r_3 &= \tfrac{1}{2}(x_1 - x_2 - x_3 + x_4)\,, \end{aligned} \tag{31}$$

which are easily solved to yield

$$\begin{aligned} x_1 &= \tfrac{1}{2}(r_1 + r_2 + r_3) \qquad x_2 = \tfrac{1}{2}(r_1 - r_2 - r_3) \\ x_3 &= \tfrac{1}{2}(-r_1 + r_2 - r_3) \qquad x_4 = \tfrac{1}{2}(-r_1 - r_2 + r_3)\,. \end{aligned} \tag{32}$$

The various possible channels emerging from the He^4 compound nucleus consist of the emission of neutrons, protons, deuterons or He^2. In accordance with the ideas expressed above for the general nuclear problem, we choose each channel surface at a separation, R, between the centers of mass of the two fragments in the channel. The resulting nuclear surface S, as found by means of (31) and (32), is shown in Fig. 4(b). It is a truncated octahedron.[36] The various channels of the system are shown on Fig. 4(c). The emission of a neutron or a proton occurs from the square channel surfaces on the three axes and the emission of two groups of two nucleons each occur from the hexagonal surfaces. The lower part of the figure identifies the various channels. Each channel is a tube in the space and the nuclear surface S is the intersection of all the possible tubes.

There are large regions of configuration space on Fig. 4 which contain neither the compound nucleus nor the channels. These regions correspond to break-up into three or more particles and are forbidden energetically. Even within each channel or tube the wave function will be small near the edge of the tube (compared to its value on the axis) if we choose the channel radii R large enough. The approximate orthogonality of the wave functions, ψ_c, corresponding to different channels arises from the same fact. We note first that each ψ_c is defined in the whole of the configuration space but is large only within its particular tube because of the energetics of the problem. Hence, if we choose a surface S large enough so that the various intersections of the channels with S do not overlap, the orthogonality condition (29) follows. The surface of Fig. 4(b) is the minimum surface of this kind. If we choose R so that the channel wave functions are large beyond their particular tubes or if we take S to be smaller than the surface of Fig. 4(b), then the orthogonality condition breaks down.

The orthogonality condition of our simple model, as just demonstrated, corresponds to the only part of the general nuclear orthogonality condition, (29), which poses a problem. That is, the orthogonality between the ψ_c of Fig. 4 is that between surface wave functions corresponding to different pairs of particles. By making the space three dimensional and increasing the number of particles, we still obtain the same kind of orthogonality for ψ_c corresponding to different pairs. The remaining orthogonality, that between channel wave functions of the same pair but with $(E_\alpha, s, l, J, m_J) \neq (E_{\alpha'}, s', l', J', m_{J'})$, is exact and due to, for example, the conventional orthogonality of angular momentum wave functions. The generalization of the example to the larger number of dimensions inherent in the nuclear problem is therefore straightforward.

[36] The occurrence of this figure in the present context leads one to the conjecture that all of Wigner's theories eventually lead to truncated octahedrons. Casual readers should here avoid the trap that the nucleus is a body-centered cubic lattice. [Cf. E. P. Wigner and F. Seitz, Phys. Rev. **43**, 804 (1933).]

IV. THE COLLISION MATRIX FOR LOW-ENERGY NUCLEAR REACTIONS

Having demonstrated the independence of the various channels, that is, the orthogonality of the channel wave functions ψ_c, we can now derive the collision matrix for low energy nuclear reactions by repeating the steps of Sec. II, replacing functions everywhere by vectors in the space of the ψ_c. First, we use Green's theorem to connect the logarithmic derivative of the total wave function on $\mathcal{S}$ to the $\mathfrak{R}$ matrix; then we match internal and external wave functions on $\mathcal{S}$ to obtain the connection between the collision matrix and the $\mathfrak{R}$ matrix.

The total wave Ψ, inside the nucleus, satisfies the same Schrödinger equation as the resonant states X_λ:

$$H\Psi = E\Psi \qquad \text{[cf. (1)]} \tag{33}$$

$$HX_\lambda = E_\lambda X_\lambda \qquad \text{[cf. (8)]}\,. \tag{34}$$

In addition, the X_λ satisfy the following boundary condition (on each $\mathcal{S}_c$) which makes them stationary:

$$r_c dX_\lambda/dr_c = b_c X_\lambda|\mathcal{S}_c \qquad \text{[cf. (9)]}\,, \tag{35}$$

where b_c is the boundary condition number appropriate to the channel c. With (35) the states X_λ form a complete set of functions inside the nucleus,

$$\Psi = \sum_\lambda C_\lambda X_\lambda \qquad \text{[cf. (12)]}\,, \tag{36}$$

where the energy dependent coefficients C_λ are:

$$C_\lambda \equiv \int_{\mathcal{V}} X_\lambda^* \Psi d\mathcal{V} \qquad \text{[cf. (13)]} \tag{37}$$

in which the integration extends over the entire compound nucleus volume $\mathcal{V}$. We multiply (33) by X_λ^*, and the complex conjugate of (34) by Ψ, subtract and integrate over the whole nuclear volume to obtain

$$\begin{aligned}(E_\lambda - E)\int_{\mathcal{V}} X_\lambda^*\Psi d\mathcal{V} &= \sum_c \int_{\mathcal{S}_c}\left(\frac{\hbar^2}{2m_c r_c}\right) \\ &\times [X_\lambda^*\Psi' - \Psi X_\lambda'^*]d\mathcal{S}_c \\ &= \sum_c (\hbar^2/2m_c r_c)^{1/2}\gamma_{\lambda c}(\phi_c' - b_c\phi_c)\,,\end{aligned} \tag{38}$$

where the prime means the derivative $r_c d/dr_c$ in the channel c and where the reduced width amplitude $\gamma_{\lambda c}$ is defined by

$$\gamma_{\lambda c} \equiv (\hbar^2/2m_c r_c)^{1/2}\int \psi_c^* X_\lambda d\mathcal{S} \qquad \text{[cf. (18)]}\,. \tag{39}$$

The ϕ_c of (38) are those of (27). From (38) and (37) we obtain

$$C_\lambda = (E_\lambda - E)^{-1}\sum_c \gamma_{\lambda c}(\phi_c' - b_c\phi_c)(\hbar^2/2m_c r_c)^{1/2} \qquad \text{[cf. (15)]}\,. \tag{40}$$

Using this value of C_λ in (36) and evaluating (36) on the surface $\mathcal{S}$, we obtain

$$\begin{aligned}(\hbar^2/2m_c r_c)^{1/2}\phi_c &= \sum_{c'} \mathfrak{R}_{cc'}[\phi_{c'}' - b_{c'}\phi_{c'}] \\ &\times (\hbar^2/2m_{c'}r_{c'})^{1/2} \qquad \text{[cf. (16)]}\,,\end{aligned} \tag{41}$$

where the $\mathfrak{R}$ matrix is

$$\mathfrak{R}_{cc'} \equiv \sum_\lambda \gamma_{\lambda c}\gamma_{\lambda c'}/(E_\lambda - E) \qquad \text{[cf. (17)]}\,. \tag{42}$$

For the development of the theory, below, it will be important that (42) is a sum of direct products of the vectors $\gamma_{\lambda c}$ and $\gamma_{\lambda c'}$.

Before connecting the $\mathfrak{R}$ matrix to the collision matrix we define the latter in terms of the wave function in the channel region of configuration space. We begin by rewriting (27) in the form

$$\Psi = \sum_c v_c - \tfrac{1}{2}(A_c I_c - B_c O_c)\psi_c\,, \tag{43}$$

where the A_c and B_c are amplitude coefficients and where I_c and O_c are, respectively, the incoming and outgoing parts of the radial wave function. That is, at large distance

$$I_c^* = O_c \approx \exp i(k_c r_c - \tfrac{1}{2}l_c\pi - \eta_c \ln 2k_c r_c)\,, \tag{44}$$

where k_c is the wave vector of the channel c, l_c the relative orbital angular momentum of the two particles in c, and

$$\eta_c \equiv Z_1 Z_2 e^2/\hbar v_c \tag{45}$$

is the Coulomb parameter which depends on the charges Z_1 and Z_2 of the pair of reaction products and on their relative velocity v_c. If the particles are uncharged or if their center-of-mass separation r_c is so large that the Coulomb fields are screened, then η_c vanishes. The factor $v_c^{-1/2}$ is included in (43) so that if A_c or B_c have unit value then the corresponding wave function has unit flux.

In a nuclear reaction we have incoming beams in some channels and outgoing beams in all channels for which the decay is allowed energetically. Whatever the incoming channels and their coefficients, the solution of the Schrödinger equation everywhere would yield all the coefficients, B_c, of the outgoing waves. This fact is expressed by writing

$$B_c = \sum_{c'} U_{cc'}A_{c'}\,, \tag{46}$$

where $U_{cc'}$ is the collision matrix. In the nuclear problem we do not attempt to solve the Schrödinger equation everywhere. Instead, we connect the collision matrix, via the $\mathfrak{R}$ matrix, to the nuclear parameters in such a way that useful approximations may be made. By means of (46), multiplication by ψ_c^*, and integration over $\mathcal{S}$, (43) becomes

$$\phi_c = v_c^{-1/2}(A_c I_c - \sum_{c'} U_{cc'}A_{c'}O_c) \quad \text{[cf. (3)(6)]}\,. \tag{47}$$

We now take the derivative of (47) and match the slope and value of the ϕ_c of (47) with those of (41) to obtain

$$U = (kr)^{1/2}O^{-1}(1 - \Re L)^{-1}(1 - \Re L^*)I(kr)^{-1/2} \quad \text{[cf. (22)]}, \tag{48}$$

where all the quantities are now matrices in channel space. In particular, $(kr)^{1/2}$ and $(kr)^{-1/2}$ are the diagonal matrices with diagonal components $(k_c r_c)^{1/2}$ and $(k_c r_c)^{-1/2}$, respectively, O is the diagonal matrix with diagonal components O_c, I is equal to O^*, and L is the diagonal matrix with diagonal components

$$L_c \equiv O_c' O_c^{-1} - b_c \equiv S_c + iP_c, \tag{49}$$

where S_c is the shift function, which leads to level shifts and P_c is the penetrability, which leads to level widths.

The result (48) for the collision matrix was the one originally obtained by Wigner and Eisenbud.[9] For reactions involving only a few channels it has been used to analyze cross sections, not only for individual levels but also for the case in which levels of the same spin and parity interfere with each other.[37] However, when the number of channels gets to be large, the inversion of the matrix $(1 - \Re L)^{-1}$, in (48) is exceedingly laborious. Since many nuclear reactions involve a large number of channels (usually, perhaps more channels than compound levels) we derive a form for the collision matrix which avoids the inversion of the channel matrix $(1 - \Re L)^{-1}$.

To obtain the new form[23] of the collision matrix, we begin by assuming that a level matrix $A_{\lambda\lambda'}$ exists satisfying the relationship

$$[(1 - \Re L)^{-1}(1 - \Re L^*)]_{cc'} = \delta_{cc'} + \sum_{\lambda\lambda'} 2iP_{c'} \times \gamma_{\lambda c}\gamma_{\lambda c'}A_{\lambda\lambda'}. \tag{50}$$

We show that (50) exists and derive the form of A. Multiplying both sides of (50), from the left, by matrix $(1 - \Re L)$, we obtain

$$\delta_{cc'} - \sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E} L_{c'}^* = \delta_{cc'} - \sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E} L_{c'} + \sum_{\lambda\lambda'} 2iP_{c'}\gamma_{\lambda c}\gamma_{\lambda' c'}A_{\lambda\lambda'} - \sum_{\lambda\lambda'}\sum_{c''} \frac{\gamma_{\lambda c}\gamma_{\lambda c''}}{E_\lambda - E} 2iP_{c'}\gamma_{\lambda'' c''}\gamma_{\lambda' c'}A_{\lambda''\lambda'}. \tag{51}$$

Upon collecting terms, we find

$$\sum_{\lambda\lambda'} 2iP_{c'} \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E} [\delta_{\lambda\lambda'} - (E_\lambda - E)A_{\lambda\lambda'} + \sum_{\lambda''} \xi_{\lambda\lambda''}A_{\lambda''\lambda'}] = 0 \tag{52}$$

where

$$\xi_{\lambda\lambda''} \equiv \sum_{c''}\gamma_{\lambda c''}\gamma_{\lambda'' c''}L_{c''} \equiv -\Delta_{\lambda\lambda'} + (i/2)\Gamma_{\lambda\lambda'}. \tag{53}$$

Since (52) must hold for arbitrary values of $\gamma_{\lambda c}$ and $\gamma_{\lambda' c'}$ we can set the square bracket of (52) equal to zero separately, yielding

$$(A^{-1})_{\lambda\lambda'} = (E_\lambda - E)\delta_{\lambda\lambda'} + \Delta_{\lambda\lambda'} - (i/2)\Gamma_{\lambda\lambda'} \tag{54}$$

thus proving the conjecture that (51) exists. The existence of the $A_{\lambda\lambda'}$ depends crucially on the fact that $\Re$ is a sum of direct products of $\gamma_{\lambda c}$ and $\gamma_{\lambda c'}$.

By means of (50) and the result (54) for the level matrix $A_{\lambda\lambda'}$ we have changed the problem of inverting the channel matrix $(1 - \Re L)^{-1}$ into the problem of inverting the level matrix A^{-1}. To complete the derivation of an especially simple form for the collision matrix we next simplify the factor, $O_c^{-1} I_{c'}$, of (48). The incoming and outgoing waves[38] of the various channels can be written in terms of the regular solution, F_c, and the irregular solution G_c, of the radial equation in the following way:

$$I_c = O_c^* = (G_c - iF_c)e^{i\omega_c}, \tag{55}$$

where ω_c is the Coulomb phase shift related to the orbital angular momentum quantum number l_c and the Coulomb parameter, η_c [Eq. (45)] by

$$\omega_c \equiv \sum_{n=1}^{l_c} \tan^{-1}(\eta_c/n). \tag{56}$$

The functions F_c and G_c are normalized by their Wronskian,

$$F_c'G_c - G_c'F_c = k_c r_c, \tag{57}$$

where the prime again means the derivative $r_c d/dr_c$. Then it follows at once from (55) and (49) that the penetrability P_c and shift function S_c are given by

$$P_c = k_c r_c [F_c^2 + G_c^2]^{-1} \tag{58}$$

$$S_c = -b_c + (F_c F_c' + G_c G_c')/(F_c^2 + G_c^2), \tag{59}$$

where both P_c and S_c are evaluated at the channel radius $r_c = R_c$.

Using these properties of the Coulomb wave functions we then also find

$$(k_c r_c)^{1/2} O_c^{-1} I_{c'} (k_{c'} r_{c'})^{-1/2} = e^{i(\omega_c + \omega_{c'})}(k_c r_c / k_{c'} r_{c'})^{1/2} \times (G_{c'} - iF_{c'})/(G_c + iF_c) = e^{i(\Omega_c + \Omega_{c'})} P_c^{1/2} P_{c'}^{-1/2} \tag{60}$$

where

$$\Omega_c \equiv \omega_c - \tan^{-1}(F_c/G_c) \tag{61}$$

[37] R. Krotkov, Can. J. Phys. **33**, 622 (1955); M. S. Moore and C. W. Reich, Phys. Rev. **118**, 718 (1960).

[38] For a more complete discussion of the channel wave functions see reference 20.

is the sum of a Coulomb phase shift and a hard-sphere phase shift.

The product of (60) and (50) yields at once the desired formula for the collision matrix components.

$$U_{cc'} = e^{i(\Omega_c+\Omega_{c'})}\{\delta_{cc'} + i\sum_{\lambda\lambda'}\Gamma_{\lambda c}^{1/2}\Gamma_{\lambda' c'}^{1/2}A_{\lambda\lambda'}\}\,, \quad (62)$$

where

$$(\Gamma_{\lambda c})^{1/2} \equiv (2P_c)^{1/2}\gamma_{\lambda c} \quad (63)$$

has the same sign as the reduced width amplitude $\gamma_{\lambda c}$.

The result (62) is useful for a large number of problems in the theory of low-energy nuclear reactions. Retaining only one level λ in the various sums over levels it leads at once to the familiar Breit-Wigner formula, regardless of the number of channels. Multilevel formulas are derived from (62) by retaining only a few levels in (62) and (54). Such formulas have been quite successful[39] in fitting nuclear cross sections which deviate considerably from the simple shape of the Breit-Wigner formula. For average cross sections the results of the statistical theory of nuclear reactions have been recently refined[20] by use of (62).

In the derivation of the results above we have completely ignored the closed channels. The closed channels are, of course, important in the consideration of threshold effects and have been treated in detail by Teichmann and Wigner[12] and by Wigner.[40] For the closed channels, the penetrability vanishes identically and we then choose the boundary condition numbers b_c, for these channels so that the shift is also zero [cf. (59)].

$$b_c = (F_cF_c' + G_cG_c')/(F_c^2 + G_c^2) \quad (64)$$
$$= W_c'/W_c \text{ (closed channels)},$$

where W_c is the Whittaker function,[20] that is, an "exponentially decaying" Coulomb wave function. With this choice of b_c the closed channels can be essentially disregarded, since L_c is identically zero for them. The simplicity of this choice of boundary condition number for the closed channels was already noted by Teichmann and Wigner[12] and has since been used by others; for example by Bloch, who, in deriving the result (62) by formal scattering theory, treated the closed channels in the same way. It is this treatment which makes our derivation of (62) like Bloch's, appear to be much simpler than the more rigorous treatments which considered the closed channel. The boundary condition (62) is energy dependent, a disadvantage which is overwhelming in the consideration of threshold effects. However, for most nuclear reactions, the use of (64) will not lead to any serious error. In fact, we show in the next section that boundary condition numbers for the open channels should also be chosen as in (64).

In the next section of this paper we discuss the physical meaning of the resonance parameters and the choice of boundary condition numbers to remove the artificial dependence of (62) on the channel radii (e.g., in the hard-sphere phase shifts).

V. EFFECT OF THE DIFFUSE NUCLEAR EDGE ON LOW ENERGY NUCLEAR REACTIONS[41]

a. Optical Model Potential

The simplest approximations to the general result (62) for the collision matrix have some undesirable properties connected with the nuclear surface. Thus, in the one-level Breit-Wigner formula, obtained from (62) by retaining only the terms corresponding to the single resonance state λ, we have, first of all, the hard-sphere phase shifts in the potential scattering. That the hard-sphere shifts should appear in this way is quite natural because the resonance theories treat the nucleus as a box with standing waves inside it. Although the total wave function can be expanded in terms of all the standing waves, when we neglect all the standing waves but one, what is left is just one resonant state plus the box or hard sphere. As we shall see, the change in the hard-sphere phase shift to a potential well phase shift is made by consideration of the many distant levels which are neglected in the usual Breit-Wigner formula. The second undesirable feature of the surface is that we have neglected all the nuclear effects in the channels.

The above criticisms of the reaction formulas have been reinforced by the recent success of the optical model of the nucleus. The elastic scattering and absorption cross sections of many nuclei can be described by a simple complex potential well. Typically the optical model well which best fits the data has the following form

$$V(r) = -(V_0 + iW_0)/[1 + e^{(r-R_0)/a}], \quad (65)$$

a potential first used in this connection by Saxon and Woods.[42] In (65), V_0, W_0, R_0, and a are constants. The potential (65) follows the nuclear density and is flat almost up to the nuclear radius R_0, at which it

[39] E. Vogt, Phys. Rev. **112**, 203 (1958); **118**, 724 (1960). J. B. French, S. Iwao, and E. Vogt, *ibid.* **122**, 1248 (1961).
[40] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).
[41] Some preliminary results for the first two parts of this section have been reported in reference 29.
[42] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954); see also C. Eckart, *ibid.* **35**, 1303 (1930).

falls off in a distance a (a is very nearly the distance from 75% of maximum potential to 25%). The fall-off parameter of the nuclear density is 0.55 ± 0.06 F, measured by the bombardment of various nuclei with high energy electrons.[43] R_0 is chosen to be about $1.25 A^{1/3}$ F where A is the atomic weight. Then V_0, for slow neutrons, is about 51 MeV and W_0 about 3 MeV. V_0 decreases slowly with energy (about half a MeV per MeV) and W increases with energy (about half a MeV per MeV below 10 MeV).[44] The shape of the imaginary part of the optical model potential is frequently altered from that of (65) to be larger in the region of the nuclear surface. Such a change has its basis in the Pauli principle (in the region of low density near the nuclear surface the Pauli principle is less effective in reducing the absorption) and is confirmed experimentally by the large absorption cross sections, compared to πR_0^2, measured in many nuclei.[45]

Analysis of many nuclear reactions with the optical model suggests that the nuclear surface is not abrupt. It will be advantageous, below, to consider the effect of the tail of the optical model potential in the channel region of configuration space. The principal effect will be to modify the nuclear penetrability. Moreover, since the average potential seen by a nucleon appears to be similar to (65) it would appear to be desirable to have the hard-sphere phase shifts of the theory change into the appropriate potential scattering phase shifts. In the remaining parts of this section we shall see how the ideas of the optical model can be incorporated in the resonance theory. We begin by taking up again the simple s-wave neutron scattering problem of Sec. II, with the square well replaced by a diffuse-edge potential.

b. The Scattering Cross Section of a Real Diffuse-Edge Potential

Unfortunately, we cannot find a simple analytic expression for the collision function (or phase shift) in the case of s-wave neutron scattering by a real diffuse-edge potential.[46] To facilitate the comparison of the results for this potential with those of the square well (Sec. II) we shall use the methods of resonance theory. We shall choose a matching radius

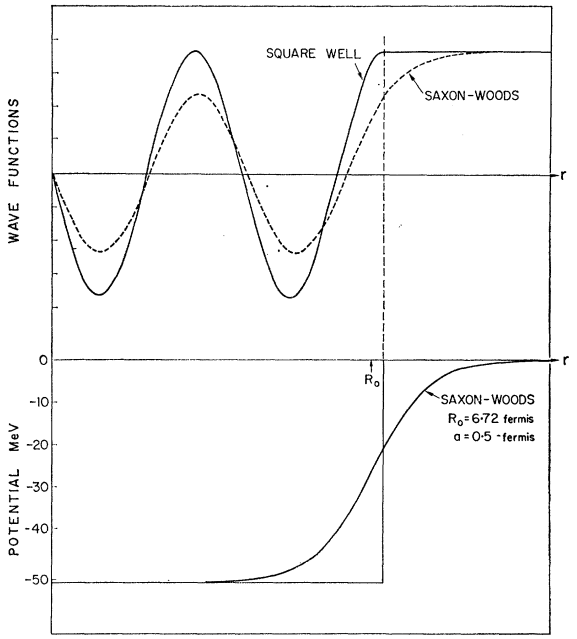

FIG. 5. Comparison of a diffuse-edge potential well [with a Saxon Woods shape, Eq. (65)] with a square well and of the resonant wave functions of the two wells.

R and a boundary condition at the matching radius in order to obtain resonance energies and reduced widths for the diffuse-edge potential. Then we shall compare the reduced widths, level energies, penetrabilities, and shift functions of the real diffuse-edge well with those of the corresponding square well.

The parameters of the diffuse-edge well are chosen to be those given in the preceding part of this section. The real square well to be compared with it is chosen to have a value of $V_1 R_1^2$ such that the lowest unbound states of the two wells coincide in energy. The radius, R_1, of the square well is chosen to be equal to the matching radius, R, of the diffuse-edge potential as discussed below. The two potentials are shown in the lower half of Fig. 5.

For the diffuse-edge potential the best choice of the matching radius is not obvious. However the properties of a low-energy resonant state are clear and are shown on Fig. 5 along with the square well resonant state on Fig. 1. Just as for the square well, a resonance occurs when the wave function has zero derivative where the potential becomes vanishingly small because the amplitude inside the potential is then a maximum. If we choose the matching radius R to be larger than R_0 by many times the surface thickness a, then the penetrability and hard-sphere phase shift are both very nearly equal to kR (just as for the square well for which the potential is exactly zero beyond the radius R_1) and the shift function is

[43] For references, see the review by D. G. Ravenhall, Revs. Modern Phys. **30**, 430 (1958).

[44] For a discussion of optical model parameters see, for example, the review papers on the optical model in the *Proceedings of the Kingston Conference on Nuclear Structure 1960* (University of Toronto Press, Toronto, 1960).

[45] V. Meyer and N. M. Hintz, reference 44, page 231.

[46] In Appendix A, some analytic results are given for a similar potential—one which decays exponentially in the surface region.

also equal to that of the square well. To understand the behavior of these functions as we bring R closer to R_0 we note that, in the low energy limit, the wave function of Fig. 5 is the irregular wave function G and that the regular wave function F is very much smaller than G. Then, from (58) and (59) the penetrability reduces to kRG^{-2} and the shift function to $-b + G'/G$, while the hard-sphere phase shift (61) remains very small, as G varies. However we then see at once, from Fig. 3, that the penetrability increases strongly as R approaches R_0 while the term G'/G in the shift function changes from the value zero, at large R, to a large value in the vicinity of R_0.

In the construction of an $\mathfrak{R}$ function for the diffuse-edge potential (with a matching radius close to R_0), the boundary condition number b has a "natural" value considerably different from the value $b = 0$ which was shown to be natural for the square well. Again, the choice of a "natural" condition is dictated by the value of b which makes the standing waves have the same logarithmic derivative, at R, as the physical resonant state of Fig. 5 [the same choice of b, again, makes the level shift function vanish and also makes the one level approximation a good one (cf. Fig. 3) and appears to simplify the $\mathfrak{R}$-function derivatives of Appendix B.]

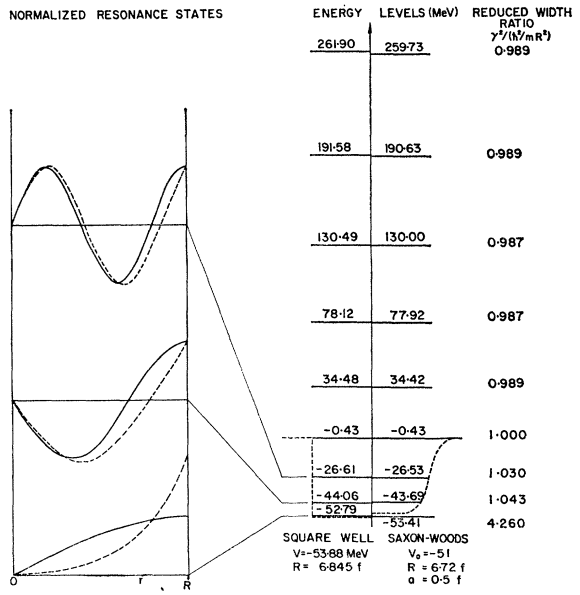

FIG. 6. Comparison of the low lying resonant states of the square well and of the corresponding diffuse-edge well. For each well, the states are defined with a natural boundary condition number ($b = 0$ for the square well and $b = 6.12$ for the Saxon-Woods well). The well parameters are fixed as discussed in the text. The wave functions given by solid lines belong to the square well and the ones given by broken lines to the Saxon-Woods well. The square well reduced widths are all equal to $\hbar^2/mR^2$ and the diffuse-edge well reduced widths are given, on the right, in terms of $\hbar^2/mR^2$.

We shall now show that with the "natural" choice of b we can make the whole $\mathfrak{R}$ function of the diffuse-edge well very nearly equal to the $\mathfrak{R}$ function of the corresponding real square well. The example with which we illustrate this point has a radius corresponding to $A = 155$ and has the $4s$ giant resonance lying very near to zero energy (cf. Fig. 3). We then choose a matching radius, R, such that the reduced width of the $4s$ level (the fourth level on Fig. 6) is equal to $\hbar^2/mR^2$ (the square well value of the reduced width with the square well radius, R_1 chosen to be equal to R). This radius is just slightly larger than R_0 and the value of b for the diffuse-edge well is 6.12. At that radius we compare, on Fig. 6, the resonant energies, reduced widths and resonant wave functions of the lowest levels in the diffuse-edge well with those of the corresponding square well. As the figure shows, all of the levels of the two potentials (except the deepest level[47] far below zero energy) have very nearly the same energies and reduced widths. Hence, the complete $\mathfrak{R}$ functions of the two wells are very closely the same. Therefore, we have found that the only important change in going from a real square well to a diffuse-edge well is the change in the penetrability.

The change of the penetrability with surface thickness has its physical explanation in the reflection properties of the surface—a square well reflects much more of the incident wave than a diffuse-edge well (or than a real nucleus). Calculations carried out on the Chalk River Datatron computer have shown that to obtain the penetrability for a diffuse edge well the square well penetrability [which is also the conventional nuclear penetrability, (58)] must be multiplied by a factor, f, which depends on the surface thickness, a, in the following way.

$$f \approx 1 + 6.7\, a^2 \qquad (66)$$

where a is in fermis. The factor f changes only slowly with the energy because the reflection does not change appreciably until the incident energy is of the same order of magnitude as the depth of the real

[47] The energy of lowest level in the diffuse-edge well is below the bottom of the well. The eigenfunction, as the figure shows, increases "exponentially" up to the matching radius and satisfies the boundary condition there. All wave functions at lower energies would increase too rapidly to satisfy the boundary condition, so the state shown is the lowest eigenstate. The fact that it lies below the bottom of the well is not so perplexing when one remembers that for $b \neq 0$ the boundary condition itself contains energy. In fact, the system of wave equation plus boundary condition of our problem is like the classical system of a vibrating string with one end attached to spring. In the lowest normal mode of the classical system, like in our present one, the energy of the whole system is positive although the string will have negative energy.

well ($\sim$ 50 MeV). In the first 10 MeV the factor appears to change by only a few percent. The calculations of Appendix A, carried out with a diffuse-edge well of slightly different shape than (65), make the above properties of the reflection factor more plausible.

The comparison of the two wells just made by means of Fig. 6 is made clearer by a comparison of the scattering cross sections, shown on Fig. 7. It is evident on the figure that the diffuse-edge well has a larger level width. In fact, since both wells have the same reduced width the total width of the diffuse-edge well should be larger by just the reflection factor $f(= 2.67$ for the surface thickness, $a = 0.5$, used on the figure.) However, because of the approximate equality of the complete $\mathfrak{R}$ functions we can obtain a good approximation to the cross section of the diffuse-edge well, holding beyond the validity of one-level approximation, by simply taking the analytic result for the square well and increase the penetrability in it by the factor $f(= 2.67)$. The resulting scattering cross section is seen to approximate the exact result quite closely off resonance as well as on.

Although we have assigned the reflection factor to the penetrability we could equally well have assigned it to the reduced width. Thus, if we had chosen R larger than R_0 by several times a then the penetrability would be the usual square-well one of kR. Now the reduced width would be larger than $\hbar^2/mR^2$ by just about the reflection factor given above. To see this we note that the reduced width (18) is proportional to the square of the value of the normalized standing wave at the matching radius. For the resonant state of Fig. 5 this means that γ_λ^2 is proportional to $G^2(R)$ so that the product of penetrability and reduced width (i.e., the actual width) is independent of G. It is the level widths of the two wells which differ by the factor f. Under various circumstances it will be advantageous to associate f with either γ_λ^2 or with the penetrability, P.

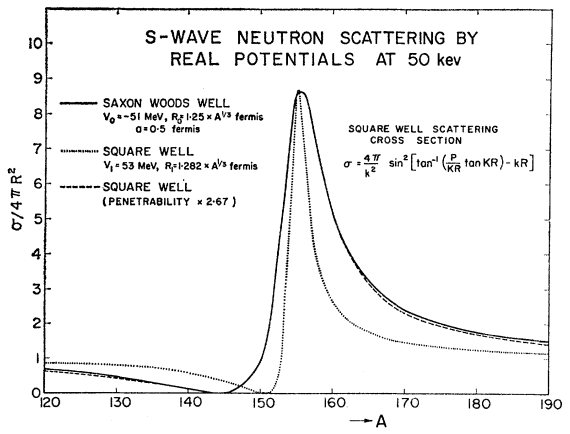

FIG. 7. The scattering cross sections of the two real equivalent wells as a function of atomic weight A. The cross section is given in units of $4\pi R^2$. The energy is fixed at 50 keV and the resonance is the 4s state well known in low energy neutron cross sections. The broken curve is an approximation to the Saxon-Woods curve obtained by simply taking the analytic result for the square well and increasing the penetrability in it by a factor of 2.67.

c. Effect of the Diffuse-Edge on the Optical Model Cross Sections

Before applying the results of the preceding section to the theory of resonance reactions, we show how the results just obtained for real potential wells can be generalized to the case of complex potential wells. The results which we shall find will also enable us to make an exceedingly accurate estimate of the surface thickness for low energy nuclear reactions. As we shall see, even for the complex wells, the principal effect of the diffuse edge is to increase the penetrability by a reflection factor.

For the complex square well the scattering cross section is still connected to the collision function by (7), [although the phase shift is no longer real and therefore (5) does not hold]. In the Schrödinger equation, (1), for the complex square well ($V \equiv -V_1 - iW_1$) we can add the constant imaginary part, iW_1, to the energy E. Then the collision function for the complex square well is still given by (22) if the $\mathfrak{R}$ function (17) is evaluated at the complex energy $E + iW_1$. That is

$$\mathfrak{R} = \sum_p \gamma_p^2/(E_p - E - iW_1) \equiv \mathfrak{R}^\infty + i\pi s\,, \tag{67}$$

where s is the strength function whose meaning for nuclei will become clear below. According to (67) we have

$$s(E) = (1/\pi) \sum_p \gamma_p^2 W_1/[(E_p - E)^2 + W_1]^2 \tag{68}$$

$$\mathfrak{R}^\infty = \sum_p (E_p - E)\gamma_p^2/[(E_p - E)^2 + W_1^2] = P\int_{-\infty}^{\infty} s(E')dE'/(E' - E)\,, \tag{69}$$

where P stands for the principal value of the improper integral. We are now using the subscript p to label the single particle states rather than λ as in Sec. II: In Sec. II we wanted the potential-well results to adapt easily to the nuclear problem; at present we want to distinguish the widely spaced giant resonances p from the closely spaced compound nucleus levels. The replacement of E by $E + iW_1$ in the one-level Breit-Wigner formula suggests that for the

complex wells the total width of the giant resonances is $\Gamma_p + 2W_1$. Since even for low-energy nuclear reactions $2W_1$ will generally be larger than Γ_p, the width of the giant resonances, for the complex wells, will not be greatly affected by the reflection factor. However the numerator of the resonance amplitude in (25) is directly proportional to Γ_p, even for the complex wells, so that the reflection factor is expected to have an appreciable effect on the magnitude of the resonance cross sections of the diffuse-edge complex wells. Before we use this fact to determine the diffuseness of the nuclear surface we show how the giant resonances of the square complex well show up in the absorption cross section and we show that the connection between diffuse and square wells is as simple for complex wells as for real ones.

The s-wave absorption cross section[48] of the complex square well is

$$\begin{aligned}\sigma_{\text{abs}} &= (\pi^2/k^2)(1 - |U|^2) \\ &= (4\pi^2/k^2)Ps/[(1 - S\mathfrak{R}^\infty + \pi Ps)^2 \\ &\quad + (P\mathfrak{R}^\infty + \pi Ss)^2], \end{aligned} \tag{70}$$

where P is the penetrability ($= kR_1$) and S the shift function. The second step follows from use of the complex-well $\mathfrak{R}$ function, (67). The denominator of (70) is very nearly equal to unity with the possible exception of energies close to giant resonance. Therefore the following approximation is usually quite good (especially at low energies where P and S both vanish)

$$\begin{aligned}\sigma_{\text{abs}} &\approx 4\pi^2 Ps/k^2 \\ &= (2\pi/k^2)\textstyle\sum_p \Gamma_p W_1/[(E_p - E)^2 + W_1^2]. \end{aligned} \tag{71}$$

The one-level approximation, obtained by retaining only one level in the sum of (67), yields

$$\begin{aligned}\sigma_{\text{abs}} = (2\pi/k^2)\Gamma_p W_1/[(E_p - E - \Delta_p)^2 \\ + \tfrac{1}{4}(\Gamma_p + 2W_1)^2], \end{aligned} \tag{72}$$

which is very similar to (71) if $2W_1 \gg \Gamma_\lambda$. It is clear from these results that both the strength function and the absorption cross section have giant resonances at the energies of the single particle resonances of the real square well.

In comparing the results for the complex square well with those of the complex diffuse-edge well we note first that for the latter we cannot make the substitution, $E \to E + iW_0$, because of the spatial dependence of the imaginary part of the diffuse-edge potential. To begin with, then, we add to the real diffuse-edge potential a square imaginary term with a radius equal to the matching radius R. From the above discussion it should be clear that we obtain a very good approximation to the results for this potential by simply using the square-well result and multiplying the penetrability in it by the reflection factor. Now we point out that the shape of W_0 has only a small effect on the cross sections. The square shape which we have just assumed might be undesirable because it can cause reflection just like the real square well does. However, W_0 is always much smaller than V_0 and hence the total energy of the particle, at the matching radius, is much larger than W_0, even for zero energy neutrons, so that no appreciable reflection from the square shape of W occurs. Calculations carried out with the Chalk River Datatron computer have shown that the phase shifts of the diffuse-edge well (65) are very closely equal to those corresponding to the potential in which the shape of W is altered to be square. Thus the complex diffuse-edge well (65) gives results differing from those of the corresponding square well mainly by the increase of Γ_p by an amount equal to the reflection factor f, of (66).

To the extent to which the approximations (71) or (72) are valid the optical model absorption cross section is everywhere proportional to the reflection factor and this proportionality constant is the only effect the diffuse edge has on the absorption cross section.

The data to which the optical model cross sections discussed above are to be compared are the average over compound nucleus resonances of the slow neutron cross sections. Near a single resonance line of spin and parity J,π, the cross section is given by the Breit-Wigner formula

$$\sigma_{\alpha\alpha'} = \frac{\pi}{k_\alpha^2} g_J \sum_{sls'l'} \Gamma_{\lambda c}\Gamma_{\lambda c'}/[(E_\lambda - E)^2 + \tfrac{1}{4}\Gamma_\lambda^2], \tag{73}$$

where

$$\begin{aligned} g_J &= (2J+1)/(2I+1)(2i+1) \\ \Gamma_\lambda &\equiv \textstyle\sum_c \Gamma_{\lambda c}. \end{aligned} \tag{74}$$

If we integrate this cross section over the entire resonance we find

$$\int_{-\infty}^{\infty} \sigma_{\alpha\alpha'}(\lambda) d(E_\lambda - E) = \frac{2\pi^2 g_J}{k_\alpha^2} \sum_{sls'l'} \Gamma_{\lambda c}\Gamma_{\lambda c'}/\Gamma_\lambda \tag{75}$$

[48] The s-wave square-well results given in the text are easily adapted to higher partial waves. We note that for higher partial waves of orbital angular momentum the "natural" boundary condition is $b = l$. Then all the reduced widths for these partial waves are again equal to $\hbar^2/mR_1^2$ and the resonance energies can be found from tables of spherical Bessel functions. The scattering and absorption cross sections for these partial waves must be multiplied by the factor $(2l + 1)$. If then in (70) for example, we use the penetrability P_l, the shift function S_l and the collision function U_l (with s_l and R_l^∞) the result given still holds for the lth partial wave.

For the average absorption cross section $\langle\sigma_\alpha(\text{abs})\rangle_{\text{av}}$ in an energy internal containing many levels we merely sum (75) over all outgoing α', divide by the average level spacing $D^{J\pi}$, and sum over all spins and parities to obtain

$$\langle\sigma_\alpha(\text{abs})\rangle_{\text{av}} = \frac{\pi}{k_\alpha^2}\sum_{J\pi sl} g_J 2\pi\langle\Gamma_{\lambda c}\rangle_{\text{av}} D^{J\pi}$$
$$= \frac{4\pi^2}{k_\alpha^2}\sum_{J\pi sl} g_J P_c s_c \qquad (76)$$

where

$$s_c \equiv \langle\gamma_{\lambda c}^2\rangle_{\text{av}}/D^{J\pi} \qquad (77)$$

is the strength function giving the average strength of the poles of the diagonal elements of the $\mathfrak{R}$ matrix. If s_c is independent of $J\pi$ and a function of l only, then (77) may be written

$$\sigma_\alpha(\text{abs}) = (4\pi^2/k_\alpha^2)\sum_l (2l+1)P_l s_l\,. \qquad (78)$$

A comparison of the s-wave part of (78) with (71) gives both the interpretation of the optical-model strength function in terms of resonance parameters and also the connection between optical model cross sections and the *average* compound nucleus cross sections. The derivation of the average absorption cross section just given is not valid for the case of overlapping levels. However, it can be shown[20] that the same results are obtained in that case.

Figure 8 shows the data[49] for the s-wave neutron strength function obtained from the average absorption cross section of various nuclei by means of (78). The figure also shows the optical model strength functions found from the optical model absorption cross sections by means of (71). In each case the plotted quantities are the strength functions multiplied by $2RA(A+1)^{-1}(0.2197)$, following the common practice in low neutron spectroscopy of obtaining reduced widths from observed widths by dividing the latter by the square of the resonance energy in eV. The fact that the radius occurs in the normalization constant will be of advantage below. As expected, the computed curves for various surface thicknesses a all lie above the square well result ($a = 0$) by the reflection factor f, of (66).

The figure also shows the so-called "black nucleus" strength function. This particular strength function has played an important role in nuclear reactions. The physical idea behind it is that the nucleus resembles a black absorbing sphere. In the absorption of s-wave neutrons this idea is formulated by assuming no nuclear forces outside a radius R, and only incoming waves, e^{-iKr} inside the nucleus (K is the wave vector inside). The $\mathfrak{R}$ function of this internal wave function is $i(KR)^{-1}$ so that the "black nucleus" strength function is

$$s = (\pi KR)^{-1} \qquad (79)$$

and the corresponding absorption cross section is

$$\sigma_{\text{abs}} = (4\pi/k^2)P(KR)^{-1}\,. \qquad (80)$$

The "black nucleus" strength function plotted on Fig. 8 is normalized by a multiplication factor proportional to the matching radius R, so that the plotted quantity is a constant independent of R.

The "black nucleus" strength function is very nearly equal to the average over giant resonances of the square-well strength function (68). To show this we note that the $\mathfrak{R}$ function (67) of the complex square well may also be written

$$\mathfrak{R} = (KR_1)^{-1}\tan KR_1\,, \qquad (81)$$

where K is the complex wave number inside the well and R_1 is the square-well radius. Then the strength function is

$$s = (1/\pi)\,\text{Im}\,[(KR_1)^{-1}\tan KR_1]\,. \qquad (82)$$

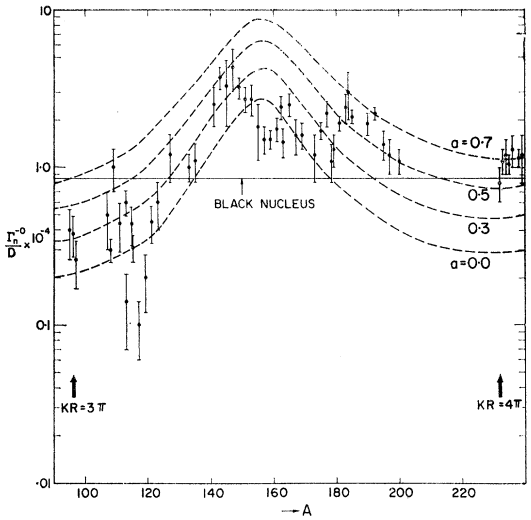

FIG. 8. The neutron strength function data of reference 49, as a function of atomic weight A, and optical model computations of the strength function for various surface thicknesses a, in fermis. Each computed curve corresponds to a Saxon-Woods potential with $W_0 = 3$ MeV, $R_0 = 1.25\ A^{1/3}$ F, and with V_0 adjusted slightly to make the $4s$ giant resonances occur at the same place for each curve. The "black nucleus" strength function is also shown. Shape deformations and different spatial distributions of the imaginary potential have been ignored on the figure because, although they produce a better fit to the data, they are largely irrelevant to the computation of the average surface thickness a from the data of the figure.

[49] D. J. Hughes, R. L. Zimmermann, and R. E. Chrien, Phys. Rev. Letters 1, 461 (1958).

where Im means the imaginary part of the square bracket. We take the average $\bar{s}$ of s over a whole giant resonance in the following way.

$$s \equiv \int_{n\pi}^{(n+1)\pi} sK_1R_1d(K_1R_1)\,, \tag{83}$$

where K_1 is the wave vector in the real part of the complex square well. The averaging integral (83) is normalized so that the "black nucleus" strength function (79) has an average value of unity. Straightforward use of (82) in (83) also yields the result unity apart from terms of the order of magnitude of $(W_1/V_1)^2$.

The "black nucleus" consists then in washing out all the giant resonance structure of the square well. However, the absorption cross section of the diffuse edge well is larger than that of the black nucleus by the reflection factor. In fact, the conventional "black nucleus" is a poor absorber because it has all the unphysical reflection of the square well.[50] At higher energies, the black nucleus is also not compatible with any simple complex well. As Austern[51] has recently pointed out the condition of incoming waves only at the nuclear surface can be achieved for one partial wave at a time by adjustment of the well parameters. The same well parameters will not usually achieve the same condition for other partial waves. The problem of achieving only incoming waves at the nuclear surface is much like the impedance matching problems of the theory of alternating current circuits. Certainly, increasing W indefinitely does not increase the absorption because a large W itself causes considerable reflection.

Just as the average over the whole giant resonance of the square-well strength function is equal to the constant black nucleus value, the average of the diffuse-edge strength functions differs from the black-nucleus value by the reflection factor. This sum rule is independent of many features of the problem which change the shape of the strength function. Thus, if W is altered from the value of 3 MeV which it has on Fig. 8 the curves become more peaked but the integral over the resonance does not change. Similarly, the collective effects which tend to split the giant resonance[49,52] do not appear to change the sum rule. If these effects had been included, a better fit to the data of Fig. 8 would have been obtained but because of the invariance to these effects of the integral (83) over the whole giant resonance we are assured of a proper estimate of the reflection factor simply by averaging the date over the whole giant resonance.

A numerical average of the strength function data of Fig. 8 over the whole giant resonance, with the data weighted as in the integral (83), is equal to $(1.42 \pm 0.06) \times 10^{-4}$ compared to the black nucleus value of 0.867×10^{-4}. The consequent value of the surface thickness obtained from the reflection factor (66) is $a = 0.31 \pm 0.02$ F. The constant value of the black nucleus strength function (multiplied by R as on Fig. 8) depends slightly on the radius. For $R_1 = 1.4\ A^{1/3}$ we would have found $a = 0.25 \pm 0.02$ and for $R_1 = 1.07\ A^{1/3}$, $a = 0.37 \pm 0.02$ so that $a = 0.31 \pm 0.06$ encompasses the reasonable variations of well parameters.

If the imaginary part of the complex well is concentrated on the surface then the reflection factor is no longer as independent of atomic weight, mainly because W is no longer so small compared to $V_0(r)$ at the matching radius. Calculated values of the strength functions for a potential well whose real part is just like that of Fig. 8 but whose imaginary part has a Gaussian shape centered at R_0 with a 1 F width and a height of 9.5 MeV at peak, are surprisingly close to those of Fig. 8. Consequently, the shape of the imaginary part of the potential does not affect, appreciably, the estimate of the surface thickness.

The surface thickness of the real part of the potential differs appreciably from the surface thickness of the nuclear density as found from the scattering of high energy electrons by various nuclei. The latter experiments yield[39] $a = 0.55 \pm 0.06$. The difference may be real and caused by the polarization of the target nucleus by a slow moving nucleon. As the nucleon approaches the nucleus it attracts the target nucleons which consequently come out to meet it. The result is that the incoming nucleon is surrounded by the normal central density of nucleons at a larger radius than the static density would indicate. If this effect is responsible for the two different values of a then we would expect the value of a for nucleons to increase with energy toward the electron scattering value at energies of the order of V_0. The most com-

---

[50] This fact was already noted by H. A. Bethe, Phys. Rev. **57**, 1125 (1940). However, he corrected it by using a potential well whose real part was square and whose imaginary part. had a diffuse edge implying $W \gg V$. We now believe the absorption of nucleons to be weaker ($W \ll V$) than was customarily believed in 1940 so that in fact, the whole point of the present section is that the unphysical reflection occurs from the real part of the potential.

[51] N. Austern, Ann. Phys. (N. Y.) **15**, 299 (1961).

[52] B. Margolis and E. S. Troubetskoy, Phys. Rev. **106**, 105 (1957).

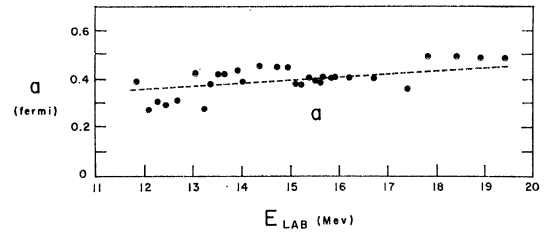

FIG. 9. The surface thickness parameter a as a function of proton energy, as found from the optical model analysis of elastic scattering of protons from C^{12} (reference 53). The line through the points is also taken from reference 53.

plete optical model fitting of angular distributions appear to show this effect. Figure 9 shows the value of a found by Nodvik *et al.*[53] for a large number of angular distributions, for the elastic scattering of protons from C^{12} at proton energies between 10 and 20 MeV. The value of a appears to increase with energy and to extrapolate reasonably to the present estimate at zero energy and to the electron scattering value at high energy.

d. Justification of the Optical Model

The many successes of the optical model, employed as a phenomenological model to describe a large number of nuclear reactions, have prompted many attempts to justify the model in terms of the formal theory of resonances. One of the earliest and most thorough of these explanations of the optical model is that due to Wigner and his collaborators.[26,54-56] In this part we review the work of Lane, Thomas, and Wigner and show how their connection between the average compound-nucleus cross sections and the cross sections of the complex square well may be extended to connect the former also to the diffuse-edge complex well.

In the work of Lane, Thomas, and Wigner one begins by constructing a set of stationary states which are different from the actual compound states X_λ, but which more closely resemble the independent particle model states of the nuclear shell model. The new set of states are eigenstates of a simpler Hamiltonian than the actual one for the nucleus: To construct the set we take, say, an incoming nucleon N and assume it interacts with the nucleus only through an average potential $\overline{V}$. Then the total wave function of the problem inside the nucleus is $\psi_c\, u\,(r_N)$ where ψ_c is the channel wave function, (28) and $u\,(r_N)$ is the radial wave function corresponding to the motion of N in $\overline{V}$; that is

$$-(\hbar^2/2m)d^2u/dr_N^2 + \overline{V}u = Eu\,, \qquad (84)$$

an equation which we already encountered in (1) above. For $l \neq 0$, $\overline{V}$ includes the centrifugal barrier. We now add a boundary condition at the channel radius R_N to make the states u into a complete orthonormal set of stationary states u_p with eigenvalues E_p. The boundary condition is [cf. (9)]

$$r_N du_p/dr_N = bu_p|_{r_N=R_N}\,. \qquad (85)$$

Although the states $\psi_c u_p$ do not satisfy the Schrödinger equation with the total nuclear Hamiltonian (the difference $V - \overline{V}$, where V is the actual interaction of N with the nucleus, has been omitted in constructing the $\psi_c\, u_p$) we can still expand the states X_λ in terms of the $\psi_c\, u_p$;

$$X_\lambda = \sum_{c,p} C_{\lambda,cp}\psi_c u_p\,, \qquad (86)$$

where c runs over all channels for nucleon emission *only* and p runs over all its possible values. Because (86) is a real orthogonal transformation, we have

$$\sum_\lambda C_{\lambda;cp}C_{\lambda;c'p'} = \delta_{cc'}\delta_{pp'} \qquad (87)$$

and

$$\sum_{cp} C_{\lambda;cp}C_{\lambda';c'p'} = \delta_{\lambda\lambda'}\,, \qquad (88)$$

which will be used below to obtain the well-known sum rules for reduced widths.

The nucleon reduced widths are simply related to the expansion coefficients $C_{\lambda;cp}$. From (86) and (39) it follows that

$$\gamma_{\lambda c}^2 = (\hbar^2/2m_cR_c)[\sum_p u_p(R_c)C_{\lambda;cp}]^2\,. \qquad (89)$$

The central idea of the work of Lane, Thomas, and Wigner[26] is that the mixing of the independent particle model states, $\psi_c u_p$, to form the actual nuclear resonances X_λ is not so complete as to remove all vestiges of the independent particle model. Thus, for each c in (89) one of the $C_{\lambda;cp}$, in the sum over p, is assumed to be much larger than the others. Then (89) may be written

$$\gamma_{\lambda c}^2 \approx (\hbar^2/2m_cR_c)u_p^2(R_c)C_{\lambda;cp}^2\,. \qquad (90)$$

In particular, if $\overline{V}$ is chosen to be a square well (or a diffuse edge well with the matching radius chosen as

[53] J. S. Nodvik, C. B. Duke, and M. A. Melkanoff, Phys. Rev. **125**, 975 (1962).
[54] E. P. Wigner, Science **120**, 790 (1954); Ann. Math. **62**, 548 (1955).
[55] E. Vogt, Phys. Rev. **101**, 1792 (1956).
[56] E. Vogt and J. Lascoux, Phys. Rev. **107**, 1028 (1957).

for Fig. 6), then $u_p^2(R_c)$ is equal to $2/R_c$ and (90) reduces to

$$\gamma_{\lambda c}^2 \approx (\hbar^2/mR_c^2)C_{\lambda;cp}^2 . \quad (91)$$

The assignment of each reduced width to a particular p group is completely in agreement with the qualitative features of the optical model strength function (68). The latter has giant resonances at the single particle energies. The strength function of Lane, Thomas, and Wigner is obtained by dividing the average of (91) by the average of the level spacing D. Then (91) agrees with (68) if we write

$$\langle C_{\lambda;cp}^2\rangle/D = (W_1/\pi) \sum_p [(E_p - E)^2 + W_1^2]^{-1} . \quad (92)$$

If (92) is valid the reduced widths, or equivalently the expansion coefficients $C_{\lambda;cp}^2$, may each clearly be assigned to the giant resonance, p, which is nearest in energy to the state λ. The square-well optical-model strength function, (68), is thus just a particular case conforming to the ideas of Lane, Thomas, and Wigner.

The attribution of the giant resonances to the reduced widths may be tested by a calculation of the configuration mixing caused by $V(r_N) - \overline{V}(r_N)$. $V(r_N)$ is the actual sum of two nucleon potentials acting on the nucleon N. According to (92) the imaginary part W_1 of the optical model potential is equal to half of the width into which the configuration mixing spreads the state $\psi_c u_p$. Early estimates[26] of the configuration mixing gave a width about an order of magnitude larger than the few MeV found for W_1 at low energies. Later estimates[55,56] included the combined effects of correlations and the Pauli principle and gave much smaller estimates of the configuration mixing, compatible both with W_1 and with the basic idea that the region into which $\psi_c u_p$ is mixed is much smaller than the spacing ($\sim 10 \rightarrow 30$ MeV, cf. Fig. 6) of the single particle states p.

The giant resonances on the right-hand side of (92) are attributed to $\langle C_{\lambda;cp}^2\rangle$ rather than to D because the configuration mixing which mixes the single particle states into the multitude of many-particle states should not appreciably affect the level density as calculated by the independent particle model. The latter exhibits no such giant resonances.

The justification of the optical model through the ideas of Lane, Thomas, and Wigner rests in assigning the particular energy variation (92) to the reduced widths. Then use of the resulting nuclear strength function in the average absorption cross section (78) gives the same absorption cross section as the complex square well (71). From the discussion in the preceding parts of this section it should be clear that in order to accommodate the diffuse nuclear surface we simply increase the penetrability both in the nuclear absorption cross section and in the square well optical-model results. Then the same strength function (92), by the same route, gives the basis of the diffuse-edge optical model.

The question arises as to whether or not the nucleon strength function has exactly the energy dependence of (92). Some very elegant calculations of Wigner[54] and Bloch[24] suggest that even far from the giant resonances the configuration mixing coefficients may be given by (92). For the analysis of nuclear data the precise energy dependence of the strength function is usually not very important; the dominant features are the positions and widths of the giant resonances.

A second question about the above basis of the optical model concerns the connection between average nuclear cross sections and the optical model cross sections. The correspondence between (78) and (70), for example, holds only to first order in πPs. The connection can be improved, particularly for the case in which there is only one outgoing channel for each total spin and parity. This situation holds for most slow neutron cross sections below a few hundred keV.

In the case of only one outgoing channel the collision matrix (48) becomes a collision function. To perform the average of the collision function over compound resonances we use a technique originally suggested by Thomas.[23] We take an averaging interval δ, containing many levels but small enough so that the functions O^{-1}, I, L, and L^*, of (48) may be regarded as energy independent. The energy variations may then be entirely attributed to the $\Re$ function. We next note that the poles of the collision function (48) (one pole associated with each level) all lie below the real axis in the complex energy plane. We therefore perform the average of U in δ by moving the averaging interval from the real axis upwards by an amount ϵ, with

$$\delta \gg \epsilon \gg D , \quad (93)$$

where D is the average spacing of the compound levels. The new average with E replaced by $E + i\epsilon$ is equal to the old because the new interval can be connected with the old forming a closed contour containing no poles: if $\delta \gg \epsilon$ holds, the integral over the two connecting ends is negligible.

When E is replaced by $E + i\epsilon$ (with $\epsilon \gg D$ as assumed) then neither the collision function nor the

$\mathfrak{R}$ function have strong fluctuations. The $\mathfrak{R}$ function of (48) is then

$$\begin{aligned}\mathfrak{R}(E+i\epsilon) &= \textstyle\sum_\lambda \gamma_\lambda^2/(E_\lambda - E - i\epsilon) \\ &\approx \int \frac{\langle\gamma_\lambda^2\rangle}{D}\,(E_\lambda - E - i\epsilon)dE_\lambda \\ &= i\pi\frac{\langle\gamma_\lambda^2\rangle}{D} + P\int \frac{\langle\gamma_\lambda^2\rangle}{D}\,(E_\lambda - E)^{-1}dE_\lambda \\ &\equiv \mathfrak{R}^\infty + i\pi s\,. \end{aligned} \tag{94}$$

Therefore, the average value of the nuclear collision function is obtained by use of (94) in (48). The real and imaginary parts of the $\mathfrak{R}$ function of the optical model (67) are connected to each other in precisely the same fashion as (94). Therefore, if the nuclear strength function is assumed to be equal to (68) then the average nuclear collision function is *exactly* equal to the square-well collision function.

The total cross section is a linear function of the collision function (for the nuclear problem or any other problem) so that the average total cross section of low-energy neutrons should be exactly equal to the square-well optical-model result if the nuclear strength function is equal to (68). This result applies also to diffuse-edge wells if we simply multiply the penetrability by the reflection factor—both in the nuclear cross section and in the optical model result. Having established the connection between the average total cross section and the model total cross section, one can split up either total cross section into parts (e.g., the absorption cross section) and make the connection separately for the various parts.

Since the result (70), including the higher order terms in the denominator, applies to average low-energy neutron cross sections one may enquire about the conditions under which the denominator deviates from unity. It appears to be usually true that $S\mathfrak{R}^\infty$, $P\mathfrak{R}^\infty$, πPs, and πSs are smaller than unity.[57] For p waves, it has recently been found[58] that the term $S_1\mathfrak{R}^\infty$ has an appreciable effect ($\sim 30\%$) on the energy variation of the absorption cross section for the diffuse edge wells, in the energy range of 10 to 100 keV. The term is large for this case because the shift function, like the penetrability, must be multiplied by the reflection factor f. Usually the shift function does not vary strongly with the energy and the "natural" boundary conditions are then chosen to make S vanish. Near the top of the conventional barrier ($\sim$ 100 keV for p waves on heavy nuclei) the shift function varies appreciably with energy (the square-well p-wave shift function decreases from 1 to 0). To see that the shift function must also be multiplied by the reflection factor we take a matching radius lying beyond R_0, of (65), by many times the surface thickness a. Then the square-well shift function applies and the reduced widths are larger than $\hbar^2/mR^2$ by the reflection factor. The level shift is then simply the product of the square-well shift function and $\hbar^2/mR^2$ multiplied by the reflection factor f. Occasionally, care must be taken to include the effects of higher order terms.

For cross sections involving many channels considerable work[20,57] has been devoted to establishing a proper connection between the optical model cross sections on the one hand and the average cross sections on the other. It is not clear to the present author that these treatments have taken higher-order terms (e.g., in πPs) into account properly, nor is it clear that these higher-order terms are important.

e. Diffuse Nuclear Surface in the Theory of Resonance Reactions

In the general theory of nuclear reactions, as developed in Sec. IV, the introduction of a diffuse nuclear surface brings about changes in nucleon sum rules, in the role of channel radii in nuclear cross sections, and in the interpretation of the boundary condition parameters of the theory.

The sum rules for nucleon reduced widths first given by Teichmann and Wigner,[12] already appeared explicitly in (91) above. Thus, if the average potential of the optical model is square (or if the matching radius of the diffuse-edge well is chosen as on Fig. 6), then no single reduced width should be larger than $\hbar^2/mR_c^2$, the original reduced width of a true single-particle state. For a diffuse-edge potential, this statement holds only if the penetrability is multiplied by the reflection factor. The usual practice, in the analysis of nuclear reactions, has been to use the "square-well" penetrabilities (58). As we showed above, this practice is equivalent to evaluating the penetrability at a matching radius (cf. Fig. 5) larger than R_0 by several times a. The resulting reduced width is larger than $\hbar^2/mR_c^2$ by the reflection factor f and also overestimates the amplitude attained by the resonant state inside the nucleus. If we accept the evidence that the average potential for nucleons has a diffuse-edge and if we persist in using the conventional penetrabilities, then a true single-particle state has a reduced width several times the Wigner-Teichmann limit. Care must be taken in comparing the data for reduced widths [obtained from observed widths by means of the conventional penetrabilities, (58)] with

[57] P. A. Moldauer, Phys. Rev. **123**, 968 (1961).
[58] A. Jain (private communication); Ph.D. thesis, Cornell University, 1962 (unpublished).

the values predicted by nuclear models, such as the shell model. The practice with the models is to equate the reduced width to the square of the amplitude $C_{\lambda;cp}$ obtained after configuration mixing. The observed reduced widths, expressed as fractions of $\hbar^2/mR^2$, are equal to $C^2{}_{\lambda;cp}$ only if the penetrability is multiplied by the reflection factor.

The principal advantage of introducing the diffuse nuclear edge into the formal theory is that it allows us to remove, at least partially, the artificial dependence of nuclear cross sections on the nuclear radii. In introducing the diffuse edge we are still splitting up configuration space as on Fig. 4, with a channel region and a compound nucleus region, but now we include in the channel region the tail of the average nuclear potential well which, in turn, modifies the penetrability. The gain is not only to remove the artificial reflection of the old picture but also to make the precise position of the matching radius less important.

Teichmann and Wigner[12] first pointed out that the choice of channel radii was made difficult by opposing conditions, one preferring a large radii well outside nuclear interactions and the other preferring a surface hugging the nucleus more closely. If the radius is chosen sufficiently large then indeed all the nuclear interactions occur inside the nucleus, the penetrabilities and shift functions are the conventional ones, and the orthogonality condition (29) is bound to be satisfied. On the other hand, then the hard-sphere phase shifts and penetrabilities vary too rapidly with the energy; the nucleus after all does not scatter like a very large hard sphere.

In the scattering of s-wave neutrons by a real diffuse-edge potential the matching radius must be chosen to be reasonably close to the mean radius R_0 of the well. Otherwise the standing waves will have a different spacing than the resonances of the potential well and even though the one-level approximation (with use of appropriate natural boundary conditions) may hold near one low energy resonance, this approximation will break down quickly for a matching radius deviating from R_0 by several times the surface thickness a.

The nuclear density and the average nuclear potential pertinent to incoming nucleons have a well-defined mean radius and a reasonably sharp surface, limiting the choice of matching radii. Recognition of the variation of reduced widths and penetrabilities with channel radius, as discussed above, allows us to choose the matching radius almost anywhere within the nuclear edge without affecting the analysis of resonance cross sections. The important point is that the initial effect of the nuclear forces, for incoming nucleons, appears to be capable of description by the average potential of the optical model. The variation of resonance parameters with channel radius, when the nucleon is moving in the average potential, is discussed in Appendix A.

One of the intrinsic difficulties of the formal resonance theories is the artificial dependence of the potential scattering phase shifts (61) on the channel radii. According to (61), the potential scattering (apart from the Coulomb scattering) is that of a hard sphere with the radius equal to the channel radius. The hard-sphere phase shifts appear in all the approximate results such as the one-level or multilevel formulas which, as already pointed out above, may be directly attributed to the manner in which the formal theories treat the nuclear configuration space. The correction of the hard-sphere phase shift to the proper potential scattering phase shift is made by consideration of the many distant levels ordinarily neglected in the one-level or multilevel formulas.

To obtain the phase shifts appropriate to the potential scattering in between resonances we could use techniques similar to those employed in (94) to obtain the average $\mathfrak{R}$ function. In between resonances the collision function doesn't fluctuate rapidly. In any case, the nearby levels which give rise to fluctuations usually exhaust only a negligible fraction of the sum rule so that the real part, $\mathfrak{R}^\infty$, of the average $\mathfrak{R}$ function arises principally from the very many distant levels. Therefore we write

$$\mathfrak{R}_{cc'} = \sum_{\lambda=1}^{N} \gamma_{\lambda c}\gamma_{\lambda c'}/(E_\lambda - E) + \delta_{cc'}\mathfrak{R}_c^\infty , \quad (95)$$

where the sum runs over the few local levels to be used in a multilevel formula and where $\mathfrak{R}_c^\infty$ is

$$\mathfrak{R}_c^\infty \equiv \sum\nolimits'_\lambda \gamma_{\lambda c}^2/(E_\lambda - E) \quad (96)$$

in which the sum runs over all the distant levels omitted in the sum of (95). Because the few local levels contribute so little to the sum rule, the value of $\mathfrak{R}_c^\infty$ should be that of the real square well if the strength function is chosen to be that of (68). In (95) and (96) the off-diagonal terms of $\mathfrak{R}_c^\infty$ have been neglected, because the sign fluctuations of the $\gamma_{\lambda c}$ make these terms small. When the $\mathfrak{R}$ matrix (95) is used in the derivation of the collision function, the previous result (62) still holds except for the following changes. First of all, the sums over levels, in (62), are now restricted to the finite sum $1,\cdots,N$. Secondly, the penetrability, shift function, and phase

shift change to $\overline{P}_c$, $\overline{S}_c$, and $\overline{\Omega}_c$ given by

$$\overline{P}_c = P_c/[(1 - \mathfrak{R}_c^\infty S_c)^2 + (\mathfrak{R}_c^\infty P_c)^2] \quad (97)$$

$$\overline{S}_c = [S_c - \mathfrak{R}_c^\infty (P_c^2 + S_c^2)]/[(1 - \mathfrak{R}_c^\infty S_c)^2 + (\mathfrak{R}_c^\infty P_c)^2] \quad (98)$$

$$\overline{\Omega}_c = \Omega_c + \tan^{-1}[\mathfrak{R}_c^\infty P_c/(1 - \mathfrak{R}_c^\infty S_c)] . \quad (99)$$

None of the correction terms introduced in these functions are very important in magnitude; they were taken to be small in the discussion of the optical model results [see (71) e.g.]. However, the change in the phase shift (99) is of heuristic value. Comparison of (99) and (19) shows that in (19) we have added to the hard-sphere phase shift that part of the potential scattering phase shift arising from the real part $\mathfrak{R}_c^\infty$ of the optical model $\mathfrak{R}$ function. Except just at giant resonance, this contribution to the phase shift changes the hard-sphere phase shift to that of the real average potential well seen by a nucleon. Thus, the nuclear scattering is not that of a hard sphere although the correction to the hard-sphere scattering is not very large. A better approximation than (62) is obtained by replacing all the phase shifts in (62) by those of the diffuse-edge real potential well of the optical model and by using, in (62), the penetrabilities appropriate to a diffuse-edge potential well.

Having shown that both the phase shifts and the penetrabilities of the formal resonance theory may be given values compatible with the physical picture of the optical model then the nuclear collision function (62) no longer has an artificial dependence on the channel radii.

The description of direct interactions in terms of the resonance theory also involves the coherent sum over distant levels, just as for the potential scattering phase shifts. In both cases the processes occur rapidly in time, compared to the lifetime of the compound states, so that it follows directly from the uncertainty principle that the very distant levels must be involved. Considerable progress toward such a description of direct interactions has been made by Bloch[24] and Brown and de Dominicis.[28] These interactions require consideration of the off-diagonal components of $\mathfrak{R}^\infty$ which are neglected in (95).

The preceding parts of this section have suggested that the boundary condition numbers of the nuclear problem have a natural value dependent on the precise choice of the matching radius within the diffuse edge of the nucleus. The "natural" boundary condition numbers are chosen in such a way that the shift-function vanishes in each channel precisely in the way prescribed above [Eq. (64)] for the "natural" boundary conditions required to eliminate the closed channels. The shift function may then plausibly be viewed as an "error function" telling us how closely the standing waves resemble the physical resonant states, if we do not choose the shift function to vanish then we are neglecting from the nuclear problem (as the discussion of Fig. 3 should indicate) effects arising from many single particle configurations lying at a great energy from the levels of interest.

The "natural" boundary condition numbers are energy dependent just as the shift function is. Since the energy on which they depend is that of the whole system it does not destroy the Hermitian orthogonality of the compound states but it can make the nuclear parameters energy dependent. Fortunately, the energy dependence is so moderate that it does not appear to cause any difficulty in the analysis of resonance reactions. The calculations of Fig. 3 show that the boundary condition numbers have to differ from their natural value by a great extent (so that the shift function becomes an appreciable fraction of the single-particle spacing) before the standing waves deviate appreciably from the physical resonant state. It is likely for this reason that resonance theories such as that of Kapur and Peierls,[15] which choose complex boundary condition numbers necessarily different from the "natural" ones, can be used to analyze nuclear cross sections without serious error.

In analyzing nuclear reactions, one usually is interested only in a small energy interval containing a few resonance lines. The boundary condition numbers should be taken as constants chosen so that the shift functions vanish somewhere in the energy interval of interest. The precise value of the constants compatible with this criterion are not likely to affect the analysis: They may affect the precise value of the nuclear parameters but not likely the energy dependence of the cross section. Even the low-energy limit of the boundary condition numbers (that is, the choice which makes the shift function vanish at zero energy) will suffice for most reactions. If the analysis depends appreciably on the precise point in the energy interval of interest at which the shift functions are chosen to vanish then the extraction of nuclear reduced widths or resonance energies is not very meaningful. Perhaps the best example of this fact is given, again, by the energy dependent boundary conditions (64), which we chose to eliminate the closed channels. The treatment described there breaks down, for cross sections near threshold, when the required boundary condition numbers vary too rapidly.

VI. SUMMARY

The principal features of the theory of resonance reactions, originally developed by Wigner and his collaborators, are described (Secs. II and III) in terms of simple examples. We give a straightforward derivation (Sec. IV) of the particular result for the collision matrix found recently to be most useful for the description of low energy nuclear reactions. This result is expressed in terms of the potential scattering phase shifts and in terms of the level widths and level shifts of the compound nucleus states.

The division of nuclear configuration space is shown to be less stringent than envisaged during the early development of nuclear reaction theory. In particular, the idea of an average nuclear potential, so successfully employed in the nuclear shell model and the nuclear optical model, is shown to be easily accommodated in the resonance theories. The original versions of the resonance theories appeared to have built into them the properties of a hard sphere. It is the artificial hard-sphere properties of the theory which are removed by the introduction of the average nuclear potential. The potential scattering phase shifts, in between resonance lines, are shown to change to the phase shifts of the real potential well of the nuclear optical model.

The levels widths $\Gamma_{\lambda c}$, formerly products $2P_c\gamma_{\lambda c}^2$ of penetrabilities P_c and reduced widths $\gamma_{\lambda c}^2$, are shown to be, more properly,

$$\Gamma_{\lambda c} = 2P_c\gamma_{\lambda c}^2 f_c\,, \tag{100}$$

where f_c is a reflection factor depending on the surface thickness a of the average nuclear potential in the way given by (65). According to (100), the entrance of a nucleon into the nucleus is a three-step process: (a) penetration of the nucleon to the nuclear surface, (b) partial reflection at the surface, and (c) acceptance into the compound state. Without the reflection factor f_c, the theory has the artificial reflectivity of the hard sphere. It is shown that the reflection factor has a value between two and three, for low-energy nuclear reactions. This factor must, for example, multiply the Wigner-Teichmann sum rule limits if the conventional penetrabilities are employed in analyzing widths. The fact that the principal effect of the average nuclear potential is to introduce the reflection factor f_c into the level widths makes possible a very accurate estimate of the nuclear surface thickness. The surface thickness parameter a of (65) is shown to have a value of 0.31 ± 0.06 for low-energy neutrons.

The level shifts are shown to vanish by choice of the boundary condition numbers. The "natural" boundary condition numbers required to accomplish this are energy dependent but under most circumstances encountered in low energy nuclear reactions this energy dependence appears to be sufficiently moderate as to not cause any difficulty.

The surprising adaptability of the resonance theory suggests that it will still be useful for a long time.

ACKNOWLEDGMENTS

This commemorative issue has afforded a great opportunity to show, in small measure, the author's appreciation of and gratitude for the advice and interest of his teacher Professor E. P. Wigner—the debt is great and still increasing rapidly.

The many computer calculations described in the text were all carried out on the Chalk River Datatron computer with programs prepared under the direction of Dr. J. M. Kennedy. His help, and that of H. H. Clayton and Dr. T. D. Newton who read the manuscript, are gratefully acknowledged.

APPENDIX A. THE EXPONENTIAL POTENTIAL

Although the Schrödinger equation (1) with the diffuse-edge potential (65) of Saxon and Woods has no simple analytic solution the exponential potential also produces a diffuse edge and is somewhat more tractable.[59] We can use this potential to show some of the more important properties of the reflection factor. For this purpose we match an exponential potential onto the Saxon-Woods potential at the radius R_0 of (65). The desired potential is

$$V(r) = -(V_0/2)e^{-(r-R_0)/2a} r > R_0\,, \tag{A1}$$

which has the same slope and value, at $r = R_0$, as (65). If we use the potential (A1) in the Schrödinger equation (1) and if we introduce the variable

$$x = 4K_0 a\, e^{-(r-R_0)/4a}\,, \tag{A2}$$

where K_0 is the wave number corresponding to $V_0/2$, then the Schrödinger equation changes to

$$d^2\phi/dx^2 + x^{-1}d\phi/dx + (1 - \lambda/x^2)\phi = 0\,, \tag{A3}$$

where

$$\lambda = 4kai\,. \tag{A4}$$

The solutions of (B3) are the Bessel functions $B_\lambda(x)$. By comparison of the asymptotic behavior ($r \to \infty$, $x \to 0$) of the Bessel functions with that of

[59] Besides being used by Bethe,[50] this potential has been employed recently by K. Kikuchi, Progr. Theoret. Phys. (Kyoto) **17**, 643 (1957), to estimate nuclear transmission functions.

the functions sin kr and cos kr we find, for the penetrability (58)

$$P \equiv kr/(F_0^2 + G_0^2) = kr|J_\lambda(x)|^{-2} . \quad \text{(A5)}$$

Evaluating the penetrability at $r = R_0$ we have

$$P = kR_0|J_{4kai}(4K_0a)|^{-2} . \quad \text{(A6)}$$

Thus the reflection factor is the reciprocal of the absolute square of the Bessel function of (A6). The order of the Bessel function, though complex, is small. For $a = 0.31$ F, $4ka$ has the value $0.27\ E^{1/2}$ where the energy E is in MeV. For low energies we can use the Bessel function J_0 in estimating the reflection factor. Equation (A6) also shows the slow change of the reflection factor with energy.

For $V_0 = 51$ MeV and $a = 0.31$ F the argument of the Bessel function in (A6) is about 1.40 and the reflection factor is about 3.2. This value is slightly larger than the value of (66) for the corresponding surface thickness, partly because of the smaller radius employed in the present calculation and partly because of the larger tail of the exponential potential, (A1). Like (66), the difference between unity and the reflection factor, $J_0^{-2}(4K_0a)$, is, to first order, quadratic in the surface thickness a.

Peaslee[29] obtained a result for the reflection factor by considering the surface thickness as small compared to the nuclear radius. The approximate result which he obtained for the reflection factor f of the Saxon-Woods potential was

$$f \approx 2^{1/2}\pi K_0 a \coth (2^{1/2}\pi K_0 a) . \quad \text{(A7)}$$

This result is in fairly good agreement with the computer calculations of Fig. 8. The average over the complete giant resonances of the curves of Fig. 8 yields, for $a = 0.3$, 0.5, and 0.7 F, respectively, $f = 1.68$, 2.62, and 3.80, whereas (A7) yields $f = 1.66$, 2.50, and 3.36 for the same value of a. Thus, (A7) appears to be too linear for the larger values of a.

APPENDIX B. DERIVATIVE OF THE R FUNCTION WITH RESPECT TO THE CHANNEL RADIUS[60]

The derivatives of reduced widths and level energies with regard to the channel radii may be found from direct differentiation of the $\mathfrak{R}$ matrix. We begin with the $\mathfrak{R}$ function of Sec. II, defined by (16) or (17). From differentiation of (16) with respect to the channel radius R, and multiplication by R, we have

$$\phi' = \mathfrak{R}'(\phi' - b\phi) + \mathfrak{R}(\phi'' - b'\phi - b\phi') , \quad \text{(B1)}$$

where primes again imply the operation $R\ d/dR$. We note that

$$\phi'' = R^2(d^2\phi/dR^2) + \phi' = (2mR^2/\hbar^2)(V - E)\phi + \phi' \quad \text{(B2)}$$

and that it follows from (16) that

$$\phi'/(\phi' - b\phi) = 1 + b\mathfrak{R} . \quad \text{(B3)}$$

Then we find at once from (B1) that

$$\mathfrak{R}' = 1 + (2b - 1)\mathfrak{R} + \mathfrak{R}^2(2mR^2/\hbar^2)(E - \hat{V}) , \quad \text{(B4)}$$

where

$$\hat{V} = V - (\hbar^2/2mR^2)(b^2 - b + b') . \quad \text{(B5)}$$

Next we rearrange the double sum occurring in the square of the $\mathfrak{R}$ function.

$$\begin{aligned}\mathfrak{R}^2 &= \sum_\lambda \gamma_\lambda^4/(E_\lambda - E)^2 + \sum_\lambda \sum_\mu{}' \gamma_\lambda^2\gamma_\mu^2/(E_\lambda - E)(E_\mu - E) \\ &= \sum_\lambda \gamma_\lambda^4/(E_\lambda - E)^2 + 2\sum_\lambda \sum_\mu{}' \gamma_\lambda^2\gamma_\mu^2/(E_\lambda - E)(E_\mu - E_\lambda) , \quad \text{(B6)}\end{aligned}$$

in which the double sum converges absolutely. The prime on the sum means that the term $\mu = \lambda$ is to be omitted. Multiplying both sums of (B6), term by term, with $(E - E_\lambda - \hat{V} + E_\lambda)$ we find

$$\begin{aligned}(E - \hat{V})\mathfrak{R}^2 &= \sum_\lambda (E_\lambda - \hat{V})\gamma_\lambda^4/(E_\lambda - E)^2 - \sum_\lambda \gamma_\lambda^4/(E_\lambda - E) + \sum_\lambda \frac{2\gamma_\lambda^2(E_\lambda - \hat{V})}{E_\lambda - E} \\ &\quad \times \sum_\mu{}' \frac{\gamma_\mu^2}{E_\mu - E_\lambda} - \sum_\lambda 2\gamma_\lambda^2 \sum_\mu{}' \frac{\gamma_\mu^2}{E_\mu - E_\lambda} .\end{aligned}$$

Using the results of (B7) in (B4) we find

$$\begin{aligned}\mathfrak{R}' &= [1 - (4mR^2/\hbar^2)\sum_\lambda \gamma_\lambda^2 \sum_\mu{}' \gamma_\mu^2/(E_\mu - E_\lambda)] \\ &\quad + \sum_\lambda [(2b - 1)\gamma_\lambda^2 - (2mR^2/\hbar^2)\gamma_\lambda^4 \\ &\quad + (4mR^2/\hbar^2)\sum_\mu{}' \gamma_\mu^2(E_\lambda - V)/(E_\mu - E_\lambda)] \\ &\quad \times (E_\lambda - E)^{-1} + [(2mR^2/\hbar^2)\sum_\lambda (E_\lambda - V) \\ &\quad \times \gamma_\lambda^4/(E_\lambda - E)^2] . \quad \text{(B8)}\end{aligned}$$

[60] The substance of Appendix B is due to Professor Wigner. The author is indebted to him for permission to use this material. However, Professor Wigner was, by no means, aware that the material would be used in the present commemorative issue. Those associated with Professor Wigner will appreciate that his direct help enters inexorably into any problem with which he is confronted.

This result is to be compared with the direct differentiation of (16):

$$\mathfrak{R}' = \sum_\lambda (\gamma_\lambda^2)'/(E_\lambda - E) - \sum_\lambda \gamma_\lambda^2 E_\lambda'/(E_\lambda - E)^2 . \quad \text{(B9)}$$

Equating those terms in (B9) and (B8) which depend on the energy in the same way yields

$$1 = (4mR^2/\hbar^2) \sum'_{\lambda\mu} \gamma_\lambda^2 \gamma_\mu^2/(E_\mu - E_\lambda) \quad \text{(B10)}$$

$$(\gamma_\lambda^2)' \equiv R d\gamma_\lambda^2/dR = (2b - 1)\gamma_\lambda^2 + (2mR^2/\hbar^2) \times [-\gamma_\lambda^4 + 2\gamma_\lambda^2(E_\lambda - \hat{V}) \sum'_\mu \gamma_\mu^2/(E_\mu - E_\lambda)] , \quad \text{(B11)}$$

$$E_\lambda' \equiv R dE_\lambda/dR = (2mR^2/\hbar^2)(\hat{V} - E)\gamma_\lambda^2 . \quad \text{(B12)}$$

The sum rule (B10) appears to be a new one and is easily verified explicitly for the case of no potential ($V = b = 0$).

It is interesting to note that in the one-level approximation we can use (B11) to arrive at the natural boundary condition numbers. Then (B11) may be approximated by

$$(\gamma_\lambda^2)' \approx (2b - 1)\gamma_\lambda^2 \quad \text{(B13)}$$

because the term proportional to γ_λ^4, in (B11), is small unless γ_λ^2 is an appreciable fraction of the sum rule limit. If we now want the total width to be independent of the matching radius we have

$$\Gamma_\lambda' = 0 = 2P_c' \gamma_{\lambda c}^2 + 2P_c(\gamma_\lambda^2)' . \quad \text{(B14)}$$

Using the definition, (58) of the penetrability (B14) leads at once then to the boundary condition number, (64), required to make the shift function vanish.

The above derivation of $(\gamma_\lambda^2)'$ and (E_λ') is not valid if the boundary condition numbers are energy dependent. As we noted in the text, the "natural" boundary condition numbers are usually only moderately dependent on the energy: for most purposes the low-energy limit (the value of b_c which makes S_c vanish at zero energy) suffices.

REVIEWS OF MODERN PHYSICS VOLUME 34, NUMBER 4 OCTOBER 1962

Recent Developments in the Theory and Technology of Chain Reactors

LAWRENCE DRESNER AND ALVIN M. WEINBERG
*Oak Ridge National Laboratory**

EUGENE Wigner's sixtieth birthday coincides, within two weeks, with the twentieth anniversary of the first fission chain reaction on December 2, 1942. Wigner's influence on the subsequent development of chain reactors has been extraordinary. There are very few aspects of the current theory that were not outlined by him nor very few of the major technological developments of the last 20 years that cannot be traced to his early ideas.

Wigner's major contributions to chain reactors occurred in three separate periods: First, from 1940 to 1945, during the last three years of which he was head of the theoretical group at The University of Chicago Metallurgical Laboratory; 1946–47, when he was research director of what is now the Oak Ridge National Laboratory; and 1952, when he was a full-time adviser to the Du Pont Company during the design of the Savannah River heavy water plutonium production reactors.

Chain reactors are heavy engineering devices; they also require, for their design, a certain theoretical and physical sophistication. Wigner's great influence on their development, both at the Metallurgical Laboratory and since, can be traced to the facility with which he could pass back and forth between engineering and physics—from a discussion of the probable distribution of energy levels in U^{235} to a critical examination of the blueprints of the concrete foundations for the Hanford reactors, or from a group-theoretical argument in transport theory to the design of aluminum fuel elements!

The full force of Wigner's skill and energy was particularly well demonstrated at the Metallurgical Laboratory during the fall of 1942 in the crucial arguments that resulted in the decision to concentrate on water-cooled, rather than on gas-cooled,

*Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.