Some Generalizations in One-Dimensional Magneto-Gas Dynamics

J. A. SHERCLIFF
Department of Engineering, University of Cambridge, Cambridge, England

I. INTRODUCTION

In a previous paper,[1] a symmetrical treatment was developed for the various processes that are possible in one-dimensional flow of arbitrary gases in thermodynamic equilibrium with properties limited only by stability and the laws of thermodynamics. It is in many ways easier to treat an arbitrary gas than a specific one such as a perfect gas which involves complicated algebraic relations. The significance of the ordinary sound speed a and the Newtonian or isothermal sound speed $a\gamma^{-\frac{1}{2}}$ emerged from the analysis. Kline and Shapiro[2] showed how it is possible to establish the nature of shock transitions via a study of the Rayleigh line, this being a simple alternative to the method which involves the Hugoniot function. The Rayleigh[3] line itself is relevant to the problem of shock structure in highly ionized gas of low Prandtl number.

It is evident that there is a worthwhile generalization to magneto-gas dynamics, a complicated subject where simple methods of analysis are much to be desired. Moreover, the development of theory valid for any gas is particularly necessary, since magneto-gas dynamics often concerns partially ionized gases of variable mean molecular weight. The speed of sound which appears in the analysis refers to the speed of waves of low enough frequency for the maintenance of equilibrium of ionization and other effects.

II. APPROXIMATE ONE-DIMENSIONAL MAGNETO-GAS DYNAMICS IN A TRANSVERSE FIELD

There exist two distinct types of one-dimensional magneto-gas dynamics under a transverse field. One type is *approximate* in the same way that the ordinary one-dimensional gas dynamics of flows with area-change, wall friction, etc., is approximate. Here one is usually dealing with flows inside a duct whose width is small in comparison with the longitudinal distances over which flow characteristics change significantly. Transverse velocities and transverse variations may then be ignored. As the author[1] and Resler and Sears[4] have pointed out, it is possible to use electromagnetic means to bring about the energy, momentum, and entropy changes that induce the basic processes of approximate one-dimensional gas dynamics, if the gas is adequately conducting.

The characteristic feature of these processes is that certain quantities have *imposed* values at all stations along the duct. Just as duct area A may be specified as a function of the longitudinal coordinate x, so may we specify a transverse magnetic field B_y as a function of x, and also the perpendicular electric field E_z, or current j_z, or perhaps a relation between them expressing the characteristics of some external load.

This imposition of values for B_y and E_z implies two conditions. One is that the magnetic Reynolds number be not so high that the imposed field B_y is seriously perturbed by the fluid motion. This condition can be obtained more easily if any net current flowing through the fluid in the z direction is brought back in solid conductors in the same cross section of the duct, between the poles of the magnets that impose the field B_y. This is standard practice in electromagnetic pumps to counteract "armature reaction."

The other condition which permits our imposing values for B_y and E_z is that they should vary only slowly in the x direction. The Maxwell equations, $\mathrm{curl}\mathbf{B}=0$ (if the effect of currents in the fluid is weak) and $\mathrm{curl}\mathbf{E}=0$ (in a steady state), cannot be satisfied unless there are also components B_x and E_x and variations of B_y and E_z across the duct, the phenomenon known as "fringing." This effect is negligible if B_y and E_z vary only slowly with x, over a characteristic length L, say, which is large in comparison with the duct width d. Then, for instance, the equation $\partial B_x/\partial y=\partial B_y/\partial x$ is satisfied by having components B_x such that B_x/d is of the same order of magnitude as B_y/L, so that B_x is small in comparison with B_y, and $\partial B_y/\partial y$ is small also (by $\mathrm{div}\mathbf{B}=0$). We can thus impose $B_y(x)$ and $E_z(x)$ [or $j_z(x)$ etc.] and ignore the other components and the need to satisfy Maxwell's equations. Henceforth we suppress the suffices x, y, and z.

The specification of $A(x)$, $B(x)$, and $E(x)$ [or $j(x)$] can result in steady gas-dynamic processes governed by the following equations, in the absence of heat transfer or viscous forces:

$$\rho vA=\text{const},$$
$$\rho v dh_0/dx=Ej, \tag{1}$$
$$\rho vT ds/dx=j^2/\sigma,$$

and

$$\rho v dv/dx+dp/dx=-jB \quad (=df/dx, \text{ if } A \text{ is const}).$$

[1] J. A. Shercliff, J. Fluid Mech. **3**, 645 (1958).
[2] S. J. Kline and A. H. Shapiro, Mém. Mécan. Fluides 171 (1954).
[3] Lord Rayleigh, Proc. Roy. Soc. (London) **A84**, 247 (1910).
[4] E. L. Resler and W. R. Sears, J. Aeronaut. Sci. **25**, 235 (1958).

h_0 is the stagnation enthalpy $h+v^2/2$ and f the impulse function $p+\rho v^2$.

Obviously, the case where $E=0$ and $A=$constant is the familiar Fanno process ($h_0=$const). E may be made zero by an external short-circuit or by the exploitation of axisymmetry. The force opposing the motion is actually distributed throughout the gas instead of originating in friction at walls as in the original Fanno process. The case where $B=0$ and $A=$constant is the Rayleigh process ($f=$const). Since the energy exchange is dissipative here, only the case of energy *supply* can occur. The case where $A=$constant and $Bv\gg j/\sigma$ corresponds to reversible work exchange at constant area ($s=$const). Resler and Sears[4] consider other possibilities.

The previous paper[1] showed how the three processes listed in the foregoing all exhibit the phenomenon of choking when v reaches the ordinary sound speed a. It needs stressing that this is not the augmented magnetic sound speed, which becomes important in flows of a highly conducting gas at high magnetic Reynolds number, when perturbation of the field by the flow is a major effect.

III. STRICT ONE-DIMENSIONAL MAGNETO-GAS DYNAMICS IN A TRANSVERSE FIELD

The other type of one-dimensional magneto-gas dynamics, with which the first type should be contrasted, is *strictly* one-dimensional. All quantities are strictly independent of the transverse coordinates y and z, and Maxwell's equations must be obeyed. This implies that the region of flow must have a transverse extent that is very large in comparison with the longitudinal distances over which quantities change significantly. Situations of this type have been widely studied in the literature,[5,6] but most of the analysis has been restricted to the perfect gas case. In general, there are longitudinal and transverse magnetic field components. Here we confine our study to the simpler case where the field is purely transverse and the operator $(\mathbf{B}\cdot\text{grad})$ vanishes.

As a result, the dynamic equation (expressed in mks units) degenerates to

$$\rho\partial\mathbf{v}/\partial t+\rho v_x\partial\mathbf{v}/\partial x+\text{grad}[p+(\mathbf{B}^2/2\mu)]=0,$$

in the absence of viscosity. Since the gradient term has only an x component, the transverse velocity components of any fluid particle are conserved. As their only effect in the presence of a transverse field is to produce electric fields in the x direction, they can be ignored. In steady flows they can be suppressed by the use of suitably moving axes.

We postulate that the magnetic Reynolds number, based on the characteristic length over which quantities very significantly, is large enough for the gas to be virtually a perfect conductor. Then the theorem of Truesdell,[7]

$$(D/Dt)(\mathbf{B}/\rho)=(\mathbf{B}\cdot\text{grad})\mathbf{v}/\rho,$$

here becomes

$$(\partial/\partial t+v_x\partial/\partial x)(\mathbf{B}/\rho)=0.$$

In steady flow $\mathbf{B}/\rho$ is constant in magnitude and direction. We assume that this is true in unsteady flows also, and take $\mathbf{B}$ as being in the y direction. The problem now concerns the simple configuration, considered by many authors, where the vectors $\mathbf{v}$, $\mathbf{B}$, and $\mathbf{E}$ (and $\mathbf{j}$) are in the x, y, and z directions, respectively. It is well known that the problem is completely analogous to the corresponding problem in ordinary gas dynamics. This is evident from the dynamic equation. The conducting gas behaves exactly like a fictitious ordinary gas whose pressure p^* is related to its density ρ and entropy s by the equation

$$p^*=p+(B^2/2\mu)=p(\rho,s)+(B/\rho)^2(\rho^2/2\mu), \qquad (2)$$

in which p is the real conducting-gas pressure and B/ρ is a constant of the motion. Obviously, the critical velocity c for these flows is given by the equation

$$c^2=(\partial p^*/\partial\rho)_s=(\partial p/\partial\rho)_s+(B^2/\mu\rho)=a^2+b^2,$$

where $b=B/(\mu\rho)^{\frac{1}{2}}$, an Alfvén velocity. This is the velocity at which simple waves propagate relative to the gas. By using a well-known result from ordinary gas dynamics, we see that simple isentropic compressive waves steepen if $(\partial^2p^*/\partial\tau^2)_s$ is positive, where τ denotes the specific volume, $1/\rho$. This condition requires that

$$(\partial^2p/\partial\tau^2)_s+(3B^2\rho^2/\mu) \text{ be positive.} \qquad (3)$$

This is true because the first term alone is positive for all known fluids. The transverse field obviously promotes steepening of compressive waves. The steepening proceeds until the smallest Reynolds number (magnetic, thermal, or viscous) becomes of order unity, so that dissipative diffusive effects resist the steepening and a shock is formed.

The expression (3) also proves to have crucial importance in the steady flows which are discussed next.

In steady one-dimensional magneto-gas dynamics, some influence must be exerted on the flow in order that any changes should occur at all. By "influence" is meant the application of additional forces to the flow (by distributed drag, actuator disks, gravity, etc.) or heat exchanges (by conduction, radiation, or release or absorption of heat by chemical, nuclear or thermonuclear reaction).

It is then possible to treat all such resultant processes in the symmetrical manner already developed for ordinary gas dynamics.[1] The only difference here is that area change is excluded, so that there is one independent variable fewer and simple processes may

[5] F. de Hoffmann and E. Teller, Phys. Rev. **80**, 692 (1950).

[6] S. A. Kaplan and K. P. Stanyukovich, Doklady **95**, 769 (1954).

[7] C. Truesdell, Phys. Rev. **78**, 823 (1950).

be defined by keeping just one flow property constant instead of two.

In the absence of forces applied to the fluid in addition to the $\mathbf{j}\times\mathbf{B}$ force, the dynamic equation

$$\rho v dv/dx + dp^*/dx = 0$$

implies that the quantity $F = p^* + \rho v^2$ is constant. Use has been made of the continuity equation

$$\rho v = \text{const} = G, \text{ say.}$$

In the absence of energy exchange of any kind other than the $\mathbf{E}\cdot\mathbf{j}$ term, the energy equation

$$\rho v dh_0/dx = Ej = (E/\mu)(dB/dx),$$

implies that $\rho v h_0 - EB/\mu$ is constant, in view of the fact that E is constant because $\mathbf{E}$ is irrotational in steady states. As the gas is a perfect conductor, $E + vB = 0$, and so the quantity

$$H = h_0 + B^2/\mu\rho = h^* + v^2/2,$$

is constant in the absence of extra energy exchange. The symbol h^* denotes $h + B^2/\mu\rho$ or $h + \rho(B/\rho)^2/\mu$. The other important property which may sometimes be constant is the entropy s.

It is now apparent that the conducting gas behaves exactly like a fictitious ordinary gas having properties ρ, T, s, p^*, h^*, which are found to be related in the correct way by the equation

$$Tds = dh^* - dp^*/\rho,$$

which must hold for substances in equilibrium. We can thus apply all the results[1] known for ordinary gasdynamic processes, expressible in terms of F, H, and s, if in addition the constants G and (B/ρ) are specified. A state of flow may be specified by any two properties, although for given values of F and H there are usually *two* possible states, corresponding to the two sides of a stationary shock. Changes in F, H, and s are not independent but are related by the usual equation

$$dF + \rho T ds = \rho dH. \tag{4}$$

Three simple processes may be defined by specifying that one of the properties F, H, and s be constant. They are

(i) F constant. Magnetic Rayleigh process. No extra forces on the fluid other than $\mathbf{j}\times\mathbf{B}$. Changes are produced by exchange or release of heat. This case has been discussed by Gross,[8] who assumed perfect-gas behavior.

(ii) H constant. Magnetic Fanno process. Extra drag forces applied to the fluid, with entropy increase due to dissipation but no energy exchange other than $\mathbf{E}\cdot\mathbf{j}$.

(iii) s constant. Extra forces on the fluid, resulting in reversible energy exchange. An example occurs when gravity operates in the direction of motion.

These processes are quite different from the processes considered in Sec. II. The critical velocity for these processes is not a but c, the magnetic sound speed. Either directly, or by analogy with the earlier work[1] using the idea of the fictitious ordinary gas, we may show that

$$(\partial F/\partial\rho)_s = c^2 - v^2 = c^2(1-M^2), \text{ etc.,}$$

where $M = v/c$, and that all such derivatives have the factor $(1-M^2)$. The results are summarized in Table I. The entries in the table are all thermodynamic properties of the real (conducting) gas, as well as the fictitious gas. This comes about because $dp = dp^*$ and $dh = dh^*$ when ρ is constant.

In particular, we observe the expected result that all the derivatives vanish when $M = 1$. The quantities $(\partial s/\partial p)_\rho$ and $(\partial s/\partial h)_\rho$ are known to be positive and finite for all fluids with positive expansion coefficients.

The next task is to establish whether the stationary values of F, H, and s when $M = 1$ are maxima or minima. From the equation

$$(\partial M^2/\partial\rho)_{F \text{ or } H} = (\partial M^2/\partial\rho)_s + (\partial M^2/\partial s)_\rho (\partial S/\partial\rho)_{F \text{ or } H},$$

it follows that $\partial M^2/\partial\rho$ takes the same value when $M = 1$ in all three processes provided $(\partial M^2/\partial s)_\rho$ is finite. But when $M = 1$,

$$(\partial M^2/\partial s)_\rho = -(\tau^2/c^2)(\partial^2 T/\partial\tau^2)_s,$$

(where $\tau = 1/\rho$) and no singularity upon isentropics, where this derivative could increase without limit, appears to be possible.

The common value of $\partial M^2/\partial\rho$ when $M = 1$ may be found, directly or from the analogy, to be equal to

$$-(\tau^4/c^2)(\partial^2 p^*/\partial\tau^2)_s,$$

which has already been seen to be negative. This shows how the derivatives in Table I pass through zero and enables us to compile Table II, assuming a positive expansion coefficient. If we assume that the variation of M, F, H, and s with ρ is continuous, it follows that

TABLE I. Derivatives expressed as multiples of $c^2(1-M^2)$.

	$\partial F/\partial\rho$	$\partial H/\partial\rho$	$\partial s/\partial\rho$
F constant (Rayleigh)	0	$-T(\partial s/\partial p)_\rho$	$-(\partial s/\partial p)_\rho$
H constant (Fanno)	$T(\partial s/\partial h)_\rho$	0	$-(1/\rho)(\partial s/\partial h)_\rho$
s constant	1	$1/\rho$	0

TABLE II.

Constant	Nature of sonic points
F (Rayleigh)	H max; s max
H (Fanno)	F min; s max
s	F min; H min

[8] R. A. Gross, J. Aero/Space Sci. **25**, 788 (1958).

only one sonic point occurs in each process, since maxima and minima must occur alternately.

Flows in ordinary gas dynamics,[1] for which the Newtonian sound speed $a\gamma^{-\frac{1}{2}}$ is critical, have obvious analogs in the case of the perfectly conducting gas under a transverse field. In particular, provided the gas has a positive expansion coefficient and

$$(\partial^2 p^*/\partial\tau^2)_T = (\partial^2 p/\partial\tau^2)_T + (3B^2\rho^2/\mu)$$

is positive, the point in a magnetic Rayleigh process where v equals the magnetic Newtonian sound speed, $(a^2/\gamma+b^2)^{\frac{1}{2}}$, is the point where the temperature is a *maximum*. Another result is that in an *isothermal* magnetic Fanno process, where high heat conductivity keeps the temperature uniform, the force opposing the motion causes the velocity to approach the Newtonian sound speed from above or below. At the Newtonian sonic point the property F is a minimum. We may remark that $(\partial^2 p/\partial\tau^2)_T$ is positive for real gases except near the critical point.

If the variation of T or F is continuous it follows that only one Newtonian sonic point occurs on Rayleigh or isothermal Fanno lines, since maxima and minima must occur alternately. This, and the corresponding result for points where $M=1$, constrains a magnetic Rayleigh line to have the form in the $T-s$ plane shown in Fig. 1, provided $(\partial^2 p/\partial\tau^2)_{s\text{ or }T}$ and the expansion coefficient are positive. Only one loop of the Rayleigh line occurs for the conducting real gas or the analogous fictitious ordinary gas. This follows from the fact that the existence of two such loops would imply that some lines of constant s or constant T would intersect the complete Rayleigh line in four places. But on the $p^*-\tau$ plane, shown in Fig. 2, the Rayleigh line is the straight line $p^*+G^2\tau=F$. Intersection of this line four times by an isothermal or isentropic is obviously incompatible with the agreed restriction that $(\partial^2 p^*/\partial\tau^2)_{s\text{ or }T}$ be positive. The $p^*-\tau$ relation is obviously continuous at constant s or T.

The subsonic branch is identified by the fact that the temperature maximum is a subsonic state since γ must be greater than unity.[1] The subsonic and supersonic branches cannot cross because this would imply two flow-sections with the same values of T and s and

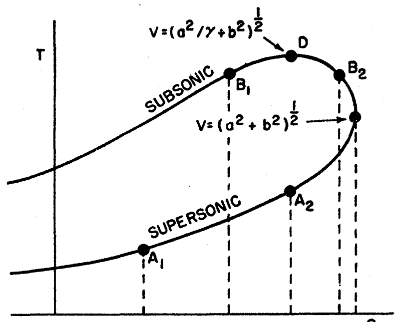
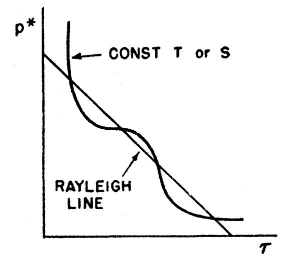


FIG. 1. Rayleigh line in the presence of a transverse magnetic field.

FIG. 2.



hence in the same thermodynamic state with the same values of ρ and c, but with different Mach numbers and velocities. This is impossible because ρv is constant.

Across normal shocks, where the magnetic field is transverse to the flow, it is well known[5] that the quantities G, B/ρ, F, and H (but not s) are conserved. Thus the two states A and B on either side of such a shock must both lie on the same magnetic Rayleigh line. In any transition between these states via a Rayleigh process as distinct from the actual shock process, we have $dH=Tds$ from (4), F being constant. Since $H_A=H_B$, then

$$\int_A^B Tds=0,$$

and clearly A and B must lie more or less as in Fig. 1, which shows the two cases A_1B_1 and A_2B_2. Evidently for each state there can be only *one* state reachable via a shock transition. It is apparent that A, on the supersonic branch, must have lower entropy and so must be upstream, whereas B is subsonic and downstream. Moreover, Table I indicates that $(\partial s/\partial\rho)_F$ is positive if $M>1$ and negative if $M<1$, and we conclude that ρ increases all the way from A to B. In particular, $\rho_B>\rho_A$.

A simple proof of the known results that transverse field shocks must be compressive[5,9] and must go from supersonic to subsonic flow is thus provided for any gas with the reasonable characteristics that $(\partial^2 p/\partial\tau^2)_s$ and the expansion coefficient are positive.

Lüst's equation[10] for shock transitions,

$$\Delta h-\bar{\tau}\Delta p+\Delta\tau(\Delta B)^2/4\mu=0,$$

in which Δ denotes changes across the shock and a bar denotes a mean value across the shock, may readily be derived from the equation,

$$\Delta h^*-\bar{\tau}\Delta p^*=0,$$

which is the Hugoniot equation for the analogous fictitious ordinary gas. An equivalent equation is Eq. (88) of de Hoffman and Teller.[5]

IV. SHOCK STRUCTURE

A. Thermal Diffusion Dominant

There is one case where the Rayleigh process actually does represent the shock process, insofar as continuum

[9] R. V. Polovin and G. I. Liubarskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 510 (1958).

[10] R. Lüst, Z. Naturforsch. **8a**, 277 (1953).

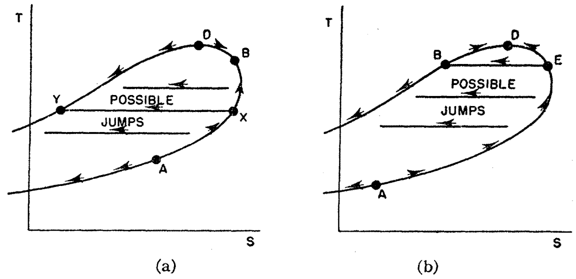


FIG. 3. (a) $v > (a^2/\gamma + b^2)^{\frac{1}{2}}$ downstream; (b) $v < (a^2/\gamma + b^2)^{\frac{1}{2}}$ downstream.

equations are valid. Rayleigh[3] himself showed this for ordinary shocks that are not too strong, in the case of a perfect gas. This is the case where the Prandtl number is very small and the thermal diffusivity $K/\rho c_p$ (K = thermal conductivity) is very large in comparison with the magnetic diffusivity $1/\mu\sigma$, and the viscous one, ν, the kinematic viscosity. This case is approached by fully ionized hot gas as Marshall[11] remarked. It is then possible to have a magnetic Rayleigh process in which the necessary heat exchange is achieved simply by longitudinal heat flow in the gas under the influence of the temperature gradients associated with the process. Radiation is ignored. Owing to the relative smallness of the magnetic and viscous diffusivities, the changes of velocity and magnetic field that occur over the same distances as do the temperature changes incur negligible magnetic or viscous diffusion. The gas is virtually the inviscid perfect electrical conductor discussed in Sec. III. Only the Reynolds number based on the thermal diffusivity (the Péclet number) is of order unity, the others being very large.

It is necessary to consider whether the postulated heat flow could occur in view of the known temperature variations along Rayleigh lines. Outside the shock, H has the fixed value H_0 upstream or downstream. Within the shock, the energy equation gives

$$G(H-H_0) = KdT/dx. \tag{5}$$

This equation shows that for points to the right of A and B (the states outside the shock) where $H > H_0$, dT is positive going downstream, and vice versa. This enables us to indicate possible downstream motions by arrows in Figs. 3(a) and 3(b), which represent the two cases $v \gtrless (a^2/\gamma + b^2)^{\frac{1}{2}}$ downstream of the shock.

There is one other possibility to be considered—the occurrence of the shock-within-a-shock, so steep that viscous and/or Ohmic dissipation can cause departure from the Rayleigh line, and so thin that the temperature cannot change across it in the face of high thermal conductivity. This has been called the isothermal shock. Thus at any stage it would be possible to jump from the supersonic to the subsonic branch of the Rayleigh line, at constant T. The jumps are only possible in the direction of decreasing entropy for the following reason. Only in adiabatic processes does the second law require the entropy to increase, whereas in the isothermal shock XY in Fig. 3(a), for instance, it simply requires that $T(s_X - s_Y)$ be less than the heat extracted per unit mass flow, which is $H_X - H_Y$. This is evidently true because the first quantity is the area beneath the straight line XY, and the second quantity exceeds this by an amount equal to the area of the loop XBDY above the straight line XY. Conversely, a jump YX would infringe the second law.

In the case shown in Fig. 3(a), starting from state A, the arrows show that any venture to the left via the Rayleigh line or jumps would preclude ever reaching state B. Thus, only the direct route AB can represent the shock process. In the case of Fig. 3(b), it is clear that state B can only be reached by an isothermal jump from state E, which itself can only be reached from A directly along the Rayleigh line. The condition for the occurrence of the second case, with its shock-within-a-shock, is $v < (a^2/\gamma + b^2)^{\frac{1}{2}}$. We thus confirm the results already known for the case of a perfect gas.[11] If the detail spatial structure of the shock were required, this would follow from the Eq. (5).

B. Magnetic Diffusion Dominant

When the gas is weakly ionized and a rather poor conductor, the magnetic diffusivity significantly exceeds the other two. Then the shock process is best regarded as belonging to the family of processes discussed in Sec. II. It is then clearer why the ordinary sound speed a is important, as Golitsyn and Stanyukovich[12] observed. This problem has been treated by Burgers[13] for the case of a perfect gas.

If we restrict the processes discussed in Sec. II to being at constant area, and resurrect Maxwell's equations, we can then consider *strict* one-dimensional flows of a finite electrical conductor. We then have, from (1), that

$$\rho v = G = \text{const}, \quad \rho v dh_0/dx = Ej,$$
and
$$df/dx = -jB, \tag{6}$$

where $df + \rho T ds = dh_0$. Instead of specifying processes by keeping one of f, h_0 and s constant, we find that the shock process is defined by relating changes of f and h_0, with the result that f, h_0, and s are again stationary when $v = a$, the ordinary sound speed. Maxwell's equations here imply that E = constant and $\mu j = dB/dx$, so that

$$f + B^2/2\mu = \text{const}$$
and
$$h_0 + (-E/\mu G)B = \text{const}. \tag{7}$$

[11] W. Marshall, Proc. Roy. Soc. (London) **A233**, 367 (1955).

[12] G. S. Golitsyn and K. P. Stanyukovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1417 (1957).

[13] J. M. Burgers, in *Magnetohydrodynamics* (Stanford University Press, Stanford, California, 1957).

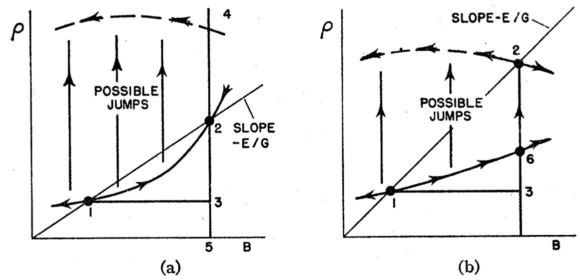


FIG. 4. Burgers line: (a) $v>a$ downstream; (b) $v<a$ downstream.

These are the equations that relate f and h_0 via the parameter B. Without loss of generality, we can arrange that outside the shock v, B, and $-E(=Bv)$ are positive quantities.

The shock structure is most conveniently discussed in terms of the relation between B and ρ, illustrated in Fig. 4, whose details emerge from the following arguments. The curve relating B and ρ, determined by (7), is called the Burgers line. Its slope may be easily determined, for

$$d\rho/dB=(\partial\rho/\partial f)_{h_0}(df/dB)+(\partial\rho/\partial h_0)_f(dh_0/dB).$$

By using values for the partial derivatives from the earlier paper,[1] we deduce that

$$(a^2-v^2)\frac{d\rho}{dB}=\frac{1}{T}\left(\frac{\partial h}{\partial s}\right)_\rho\left(-\frac{B}{\mu}\right)-\frac{1}{T}\left(\frac{\partial p}{\partial s}\right)_\rho\left(\frac{E}{\mu G}\right)$$
$$=-\frac{1}{\mu T}\left(\frac{\partial h}{\partial s}\right)_\rho\left\{B+\frac{E}{G}\left(\frac{\partial p}{\partial h}\right)_\rho\right\}. \qquad (8)$$

Now $(\partial h/\partial s)_\rho$ is positive, finite, and nonzero if the expansion coefficient is positive. Also

$$\rho(\partial h/\partial p)_\rho=1+\rho T(\partial s/\partial p)_\rho>1$$

if the expansion coefficient is positive. Thus, for $d\rho/dB$ to be zero, $B/\rho=(-E/G)(1/\rho)(\partial p/\partial h)_\rho<-(E/G)$. We conclude that, if B/ρ is not sufficiently less than $-E/G$, $d\rho/dB$ and (v^2-a^2) have the same sign and $d\rho/dB$ is not zero.

The two states 1 and 2 on either side of the shock are represented by the intersections of the Burgers line with the line $B/\rho=-E/G$, which must be two in number, since the shock transition is unique as established in Sec. III. The upstream state 1 is identifiable as the one of lower density. At these points, (8) gives

$$(a^2-v^2)(d\rho/dB)=-(B/\mu).$$

At the upstream one we know that $v^2>a^2+B^2/\mu\rho$. Hence

$$0<d\rho/dB<\rho/B.$$

If the downstream state is still supersonic in the *ordinary* sense, then $a^2<v^2<a^2+B^2/\mu\rho$ and $d\rho/dB>\rho/B$. Consider now the possible paths of the Burgers line in this case [Fig. 4(a)], starting from the upstream state and letting B increase. Since $d\rho/dB$ is positive, the curve cannot cross the line 13, nor the line 12 except at 1 or 2. It can cross the line 5324 somewhere, since for any value of B, Eqs. (7) indicate that there are two possible values of ρ, which correspond to the two sides of an ordinary shock, specified by values of f, h_0, and G. But one of these states is at 2, which is supersonic, and so the other crossing of 5324 would be at higher density. Thus the Burgers line cannot cross 23, except at 2, and must take the form shown in Fig. 4(a), assuming no outright discontinuity. S-bends, etc., are impossible because only two values of ρ occur for a given value of B. For each point on the branch $1\rightarrow 2$, there is another subsonic point, corresponding to the other side of an ordinary shock. This subsonic branch of the Burgers line is shown dotted in Fig. 4(a).

Figure 4(b) shows the case where the downstream state 2 is subsonic in the ordinary sense, $d\rho/dB$ being negative there. Arguments similar to those used previously show that the Burgers line can cross 23 at some supersonic point 6, but cannot cross 13 or 12 except at 1 and 2. The branch 16 must therefore be as shown.

It now remains to consider possible directions of travel along the Burgers line in a shock process. We have

$$\rho Tds=\rho dh_0-df=\rho(dB/\mu)[(E/G)+(B/\rho)].$$

Heat transfer being negligible, the entropy must increase in the downstream direction. Thus, for points to the right of the line $B/\rho=-E/G$, B increases in a downstream direction, and vice versa. These facts enable the arrows to be added to Figs. 4(a) and 4(b) to denote motion downstream. There is one other possibility to be considered—the occurrence of the shock-within-a-shock, so steep that viscous and/or thermal action can cause departure from the Burgers line, and so thin that its current content cannot change the field B significantly. This has been called the isomagnetic shock,[12] and is an ordinary gas-dynamic shock. At any stage it would be possible to jump from the supersonic branch to the subsonic branch of the Burgers line, at constant B.

In the case $v>a$ (downstream), starting from state 1, any venture to the left of or above the line $B/\rho=-E/G$ would preclude reaching state 2. Thus only the curve $1\rightarrow 2$ can represent the shock process. In the case $v<a$ (downstream), it is clear that state 2 can only be reached via an isomagnetic shock from state 6, which itself can only be reached from 1 along the supersonic Burgers line. We thus confirm the results already known for the special case of a perfect gas.[13] If the spatial structure of the shock were required in detail, this would follow from the equation

$$dB/dx=\mu\sigma(E+Bv).$$

It is easy to establish other geometrical features of the Burgers line If $(\partial^2p/\partial\tau^2)_s>0$, as already assumed, the two branches connect via a sonic point which is a maximum for s and B and a minimum for f and h_0. In contrast to the Rayleigh line, it is difficult to imagine means for traversing the Burgers line other than by an actual shock process, and so reaching this region of the curve.

C. Viscous Diffusion Dominant

The purely viscous magnetic shock, where the viscous diffusivity ν exceeds all others, does not appear to be approached in reality, and is also complicated to examine fully. Some simple remarks may, however, be made. This shock process belongs to the family discussed in Sec. III, where the electrical conductivity is virtually infinite. Here again the process is not defined by keeping one of F, H, or s constant, but by relating F and H. However, the point where $v=c$ fails to be significant in this case. From Burger's equations[13] we may relate F and H, denoting by τ' the departure of the normal stress in the x direction from the thermodynamic pressure p. Then

$$F-F_0=\rho(H-H_0)=\tau',$$

where subscript 0 denotes the constant value outside the shock. It follows that $\rho Tds=(H_0-H)d\rho$. For given values of F, G, and H, only two states are possible since shock transitions are unique. Thus H cannot equal H_0 (and F equal F_0) within the shock, and so H_0-H has a constant sign, which must be positive since s and ρ both increase across the shock. We see that ρ and s increase in unison through the shock. There is no extremum for ρ or s which might have indicated a tendency to form a shock-within-a-shock. In any case, such a steeper discontinuity would have to be an "isovelocity" shock, viscosity being so strong. Across it v, ρ, B, $F-\tau'$, and $\rho H-\tau'$ would be constant. It follows that $h-p/\rho$ or u would be unchanged also. There can thus be no such change, since the existence of two different states with the same values of ρ and u involves the impossible occurrence of negative specific heats at constant volume.

It therefore appears that viscosity alone could produce shock transitions of all strengths, as Rayleigh[3] concluded for an ordinary, perfect gas.

V. CONCLUSION

The results of the preceding sections may be summarized by saying that the behavior is throughout qualitatively like that of a perfect gas provided the gas obeys the realistic conditions that $(\partial^2 p/\partial\tau^2)_{s \text{ or } T}$ and the expansion coefficient be positive. The assumption that the gas is in equilibrium throughout is rather less realistic in the shock processes.

The more formidable problem of one-dimensional flows of an arbitrary gas in the presence of a magnetic field with a longitudinal component must be deferred to a later paper.

DISCUSSION

Session Reporter: N. H. Kemp

A. R. Kantrowitz, *Avco-Everett Research Laboratory, Everett, Massachusetts:* I should like to ask about the stability of the shock wave. Can the question be phrased this way: If a compression wave steepens to form a shock, then does that mean that the shock of that type of wave is stable?

H. Grad, *Institute of Mathematical Sciences, New York University, New York, New York:* This definition of stability is equivalent to another criterion. There is a very general theory of shock waves in hyperbolic partial differential equations of conservation form by Peter Lax.[a] The definition of stability that Lax gives involves counting how many characteristics hit each side of the shock. You may pose a shock problem in the following way: solve a smooth flow problem on the left; solve a smooth flow problem on the right, and use jump conditions in order to determine the motion of the assumed shock. In order for this to be a well-posed problem, the number of characteristics on one side has to be related in a certain very simple way to the number of characteristics on the other side. In a certain context, this is identical to the steepening criterion for shock stability, because if the characteristics converge to form an envelope, the characteristics after formation of the shock have the proper orientation. The stability with respect to the steepening is identical to this characteristic counting and is probably equivalent to the theory given by Dr. Germain. An interesting point is that "switch-on" shocks are singular in the Lax theory and have not yet been treated.

P. Germain, *University of Paris, Paris, France:* I want to just point out that the Russians I have quoted this morning use a kind of argument which is quite similar to the Peter Lax theory that Professor Grad mentioned. They count the characteristics which go one way and the other way.

A. R. Kantrowitz: Then you do agree it just is the steepening criterion also, or not?

J. A. Shercliff: I want to say that steepening of compressive waves would, in fact, only form these stable waves, the fast and the slow waves. I cannot imagine the other waves ever being formed by a steepening process.

[a] P. D. Lax, Communs. Pure Appl. Math. **10**, 537 (1957).