On Gas-Ionizing Magnetohydrodynamic Shock Waves

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CONSIDERATION is given to shock waves propagating in a nonconducting gas which is situated in an electromagnetic field. For the sake of simplicity it is assumed that the electric and magnetic fields are mutually perpendicular and parallel to the plane of the shock wave. Within the framework of the theory of a continuous medium, a study is made of the structure of shock waves of this type. From a consideration of the structure of these shock waves we derive a supplementary relation (connecting the quantities on the shock wave) which is necessary so that the shock wave may be regarded as the limit of a continuous flow when the dissipative coefficients tend to zero. This relation, and, consequently, the variation of all quantities on the shock wave, is essentially dependent upon the ratios of dissipative coefficients (viscosity, thermal conduction and magnetic viscosity) in the transition zone. The supplementary relation that is derived permits finding the sole solution for such nonsteady problems as the piston problem in an ideal gas.

In the transition through a shock wave that is ionizing a gas, the tangential component of the electric field, and also the flows of mass, momentum, and energy must be continuous. These conditions permit finding all the quantities beyond the shock wave and its rate of motion relative to the gas in front of it, if we know the state of the gas p,ρ and the electromagnetic field $\mathbf{E},\mathbf{H}$ directly in front of the shock wave, and the rate of motion of the gas or any other quantity (density, pressure, etc.) beyond the shock wave.[1] However, if $\sigma=0$ ahead of the shock wave, in nonsteady problems, an electromagnetic wave in which an electromagnetic field is varying can propagate in front of a shock wave. Thus, if conservation laws and the condition of continuity of the tangential component of the electric field are employed as boundary conditions on a shock wave, nonsteady problems (such as the piston problem) do not have a unique solution.[2] (This solution depends upon the intensity of the electromagnetic wave.) To obtain a supplementary boundary condition on a shock wave, which condition permits finding the only solution of such problems, we consider now the structure of shock waves ionizing a gas.

We assume that the electrical conductivity of the gas is a function of the temperature $\sigma=\sigma(T)$, and $\sigma=0$ when $T<T^*$ and $\sigma>0$ when $T>T^*$. We consider the structure of a magnetohydrodynamic shock wave moving in a gas, the temperature of which is $T_1<T^*$. The structure of shock waves (when the conductivity is everywhere different from zero) was considered earlier.[3,4] For the sake of simplicity we confine ourselves here to cases when only two dissipative coefficients—magnetic viscosity and viscosity, or magnetic viscosity and thermal conduction—differ from zero, and we assume that the magnetic and electrical fields are mutually perpendicular and parallel to the plane of the wave front. The equations of magnetohydrodynamics that describe one-dimensional steady gas motions may be written as, in the first case,

$$
\begin{aligned}
&\nu_m dH/dx = Hv - c_1\\
&\mu\frac{dv}{dk}=\frac{\gamma+1}{2}jv-\gamma\left(J-\frac{H^2}{8\pi}\right)+(\gamma-1)\cdot\left(\mathcal{E}-\frac{c_1}{4\pi}H\right)v^{-1} \qquad (1)\\
&jc_vT=\tfrac{1}{2}jv^2-\left(J-\frac{H^2}{8\pi}\right)v+\left(\mathcal{E}-\frac{c_1}{4\pi}H\right);
\end{aligned}
$$

in the second case,

$$
\begin{aligned}
&\nu_m dH/dx = Hv - c_1\\
&\kappa\frac{dT}{dy}=-\frac{\gamma+1}{(\gamma-1)^2}jv^2+\frac{\gamma}{\gamma-1}\left(J-\frac{H^2}{8\pi}\right)v-\left(\mathcal{E}-\frac{c_1}{4\pi}H\right) \qquad (2)\\
&jRT=\left(J-\frac{H^2}{8\pi}\right)v-jv^2,
\end{aligned}
$$

where $\nu_m=c^2/4\pi\sigma$, $\mu=\nu\rho=\frac{3}{4}\eta+\zeta$ is the coefficient of viscosity, $\kappa=c_p\rho\chi$ is the coefficient of thermal conduction, and R is the gas constant. In deriving these equations, advantage is taken of the fact that in one-dimensional steady gas motion the electric field $E=c_1/c$ (c is the velocity of light), the flow of mass j, the flow of momentum J, and the energy flow $\mathcal{E}$ are independent of coordinate x. The gas in Eqs. (1) and (2) was considered to be perfect. Imperfection of the gas does

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[2] A. G. Kulikovskii and G. A. Lyubimov, Izvest. Akad. Nauk S.S.S.R. Ser. Tekh. Nauk No. 4 (1959).
[3] W. Marshall, Phys. Rev. **103**, 1900 (1956).
[4] G. S. S. Ludford, J. Fluid Mech. **5**, 67 (1959).

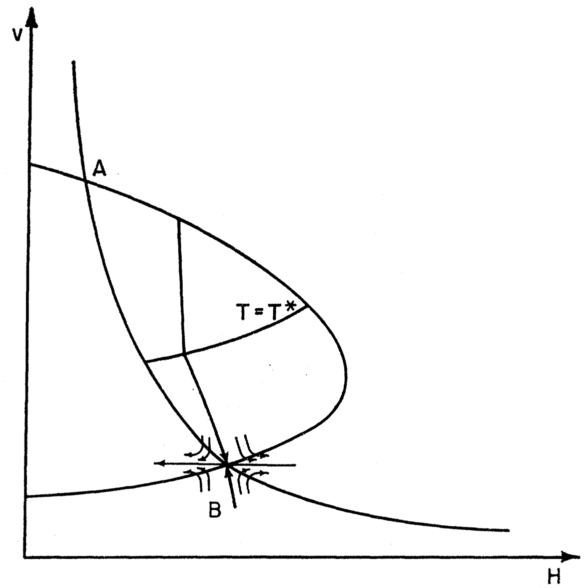

FIG. 1.

not lead to qualitative changes in the subsequent derivations.

In region $T>T^*$ on the plane H, v, where the conductivity is $\sigma>0(\nu_m\neq\infty)$, the differential equations (1) and (2) determine the system of integral curves (see Fig. 1).

In region $T<T^*$ we have $H=$constant, because in this region $\sigma=0$. The shock wave may be represented by those solutions which pass into a translational flow when $x=\pm\infty$, that is, solutions in which all derivatives tend to zero when $x\rightarrow\pm\infty$. Since on the supposition that $T_1<T^*<T_2$, where T_2 is the temperature beyond the shock wave, there must (on the plane H,v) correspond to the state for $x=-\infty$ a certain point of the curve

$$jv^2-\frac{2\gamma}{\gamma+1}\left(J-\frac{H^2}{8\pi}\right)v+\frac{2(\gamma-1)}{\gamma+1}\left(\mathcal{E}-\frac{c_1}{4\pi}H\right)=0, \quad (3)$$

and to the state for $x=\infty$ there corresponds a point which is the intersection of the hyperbola $Hv=c_1$ and curve (3). There are either three or one such points of intersection. In the first case it may be shown that the point of intersection, which corresponds to the smallest value of v, lies either in the region $v<0$ or in the region $T<0$, and for this reason is unattainable. Point A (Fig. 1), which either corresponds to the greatest value of v or is only one, can be only a node for the systems of Eqs. (1) and (2); it is noteworthy that the integral curves emerge from point A when x increases. Thus, when $x=\infty$, it is only point B (Fig. 1) that can correspond to the flow. For system (1) point B is a saddle point.[4] If the inequality $jv>\frac{1}{2}[J-(H^2/8\pi)]$ is fulfilled at point B, then point B is a saddle point also for system (2). In the case of the inverse sign of the inequality, point B is the node, and the integral curves of system (2) emerge from this point when x increases.

In the latter case, there is formed (within the region of the flow) an isothermal isomagnetic discontinuity.

If from point B, which corresponds to the flow when $x=\infty$, we move along one of the two integral curves in the direction corresponding to diminution of x, the integral curve must reach the line $T=T^*$, because $T_1<T^*$. In this case, the conduction becomes zero, and further motion occurs when $H=$constant until the integral curve reaches a certain point of curve (3). On the last section, the motion is purely gas dynamical. It is easy to see that in order to arrive, in this way, at a certain point of curve (3) it is necessary to emerge from point B along the integral curve passing in the region where

$$dv/dx\leq 0,\quad dT/dx\geq 0,\quad dH/dx\geq 0. \quad (4)$$

These inequalities are fulfilled during the entire motion. Thus, a gas moving from $x=-\infty$ first experiences gas-dynamic compression (which, however, is not a completed gas-dynamic shock wave), and then, when $T=T^*$, it begins to interact with the magnetic field, and arrives at state B.

The change of the magnetic field in wave H_2-H_1 is determined by the point of intersection of the integral curve with line $T=T^*$. This point of intersection depends not only on the characteristics of the oncoming flow, but also on the ratios of the dissipative coefficients ν_m, ν, and χ to each other inside the transition zone which represents the shock wave. If we integrate the equations it is possible to express H_1 in terms of the flow parameters when $x=\infty$, and since the latter are uniquely related to the quantities for $x=-\infty$, we then have

$$H_1=H_1(E_1,\rho_1,T_1,v_1). \quad (5)$$

This relation can serve as an additional boundary condition when replacing a continuous flow by a shock wave. Condition (5) is independent of conditions expressing the continuity of flows of mass, momentum, and energy, and also of the tangential component of the electric field. When magnetohydrodynamic shock waves propagate in a conducting gas, relation (5) replaces the well-known relation

$$H_1=-(c/v_1)E_1.$$

If relation (5) is not fulfilled, a continuous steady motion representing a shock wave is impossible. Thus not all discontinuity surfaces which satisfy the conservation laws can be considered as the limit of continuous flows representing a shock wave structure.

If in the transition zone one of the dissipative coefficients is much larger than the others, the supplementary boundary condition (5) may be found in an explicit form.

Let there be fulfilled everywhere in the transition zone

$$\nu_m \ll \nu, \quad \chi = 0 \tag{6}$$

or

$$\nu_m \ll \chi, \quad \nu = 0. \tag{7}$$

From Eqs. (1) and (2) we find that when $T = T^*$ the velocity of the gas is, in the case of (6),

$$v^* = j^{-1}\left\{\left(J - \frac{H_1^2}{8\pi}\right) - \left[\left(J - \frac{H_1^2}{8\pi}\right)^2 - 2j\left(\mathcal{E} - \frac{c_1}{4\pi}H - jc_v T^*\right)\right]^{\frac{1}{2}}\right\}; \tag{8}$$

in the case of (7),

$$v^* = j^{-1}\left\{\frac{1}{2}\left(J - \frac{H_1^2}{8\pi}\right) + \left[\frac{1}{4}\left(J - \frac{H_1^2}{8\pi}\right)^2 - j^2 RT^*\right]^{\frac{1}{2}}\right\}. \tag{9}$$

Since upon fulfillment of conditions (6) and (7) the integral curves of systems (1) and (2) entering point B pass (when $T = T^*$) as close as desired to the hyperbola $Hv = c_1$, we may take as the supplementary boundary condition

$$H_1 v^* = cE_1 (\equiv c_1 = H_2 v_2). \tag{10}$$

Let us now consider cases when the following conditions are fulfilled:

$$\nu_m \gg \nu, \quad \chi = 0 \quad \text{or} \quad \nu_m \gg \chi, \quad \nu = 0. \tag{11}$$

The velocities v^*, which correspond to $T = T^*$, are expressed exactly as in the previous case by equality (8) or (9). Upon fulfillment of inequalities (11), the integral boundary systems (1) and (2) entering point B, first proceed near curve (3) and then near the straight line $H = H_2$. Since the differences $v_1 - v^*$ are finite quantities for fixed T_1 and T^*, and the integral curves [in the case of inequalities (11) being reinforced] can pass as close as desired to curve (3), then in this case, the supplementary boundary condition is

$$H_1 = H_2. \tag{12}$$

In this case, all the other quantities vary as in a gas-dynamic shock wave.

The width of the shock waves is determined by the largest of the dissipative coefficients. An exception is case (11), when the width of the shock wave can be evaluated by the methods of ordinary gas dynamics.

The piston problem has been solved by the considerations presented here. The amplitude of the electromagnetic wave was calculated for the simplest cases. If we indicate the electromagnetic field by the subscript 0 at infinity, and the subscript 1 ahead of the shock wave, then

$$E_1 = \frac{H_0 - E_0}{1 - V/c}\,\frac{V}{c}, \quad H_1 = \frac{H_0 - E_0}{1 - V/c},$$

where V is a certain characteristic velocity.

DISCUSSION

Session Reporter: J. E. McCune

A. R. Kantrowitz, *Avco-Everett Research Laboratory, Everett, Massachusetts:* I would like to make the point that for very strong shock waves, it is quite interesting that the magnetic field compression observed is precisely the same as the density ratio observed. This indicates that in many practical situations the ionization actually occurs ahead of the shock wave, presumably by the transport of ultraviolet light, which gives enough ionization to give a high conductivity in a still cold gas.