Hydromagnetic Waves of Finite Amplitude in a Plasma with Isotropic and Nonisotropic Pressure Perpendicular to a Magnetic Field

K. HAIN, R. LÜST, AND A. SCHLÜTER

Max-Planck-Institut für Physik und Astrophysik, Munich, Germany

In recent years the problem of the structure of hydromagnetic shock waves, in particular, their thickness as compared to the kinetic mean free path and the gyroradius, has become of great interest and importance. For this reason, the special case of a stationary plane wave propagating in the absence of collisions perpendicular to a magnetic field has been investigated. In a first investigation[1] it was assumed that the pressure of the plasma was zero. Here, more general conditions are discussed: first, that the gas pressure perpendicular to the magnetic field is isotropic; second, that the gas pressure perpendicular to the magnetic field is nonisotropic.

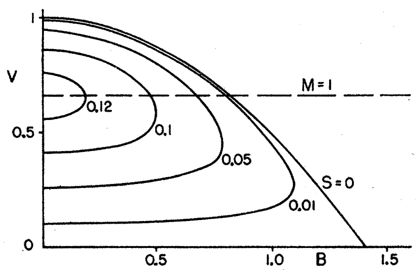

FIG. 1. Velocity v as a function of B for different values of S.

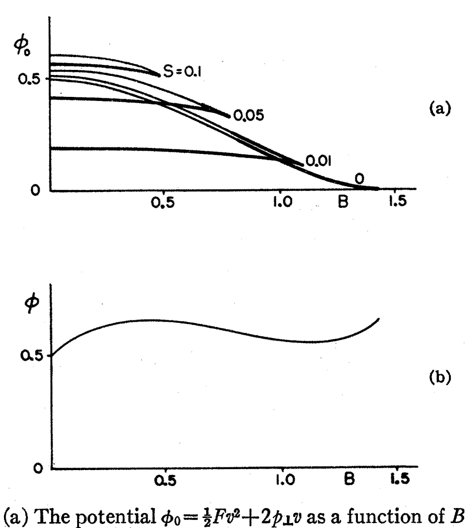

FIG. 2. (a) The potential $\phi_0=\frac{1}{2}Fv^2+2p_\perp v$ as a function of B for different values of S. (b) General form of the potential $\phi=\phi_0+cEB$.

1. ISOTROPIC PRESSURE

For the description of the plasma we use the macroscopic equations. The state of the plasma is characterized by the density ρ and by the pressure parallel and perpendicular to the magnetic field, $p_\parallel$ and $p_\perp$. The magnetic field has only one component B in the z direction and the velocity has only one component in the x direction. We assume stationary conditions, and that all quantities depend only on the x coordinate. Therefore, the electric current has only a y component j. Thus, the electric field has a y component E, which must be a constant, and also a component in the x direction owing to possible charge separation.

With these assumptions, the two-fluid equations

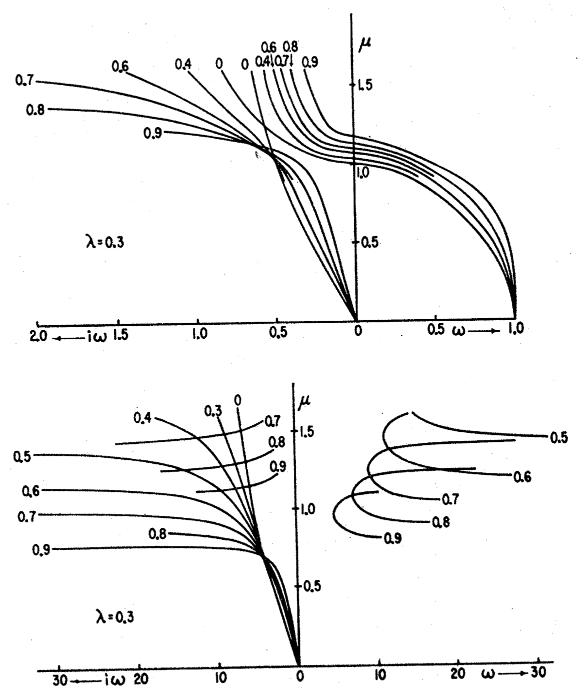

FIG. 3. Eigenfrequencies ω for different values of $\mu=M^{-1}$ and $\mu_0=M_0^{-1}$. The abscissa is ω or $i\omega$, the ordinate is μ, and the parameter of the different curves is μ_0.

[1] L. Davis, R. Lüst, and A. Schlüter, Z. Naturforsch. 13a, 916 (1958); see also J. H. Adlam and J. E. Allen, Phil. Mag. 3, 448 (1958); C. S. Gardner, H. Goertzel, H. Grad, C. S. Morawetz, M. H. Rose, and H. Rubin, Institute of Mathematical Sciences, New York University Rept. NYO-2538 (January 12, 1959); C. S. Morawetz, Institute of Mathematical Sciences, New York University Rept. NYO-8677 (January 12, 1959).

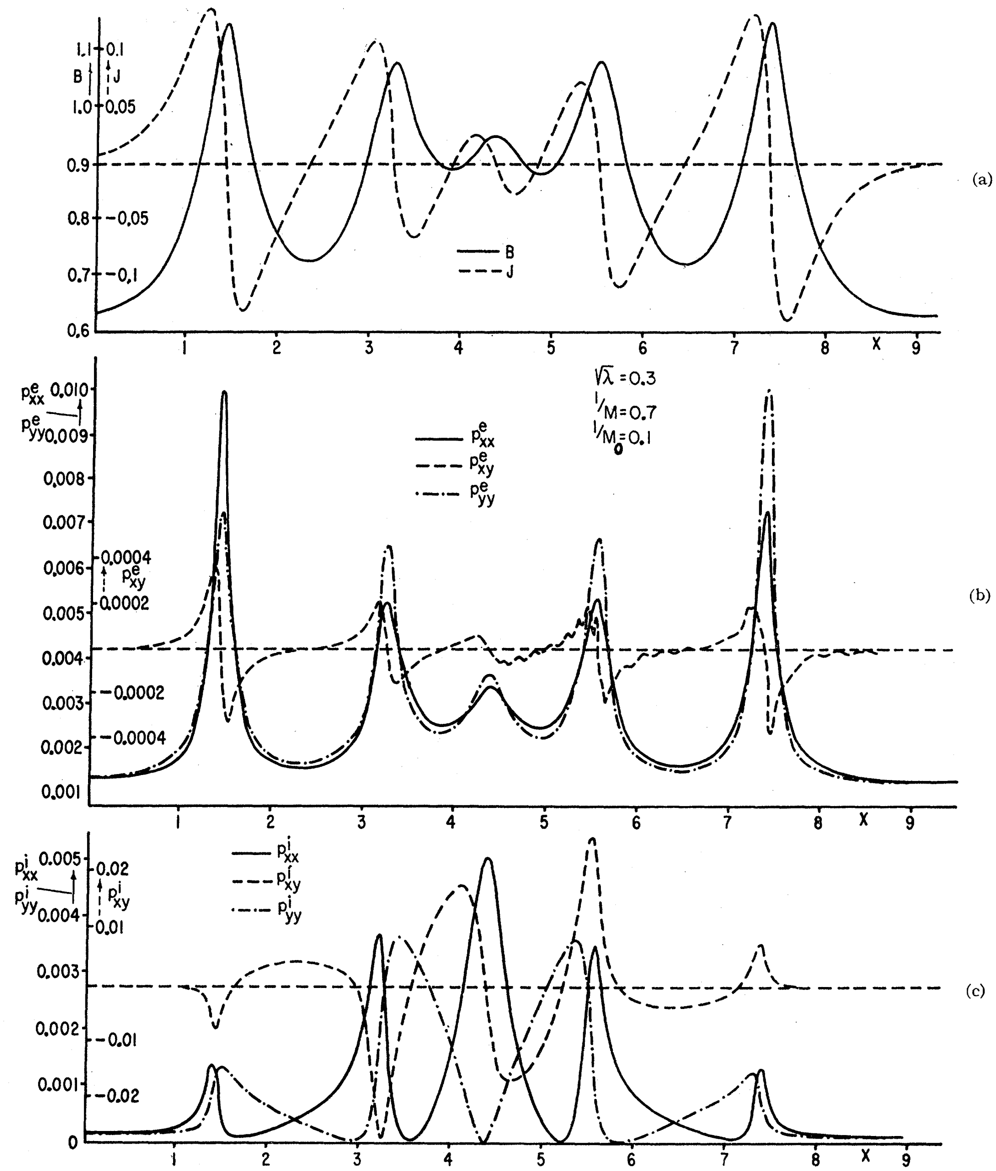

FIG. 4. For electron and ion mass ratio, $\lambda^{\frac{1}{2}}=0.3$, $M^{-1}=0.7$, and $M_0^{-1}=0.1$ is plotted as function of the space coordinate x: (a) magnetic field B and the "electric current" J, (b) pressure components for the electrons, (c) pressure components for the ions.

simplify. From the continuity equation,

$$\rho v=F, \tag{1}$$

where the flux of matter F is constant. From the equation of momentum,

$$\rho v^2+p_\perp+(1/8\pi)B^2=\Pi, \tag{2}$$

where the flux of momentum is a constant. From the generalized Ohm's law,

$$(m^{(i)}m^{(e)}/e^2)v(d/dx)(j/\rho)=E-c^{-1}vB, \tag{3}$$

where $m^{(i)}$ and $m^{(e)}$ are the masses of the ions and of the electrons, respectively, e is the electric charge of the electrons, and c is the velocity of light;

$$E_x=\frac{m^{(e)}-m^{(i)}}{e\rho}\frac{d}{dx}\left(\frac{1}{8\pi}B^2+\frac{1}{2}p_\perp\right). \tag{4}$$

Equation (4) determines the electric field in the x direction. Here it is assumed that the pressure of the ions $p_\perp^{(i)}$ and the pressure of the electrons $p_\perp^{(e)}$ are equal; that is, $p_\perp^{(i)}=p_\perp^{(e)}=\frac{1}{2}p_\perp$. From the Maxwell equations, we obtain

$$dB/dx=-(4\pi/c)j \tag{5}$$

and

$$E=\text{const.} \tag{6}$$

Finally, using the Chew-Goldberger-Low approximation, it follows that

$$p_\perp v^2 = S \tag{7}$$

and

$$p_{\parallel} v = \bar{S}, \tag{8}$$

where S and $\bar{S}$ are two constants.

On combining Eqs. (3) and (5) and introducing a new independent variable τ defined by

$$d\tau = [4\pi F/m^{(i)}m^{(e)}]^{\frac{1}{2}}(e/c)v^{-1}dx, \tag{9}$$

we obtain

$$\ddot{B} = vB - cE, \tag{10}$$

where the dots denote the derivative with respect to τ. Equation (10) can be integrated and leads to

$$(1/8\pi)(\dot{B})^2 + \tfrac{1}{2}Fv^2 + 2p_\perp v + (1/4\pi)cEB = \epsilon, \tag{11}$$

where the flux of energy ϵ is a constant.

As a consequence of the energy equation (11), it is possible to write (10) in the form

$$\ddot{B} = -[d\phi(B)/dB], \tag{12}$$

with

$$\phi(B) = \tfrac{1}{2}Fv^2 + 2p_\perp v + (1/4\pi)cEB. \tag{13}$$

Here v and $p_\perp$ can be eliminated in principle by use of Eqs. (2) and (7). In an explicit way this is possible only for $S=0$. In Fig. 1, v is plotted as a function of B for different values of S.

Equation (12) is equivalent to the equation for a one-dimensional anharmonic oscillator. The essential features of the solutions are readily obtained by studying the potential $\phi(B)$. In Fig. 2 the general form of $\phi(B)$ is plotted. As for the zero-pressure case, there exist solitary waves and wave trains as solutions. The characteristic lengths of these waves are of the order of the geometric mean of the Larmor radii of the ions and the electrons.

These solutions depend essentially on three parameters, for instance, S, E, and ϵ, since the other two constants F and B can be eliminated by a scale transformation. In the case of a solitary wave, the solutions are characterized by two parameters, for instance, S and ϵ. The flux of energy ϵ is directly related to the Mach number M at infinity (ratio of the velocity to the velocity of a hydromagnetic wave at infinity), and S is proportional to β, the ratio of the gas pressure to the magnetic pressure. But there exist solutions only for a limited range of the value of the different parameters. For instance, the maximum Mach number M that can be reached has the value 2 (for $S=0$). With increasing S the maximum value of M decreases.

2. NONISOTROPIC PRESSURE

We now assume that the pressures parallel and perpendicular to the magnetic field are different, and in addition that the pressure is nonisotropic in the plane perpendicular to the magnetic field. In this case, we have a pressure tensor for the ions as well as for

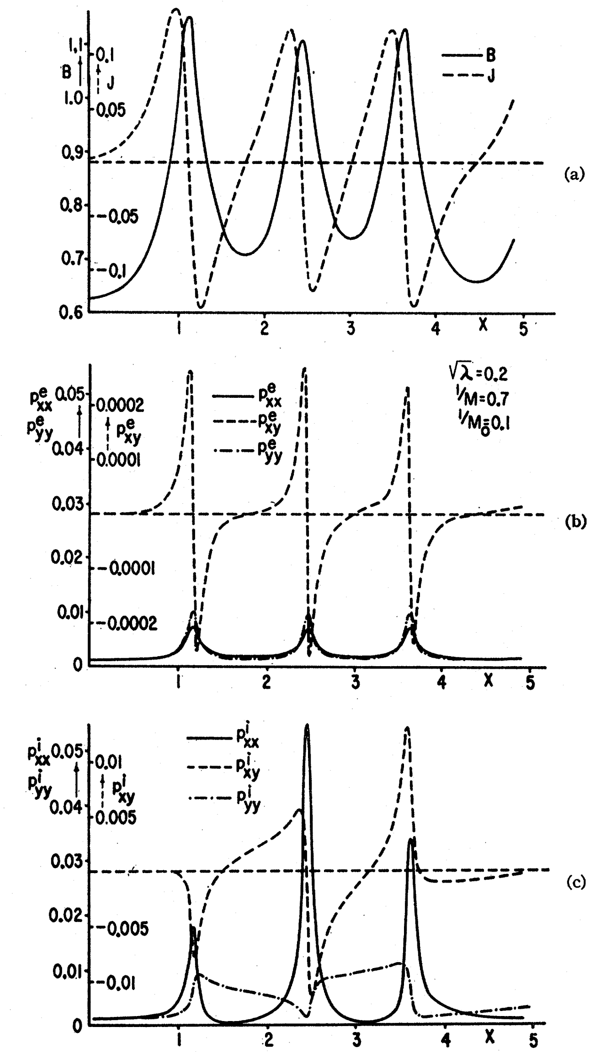

FIG. 5. For $\lambda^{\frac{1}{2}}=0.2$, $M^{-1}=0.7$, and $M_0^{-1}=0.1$: (a) magnetic field B and the "electric current" J, (b) pressure components for the electrons, (c) pressure components for the ions.

the electrons, with the components p_{xx}, p_{yy}, and p_{xy} (which are perpendicular to the magnetic field again in the z direction). We make the same assumption as before, namely, that we have stationary conditions and that all quantities depend only on the x coordinate.

In deriving the equations of motion, all moments of the collision-free Boltzmann equation up to the second order in the velocity for the two different kinds of particles with opposite charge have been taken into account. Only those terms which correspond to the heat conductivity have been neglected. In this way we finally obtain

$$\rho v = F \tag{14}$$

and

$$Fv + p_{xx}^{(i)} + p_{xx}^{(e)} + (1/8\pi)B^2 = \Pi, \tag{15}$$

where the indices (i) and (e) refer to the ions and electrons, respectively. F and Π are again constants.

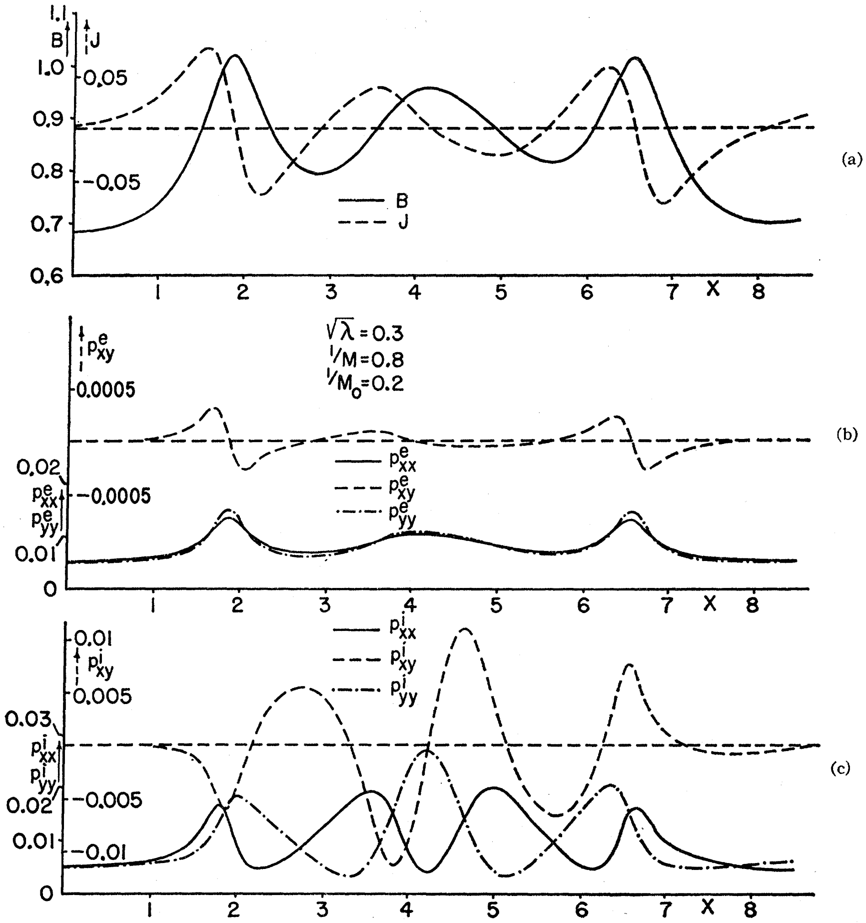

FIG. 6. For $\lambda^{\frac{1}{2}}=0.3$, $M^{-1}=0.8$, and $M_0^{-1}=0.2$: (a) magnetic field B and the "electric current" J, (b) pressure components for the electrons, (c) pressure components for the ions.

Instead of Eq. (10) we here obtain

$$\dot{B}=(4\pi/F)^{\frac{1}{2}}[\lambda^{\frac{1}{2}}p_{xy}{}^{(i)}-\lambda^{-\frac{1}{2}}p_{xy}{}^{(e)}]+I, \tag{16}$$

$$\dot{I}=vB-cE \quad \text{with} \quad \lambda=m^{(e)}/m^{(i)}, \tag{17}$$

where the dot denotes the derivative with respect to τ. Instead of Eq. (7), we have now six equations for the six components of the pressure tensors of the ions and the electrons:

$$\dot{p}_{xx}{}^{(i)}+3p_{xx}{}^{(i)}\frac{\dot{v}}{v}=\frac{1}{(\pi F)^{\frac{1}{2}}}\lambda^{\frac{1}{2}}p_{xy}{}^{(i)}B, \tag{18a}$$

$$\dot{p}_{yy}{}^{(i)}+p_{yy}{}^{(i)}\frac{\dot{v}}{v}=-\frac{1}{v}p_{xy}{}^{(i)}\left[\frac{1}{(\pi F)^{\frac{1}{2}}}\lambda^{\frac{1}{2}}cE-(1+\lambda)\frac{1}{F}p_{xy}{}^{(i)}\right], \tag{18b}$$

$$\dot{p}_{xy}{}^{(i)}\left[1-(1+\lambda)\frac{1}{Fv}p_{xx}\right]+2p_{xy}{}^{(i)}\frac{\dot{v}}{v}=\frac{1}{(4\pi F)^{\frac{1}{2}}}\lambda^{\frac{1}{2}}\left[-\frac{c}{v}p_{xx}{}^{(i)}E+p_{yy}{}^{(i)}B\right], \tag{18c}$$

and

$$\dot{p}_{xx}{}^{(e)}+3p_{xx}{}^{(e)}\frac{\dot{v}}{v}=-\frac{1}{(\pi F)^{\frac{1}{2}}}\left(\frac{1}{\lambda}\right)^{\frac{1}{2}}p_{xy}{}^{(e)}B, \tag{19a}$$

$$\dot{p}_{yy}{}^{(e)}+p_{yy}{}^{(e)}\frac{\dot{v}}{v}=\frac{1}{v}p_{xy}{}^{(e)}\left[\frac{1}{(\pi F)^{\frac{1}{2}}}\left(\frac{1}{\lambda}\right)^{\frac{1}{2}}cE+(1+\lambda)\frac{1}{F}p_{xy}{}^{(e)}\right], \tag{19b}$$

$$\dot{p}_{xy}{}^{(e)}\left[1-(1+\lambda)\frac{1}{Fv}p_{xx}\right]+2p_{xy}{}^{(e)}\frac{\dot{v}}{v}=\frac{1}{(4\pi F)^{\frac{1}{2}}}\left(\frac{1}{\lambda}\right)^{\frac{1}{2}}\left[\frac{c}{v}p_{xx}{}^{(e)}E-p_{xy}{}^{(e)}B\right]. \tag{19c}$$

With Eqs. (18a)–(18c) or (19a)–(19c) one can show that there exist the relations

$$[p_{xx}{}^{(i)}p_{yy}{}^{(e)}-p_{xy}{}^{(i)2}]v^4=S^{(i)2} \tag{20}$$

and

$$[p_{xx}{}^{(e)}p_{yy}{}^{(e)}-p_{xy}{}^{(e)2}]v^4=S^{(e)2}, \tag{21}$$

with $S^{(i)}$ and $S^{(e)}$ as constants. These equations correspond to the adiabatic invariant of Eq. (7) in the case of an isotropic pressure. It is also possible to derive from this set of equations an energy-flux equation:

$$\frac{1}{2}Fv^2+\frac{1}{8\pi}I^2+\frac{3}{2}v[p_{xx}^{(i)}+p_{xx}^{(e)}]+\frac{1}{2}v[p_{yy}^{(i)}+p_{yy}^{(e)}]$$
$$-\frac{1}{2}\frac{m}{F}\left[\frac{1}{m^{(i)}}p_{xy}^{(i)2}+\frac{1}{m^{(e)}}p_{xy}^{(e)2}\right]+\frac{c}{4\pi}EB=\epsilon. \quad (22)$$

One important difference is noted in a comparison to the equation of isotropic pressure, namely, that here the ratio of masses of the two particles enters into the equations.

Special solutions were obtained numerically for different ratios of the masses. For these special solutions it was assumed that all quantities are constant at $-\infty$ and the pressure isotropic [$p_{xy}^{(i)}=p_{xy}^{(e)}=0$; $p_{xx}^{(i)}=p_{xx}^{(e)}=p_{yy}^{(i)}=p_{yy}^{(e)}=(p/2)$; $cE=Bv$. In addition to the mass ratio, the solutions depend on the Mach number M and on the ratio of the gas pressure to the magnetic pressure at infinity, β.

For starting the computation at $-\infty$ the differential equations were linearized. In this way the eigenfrequency (or the growth rates) ω at infinity can be obtained. In general, there exist three different solutions for ω^2, and all of them are real and not zero. If the pressure is zero, two possible eigenfrequencies are imaginary (periodic solutions) and are equal to twice the gyrofrequency of the ions and the electrons, respectively. The other solution of ω is real (exponential increase) and leads to the solitary wave treated previously.

In Fig. 3 the different roots for ω^2 are shown as functions of M^{-1}, where

$$M^2=v^2/[3(p/\rho)+(1/4\pi)(B^2/\rho)],$$

and M is the Mach number at $-\infty$. The parameter of the different curves is M_0^{-1}, with

$$M_0^2=v^2/3(p/\rho)$$

(M_0^{-2} is proportional to β).

The ω solutions connected with the solitary wave are the only ones which are not sensitively dependent on the ration λ^2 (mass of the electron to mass of the ion).

In Figs. 4–7 the different components of the pressure and the magnetic field are plotted as in function of x for different values of λ, M^2, and M_0^2. If the mass ratio is different from unity, the solutions are similar to a solitary wave but show three maxima, while for equal masses there exists only one maximum. The pressure in these waves becomes highly anisotropic, especially for the ions. Even while the anisotropy for the electrons is small, it has a strong effect on the magnetic field, since in Eq. (16) $p_{xy}^{(e)}$ is multiplied by the factor $\lambda^{-\frac{1}{2}}$. The simple limits of the mass ratio, approaching zero or infinity, seem not to exist. One unexpected and interesting feature of these solutions is that they are symmetrical and that all quantities come back to their original values at $-\infty$.

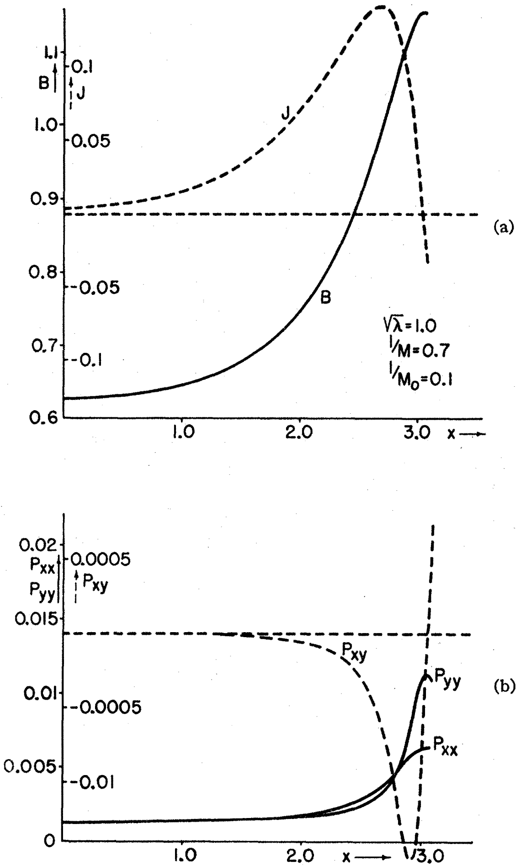

FIG. 7. For $\lambda^{\frac{1}{2}}=1.0$, $M^{-1}=0.7$, and $M_0^{-1}=0.1$: (a) magnetic field B and the "electric current" J, (b) pressure components for the electrons and ions, which are equal.

DISCUSSION

Session Reporter: J. E. McCune

P. Germain, *University of Paris, Paris, France:* I would like to mention that I have also tried to study shocks using the two-fluid model and looked into the possibility of using a singular perturbation method to go from the one-fluid theory to the two-fluid theory. But there is apparently no way to continuously go over from the one-fluid picture to this model, since your particles move isentropically.

R. Lüst: This theory is very similar to what C. S. Morawetz did at New York University. The particles move isentropically in the absence of collisions, but oscillations are set up which could later damp out to give the entropy increase.

E. L. Resler, Jr., *Cornell University, Ithaca, New York:* The equations you use are the same as those used by Dr. Petschek to get the waves he later uses for his postulated wave mixing. In your results you obtain oscillations for a while and then come back to the original state. My question is: How can the mechanism proposed by Dr. Petschek be put into the moment equation theory—am I to understand you have to take higher moments?

A. R. Kantrowitz, *Avco-Everett Research Laboratory, Everett, Massachusetts:* The theories so far developed can be put into two different classes and distinguished according to whether they describe an essentially laminar or a turbulent process. In the turbulent theories one gets dissipation either by wave mixing as we have suggested or by some plasma oscillation mechanism such as described by Dr. Colgate, Dr. Parker, and others. I think the only dissipative process suggested in the laminar theories is this phase mixing. So this is the fundamental point that has to be settled. I would like to suggest that we try to define experimentally accessible differences between these viewpoints—other than shock thickness—such as electron and ion temperatures and try to settle the matter experimentally. Along these lines I might add that the situation with regard to the experiment done by Patrick is not yet completely clear and one must not take the agreement between our theory and experiment too seriously.