Energy Spectra in Magneto-Fluid Dynamic Turbulence

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1. INTRODUCTION

RECENTLY the problem of turbulence in an electrically conducting fluid has begun to attract general attention. Astronomical phenomena, with the enormous magnitude of their characteristic Reynolds numbers, are usually supposed to be associated with a turbulent velocity field. In the newly developed field of the controlled thermonuclear reaction, the cause of the unexpectedly high rate of energy dissipation in the apparatus was attributed to the presence of some kind of turbulence in the plasma.

The meanings of the word "turbulence" used on these occasions, however, were not always identical. What was sometimes meant by "turbulence" happened to be quite different a phenomenon from that defined in ordinary fluid mechanics, and there has been a considerable gap of knowledge between the unknown turbulence-like phenomena and the hydrodynamic turbulence which has its own definition and fairly solid contents. One of the effective ways of filling up this gap may be to make our knowledge of turbulence as concrete and far-reached as possible in the framework of magneto-fluid dynamics.

One of the interesting problems with which previous work on this subject has been concerned is that of the "turbulent dynamo," that is, the question of whether a random magnetic field can be generated and maintained by the turbulent velocity field. Batchelor[1] based his theory upon the analogy between the magnetic field and the vorticity, and concluded that a spontaneous magnetic field is maintained when a relationship is satisfied between the viscosity, the conductivity, and the magnetic permeability. This analogy, however, cannot be pushed very far because, unlike the vorticity, the magnetic field is not directly connected with the velocity and, therefore, it is essentially governed by a linear equation.

Later, Chandrasekhar[2] extended Heisenberg's theory of ordinary turbulence to the magneto-fluid dynamic case, and found that the intensities of the kinetic and magnetic energy spectra are of the same order. In this extension, however, the weak point of Heisenberg's theory, which did not take account of the energy transfer from larger to smaller wave-number components, is amplified, since the magnetic field has itself no means of energy adjustment within its own spectrum, and thus the conclusion of this theory should be taken with some reservation.

Recently, the theory of ordinary turbulence based on the assumption of zero fourth-order cumulants[3,4] was extended by Roberts and Tatsumi[5] to magneto-fluid dynamic turbulence. Fundamental equations for determining the kinetic and magnetic energy spectra were derived and a calculation was made of the decay of the energy spectra from their initial forms of a single-line type. Here, some universal aspects of turbulence are investigated on this ground. The steady form of the energy spectrum for infinitely large Reynolds numbers are derived in the nonmagnetic case. Possible forms of the steady spectrum are suggested for the magneto-fluid dynamic case, and some discussion is given on the possibility of the "turbulent dynamo."

2. FUNDAMENTAL EQUATIONS

The general dynamics of an incompressible, viscous, electrically conducting fluid is described by the Navier-Stokes equation

$$(\partial u_i/\partial t)+(\partial/\partial x_j)(u_iu_j-h_ih_j)=-(\partial\bar{\omega}/\partial x_i)+\nu\nabla^2 u_i, \quad (1)$$
$$\bar{\omega}=p/\rho+\tfrac{1}{2}h_i^2,$$

the induction equation

$$(\partial h_i/\partial t)+(\partial/\partial x_j)(h_iu_j-u_ih_j)=\lambda\nabla^2 h_i, \quad (2)$$
$$\lambda=1/(4\pi\eta\sigma),$$

and the equation of continuity

$$\partial u_i/\partial x_i=0, \quad \partial h_i/\partial x_i=0, \quad (3)$$

where u_i are the components of the turbulent velocity, h_i the components of the magnetic field divided by $(4\pi\rho/\eta)^{\frac{1}{2}}$, ρ the density, p the pressure, η the magnetic permeability, ν the kinetic viscosity, and σ the electric conductivity. All electromagnetic quantities are measured in emu, and the summation convention over the suffixes $i=1, 2, 3$ are employed throughout.

The velocity and magnetic fields of homogeneous turbulence may be expressed statistically in terms of the various correlation tensors

$$\begin{aligned}\mathbf{B}_{ij}^{(1)}(\mathbf{r})&=\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle,\\ \mathbf{B}_{ij}^{(2)}(\mathbf{r})&=\langle u_i(\mathbf{x})h_j(\mathbf{x}+\mathbf{r})\rangle,\\ \mathbf{B}_{ij}^{(3)}(\mathbf{r})&=\langle h_i(\mathbf{x})h_j(\mathbf{x}+\mathbf{r})\rangle,\end{aligned} \quad (4)$$

[1] G. K. Batchelor, Proc. Roy. Soc. (London) **A201**, 405 (1950).
[2] S. Chandrasekhar, Proc. Roy. Soc. (London) **A233**, 322, 330 (1955).
[3] T. Tatsumi, Proc. Roy. Soc. (London) **A239**, 16 (1957).
[4] T. Tatsumi, Proc. IXth Internatl. Congr. Appl. Mech. 396 (1957).
[5] P. H. Roberts and T. Tatsumi, J. Math. Mech. **9**, 697 (1960).

and their Fourier transforms

$$\mathbf{\Phi}_{ij}^{(n)}(\boldsymbol{\kappa})=\int \mathbf{B}_{ij}^{(n)}(\mathbf{r})\exp(-i\boldsymbol{\kappa}\mathbf{r})d\mathbf{r},\quad (n=1,2,3). \tag{5}$$

If we assume the turbulent fields to be isotropic, $\mathbf{\Phi}_{ij}^{(1)}$ and $\mathbf{\Phi}_{ij}^{(3)}$, each being a second-order polar tensor orthogonal to the wave-number vector $\boldsymbol{\kappa}$, must be expressed in the following form[6]:

$$\mathbf{\Phi}_{ij}^{(1),(3)}(\boldsymbol{\kappa})=\Phi^{(1),(3)}(\kappa)\left(\frac{\kappa_i\kappa_j}{\kappa^2}-\delta_{ij}\right), \tag{6}$$

where $\Phi^{(1)}$ and $\Phi^{(3)}$ are arbitrary scalar functions of $\kappa=|\boldsymbol{\kappa}|$. They are related to the energy spectrum function $E(\kappa)$ and $H(\kappa)$ for the kinetic and magnetic energies, respectively, by

$$\begin{pmatrix}E\\H\end{pmatrix}(\kappa)=4\pi\kappa^2\begin{pmatrix}\Phi^{(1)}\\\Phi^{(3)}\end{pmatrix}(\kappa). \tag{7}$$

On the other hand, $\mathbf{\Phi}_{ij}^{(2)}$ is a skew-isotropic tensor of the second-order since the magnetic field $\mathbf{h}$ is a skew vector. Here, we assume that

$$\mathbf{\Phi}_{ij}^{(2)}(\boldsymbol{\kappa})\equiv 0, \tag{8}$$

that is, equivalent to require that the turbulent fields have the reflexional symmetry.

Dynamic equations for $\Phi^{(1)}$ and $\Phi^{(3)}$ may be obtained from Eqs. (1) and (2) by multiplying them by the components of the velocity and magnetic fields at other points in space, averaging them over ensembles, and lastly taking their Fourier transforms. The equations thus obtained, however, are not closed themselves, owing to the nonlinearity of Eqs. (1) and (2), and in order to make them determinate, it is necessary to introduce some assumption for expressing the nonlinear terms as functions of $\Phi^{(1)}$ and $\Phi^{(3)}$. For this purpose use is made of the assumption of zero fourth-order cumulants in the same manner as in a previous paper on the ordinary hydrodynamic turbulence.[3,4] This assumption enables us to express a fourth-order product mean value of four random variables A, B, C, and D in terms of the second-order ones in such a manner that

$$\langle ABCD\rangle=\langle AB\rangle\langle CD\rangle+\langle AC\rangle\langle BD\rangle+\langle AD\rangle\langle BC\rangle. \tag{9}$$

This relation, being exactly true if A, B, C, and D are normally distributed, is expected to be valid for considerably wide classes of the random variables. There was some experimental support for the relation (9) in the ordinary hydrodynamic case in which A, B, $\cdots$ denoted various velocity components at different points, and here we may expect that (9) is equally valid for the magneto-fluid dynamic case although there has so far been reported no experimental evidence for this assumption. Recently, Kraichnan[7,8] adversely criticized this type of assumptions (the "quasi-normality" hypothesis according to his terminology) as violating several consistency relationships in the equations of motion, but his criticism is not applicable to the assumption as employed here as well as in previous papers.[3,4]

On making use of the relation (9) with the choice of A, B, $\cdots$ as various components u_i and h_j at different points, we can derive a pair of closed dynamic equations for $\Phi^{(1)}$ and $\Phi^{(3)}$. They may be expressed in nondimensional form as follows:

$$\begin{aligned}\phi^{(1)}(s,\tau)=\;&\phi^{(1)}(s,0)\exp(-2s^2\tau)\\&+\frac{R^2}{2}\int_0^\tau\int_0^\infty\int_{-1}^1[\exp\{-2s^2(\tau-\tau')\}\\&-\exp\{-(s^2+s'^2+s''^2)(\tau-\tau')\}]\\&\times\{\phi^{(1)}(s,\tau')-\phi^{(1)}(s',\tau')\}\phi^{(1)}(s'',\tau')\\&\times\left(\frac{s^2s'^2}{s''^2}+\mu ss'\right)\frac{s'^2}{s^2-s'^2-s''^2}(1-\mu^2)d\tau'ds'd\mu\\&-\frac{R^2}{2}\int_0^\tau\int_0^\infty\int_{-1}^1[\exp\{-2s^2(\tau-\tau')\}\\&-\exp\{-(s^2+\gamma(s'^2+s''^2))(\tau-\tau')\}]\\&\times\{\phi^{(1)}(s,\tau')-\phi^{(3)}(s',\tau')\}\phi^{(3)}(s'',\tau')\mu ss'\frac{s'^2}{s''^2}\\&\times\frac{s'^2}{s^2-\gamma(s'^2+s''^2)}(1-\mu^2)d\tau'ds'd\mu,\end{aligned} \tag{10}$$

$$\begin{aligned}\phi^{(3)}(s,\tau)=\;&\phi^{(3)}(s,0)\exp(-2\gamma s^2\tau)\\&+\frac{R^2}{2}\int_0^\tau\int^\infty\int_{-1}^1[\exp\{-2\gamma s^2(\tau-\tau')\}\\&-\exp\{-(s''^2+\gamma(s^2+s'^2))(\tau-\tau')\}]\\&\times\{\phi^{(3)}(s,\tau')-\phi^{(3)}(s',\tau')\}\phi^{(1)}(s'',\tau')\frac{s^2s'^2}{s''^2}\\&\times\frac{s'^2}{\gamma(s^2-s'^2)-s''^2}(1-\mu^2)d\tau'ds'd\mu\\&-\frac{R^2}{2}\int_0^\tau\int_0^\infty\int_{-1}^1[\exp\{-2\gamma s^2(\tau-\tau')\}\\&-\exp\{-(s'^2+\gamma(s^2+s''^2))(\tau-\tau')\}]\\&\times\{\phi^{(3)}(s,\tau')-\phi^{(1)}(s',\tau')\}\phi^{(3)}(s'',\tau')\mu ss'\frac{s^2}{s''^2}\\&\times\frac{s'^2}{\gamma(s^2-s''^2)-s'^2}(1-\mu^2)d\tau'ds'd\mu,\end{aligned} \tag{11}$$

[6] H. P. Robertson, Proc. Cambridge Phil. Soc. **36**, 209 (1940).
[7] R. H. Kraichnan, Phys. Rev. **107**, 1485 (1957).
[8] R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).

where

$$\begin{pmatrix}\phi^{(1)}\\ \phi^{(3)}\end{pmatrix}(s,\tau)=\frac{8\pi\kappa_0^3}{\langle u^2\rangle_0}\begin{pmatrix}\Phi^{(1)}\\ \Phi^{(3)}\end{pmatrix}(\kappa,t)=\frac{2\kappa_0^3}{\langle u^2\rangle_0\kappa^2}\begin{pmatrix}E\\ H\end{pmatrix}(\kappa,t), \quad (12)$$

$$s''^2=s^2+s'^2+2\mu ss',$$

$s=\kappa/\kappa_0$ is the nondimensional wave number, $\tau=\nu\kappa_0^2 t$ the nondimensional time, κ_0 the characteristic wave number, $R=\langle u^2\rangle_0^{\frac{1}{2}}/\nu\kappa_0$ the Reynolds number of turbulence, $\frac{1}{2}\langle u^2\rangle_0$ the initial value of the kinetic energy per unit mass, and $\gamma=\lambda/\nu$ the ratio of the magnetic viscosity to the kinetic viscosity. It should be noted that in (10) and (11) the initial rate of energy transfer is assumed to be zero for simplicity.

3. STEADY STATES OF THE ENERGY SPECTRUM

The set of equations (10) and (11) is sufficient for calculating the kinetic and magnetic energy spectra $\phi^{(1)}$ and $\phi^{(3)}$ as functions of the wave number and time provided their initial values are prescribed. In this sense it may be said that the problem of the decaying isotropic turbulence is mathematically solved within the framework of the assumption (9). For the ordinary hydrodynamic case the change of the energy spectrum with time from the initial single-line spectrum was calculated by the author.[3,4] A similar calculation has been carried out for magneto-fluid dynamic turbulence by Roberts and Tatsumi.[5] These calculations, however, give the solutions in ascending powers of R, so that their convergence becomes poor for large Reynolds numbers. Therefore, there still remains the practically important problem of finding solutions of (10) and (11) for large values of R, with which the theory of turbulence undoubtedly has primary concern.

Moreover, we are more interested in a simple solution than a set of numerical curves which could only be obtained after laborious computations, and we are tempted to look for "universal" aspects of the turbulence under which such simple situations may be expected. It is shown in the following that there really exist such universal situations at very large Reynolds numbers.

For large values of R the nonlinear terms in (10) and (11) become dominant over other terms. Then if we first assume $\phi^{(1)}$ and $\phi^{(3)}$ to be steady for small values of τ, (10) and (11) reduce to (case I: $\tau\ll 1$)

$$-\frac{R^2\tau^2}{4}\int^{\infty}\int_{-1}^{1}\left[F_1\left(\frac{s^2s'^2}{s''^2}+\mu ss'\right)s'^2-F_2\mu ss'\frac{s'^4}{s''^2}\right]\times(1-\mu^2)ds'd\mu=0, \quad (13)$$

$$-\frac{R^2\tau^2}{4}\int_0^{\infty}\int_{-1}^{1}\left[F_3\frac{s^2s'^2}{s''^2}-F_4\mu ss'\frac{s^2s'^2}{s''^2}\right]\times(1-\mu^2)ds'd\mu=0, \quad (14)$$

where

$$\begin{aligned}F_1&=\{\phi^{(1)}(s)-\phi^{(1)}(s')\}\phi^{(1)}(s''),\\ F_2&=\{\phi^{(1)}(s)-\phi^{(3)}(s')\}\phi^{(3)}(s''),\\ F_3&=\{\phi^{(3)}(s)-\phi^{(3)}(s')\}\phi^{(1)}(s''),\\ F_4&=\{\phi^{(3)}(s)-\phi^{(1)}(s')\}\phi^{(3)}(s'').\end{aligned} \quad (15)$$

If, on the other hand, we assume that $\phi^{(1)}$ and $\phi^{(3)}$ do not change with time for large values of τ, (10) and (11) become, after partial integrations (case II: $\tau\gg 1$),

$$-\frac{R^2}{2}\int^{\infty}\int_{-1}^{1}\left\{F_1\left(\frac{s^2s'^2}{s''^2}+\mu ss'\right)\frac{s'^2}{s^2+s'^2+s''^2}-F_2\mu ss'\frac{s'^4}{s''^2[s^2+\gamma(s'^2+s''^2)]}\right\}(1-\mu^2)ds'd\mu=0, \quad (16)$$

$$-\frac{R^2}{2}\int_0^{\infty}\int_{-1}^{1}\left\{F_3\frac{s^2s'^4}{s''^2[s''^2+\gamma(s^2+s'^2)]}-F_4\mu ss'\frac{s^2s'^2}{s''^2[s'^2+\gamma(s^2+s''^2)]}\right\}(1-\mu^2)ds'd\mu=0. \quad (17)$$

In both cases I and II it is assumed that the value of γ is such that $\gamma\tau\ll 1$ or $\gg 1$ according as $\tau\ll 1$ or $\gg 1$, that is, γ is of order of unity. There can be other combinations of the values of $\gamma\tau$ and τ, but with such combinations the coefficient of each F has a different time dependence so that the time-independent solutions cannot be expected.

4. STEADY ENERGY SPECTRA IN NON-MAGNETIC TURBULENCE

First we consider the nonmagnetic case. Putting $\phi^{(3)}=0$, in Eqs. (13)–(17), we have, for case I,

$$\int_0^{\infty}\int_{-1}^{1}F_1\left(\frac{s^2s'^2}{s''^2}+\mu ss'\right)s'^2(1-\mu^2)ds'd\mu=0; \quad (18)$$

for case II,

$$\int_0^{\infty}\int_{-1}^{1}F_1\left(\frac{s^2s'^2}{s''^2}+\mu ss'\right)\frac{s'^2}{s^2+s'^2+s''^2}(1-\mu^2)ds'd\mu=0. \quad (19)$$

If we replace the variables s' and s'' by $\theta=s'/s$ and $\theta'=s''/s=(1+\theta^2+2\mu\theta)^{\frac{1}{2}}$, respectively, (18) and (19) become (case I)

$$s^5\int_0^{\infty}\int_{-1}^{1}F_1\left(\frac{\theta}{\theta'^2}+\mu\right)\theta^3(1-\mu^2)d\theta d\mu=0, \quad (20)$$

(case II)

$$s^3\int_0^{\infty}\int_{-1}^{1}F_1\left(\frac{\theta}{\theta'^2}+\mu\right)\frac{\theta^3}{1+\theta^2+\theta'^2}(1-\mu^2)d\theta d\mu=0. \quad (21)$$

In order that these equations be satisfied, F_1 must be expressed as a product of a function of s and another function of θ and μ, so that $\phi^{(1)}(s)$, $\phi^{(1)}(s')$, and $\phi^{(1)}(s'')$ must involve the same powers of s. Thus, $\phi^{(1)}$ is a power function of s:

$$\phi^{(1)}(s)=As^{-q}, \tag{22}$$

where A and q are numerical constants.

Now, on substituting the expression (22) into (18), we have

$$\begin{aligned}0&=\int_0^\infty\int_{-1}^1(1-\theta^{-q})\theta'^{-q}\left(\frac{\theta}{\theta'^2}+\mu\right)\theta^3(1-\mu^2)d\theta d\mu\\&=\int_0^1\int_{-1}^1(1-\theta^{-q})(1-\theta^{2q-8})\left(\frac{\theta}{\theta'^2}+\mu\right)\theta^3(1-\mu^2)d\theta d\mu,\end{aligned}$$

from which it immediately follows that $q=0$ or 4. Similarly, substitution of (22) into (19) gives

$$\begin{aligned}0&=\int_0^\infty\int_{-1}^1(1-\theta^{-q})\theta'^{-q}\left(\frac{\theta}{\theta'^2}+\mu\right)\frac{\theta^3}{1+\theta^2+\theta'^2}(1-\mu^2)d\theta d\mu\\&=\int_0^1\int_{-1}^1(1-\theta^{-q})(1-\theta^{2q-6})\left(\frac{\theta}{\theta'^2}+\mu\right)\\&\qquad\times\frac{\theta^3}{1+\theta^2+\theta'^2}(1-\mu^2)d\theta d\mu,\end{aligned}$$

so that $q=0$ or 3. Thus, the following steady solutions are found to be possible for these limiting cases:

$$\text{case I,}\quad \phi^{(1)}(s)=A \quad\text{or}\quad As^{-4}, \tag{23}$$

$$\text{case II,}\quad \phi^{(1)}(s)=A \quad\text{or}\quad As^{-3}. \tag{24}$$

The constant solutions in (23) and (24) represent the state of equipartition of kinetic energy over the whole wave-number space. In this context it may be interesting to note that the equipartition of energy was shown to be the *only* possible steady solution for the normally distributed turbulent velocity field in the limit of infinite Reynolds number.[9] In the present field, in which the normality relation (9) was used only for relating the fourth-order velocity product mean values to those of the second-order, (23) and (24) have shown that another steady solution is possible for each case besides that of equipartition.

We are not interested in the equipartition solution because it represents the "dead" state of turbulence which is attained after infinitely long decay time. On the other hand, the inverse power spectra in (23) and (24) correspond to the state of dynamic balancing between the in- and out-flows of the energy over all wave numbers. Unlike Kolmogoroff's spectrum[10] and any other spectra[11-13] relating to the notion of the universal equilibrium range at large wave numbers, these solutions give the energy spectrum function

$$E(\kappa)\propto\begin{cases}\kappa^{-2} & \text{(case I)},\\ \kappa^{-1} & \text{(case II)},\end{cases} \tag{25}$$

throughout the whole range of the wave number. These spectra, involving infinite amounts of energy, have themselves no practical meaning, and they should be interpreted as the limiting forms to which nonsteady real spectra asymptotically approach as $R\to\infty$ and $\tau\to 0$ or $\tau\to\infty$. The relationship of these limiting solutions to the realistic spectra with very large but finite Reynolds numbers will be discussed elsewhere.

5. STEADY ENERGY SPECTRA IN MAGNETO-FLUID DYNAMIC TURBULENCE

By applying the same argument as that used for deriving the spectrum (22) to Eqs. (13) to (17), we see that the steady solutions of the magneto-fluid dynamic case must be of the form

$$\phi^{(1)}(s)=As^{-q},\quad \phi^{(3)}(s)=Bs^{-q}, \tag{26}$$

where A, B, and q are numerical constants.

Now we investigate this type of solution for case I and case II separately.

Case I: $R\to\infty$, $\tau\ll 1$

Substituted from (26), Eqs. (13) and (14) become

$$\begin{aligned}-A^2&\int_0^1\int_{-1}^1(1-\theta^{-q})(1-\theta^{2q-8})\theta'^{-q}\left(\frac{\theta}{\theta'^2}+\mu\right)\\&\times\theta^3(1-\mu^2)d\theta d\mu+B\int_0^1\int_{-1}^1[(A-B)\\&\times(1+\theta^{q-10})+B(1-\theta^{-q})(1-\theta^{2q-10})]\\&\qquad\times\theta'^{-q}\mu\theta\frac{\theta^4}{\theta'^2}(1-\mu^2)d\theta d\mu=0,\end{aligned} \tag{27}$$

$$\begin{aligned}-AB&\int_0^1\int_{-1}^1(1-\theta^{-q})(1-\theta^{2q-8})\theta'^{-q}\frac{\theta^4}{\theta'^2}\\&\times(1-\mu^2)d\theta d\mu+B\int_0^1\int_{-1}^1[(B-A)(1+\theta^{q-6})\\&+A(1-\theta^{-q})(1-\theta^{2q-6})]\\&\qquad\times\theta'^{-q}\mu\theta\frac{\theta^2}{\theta'^2}(1-\mu^2)d\theta d\mu=0.\end{aligned} \tag{28}$$

[9] E. Hopf, J. Ratl. Mech. Anal. 1, 87 (1952).

[10] A. N. Kolmogoroff, Compt. rend. acad. sci. U.R.S.S. 30, 301; 32, 16 (1941).
[11] W. Heisenberg, Z. Physik 124, 628 (1948).
[12] A. M. Obukhoff, Compt. rend. acad. sci. U.R.S.S. 32, 19 (1941).
[13] L. S. G. Kovásznay, J. Aeronaut. Sci. 15, 745 (1948).

It may be shown without difficulty that, unlike in the nonmagnetic case, $q=4$ does not give a steady solution in this case. If we put $q=4$, Eqs. (27) and (28) degenerate into a single equation

$$B\int_0^1 [A(1+\theta^{-6})-B\theta^{-2}(1+\theta^{-2})]\theta^5 d\theta \times\int_{-1}^1 \frac{\mu}{\theta'^6}(1-\mu^2)d\mu=0. \quad (29)$$

Since the integral with respect to μ shows a singularity at $\theta=1$, we must require that $A=B$, and thus the left-hand side of (29) does not vanish unless $B=0$ identically (the nonmagnetic case).

Now the question is whether there exists any steady solution in this case, that is, whether any choice of the parameters A, B, and q can give solutions of equations (27) and (28). The answer which is presented in the following is, however, negative. The integrals

$$\int_{-1}^1 (1-\mu^2)d\mu \quad \text{and} \quad \int_{-1}^1 \mu(1-\mu^2)d\mu$$

in (27) and (28) have positive and negative definite values, respectively, since $1-\mu^2\geq 0$ and $1+\theta^2+2|\mu|\theta\geq 1+\theta^2-2|\mu|\theta$ for $-1\leq\mu\leq 1$. Although the first term in (27) involves both positive and negative parts, the positive contribution is found always to exceed the negative one. Thus, the first terms of (27) and (28) are both negative for $0<q<4$. Hence, in order that (27) and (28) be satisfied, the second terms of these equations must be positive, so that

$$A<B \quad \text{and/or} \quad q>5,$$

and

$$A>B \quad \text{and/or} \quad q>3,$$

from which immediately follows (for $0<q<4$)

$$A<B \quad \text{and} \quad 3<q<4. \quad (30)$$

For $q>4$, on the other hand, the first terms of (27) and (28) are positive, and hence we require that

$$A>B \quad \text{and/or} \quad 0<q<5,$$

and

$$A<B \quad \text{and/or} \quad 0<q<3,$$

so that

$$A<B \quad \text{and} \quad 4<q<5. \quad (31)$$

It follows from (30) and (31) that in any case $A<B$, that is, the intensity of the kinetic energy spectrum is less than that of the magnetic energy spectrum. However, since the singularities of the integrals in (27) and (28) at $\theta=1$, $\mu=-1$ do not generally vanish for the range $3<q<5$, there does not seem to exist any steady solution in case I.

Case II: $R\rightarrow\infty$, $\tau\gg 1$

For the special case of $\gamma=1$, Eqs. (16) and (17) become, after being substituted from (26),

$$-A^2\int_0^1\int_{-1}^1 (1-\theta^{-q})(1-\theta^{2q-6})\theta'^{-q}\left(\frac{\theta}{\theta'^2}+\mu\right) \times\frac{\theta^3}{1+\theta^2+\theta'^2}(1-\mu^2)d\theta d\mu+B\int_0^1\int_{-1}^1 [(A-B) \times(1+\theta^{q-8})+B(1-\theta^{-q})(1-\theta^{2q-8})] \times\theta'^{-q}\mu\theta\frac{\theta^4}{\theta'^2(1+\theta^2+\theta'^2)}(1-\mu^2)d\theta d\mu=0, \quad (32)$$

$$-AB\int_0^1\int_{-1}^1 (1-\theta^{-q})(1-\theta^{2q-6})\theta'^{-q}\frac{\theta^4}{\theta'^2(1+\theta^2+\theta'^2)} \times(1-\mu^2)d\theta d\mu+B\int_0^1\int_{-1}^1 [(B-A)(1+\theta^{q-4}) +A(1-\theta^{-q})(1-\theta^{2q-4})] \times\theta'^{-q}\mu\theta\frac{\theta^2}{\theta'^2(1+\theta^2+\theta'^2)}(1-\mu^2)d\theta d\mu=0. \quad (33)$$

A similar argument to that leading to (30) and (31) shows that Eqs. (32) and (33) are satisfied only if A, B, and q satisfy the inequalities

$$A<B \quad \text{and} \quad 2<q<4. \quad (34)$$

Since, however, the integrals in (32) and (33) have singularities at $\theta=1$, $\mu=-1$ for the range $3\leq q$, the only possible case is

$$A<B \quad \text{and} \quad 2<q<3. \quad (35)$$

It is yet to be clarified whether any set of the values of A, B, and q which lie within the range (35) can really satisfy Eqs. (32) and (33), but, even when such a set really exists, it is obvious that it does not give a "turbulent dynamo," which is necessarily associated with an energy transfer from the kinetic to the magnetic energy spectrum, because then the first term of (32) is negative, and hence there is an inflow of energy *into* the kinetic energy spectrum.

Nothing definite has yet been established for the general case $\gamma\neq 1$. But, in view of the fact that no solution has been found at $q=2$ for all values of γ, the foregoing conclusion for $\gamma=1$ may be supposed to be equally valid for $\gamma\neq 1$.

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DISCUSSION

Session Reporter: W. H. Reid

A question from Professor **J. M. Burgers** (*University of Maryland, College Park, Maryland*) brought out the fact that all correlations between the velocity and magnetic fields of the type $\langle u_i h_j' \rangle$ had been assumed to vanish identically for isotropic turbulence. The remaining discussion centered on the divergencies in the power-law spectra obtained and it eventually emerged that they must be understood in a sense analogous to the Kolmogoroff spectrum, i.e., that for sufficiently large Reynolds numbers they represent those stationary parts of the spectra which lie between the energy-containing and dissipation ranges of wave numbers.