Inviscid Flow past a Body at Low Magnetic Reynolds Number*

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1. INTRODUCTION

THE steady flow of an incompressible, inviscid, electrically conducting fluid past an obstacle, in the presence of an applied magnetic field, depends on two parameters: the magnetic Reynolds number R_M, and β, the ratio of a representative magnetic pressure to the dynamic pressure of the free stream. We consider two different types of flow for which R_M is small.

Section 2 contains a short discussion of the equations of motion. In Sec. 3 we assume β to be small: The magnetic field which, for simplicity, is supposed to be caused by a distribution within the body itself, is weak. The first-order effects in β and R_M are determined. The perturbation in R_M is not regular at infinity: Exponential decay factors, of the same type as those occurring in Oseen's approximation for slow viscous flow, must be introduced. Two paraboloidal wakes (of magnetic intensity and vorticity) are formed.

We may picture the fluid particles as moving with the potential flow past the body, acquiring rotation through the action of the nonconservative electromagnetic force. As a general consequence, at least in plane and axially symmetric cases, the vorticity becomes logarithmically infinite as any point on the surface of the body is approached. The section ends with a discussion of the drag coefficient, which is proportional to βR_M.

Secondly (Sec. 4), we consider the effect of a very strong magnetic field: βR_M large. The body is assumed to be nonconducting. The applied magnetic field, which is uniform and directed at right angles to the free stream, is relatively insensitive to the flow disturbance. On the other hand, the flow is strongly affected by the field.

To a first approximation, change in the flow across the lines of force is prohibited. However, the fluid may slip freely along these lines, which it does to an extent just sufficient to circumnavigate the obstacle. To obtain a more accurate description, the coordinate along the lines of force is compressed by a factor $(\beta R_M)^{\frac{1}{2}}$: This makes the inertia forces comparable with the electromagnetic. The results should be compared with those of Stewartson.[1] Finally, the drag coefficient, which is proportional to $(\beta R_M)^{\frac{1}{2}}$, is given for a circular cylinder.

2. EQUATIONS OF MOTION

The steady motion of an incompressible, inviscid, electrically conducting fluid is governed by the equations

$$\begin{aligned} \mathbf{v}\cdot\mathrm{grad}\mathbf{v} &= -(1/\rho_0)\,\mathrm{grad}p + (\mu/\rho_0)\,\mathrm{curl}\mathbf{H}\times\mathbf{H}, \\ \mathrm{curl}\mathbf{H} &= \sigma(\mathbf{E}+\mu\mathbf{v}\times\mathbf{H}), \end{aligned} \tag{1}$$

where

$$\mathrm{div}\mathbf{v}=0, \quad \mathrm{curl}\mathbf{E}=0, \quad \mathrm{div}\mathbf{H}=0. \tag{2}$$

Let $\mathbf{U}_0$ be the velocity of the uniform stream at infinity, let a be a representative length in the fixed obstacle, and let h be a representative magnetic intensity. An appropriate choice for h depends on the particular problem considered (see later sections). Then, when $\mathbf{v}$, $\mathbf{r}$, and $\mathbf{H}$ are made dimensionless by referring them to U_0, a, and h, respectively, Eqs. (1) become

$$\mathbf{v}\cdot\mathrm{grad}\mathbf{v} = -\mathrm{grad}p + \beta\,\mathrm{curl}\mathbf{H}\times\mathbf{H}, \tag{3a}$$

$$\mathrm{curl}\mathbf{H} = R_M(\mathbf{E}+\mathbf{v}\times\mathbf{H}), \tag{3b}$$

while (2) are unchanged. Here $R_M=U_0a\mu\sigma$ is the magnetic Reynolds number and $\beta=\mu h^2/\rho_0U_0^2$ is the ratio of a representative magnetic pressure to the free-stream dynamic pressure; the electric field is now given by $(\mu U_0h)\mathbf{E}$ and the pressure by $(\rho_0U_0^2)p$.

For an axially symmetric flow in which $\mathbf{v}$ and $\mathbf{H}$ lie in the meridian plane and are independent of the azimuthal angle, the conduction equation (3b) shows that $\mathbf{E}$ is perpendicular to this plane and also independent of the azimuthal angle. The second of Eqs. (2) then shows that

$$\mathbf{E}=0,$$

if it is to be finite at the axis. Similarly, for the corresponding plane motion, we have

$$\mathbf{E}=\mathbf{E}_0(\mathrm{const}),$$

where $\mathbf{E}_0$ is perpendicular to the plane.

The character of the motion depends on the parameters β and R_M appearing in Eqs. (3). In the following sections we suppose that R_M is small, which means that the applied magnetic field is only slightly perturbed by the motion of the fluid.

3. WEAK MAGNETIC FIELD

First we consider β small. Then Eq. (3a) shows that, in the limit $\beta\to 0$, we have $\mathbf{v}\to\mathbf{v}_0$, the ordinary potential flow past the obstacle. The magnetic field is convected with this flow.

For definiteness, we assume that the field is due to a magnetic distribution within the body itself. As $R_M\to 0$, it tends to the undisturbed field $\mathbf{H}_0$ of the distribution ($\mathrm{curl}\mathbf{H}_0=0$), while $\mathbf{E}\to 0$. In any finite region the ap-

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[1] K. Stewartson, Proc. Cambridge Phil. Soc. 52, 301 (1956).

proximation is uniform, the disturbance field $\mathbf{H}_1$ being given by

$$\text{curl}\mathbf{H}_1 = R_M \mathbf{v}_0 \times \mathbf{H}_0, \tag{4}$$

see (3b). At large distances, however, the magnetic field dies out more quickly than $\mathbf{H}_0$ by a factor $\exp[-\frac{1}{2}R_M r(1-\cos\theta)]$, where θ is the polar angle measured from the undisturbed stream direction.

This is easily seen by taking the curl of (3b) and using (2). Then

$$\nabla^2\mathbf{H} = R_M(\mathbf{v}_0\cdot\text{grad}\mathbf{H} - \mathbf{H}\cdot\text{grad}\mathbf{v}_0),$$

and, when $\mathbf{v}_0$ is replaced by its value at infinity, this is of the same form as Oseen's equation for the vorticity in the slow flow of a viscous fluid. Thus the magnetic field is swept into a paraboloidal wake (parabolic in the plane case), given by the exponential factor just mentioned. Correspondingly, the vorticity of the velocity disturbance caused by $\mathbf{H}_1$ (which we are just about to discuss) is confined to a similar wake of half the size.

For present purposes it is sufficient to take (4), and when this is used in (3a) the disturbance velocity $\mathbf{v}_1$, due to $\mathbf{H}_1$, is found to satisfy

$$\mathbf{v}_0\cdot\text{grad}\mathbf{v}_1 + \mathbf{v}_1\cdot\text{grad}\mathbf{v}_0 = -\text{grad}p_1 + \beta R_M(\mathbf{v}_0\times\mathbf{H}_0)\times\mathbf{H}_0. \tag{5}$$

Of particular interest is the production of vorticity in the flow, for which an equation is given by the curl of (5). For the axially symmetric case of Sec. 2 the equation reads

$$(d/dt)(\omega/y) = (\beta R_M/y)\,\text{curl}[(\mathbf{v}_0\times\mathbf{H}_0)\times\mathbf{H}_0], \tag{6}$$

where ω is the single azimuthal component of the vortex vector, y denotes distance from the axis, and the single nonzero azimuthal component is to be understood for the curl on the right side. The d/dt is to be taken with respect to $\mathbf{v}_0$.

The picture is now clear. The particles effectively move along the streamlines of the potential flow with velocity $\mathbf{v}_0$, acquiring vorticity according to (6). The value of ω itself is obtained when (6) is integrated along these streamlines.

Now the time that a fluid particle takes to pass through the neighborhood of the front stagnation point tends to infinity logarithmically with its closest approach to that point. The same is true of the vorticity which the particle acquires, provided the right-hand side of (6) does not vanish at the front stagnation point. [This happens only when either (i) $\mathbf{H}_0$ or (ii) all second derivatives of $\mathbf{v}_0$ vanish at the stagnation point.] It follows that ω becomes logarithmically infinite as any point on the surface of the body is approached.

For the plane motion of Sec. 2, Eq. (6) is replaced by

$$d\omega/dt = \beta R_M\,\text{curl}[(\mathbf{v}_0\times\mathbf{H}_0)\times\mathbf{H}_0],$$

and a similar conclusion holds. [In this case the exceptions are that at the stagnation point: (i) $\mathbf{H}_0$ is perpendicular or parallel to the obstacle; (ii) all first derivatives of $\mathbf{v}_0$ vanish.]

In computing drag, Chopra and Singer[2] have noticed that the rate of doing work, i.e., U_0 times the drag D, is equal to the Joule dissipation. For the present approximation this leads to the result

$$\frac{D}{\rho_0 U_0^2 a^2} = \beta R_M \int (\mathbf{v}_0\times\mathbf{H}_0)^2 dV,$$

the integration being taken over the whole volume occupied by the fluid.

When the obstacle is a sphere of radius a and the field is due to a dipole, of moment M directed along the free-stream direction, at its center, this gives

$$C_D = D/\tfrac{1}{2}\pi\rho_0 U_0^2 a^2 = [144\mu'^2/5(2\mu+\mu')^2]\beta R_M.$$

Here μ' is the permeability of the sphere and h has been taken to be M/a^3. Other examples have been given by Chopra and Singer,[2] Ludford and Murray,[3] and Chi.[4] The essential point to notice is that the drag can be computed directly from the undisturbed potential flow and magnetic field.

If one wishes to know what part of the drag is due to pressure and what part is due to the Maxwell stress, further analysis is necessary. In the case just given, all is due to the latter. For details a previous paper[5] of Ludford and Murray may be consulted; there also is found a more systematic treatment in which the present approximation appears as the first in a sequence. The corresponding treatment of plane flow past a circular cylinder, which encounters certain interesting mathematical difficulties, has been developed by Chi.[4]

With minor modifications, the same approach may be used when the magnetic field is applied at infinity.

4. VERY STRONG MAGNETIC FIELD

Secondly, we take β so large that βR_M is also large (even though R_M is, as before, assumed to be small). For simplicity, we consider plane flow past a cylinder of arbitrary cross section, the magnetic field being applied at infinity in the y direction, i.e., perpendicular to the free stream in the x direction. (Here we set $h = H_0$, the strength of the applied field.) There is now a constant induced electric field in the z direction.

In the limit $R_M \to 0$, Eq. (3b) shows that the magnetic field is undisturbed, i.e., $\mathbf{H} = \mathbf{j}$ (since we assume, for simplicity, that the permeability of the cylinder is the same as that of the fluid). On substituting (3b) in

[2] K. P. Chopra and S. F. Singer, *Heat Transfer and Fluid Mechanics Institute* (Stanford University Press, Stanford, California, 1959), p. 166.

[3] G. S. S. Ludford and J. D. Murray, *Proceedings of the Sixth Midwestern Conference on Fluid Mechanics, Austin* (University of Texas Printing Division, Austin, Texas, 1959), p. 457.

[4] L. Chi, Department of Mathematics, University of Maryland, Tech. Rept. 46 (June, 1960).

[5] G. S. S. Ludford and J. D. Murray, J. Fluid Mech. 7, 516 (1960).

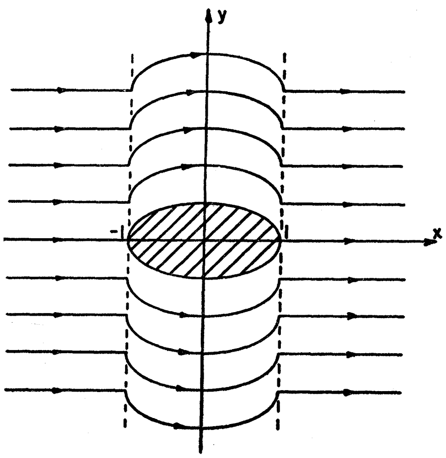

FIG. 1. Limiting form of the streamlines for a very strong magnetic field perpendicular to the free stream.

Eq. (3a), taking the limit $\beta R_M \to \infty$, and using this last result, we find

$$0=\text{grad}(p/\beta R_M)+[\mathbf{v}-(\mathbf{v}\cdot\mathbf{j})\mathbf{j}], \tag{7}$$

where $\mathbf{v}=(u,v)$ is now the disturbance velocity and we have allowed for the possibility that p becomes infinite with βR_M.

The only acceptable solution of these equations is illustrated in Fig. 1. (To avoid unessential complications we restrict attention to cross sections which are symmetric with respect to the x axis.) There is no variation in the y direction, the streamlines forming two families of congruent curves. Only the y component of velocity is disturbed: The fluid slips along the magnetic lines of force to an extent sufficient to circumnavigate the obstacle. Even this solution breaks down on the vertical lines $x=\pm 1$, since an infinite vertical velocity is required there.

To obtain a better approximation, account must be taken of the inertia forces, whose effect, though small in any finite region, is appreciable by accumulation over large distances. Thus we compress the coordinate y, with respect to which there are no variations in the solution just given, by writing

$$Y=y/(\beta R_M)^{\frac{1}{2}},$$

and, correspondingly, magnify the x component of the disturbance velocity by a factor $(\beta R_M)^{\frac{1}{2}}$. Then (7) is replaced by the equations

$$0=\partial P/\partial x+U, \quad \partial v/\partial x=-\partial P/\partial Y, \tag{8}$$

where $U=(\beta R_M)^{\frac{1}{2}}u$ and $P=p/(\beta R_M)^{\frac{1}{2}}$. To these must be added the continuity equation

$$\partial U/\partial x+\partial v/\partial Y=0.$$

The flow above the y axis is identical with that below. For the former we require a solution of

$$\partial^3 v/\partial x^3+\partial^2 v/\partial Y^2=0 \tag{9}$$

taking on the values

$$v=\begin{cases} 0, & |x|>1 \\ f'(x), & |x|<1 \end{cases} \quad \text{as} \quad Y\to +0, \tag{10}$$

where $y=f(x)$ gives the shape of the upper surface of the cylinder.

The problem is quite similar to the initial-value problem of one-dimensional heat flow. It would seem, however, that the initial values of $\partial v/\partial Y$ should also be prescribed, but this is replaced by the requirement that the disturbance vanishes as $Y\to\infty$. We build up the complete solution from the one having the unit step function for its initial values; this corresponds to the error-function solution of heat conduction.

Consider the solution

$$v_0=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(\lambda Y-\lambda^{\frac{2}{3}}x)\frac{d\lambda}{\lambda}, \tag{11}$$

which is clearly a function of

$$\eta=x/Y^{\frac{2}{3}}$$

alone. In the first instance v_0 is defined for positive x only, but by deforming the path of integration in the complex λ plane, it can be continued analytically to all values of x. We easily find the asymptotic expansions:

$$\frac{3}{2}\left[1+\frac{1}{|\eta|^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!\,\Gamma(-\frac{1}{2}-3n)}\cdot\frac{1}{|\eta|^{3n}}\right] \quad \text{for} \quad \eta<0, \tag{12a}$$

$$\frac{9\exp[-(4/27)\eta^3]}{4(\pi\eta^3)^{\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{A_n}{\eta^{3n}} \quad \text{for} \quad \eta>0, \tag{12b}$$

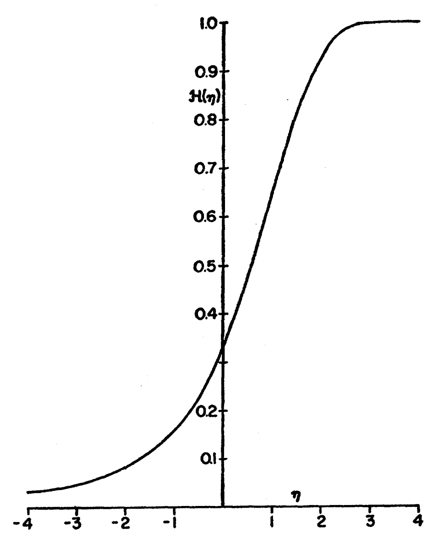

FIG. 2. The step-function solution of Eq. (9): $\psi=H(\eta)$, $\eta=x/Y^{\frac{2}{3}}$.

where the coefficients A_n are determined by the recurrence relation

$$A_{n+1}+\frac{27}{4}(n+\tfrac{1}{2})A_n=\frac{\Gamma(\frac{3}{2})}{(2n+2)\,!\Gamma(-\frac{3}{2}-3n)},\quad A_0=1.$$

It follows that

$$\mathcal{H}(\eta)=1-\tfrac{2}{3}v_0(\eta)$$

is the required step-function solution.

A graph of $\mathcal{H}(\eta)$ is given in Fig. 2. In addition to (12), the series expansion

$$\mathcal{H}(\eta)=\tfrac{1}{3}+\tfrac{2}{3}\eta\sum_{n=0}^{\infty}\frac{(-1)^n}{(3n+1)\,!\Gamma(\frac{1}{3}-2n)}\eta^{3n}+\tfrac{2}{3}\eta^2\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(3n+2)\,!\Gamma(-\frac{1}{3}-2n)}\eta^{3n}$$

was used in constructing this figure. Note the gradual (algebraic) decay of $\mathcal{H}$ for negative η[cf. (12a)] in contrast to the sharp (exponential) rise to its limiting value when η is positive [cf. (12b)].

The solution of Eq. (9) which satisfies the initial conditions (10) is

$$v=\frac{1}{Y^{\frac{2}{3}}}\int_{-1}^{1}f'(\xi)\mathcal{H}'\left(\frac{x-\xi}{Y^{\frac{2}{3}}}\right)d\xi,$$

since the x derivative of $\mathcal{H}$ is, by construction, the source solution.

Of main interest are the values of P on the surface of the cylinder, i.e., as $Y\rightarrow +0$, which follow from (8) and (12). A unit source at ξ contributes nothing to the pressure at x when x is greater than ξ, while for $x<\xi$ it provides an amount

$$\pi^{-\frac{1}{2}}(\xi-x)^{-\frac{1}{2}}$$

for P. On integrating we have

$$P=\frac{1}{\sqrt{\pi}}\int_x^1\frac{f'(\xi)}{(\xi-x)^{\frac{1}{2}}}d\xi\quad\text{for}\quad |x|<1.$$

The total drag D on the cylinder is therefore given by

$$\frac{D}{\rho_0U_0^2a}=2\left(\frac{\beta R_M}{\pi}\right)^{\frac{1}{2}}\int_{-1}^{1}f'(x)dx\int_x^1\frac{f'(\xi)}{(\xi-x)^{\frac{1}{2}}}d\xi,$$

where the factor 2 corresponds to equal contributions from the upper and lower surfaces.

For a circular cylinder of radius a, $f(x)=(1-x^2)^{\frac{1}{2}}$ and the drag coefficient is

$$C_D=\frac{D}{\frac{1}{2}\rho_0U_0{}^22a}=2\left(\frac{\beta R_M}{\pi}\right)^{\frac{1}{2}}\times\int_{-1}^{1}dx\int^{1}\frac{x\xi}{[(\xi-x)(1-x^2)(1-\xi^2)]^{\frac{1}{2}}}d\xi$$

$$=(18\pi)^{-\frac{1}{2}}B^2(\tfrac{1}{4},\tfrac{1}{2})(\beta R_M)^{\frac{1}{2}}=3.656(\beta R_M)^{\frac{1}{2}}.$$

Flow past a three-dimensional body may be treated in a similar fashion. The dimensionless drag $D/\rho_0U_0{}^2a^2$ is again proportional to $(\beta R_M)^{\frac{1}{2}}$.

Details are given in a separate paper.[6]

[6] G. S. S. Ludford, Department of Mathematics, University of Maryland, Tech. Rept. 45 (April, 1960).