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The Theory of the Anelastic Fluid*

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The general theory of anelastic substances is simplified for the case of an anelastic fluid. It is shown that slow steady motions and the diffusion of vorticity obey the same laws as in the Stokes-Navier theory of a viscous fluid. A criterion for the validity of these equations is derived.

The propagation and absorption of longitudinal isentropic waves exhibits new phenomena, however. There is hysteresis in the response of the density to the pressure. This hysteresis vanishes for very low and very high frequencies. The compressibility for these two limiting cases approaches different, real, values. For intermediate values the compressibility is a complex number, corresponding to the hysteresis. The velocity of propagation is also a complex number, which is a function of the frequency. The roots of this function are two in number, and determine two relaxation times. One of these is associated with the hysteresis mentioned above, the other with viscosity.

THE experimental and theoretical study of viscosity is farther advanced than any other branch of anelasticity. This is partly because the precise measurement of viscosity, and the theoretical discussion of the motion of spheres through viscous media, were strongly stimulated by Dr. Millikan's oil-drop experiment.

In another place,[1] a theory of elasticity and anelasticity has been developed. In this theory, the distinction between solids and fluids becomes a quantitative, rather than a qualitative one. In the case of fluids, it is predicted that there will be other dissipative phenomena than viscosity. Any modified theory of dissipative phenomena in fluids must be compatible with the existing data on viscosity. It is not obvious that this is the case for the equations developed in IV; one purpose of the present paper is to show that they lead to the same steady states of flow as do the Stokes-Navier equations. Consequently, the viscosimeter formulae, and those for the terminal velocity of falling spheres, can be taken over unchanged from the older theory. However, it will be shown that the Reynold's number criterion is merely necessary, and not sufficient, for the validity of the approximations involved in these equations.

The theory proposed in IV differs from the traditional theory largely in the equations for the propagation and dissipation of longitudinal waves. It has become customary to describe anomalous propagation in terms of the relaxation times of the dissipative processes. In the case of an ideal gas, there is only one relaxation time: the mean free time between molecular collisions. In real gases, whose molecules are capable of existing in excited states, the mean life time of the excited state is a second relaxation time. In both gases and liquids, chemical reactions can introduce other "lines" into the relaxation "spectrum." The present theory ignores all of these possible causes of dissipation except the collision process, which results in the viscous forces. It considers a different kind of dissipative process that is characteristic of anelastic media, either solid or fluid. This process depends on the fact that anelastic changes in the state of strain can occur independently of changes in density. The term "independent" is here used in the geometric or kinematic sense. Physically, the changes in strain and density must be related by a definite law; this is not, however, the simple equality that exists in the case of elastic changes. The particular relation that has been assumed in IV is in a sense the simplest one consistent with the second law of thermodynamics, and has no other foundation than this. For this reason, it is not yet possible to give a molecular picture of the processes involved. It may be that these processes do not

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[1] Carl Eckart, "The thermodynamics of irreversible processes. IV," Phys. Rev. 73, 373 (1948). Hereafter cited simply as IV.

occur in an ideal gas; in that case it will continue to be difficult to work out the molecular details.

THE EQUATIONS OF MOTION FOR AN ANELASTIC FLUID

The equations will not be derived here; instead, the equations of IV will be reproduced, with the simplifications appropriate when fluids rather than solids are involved. It is supposed that the fluid departs only slightly from a state of constant density, ρ_0, so that, if v is its actual specific volume, the condensation

$$s = 1 - \rho_0 v$$

is a small number. The components of the strain are denoted by σ_{kl}, those with unequal indices being the shear strains. In the case of a medium incapable of anelastic deformation, the quantity

$$\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

is related to s by the simple equation $s + \sigma = 0$. As remarked above, this relation fails in the case of anelastic media, and s and σ must be treated as independent variables.

For isentropic motions, the internal energy, ϵ, of the fluid will be a function only of s and σ, and independent of the shear strains. For present purposes, it may be taken to be

$$\epsilon = (1/2\rho_0)(\lambda_1\sigma^2 + 2\lambda_2 s\sigma + \lambda_3 s^2), \tag{1}$$

with $\lambda_1 > 0$, $\lambda_1\lambda_3 > \lambda_2^2$ to insure stability. If $\mathbf{u}$ is the vector velocity of the fluid, and it as well as s and σ can be treated as small, the equation of continuity is

$$\frac{\partial s}{\partial t} + \nabla \cdot \mathbf{u} = 0, \tag{2}$$

while Eq. (52) of IV becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = (\lambda_2 - \lambda_3)\nabla s + (\lambda_1 - \lambda_2)\nabla\sigma + N\nabla^2\mathbf{u} + (N + N')\nabla\nabla \cdot \mathbf{u}. \tag{3}$$

Clearly

$$P = (\lambda_3 - \lambda_2)s + (\lambda_2 - \lambda_1)\sigma \tag{4}$$

is the total elastic pressure, while the remaining terms on the right are the viscous forces, N and N' being two coefficients of viscosity, such that $N > 0$, $3N' + 2N > 0$.

Equation (55) of the previous paper relates σ to s, and is the generalization of the equation $s + \sigma = 0$. For present purposes, it can be reduced to

$$\frac{\partial}{\partial t}(s + \sigma) = -3(3M' + 2M)(\lambda_2 s + \lambda_1 \sigma), \tag{5}$$

where M and M' are the coefficients of anelasticity. In the case of liquids, they enter the equations only in the combination

$$A = 3(3M' + 2M) > 0.$$

It will be important, below, to note that these equations are approximate, having been simplified by the neglect of all transport phenomena (i.e., omission of the operator $\mathbf{u} \cdot \nabla$ from the derivative D/Dt).

Equation (53) of IV also determines the rate of change of the shear strains, but since these do not enter the internal energy, they are of little interest here. Moreover, the shear strains in fluids are large, increasing without limit in many cases. The neglect of the transport terms in Eq. (53) of IV makes them unsuitable for the discussion of the large shear strains in any case.

STEADY MOTIONS

In the case of steady motion the above equations reduce to

$$\nabla \cdot \mathbf{u} = 0, \tag{2a}$$

$$N\nabla^2\mathbf{u} = \nabla P, \tag{3a}$$

$$\lambda_2 s + \lambda_1 \sigma = 0. \tag{5a}$$

The first two of these have precisely the form taken by the Stokes-Navier equations when all inertial terms are neglected.[2] They lead to the accepted formulae for the flow of viscous fluid through a tube, Stokes' formula for the motion

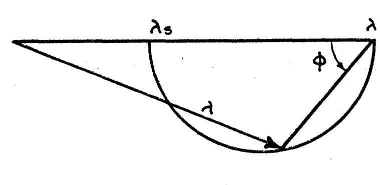

FIG. 1. Circle diagrams showing variation of λ with frequency.

[2] H. Lamb, *Hydrodynamics* (Cambridge University Press, Teddington, England, 1932), sixth edition, p. 595, *et seq.*

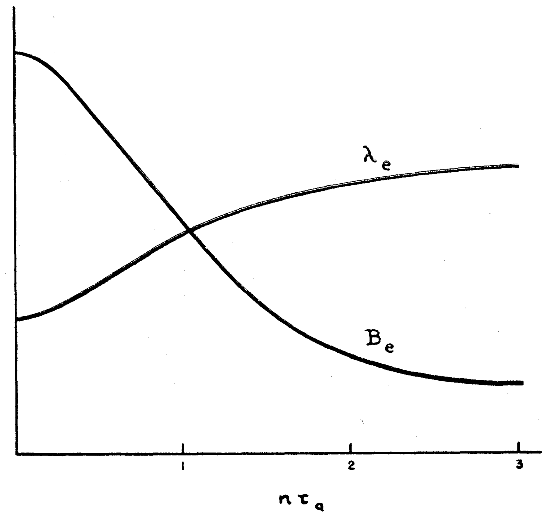

FIG. 2. Variation of λ_e and B with $n\tau_a$.

of a sphere, etc. By introducing the inertial terms, Oseen's linearized equations,[2] on which his critical investigation of Stokes' law was based, can be obtained. However, in the steady state, the inertial terms are of the form $\rho_0\mathbf{u}\cdot\nabla\mathbf{u}$. In order to be consistent, other transport terms must be included as well. Thus the present theory leads to a generalization of Oseen's equations. The physical significance of this will be discussed below.

By combining Eqs. (4) and (5a), the equation

$$P=s(\lambda_1\lambda_3-\lambda_2^2)/\lambda_1 \tag{4a}$$

is obtained. The quantity $1/\lambda_s$, where

$$\lambda_s=\lambda_3-\lambda_2^2/\lambda_1,$$

is thus the steady-state compressibility of the fluid. It will be seen below, that for simple harmonic motions, the compressibility depends on the frequency.

THE DIFFUSION OF VORTICITY

The vorticity

$$\mathbf{R}=\nabla x\mathbf{u} \tag{6}$$

also satisfies the same equation as in the Stokes-Navier theory. For, on taking the curl of both sides of Eq. (3), the gradients are eliminated, and the resulting equation is

$$\rho_0\frac{\partial\mathbf{R}}{\partial t}=N\nabla^2\mathbf{R}. \tag{7}$$

Thus, vorticity diffuses through an anelastic fluid according to the same law as through an elastic fluid, and the constant N is seen to be the ordinary coefficient of shear viscosity.

THE PROPAGATION OF LONGITUDINAL WAVES

On taking the divergence of both sides of Eq. (3) and using Eq. (2), the longitudinal wave equation is obtained in the form

$$\rho_0\frac{\partial^2 s}{\partial t^2}=(\lambda_3-\lambda_2)\nabla^2 s+(\lambda_2-\lambda_1)\nabla^2\sigma+B\nabla^2\frac{\partial s}{\partial t}, \tag{8}$$

where $B=(2N+N')>0$. This must be solved simultaneously with Eq. (5).

The plane wave solutions, for which

$$s,\ \sigma\sim\exp in(x/c-t),$$

are readily obtained by solving the simultaneous linear equations

$$\begin{aligned}(\lambda_2A-in)s+(\lambda_1A-in)\sigma&=0,\\ (\lambda_3-\lambda_2-Bin)s+(\lambda_2-\lambda_1)\sigma&=\rho_0c^2s.\end{aligned} \tag{9}$$

Combining the first of these with Eq. (4) results in

$$P/s\equiv\lambda=(\lambda_s-in\tau_a\lambda_f)/(1-in\tau_a), \tag{10}$$

where

$$\lambda_f=\lambda_1-2\lambda_2+\lambda_3>\lambda_s,$$

and

$$\tau_a=1/A\lambda_1.$$

It is easily seen that when $n\tau_a\ll 1$, the compressibility is $1/\lambda_s$, while for $n\tau_a\gg 1$, it is $1/\lambda_f$. For moderate values of $n\tau_a$, the compressibility is a complex number; this means that there is hysteresis in the response of the density of the liquid to the existing pressure. The condensation lags behind the pressure since $\lambda_f>\lambda_s$. Figure 1 is a circle diagram showing the variation of λ with frequency.

Returning to Eq. (9): the quantity c is determined by the vanishing of their determinant, i.e., by the equation

$$\rho_0c^2=\lambda(n)-Bin. \tag{11}$$

The real part of c is the phase velocity of the waves, while the imaginary part of n/c is their absorption coefficient. It is seen that both the phase velocity and the absorption coefficient depend on the frequency.

The Eq. (11) may also be written

$$\rho_0 c^2 = \lambda_e - B_e in, \tag{11a}$$

where λ_e and B_e are real and may be called the effective bulk coefficients of elasticity and viscosity, respectively. Figure 2 shows graphs of λ_e and B_e.

The fact that B_e has a maximum for zero frequency may seem paradoxical in view of the fact that the equations of steady flow are identical with the Stokes-Navier equations. The reason for this is that the neglect of the transport terms is equivalent to assuming that there is no hysteresis in the response of the density of the existing pressure. When these terms are not neglected, dissipative forces proportional to B_e and to the square of the mean velocity are encountered. The ratio of these forces to the elastic forces will be of the order of magnitude

$$\frac{U}{L}\left[\tau_a\left(\frac{\lambda_f}{\lambda_s}-1\right)+\tau_v\right],$$

where

$$\tau_v = B/\lambda_s,$$

while U is the mean velocity and L the distance over which the pressure field is not constant. Only when both ratios $U\tau_v/L$ and $U\tau_a/L$ are small, can one justify the neglect of the transport terms and the treatment of the fluid as incompressible. The customary requirement is that the inertial term $\rho_0 \mathbf{u}\cdot\nabla\mathbf{u}$ be small compared to the viscous term $N\nabla^2\mathbf{u}$; this leads to Reynold's criterion $\rho_0 UL/N \ll 1$. The ratio of the dimensionless number $U\tau_v/L$ to Reynold's number is very small, being of the order of magnitude $(l/L)^2$, where l is the mean free path of the molecules. Reynold's criterion is thus the more stringent of these two. The magnitude of $U\tau_a/L$ for actual liquids is unknown.

The equation

$$c(n) = 0$$

has two roots for real values of in. The reciprocals of these roots are the relaxation times of the medium. The equation for the relaxation times is, therefore,

$$\lambda_s\tau^2 - (\lambda_s\tau_v + \lambda_f\tau_a) + \lambda_s\tau_v\tau_a = 0. \tag{12}$$

If $\tau_v = 0$ (non-viscous fluid), the roots are $\tau = 0$ and $\tau_a(\lambda_f/\lambda_s)$. If τ_v approaches infinity, the roots approach $\tau = \infty$ and τ_a. In general, one of the roots will have the order of magnitude τ_a, while the other will have the magnitude of τ_v. The first is associated with the anelasticity, the second with the viscosity of the medium. In the case of an ideal gas, τ_v is of the order of magnitude of the mean free time of the molecules as has been remarked above.