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# Narrow Gaps in Microwave Problems

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An explicit expression, $E_x=f(x)$, is given for the $x$ component of the electric field on the $y=0$ plane in the gap between two infinite, oppositely charged conducting masses bounded by the $y=0$, $x=b$ planes and the $y=0$, $x=-b$ planes, respectively. The formula can be handled analytically and its maximum deviation from the rigorous value is one part in 5000. The coefficients of the Fourier expansion of $E_x$ and $V$ are given and used to write down a rapidly convergent series for the potential in the narrow gap between two right circular, coaxial conducting cylindrical electrodes. The resonant frequency of a re-entrant cylindrical cavity with a narrow gap is computed to about 0.03 percent. The radiation pattern from the open end of a rectangular wave guide terminating in an infinite conducting plane and transmitting only the $TE_{10}$ mode is calculated with four-place accuracy.

## I. A FORMULA FOR THE TANGENTIAL FIELD IN A GAP

In many electromagnetic problems the field space may be broken up into a number of regions of simple geometrical shape partially enclosed by perfectly conducting boundaries. In each region orthogonal expansions or integrals for the relevant fields or potentials often can be written down, but the determination of the coefficients so as to match the fields across the common boundaries is a formidable task, involving in most cases the solution of one or more infinite determinants.[1] If an exact expression for the tangential electric field on the interfaces is known, such difficulties vanish and a rigorous solution is possible. Often they are slits which encircle the region internally or externally and are narrow compared with curvatures and other geometrical features and, if the field is time dependent, with the wave-length. A conformal transformation then gives the field near the slit. Consider the slit section shown in Fig. 1. The stream function $U$ and the potential function $V$ are connected with $x$ and $y$ by the parametric equations[2]

$$\pi(U+jV)=2V_0(\tfrac{1}{2}j\pi-\ln z_1), \tag{1}$$

$$\begin{aligned}\pi(x+jy)&=2b[(z_1^2-1)^{\frac{1}{2}}+\sin^{-1}(1/z_1)],\\ &=2jb\{(1-z_1^2)^{\frac{1}{2}}-\ln[1+(1-z_1^2)^{\frac{1}{2}}]\\ &\qquad-\tfrac{1}{2}j\pi+\ln z_1\},\end{aligned} \tag{2}$$

where $z_1$ is a complex parameter. It is easily verified that this expression, obtained by a simple Schwarz transformation, satisfies the conditions: $V=V_0$, $U<0$ when $y=0$, $x>b$ or $y_1=0$, $x_1>1$; $V=V_0$, $U>0$, when $y<0$, $x=b$ or $y_1=0$, $0<x_1<1$; $V=0$, $-\infty<U<+\infty$, when $x=0$ or $x_1=0$, and that $V$ has the same sign as $x$. The tangential electric field is

$$E_x=\left|\frac{\partial V}{\partial x}\right|_{y=0}=\left|\frac{V_0}{b(z_1^2-1)^{\frac{1}{2}}}\right|_{y=0}. \tag{3}$$

It is impossible to solve explicitly for $E_x$ as a function of $x$, so values of $x_1$ and $y_1$ are found by trial that make (2) real, thus giving $x$. These numbers, $x_1$ and $y_1$, are substituted in (3) to give $E_x$. Typical values when $y=0.000000\pm0.000002$ are shown in the first four columns of Table I. All numbers were calculated to one more place than the number given, so that the last digit is correct. To be useful $E_x$ must be expressed as a function of $x$ that can be handled analytically. If $r$ is the distance from an edge of the slit, we would expect the field to become infinite as $r^{-\frac{1}{3}}$ when $r\to0$ as at the edge of a

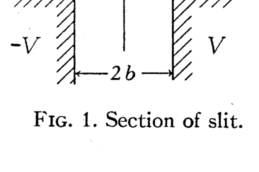


FIG. 1. Section of slit.

[1] W. C. Hahn, J. App. Phys. 12, 62 (1941).
[2] In this paper m.K.S. units are used and $j=(-1)^{\frac{1}{2}}$.

TABLE I.

| $x_1$ | $y_1$ | $x/b$ | $bE_x/V_0(3)$ | $bE_x/V_0(4)$ | $10^5\delta$ |
|---|---|---|---|---|---|
| 0.00 | 0.66275 | 0.00000 | 0.83355 | 0.83346 | −9 |
| 0.20 | 0.64162 | 0.22992 | 0.84290 | 0.84293 | +3 |
| 0.40 | 0.57629 | 0.45621 | 0.87544 | 0.87552 | +8 |
| 0.55 | 0.49451 | 0.62055 | 0.92611 | 0.92598 | −13 |
| 0.70 | 0.37886 | 0.77566 | 1.02728 | 1.02718 | −10 |
| 0.80 | 0.27852 | 0.86992 | 1.16683 | 1.16706 | +23 |
| 0.90 | 0.15457 | 0.95059 | 1.51429 | 1.51419 | −10 |
| 1.00 | 0.00000 | 1.00000 | ∞ | ∞ | — |

charged rectangular wedge. A suitable expression is

$$E_x=(V_0/b)\{A+B\cos(\pi x/b)+C(x/b)^2 +D[\sec.(\tfrac{1}{2}\pi x/b)]^{.355}\}, \quad (4)$$

where $A=0.34091$, $B=-0.00656$, $C=-0.07785$, and $D=0.49910$. The field values given by this formula are shown in the fifth column of Table I. It will be observed that the maximum difference between this and the true value is one part in five thousand. The exponent 0.355, which differs slightly from the expected value 0.333, is chosen to make

$$\int_0^b E_x ds=1.00000\,V_0. \quad (5)$$

## II. FIELD BETWEEN TWO PLANE CYLINDRICAL POLE PIECES

Many applications of (4) require expansions of the form

$$E_x=\frac{V_0}{b}\left(1+\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{b}\right), \quad (6)$$

$$V=V_0\left(\frac{x}{b}+\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{b}\right), \quad (7)$$

valid when $-b<x<+b$. Because $E_x$ is infinite at $x=b$ it is better to use the value of $E_x$ an infinitesimal distance $b\epsilon/\pi$ above or below the $x$ axis which, in all cases that we consider, will

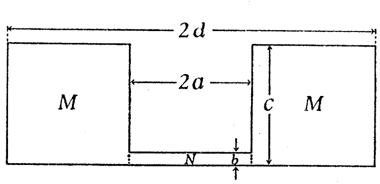


FIG. 2. Axial section of re-entrant cavity.

TABLE II.

| $n$ | $A_n$ | $B_n$ | $n$ | $A_n$ | $B_n$ |
|---|---|---|---|---|---|
| 1 | −0.27068 | −0.086159 | 11 | −0.06356 | −0.001839 |
| 2 | 0.18314 | 0.029148 | 12 | 0.06015 | 0.001595 |
| 3 | −0.14387 | −0.015265 | 13 | −0.05714 | −0.001399 |
| 4 | 0.12054 | 0.009592 | 14 | 0.05450 | 0.001239 |
| 5 | −0.10486 | −0.006676 | 15 | −0.05214 | −0.001106 |
| 6 | 0.09349 | 0.004960 | 16 | 0.05002 | 0.000995 |
| 7 | −0.08480 | −0.003856 | 17 | −0.04811 | −0.000901 |
| 8 | 0.07791 | 0.003100 | 18 | 0.04638 | 0.000820 |
| 9 | −0.07228 | −0.002556 | 19 | −0.04480 | −0.000750 |
| 10 | 0.06758 | 0.002151 | 20 | 0.04335 | 0.000690 |

introduce the factor $e^{-n\epsilon}$ under the summation in (6) and insure its convergence. This is true because of the relations

$$\frac{I_p(n\pi\rho/b)}{I_q(n\pi\rho_0/b)}\underset{n\to\infty}{\longrightarrow}\frac{K_p(n\pi\rho_0/b)}{K_q(n\pi\rho/b)} \underset{n\to\infty}{\longrightarrow}\left(\frac{\rho_0}{\rho}\right)e^{-n\pi(\rho-\rho_0)/b}\underset{\rho\to\rho_0}{\longrightarrow}e^{-n\epsilon}. \quad (8)$$

If $\epsilon$ is taken sufficiently small, the physical set-up is unchanged and the exponential factor becomes unity for as many terms as required. The values of $A_n$ for $0<n\leqslant 20$ were found in the usual way by the substitution of (4) for $E_x$ in (6), multiplication through by $\cos(n\pi x/b)$, and integration from $x=-b$ to $x=+b$. The values of $B_n$ were found by the integration of (6) from 0 to $x$. The results are shown in Table II. For $n>1$ the general expression for $A_n$ or $n\pi B_n$ is

$$A_n=(-1)^n\left[0.299856\times\frac{\Gamma(n+0.1775)}{\Gamma(n+0.8225)}-\frac{0.031552}{n^2}\right]. \quad (9)$$

When $n=1$ we must subtract 0.006556 from (9). The gamma-functions arise from the integration of terms involving $[\sec.(\frac{1}{2}\pi x/b)]^{.355}$ for which the integral at the top of page 263 in Whittaker and Watson's *Modern Analysis* was used.

We can now write out the solution of an experimentally important static problem. Suppose the section between $z=+b$ and $z=-b$ is removed from the solid circular cylinder whose surface is $\rho=(x^2+y^2)^{\frac{1}{2}}=a$ and the two halves are charged to potentials $+V_0$ and $-V_0$. If $b\ll a$, the potential in the gap is

$$V = V_0\left[\frac{z}{b} + \sum_{n=1}^{\infty} B_n \frac{I_0(n\pi\rho/b)}{I_0(n\pi a/b)} \sin\frac{n\pi z}{b}\right]. \quad (10)$$

Except near the edge $\rho = a$ this series converges very rapidly because the ratio of the modified Bessel functions decreases exponentially with $n$. The field derived from this also applies to the region between the plane circular pole pieces of an electromagnet if their permeability is so high that $\mathbf{B}$ is normal to their faces.

## III. RESONANT FREQUENCY OF RE-ENTRANT CAVITY

The accuracy of (4) makes it possible to improve on Hansen's[3] original method of calculating the lowest resonant frequency of the common re-entrant circular cylindrical cavity shown in Fig. 2. The axially symmetrical field[4] components tangential to a cylinder in the $M$ region, which satisfy the boundary conditions at $\rho = b$, $z = 0$, and $z = c$, are, if the frequency is such that $\beta < \pi/c$,

$$E_z = \left[C_0 \frac{R_0(\beta_0\rho)}{R_0(\beta_0 a)} + \sum_{m=1}^{\infty} C_m \frac{R_0(\beta_m\rho)}{R_0(\beta_m a)} \cos\frac{m\pi z}{c}\right] e^{j\omega t}, \quad (11)$$

$$B_\phi = \frac{\beta_0^2}{j\omega}\left[\frac{C_0 R_0'(\beta_0\rho)}{\beta_0 R_0(\beta_0 a)} + \sum_{m=1}^{\infty} \frac{C_m R_0'(\beta_m\rho)}{\beta_m R_0(\beta_m a)} \cos\frac{m\pi z}{c}\right] e^{j\omega t}, \quad (12)$$

where

$$\begin{aligned} R_0(\beta_0\rho) &= J_0(\beta_0 d) Y_0(\beta_0\rho) - Y_0(\beta_0 d) J_0(\beta_0\rho), \\ R_0(\beta_m\rho) &= I_0(\beta_m d) K_0(\beta_m\rho) - K_0(\beta_m d) I_0(\beta_m\rho). \end{aligned} \quad (13)$$

In the $N$ region, fields which are finite on the axis and satisfy the boundary conditions at $z = 0$ and $z = b$, are

$$E_z = \left[A_0 \frac{J_0(\beta_0\rho)}{J_0(\beta_0 a)} + \sum_{n=1}^{\infty} A_n \frac{I_0(\beta_n\rho)}{I_0(\beta_n a)} \cos\frac{n\pi z}{b}\right] e^{j\omega t}, \quad (14)$$

$$B_\phi = \frac{\beta_0^2}{j\omega}\left[\frac{A_0 J_0'(\beta_0\rho)}{\beta_0 J_0(\beta_0 a)} + \sum_{n=1}^{\infty} \frac{A_n I_0'(\beta_n\rho)}{\beta_n I_0(\beta_n a)} \cos\frac{n\pi z}{b}\right] e^{j\omega t}. \quad (15)$$

The wave numbers $\beta_n$ and $\beta_m$ in these equations are given by

$$\begin{aligned} \beta_m^2 = \left|\beta_0^2 - \frac{m^2\pi^2}{c^2}\right|, \quad \beta_n^2 = \left|\beta_0^2 - \frac{n^2\pi^2}{b^2}\right|, \\ \beta_0^2 = \beta^2 = \omega^2\mu\epsilon = 4\pi^2\lambda^{-2}. \end{aligned} \quad (16)$$

The hypothesis that $b \ll \lambda$ and $b \ll a$ means that when $\rho = a$ (14) and (6) must be identical so that, if the excitation is adjusted so that $V_0/b$ is $e^{j\omega t}$, the $A_n$'s in (14) and (15) are those given in Table II and $A_0$ is one. In (11) and (12) $C_m$ can be determined by writing $a + \epsilon$ for $\rho$ in (11), multiplying both sides by $\cos(m\pi z/c)$ and integrating from $-c$ to $+c$. When $-b < z < +b$, $E_z$ is given by (4) and when $b < |z| < c$ it is zero. Thus we find $C_0$ to be $V_0/c$ and

$$C_m = \frac{V_0}{b}\left[\left\{\frac{0.16747}{m} - \frac{0.004174b^2}{c^2 - m^2b^2} + \frac{0.0100435c^2}{m^3b^2} + \frac{0.48081b}{c} \frac{\Gamma[(mb/c)+0.1775]}{\Gamma[(mb/c)+0.8225]}\right\} \sin\frac{m\pi b}{c} - \left\{\frac{0.031553c}{m^2 b} - \frac{0.29986b}{c} \frac{\Gamma[(mb/c)+0.1775]}{\Gamma[(mb/c)+0.8225]}\right\} \cos\frac{m\pi b}{c}\right], \quad (17)$$

when $V_0/b$ is one this gives for the special case $c = 10b$ the values of Table III.

It is clear that although (11) and (14) are matched at $\rho = a$ for all values of $\beta_0$ only certain values will match (12) and (15) in the range $0 < z < b$. It follows from (16) that if any term in (12) or (15) is known, all are known, provided the initial hypothesis is valid. The simplest relation is obtained by equating (12) to (15) at

[3] W. W. Hansen, J. App. Phys. **10**, 38 (1939).

[4] Ramo and Whinnery, *Fields and Waves in Modern Radio* (John Wiley and Sons, Inc., New York, 1945), p. 326.

TABLE III.

| $m$ | $C_m$ | $P_m$ | $m$ | $C_m$ | $P_m$ |
|---|---|---|---|---|---|
| 0 | 0.10000 | −0.11930 | | | |
| 1 | 0.19602 | −0.09635 | 11 | −0.04283 | −0.00013 |
| 2 | 0.18438 | −0.03598 | 12 | −0.05294 | −0.00025 |
| 3 | 0.16583 | −0.01878 | 13 | −0.05705 | −0.00032 |
| 4 | 0.14160 | −0.01033 | 14 | −0.05520 | −0.00031 |
| 5 | 0.11325 | −0.00547 | 15 | −0.04827 | −0.00025 |
| 6 | 0.08259 | −0.00261 | 16 | −0.03736 | −0.00016 |
| 7 | 0.05152 | −0.00101 | 17 | −0.02382 | −0.00008 |
| 8 | 0.02190 | −0.00024 | 18 | −0.00910 | −0.00002 |
| 9 | −0.00456 | +0.00002 | 19 | +0.00538 | +0.00001 |
| 10 | −0.02749 | 0.00000 | 20 | +0.01832 | 0.00000 |

TABLE IV.

| $\theta$ | $\phi=0$ (26), (24) | Schel. | (26) | $\phi=\frac{1}{2}\pi$ (24) | Schel. |
|---|---|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 10° | 0.9006 | 0.9097 | 0.9902 | 0.9919 | 0.9821 |
| 20° | 0.6598 | 0.6877 | 0.9624 | 0.9690 | 0.9312 |
| 30° | 0.3963 | 0.4373 | 0.9204 | 0.9342 | 0.8543 |
| 40° | 0.1975 | 0.2388 | 0.8710 | 0.8938 | 0.7626 |
| 50° | 0.0886 | 0.1236 | 0.8221 | 0.8520 | 0.6650 |
| 60° | 0.0293 | 0.0508 | 0.7773 | 0.8141 | 0.5702 |
| 70° | 0.0085 | 0.0225 | 0.7422 | 0.7842 | 0.4837 |
| 80° | 0.0016 | 0.0101 | 0.7199 | 0.7651 | 0.4082 |
| 90° | 0.0000 | 0.0032 | 0.7123 | 0.7587 | 0.3751 |

$\rho=a$ and integrating from $z=-b$ to $z=+b$. This gives

$$1=\frac{J_0(\beta_0 a)}{J_0'(\beta_0 a)}\left[C_0\frac{R_0'(\beta_0 a)}{R_0(\beta_0 a)}+\frac{\beta_0 c}{\pi b}\sum_{m=1}^{\infty}\frac{C_m}{m\beta_m}\frac{R_0'(\beta_m a)}{R_0(\beta_m a)}\sin\frac{m\pi b}{c}\right]. \quad (18)$$

From (16) $\beta_m$ is proportional to $m$ when $m$ is large, so the terms decrease slightly faster than $m^{-2}$. This is analogous to Hahn's[1] Eq. (23). The value of $\beta_0$ that satisfies (18) is found by trial. The process is simplified by the fact that when $m>1$ $K_0'/K_0$ may be written for $R_0'/R_0$ to the sixth significant figure and that when $m>3$ the terms after the summation are independent of $\beta_0$ to the fifth figure. Near the correct value in the example to be considered the ratio in front of the bracket changes 13 times as fast with $\beta_0$ as the value of the bracket which, therefore, need be computed only for an approximate $\beta_0$. The values $P_m$ of the terms in the bracket when the dimensions in meters are $a=0.05$, $b=0.01$, $c=0.10$, and $d=0.15$ are given in Table III for $\beta_0=11.20$. The calculated value of $A_0$ is 0.9999.

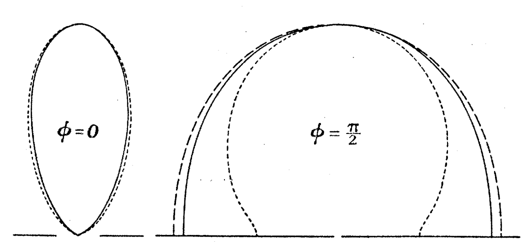


FIG. 3. Angular intensity distribution of radiation from rectangular wave guide. The solid line is rigorous, the dashed line is from the assumption of an unperturbed field, and the dotted line is from Schelkunoff's formula.

The periodicity of $P_m$ in $m$ is ten. An accuracy of 0.3 percent in $\beta_0$ is obtained with eight terms and of 0.03 percent with 16 terms. Narrower gaps give longer periodicity. In the present method the value of each $C_m$ is calculated independently. In Hahn's method the value of each depends on all the others and on the $A$'s so that one must compute more than twenty $C_m$'s to get twenty correct values. Of course less accuracy is required for large $m$ values. If $A_1$ is calculated from the $C_m$'s as $A_0$ was, it is immediately evident that twenty $C_m$'s are quite inadequate. The sum of the first four terms does not even have the right sign. It is believed, therefore, that for narrow gaps the present method will give more exact values for any specified number of terms than Hahn's method.

## IV. RADIATION FROM END OF RECTANGULAR WAVE GUIDE

Starting with Eq. (4) one can calculate almost rigorously the radiation pattern from a rectangular wave guide whose open end is surrounded by an infinite plane, perfectly conducting flange normal to its axis and which transmits only the $TE_{10}$ mode. The solution is rigorous when the guide height, $2b$, in the direction of the electric field is very small compared with the width $2a$ normal to it. For this mode the electric field is proportional to $\cos(\frac{1}{2}\pi x/a)$ so that, if the radiation were confined between perfectly conducting planes at $x=\pm a$, the field would be two dimensional in that **E** would always be parallel to these planes. If $b$ is small compared with $\lambda$, then **E** would be given rigorously by the conformal transformation (1) and (2). If the planes are removed, the field in the opening is very slightly affected because it is still zero at $x=\pm a$. It can be shown[5] that the vector potential of the steady-state radiation from an orifice in a conducting plane is given rigorously by

$$\mathbf{A}=\frac{1}{2\pi\omega}e^{j\omega t}\int_S \frac{(\beta r-j)(\mathbf{n}\times\mathbf{E})\times\mathbf{r}_1}{r^2}e^{-j\beta r}dS, \quad (19)$$

where $S$ is the area of the plane opening, $\mathbf{E}$ the electric field tangential to it, $\mathbf{r}_1$ a unit vector along $\mathbf{r}$ measured from $dS$ to the field point, $\omega$ the angular frequency, and $\beta=\omega(\mu\epsilon)^{\frac{1}{2}}$ the wave number. We shall calculate the field at a great distance so that

$$r\approx R\left(1-\frac{xx_0}{R^2}-\frac{yy_0}{R^2}\right), \quad (20)$$

where $x_0$, $y_0$, 0 are the coordinates of $dS$, and $x$, $y$, $z$ those of the field point, and $R$ is measured from the center of the opening to the field point. The tangential $\mathbf{E}$ is in the $y$ direction and depends on $x_0$ and $y_0$ so that

$$(\mathbf{n}\times\mathbf{E})\times\mathbf{r}=[\mathbf{j}z-\mathbf{k}(y-y_0)]\times E(y_0)\cos(\tfrac{1}{2}\pi x_0/a). \quad (21)$$

Where $r$ is large it may be replaced by $R$ except in the exponent, the $j$ term may be neglected, and $y_0$ is negligible compared with $y$. Multiplication by $dx_0dy_0$ and integration over the ranges $-a<x_0<a$ and $-b<y_0<b$ give, for $\mathbf{A}e^{-j(\omega t-\beta R)}$ and after completion of the $x$ integration,

$$\frac{2\beta a(\mathbf{j}z-\mathbf{k}y)\cos(\beta ax/R)}{\omega(\pi^2R^2-4\beta^2x^2a^2)}\times\int_{-b}^{b}E(y_0)e^{j\beta yy_0/R}dy_0. \quad (22)$$

By use of the relations

$$x=R\sin\theta\cos\phi,\quad y=R\sin\theta\sin\phi,\quad z=R\cos\theta \quad (23)$$

it is evident that

$$\mathbf{j}z-\mathbf{k}y=R(\boldsymbol{\theta}_1\sin\phi+\boldsymbol{\theta}_1\cos\theta\cos\phi), \quad (24)$$

so that the distant field is normal to $\mathbf{R}$. If the assumption is made at this point that the field of the $TE_{10}$ in the opening is unperturbed so that $E(y_0)$ has the constant value $E_0$, a simple integration gives $\mathbf{E}_\theta\times\mathbf{E}_\phi^*$ to be

$$\mathbf{r}_1\frac{16\beta^2a^2E_0^2(1-\sin^2\theta\cos^2\phi)\sin^2(\pi N)}{\pi^6R^2(1-4M^2)^2\sec^2(\pi M)N^2}, \quad (25)$$

where

$$M=\pi^{-1}\beta a\sin\theta\cos\phi,\qquad N=\pi^{-1}\beta b\sin\theta\sin\phi. \quad (26)$$

If the nearly exact expression (4) is used, a more complicated integration gives

$$\mathbf{E}_\theta\times\mathbf{E}_\phi^*=\mathbf{r}_1\frac{16\beta^2a^2b^2E^2(1-\sin^2\theta\cos^2\phi)[f(N)]^2}{\pi^6R^2(1-4M^2)^2\sec^2(\pi M)}, \quad (27)$$

where

$$f(N)=\left[\frac{2C}{\pi N^2}+0.94373D\frac{\Gamma(N+0.1775)}{\Gamma(N+0.8225)}\right]\cos(\pi N)+\left[\frac{A+C}{N}-\frac{2C}{\pi^2N^3}+\frac{BN}{1-N^2}+1.51322D\frac{\Gamma(N+0.1775)}{\Gamma(N+0.8225)}\right]\sin(\pi N). \quad (28)$$

The mean energy per square meter per second is given by Poynting's vector

$$\overline{\mathbf{\Pi}}=\frac{\mathbf{E}\times\mathbf{B}^*}{2\mu}=\frac{\mathbf{E}_\theta\times\mathbf{E}_\phi^*}{\mu c}. \quad (29)$$

Table IV and Fig. 3 show the relative intensity as a function of $\theta$ in the plane normal to the field ($\phi=0$) and in the plane parallel to it ($\phi=\frac{1}{2}\pi$). For comparison with the accurate values of (27) and the approximate ones of (25) the values using Schelkunoff's[6] formula are given. This formula ignores all the boundary conditions outside the orifice and has essentially the same basis as Kirchhoff's method.[7] It is like (25) except for a more complicated factor in place of $(1-\sin^2\theta\cos^2\phi)$. It is clearly a much poorer approximation than the simpler (25). These results confirm previous findings[8] that the use

[5] W. R. Smythe, Phys. Rev. **72**, 1066 (1947).

[6] S. A. Schelkunoff, *Electromagnetic Waves* (D. Van Nostrand Company, Inc., New York, 1943), p. 359.

[7] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 462.

[8] J. A. Stratton and L. J. Chu, Phys. Rev. **56**, 106 (1939).

of unperturbed values in (19) gives useful results. In the present case the fact that (25) gives results within 6 percent of (27), in spite of the erroneous orifice field values, indicates that the results of (27), where the field values are almost exact, are probably good to the last place given.

To get the maximum intensity at $\theta=0$ it is easiest to put $x=0$ and $y=0$ in (22) and to remember from (5) that the integral is then $2bE_0$. Since the electrical intensity is $-j\omega A_y$ we have, with the aid of (29),

$$\overline{\Pi}_0=\frac{8\beta^2a^2b^2\epsilon^{\frac{1}{2}}}{\pi^4\mu^{\frac{1}{2}}R^2}E_0^2. \tag{30}$$

Schelkunoff's formula gives a value 1.816 times as great. For the numerical example of Table IV the maximum intensity is

$$\overline{\Pi}_0=7.026\times10^{-6}E_0^2R^{-2}\text{ watts/m}^2, \tag{31}$$

where $E_0$ is in volts per meter.

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# The Parastatic Moving Needle Galvanometer*

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## 1. INTRODUCTION

THE principle of resonance radiometry, devised by Pfund[1] and developed by Hardy,[2] affords an attractive method for the detection and recording of infra-red radiations. This method is attractive because of the possibility of using a single vibration galvanometer, in conjunction with a bolometer or thermopile, near to the natural limit set by Johnson noise.

Such a use is contingent on the vibration galvanometer satisfying the following conditions: 1. The vibration system of the galvanometer should be mounted on a fiber which is delicate enough to manifest the natural statistical fluctuations. 2. The mounting fiber should be free from internal mechanical friction and hysteresis. 3. The electromagnetic coupling between the mechanical vibrating system and the driving voltage, generated by a bolometer or thermopile, should be sufficient so that spurious deflections of the system produced by Johnson (or electrical) noise in the electrical circuit will be of the same order of magnitude as spurious deflections produced by the Brownian (or mechanical) noise of the galvanometer itself. 4. It must be possible to isolate the galvanometers suspended system from

(a) external spurious electrical or magnetic fields, and
(b) external spurious mechanical vibrations.

I have devised a new galvanometer which satisfied these conditions. It compares with previous types of galvanometers as follows: My galvanometer, in contrast with the D'Arsonval galvanometer, is provided with a quartz suspension which is quite free from mechanical hysteresis, and which has very high strength. It is possible thus to satisfy conditions 1 and 2 because the new galvanometer is of the moving magnetic needle type. If suspension fibers are required to conduct the measured electrical current, as is the case with the D'Arsonval galvanometer, these conditions cannot be met satisfactorily.

In contrast with the Kelvin astatic moving needle galvanometer, the needle system of the new galvanometer is not only astatic but immune to space gradients of magnetic field. My galvanometer has four magnets comprising two astatic pairs instead of two magnets comprising one astatic pair. Instead of four field coils, as with Kelvin's galvanometer, each being wound after the specifications of Maxwell[3] with several sizes

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[1] A. H. Pfund, Science 69, 11 (1929).

[2] James Daniel Hardy, Rev. Sci. Inst. 1, 429 (1930).

[3] J. C. Maxwell, *Electricity and Magnetism* (Clarendon Press, Oxfrod, 1881), Vol. 2, §§717 *et. seq.*