

Mixed-State Quantum Anomaly and Multipartite Entanglement

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Quantum entanglement measures of many-body states have been increasingly useful to characterize phases of matter. Here, we explore a surprising connection between mixed-state entanglement and 't Hooft anomaly. More specifically, we consider lattice systems in d space dimensions with anomalous symmetry G where the anomaly is characterized by an invariant in the group cohomology $H^{d+2}[G, U(1)]$. We show that any mixed state ρ that is strongly symmetric under G , in the sense that $G\rho \propto \rho$ is necessarily $(d+2)$ -nonseparable, i.e., is not the mixture of tensor products of $d+2$ states in the Hilbert space. Furthermore, such states cannot be prepared from any $(d+2)$ -separable states using finite-depth local quantum channels, so the nonseparability is long-ranged in nature. We provide proof of these results in $d \leq 1$ and plausibility arguments in $d > 1$. The anomaly-nonseparability connection, thus, allows us to generate simple examples of mixed states with nontrivial long-ranged multipartite entanglement. In particular, in $d = 1$ we find an example of *intrinsically mixed* quantum phase, in the sense that states in this phase cannot be two-way connected to any pure state through finite-depth local quantum channels. We also analyze a mixed anomaly involving both strong and weak symmetries, including systems constrained by the Lieb-Schultz-Mattis type of anomaly. We find that, while strong-weak mixed anomaly, in general, does not constrain quantum entanglement, it does constrain long-range correlations of mixed states in nontrivial ways. Namely, such states are not symmetrically invertible and not gapped Markovian, generalizing familiar properties of anomalous pure states.

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I. INTRODUCTION

A fundamental characterization of global symmetry in a local quantum system is its associated 't Hooft anomaly [1–3]. The defining feature of a nontrivial anomaly is that it forbids a symmetric and completely trivial state in the system; thus, any state preserving the anomalous symmetry must be “nontrivial.” More precisely, it is long-range entangled [4]. Constraining the behaviors of quantum systems using anomaly (also known as “anomaly matching”) has become an extremely powerful tool in the theoretical study of strongly interacting quantum systems in both condensed matter and high-energy physics.

Heuristically, the anomaly describes intrinsic obstructions to “localize” the symmetry action. In particular, an anomalous symmetry action cannot be “on site.” A common physical setup for anomalous symmetry is the boundary states of nontrivial symmetry-protected topological (SPT) phases [5,6]. They are short-range entangled (SRE) but cannot be adiabatically connected to a trivial symmetric product state without breaking the protecting symmetry. Because of their topological nature, the symmetry has to act anomalously on the spatial boundary of an SPT state, leading to a variety of interesting topological phenomena. It is, however, worth noting that anomalies are not limited to the boundary of SPT states. A large class of “mixed anomalies” between internal and spatial symmetries have been identified in lattice systems, which generalizes the celebrated Lieb-Schultz-Mattis(-Oshikawa-Hastings) theorem in spin models [7–13]. Many examples of lattice models realizing anomalous internal symmetries have also been constructed and studied [14–16].

While the implications of quantum anomaly have been extensively studied for ground states and low-energy dynamics, real-world systems, however, are best described by mixed states due to the inevitable presence of noise,

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which correlates the system of interest with inaccessible external degrees of freedom. One natural question is whether we can generalize the notion of SPT phases and 't Hooft anomaly to the mixed-state setting. The main objective of this work is to formulate a set of (information-theoretic) constraints of 't Hooft anomaly on mixed states, in terms of spatial separability and symmetric invertibility. As a by-product, we also identify examples of nontrivial mixed-state phases of matter, even in $(1+1)\text{D}$.

To discuss this problem, it is crucial to first define appropriate generalizations of the notion of global symmetry to open quantum systems. Depending on the couplings to the environment, there can be two types of symmetries [17–22]: When the couplings preserve the symmetry of the system, the symmetry is said to be *strong* (or *exact*). In other words, the couplings do not transfer symmetry charges to the environment—in the language of statistical mechanics, the strong symmetry corresponds to a canonical (instead of a grand-canonical) ensemble. If instead only the total charge of the system and environment is conserved, the symmetry becomes *weak* (or *average*). Formally, a mixed state ρ is strongly symmetric under a unitary U if $U\rho = e^{i\theta}\rho$ for some phase factor $e^{i\theta}$ and is weakly symmetric if $U\rho U^{-1} = \rho$. In both cases, ρ can be decomposed into eigenstates of U , but the strong symmetry condition further requires that all states share the same eigenvalue, i.e., the same charge.

From the definition, it is quite clear that weak symmetry has very little constraining power on the entanglement structure of states: The maximally mixed state $\mathbb{1}/\dim(\mathcal{H})$ is always weakly symmetric under any symmetry, regardless of whether the symmetry is anomalous or not. This is consistent with the classification of mixed-state SPT phases recently established in Refs. [18,19,22], where the classification is trivial for weak symmetry alone. However, as we see, the case involving strong symmetry is fundamentally different.

Here, we primarily study the universal features of mixed states ρ that are strongly symmetric under an anomalous symmetry. We focus on systems made of bosons or spins (so no physical fermions) and consider 't Hooft anomalies of the group-cohomology type [5]. We prove that, in $1d$, such states cannot be tripartite separable. In other words, for any spatial tripartition of the system into connected regions $A \cup B \cup C$ without boundary, a state strongly symmetric under an anomalous symmetry cannot be written as a convex sum of states of the form $\rho_A \otimes \rho_B \otimes \rho_C$. Furthermore, we prove that such tripartite entanglement is long-ranged: A symmetric state ρ cannot be prepared from a tripartite-separable state ρ_0 using a finite-depth local quantum channel. We further conjecture that such state in d dimensions is not $(d+2)$ -separable and establish it for a special class of symmetry transformations. We call this result the *anomaly-nonseparability connection*. It constrains the multipartite entanglement structure of

anomalous many-body states in an inherent way that may not be captured by the usual bipartite entanglement measures, such as the von Neumann entropy or negativity. Furthermore, it is nonperturbative and is valid even for highly entangled (e.g., volume-law) or highly mixed (e.g., infinite-temperature) states.

Mixed anomalies involving both strong and weak symmetries appear to be more subtle and are discussed briefly in Secs. V and VI. As we see, mixed strong-weak anomaly, in general, does not preclude separable states, unlike the strong-strong case considered in prior sections. Instead, such mixed anomaly constrains the long-range correlation in some nontrivial ways: A state with mixed strong-weak anomaly is necessarily (a) not *symmetrically invertible* (Sec. VB) and (b) not *gapped Markovian* (Sec. VC). Similar statements are, in fact, defining features of standard pure state SPT edge states. Curiously, non-Abelian continuous symmetries appear to allow for stronger results. In particular, for a spin-1/2 chain with strong $\text{SO}(3)$ and weak lattice translation symmetries (which features the celebrated Lieb-Schultz-Mattis anomaly), a symmetric state cannot be bipartite separable or be prepared from a bipartite-separable state using a finite-depth local quantum channel.

The rest of this paper is structured as follows. We start in Sec. II by reviewing the three main concepts used here: partial separability (Sec. II A), strong and weak symmetries (Sec. II B), and anomalies (Sec. II C). Before proving the anomaly-nonseparability connection for generic strongly symmetric anomalous systems, we illustrate it with two examples: In Sec. III A, we review how the long-range entanglement between the edge modes of a $(1+1)\text{D}$ SPT system result from the projectiveness of the boundary symmetry; then, in Sec. III B 1, we provide an elementary proof that, in a $1d$ qubit chain, states strongly symmetric under the anomalous symmetry of the $\mathbb{Z}_2$ CZX model are not tripartite separable. We then provide the general proof of the long-ranged anomaly-nonseparability connection in $(1+1)\text{D}$ in Secs. IV A and IV B. Using such proof, we argue in Sec. IV C that the maximally mixed CZX-symmetric state is part of an *intrinsically mixed* phase of matter. In Sec. IV D, we extend our statement to higher dimensions in the form of a conjecture. In Sec. V, we discuss a weaker result for mixed anomalies involving both strong and weak symmetries, in terms of noninvertibility (Sec. VB) and Markovian gaplessness (Sec. VC), with the main tool being the obstruction to symmetry localization due to the strong-weak anomaly (Sec. VA). In Sec. VI, we discuss a special type of mixed anomaly, namely, a spin-1/2 chain with strong $\text{SO}(3)$ [or $\text{O}(2)$] and weak translation symmetries, and show that a symmetric state in such systems is necessarily long-range bipartite entangled. We finish with a brief exposition to the relationship with other works in Sec. VII and with other discussions in Sec. VIII. Several Appendixes contain peripheral details.

II. PRELIMINARIES

A. Partial separability

Multipartite quantum systems have more complex entanglement structure than their bipartite counterparts [23]. Famously, there are two inequivalent ways three qubits can be genuinely entangled under stochastic local operations and classical communication [24]. For more than three parties, there is a continuum of such classes, and a complete characterization of their entanglement is much harder [24,25]. Here, we review the characterization of the entanglement of multipartite systems by their partial separability: whether or not they can be split into k disentangled parts. This gives a discrete gradation of separability between fully separable states and genuinely bipartite-entangled states that is the key to characterize anomalous mixed states.

First, some basic definitions. Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{Q}(\mathcal{H})$ the set of quantum states of $\mathcal{H}$, i.e., of trace-one positive semidefinite linear operators $\rho: \mathcal{H} \rightarrow \mathcal{H}$. We are mostly interested in lattice models, where the Hilbert space $\mathcal{H}$ of the whole system is the tensor product of Hilbert spaces from each “site”:

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N, \quad (1)$$

where N is the total number of sites. We say that the system is N -partite. In this work, $\mathcal{H}_n$ are finite dimensional for all n , and for simplicity we assume they all have the same dimension.

Denote by $I = \{1, 2, \dots, N\}$ the set of site indices. A collection of pairwise disjoint nonempty subsets $\{A_i\}_{i=1}^k \subset 2^I$ is a k partition of I if $\cup_{i=1}^k A_i = I$. Each subset A_i is associated with a subsystem Hilbert space $\mathcal{H}_{A_i} = \bigotimes_{l \in A_i} \mathcal{H}_l$. We are mostly interested in geometrically local partitions, namely, dividing the system into several disjoint regions of space.

We now state the definition of partial separability with respect to a given partition as found in the literature [23,26].

Definition 1 (pure state partial separability). A pure state $|\psi\rangle \in \mathcal{H}$ is *separable with respect to a partition* $\{A_i\}_{i=1}^k$ if it is a tensor product state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_k\rangle$, where $|\psi_i\rangle \in \mathcal{H}_{A_i}$.

We might not want to choose a specific partition but only the number of sets k . In that case, we simply say $|\psi\rangle$ is *k-separable*. The negation of *k-separable* is *k-nonseparable* or *k-entangled*. We call an N -separable state *fully separable*, which is sometimes simplified to just “separable,” with the negation being just *entangled*. On the opposite direction, 2-nonseparable states are sometimes called *genuinely entangled*. See Table I for examples.

The definition of *k-separable* pure states can be naturally generalized to mixed states by taking its convex hull.

Definition 2 (mixed-state partial separability). A mixed state $\rho \in \mathcal{Q}(\mathcal{H})$ is *separable with respect to a partition*

TABLE I. Examples of multipartite qubit states and whether they are k -separable. $|\Psi_{00}\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ is the Bell state, SHIFTS stands for the mixed state defined in Ref. [27] (see the main text), and the $T = \infty$ CZX states are defined in Eq. (26).

State	2-separable? (partition)	3-separable?
$ 000\rangle$	Yes (all)	Yes
$ \Psi_{00}\rangle \otimes 0\rangle$	Yes (12 3)	No
GHZ, W	No (all)	No
SHIFTS	Yes (all)	No
$T = \infty$ CZX	Yes (all)	No

$\mathcal{P} = \{A_i\}_{i=1}^k$ if it is the mixture of pure states separable under the same partition $\mathcal{P}$:

$$\rho = \sum_{\alpha} p_{\alpha} |\psi_1^{(\alpha)}\rangle \langle \psi_1^{(\alpha)}| \otimes \cdots \otimes |\psi_k^{(\alpha)}\rangle \langle \psi_k^{(\alpha)}|, \quad (2)$$

with $|\psi_i^{(\alpha)}\rangle \in \mathcal{H}_{A_i} = \bigotimes_{l \in A_i} \mathcal{H}_l$ and $\sum_{\alpha} p_{\alpha} = 1$.

Notably, deciding whether a generic mixed state ρ is separable is a computationally hard problem [28–31] as we need to find the optimal decomposition $\rho = \sum_{\alpha} p_{\alpha} |\Psi^{(\alpha)}\rangle \langle \Psi^{(\alpha)}|$ satisfying the separability condition. This layer of complexity makes studying entanglement much more challenging for mixed states compared to pure states.

One can also define separability in a broader sense [23,26], called just *k-separable*, by allowing different pure states $|\Psi^{(\alpha)}\rangle$ in a decomposition $\rho = \sum_{\alpha} p_{\alpha} |\Psi^{(\alpha)}\rangle \langle \Psi^{(\alpha)}|$ to be separable with respect to different k partitions $\{A_i^{(\alpha)}\}_{i=1}^k$. We focus on the narrower definition in Definition 2 in this work for simplicity, but we note that some of our key results such as the anomaly-nonseparability connection (Theorem 1 and Conjecture 1) apply equally well for the broader definition.

If a state is *k-separable*, then it is *l-separable* for every $l < k$ (see Fig. 1). The converse does not hold: The state $|\Psi_{00}\rangle \otimes |0\rangle$, where $|\Psi_{00}\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ is the

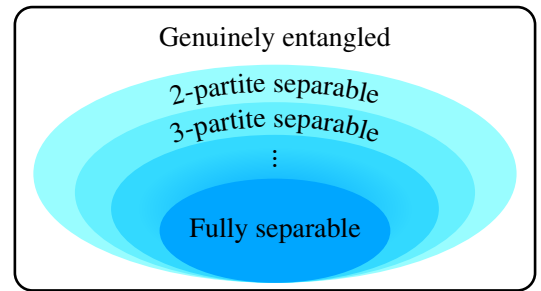


FIG. 1. Inclusion relations of multipartite states with respect to separability: Any k -separable with $k > l$ is also l -separable, and genuinely entangled states are not 2-partite separable.

Bell state, is bipartite separable under the 12|3 partition, but it is not tripartite separable. If, however, an n -partite pure state $|\psi\rangle$ is bipartite separable under *all* partitions, then it is easy to check that it is also fully separable [32]. Surprisingly, the same does not hold for mixed states. Indeed, Bennett *et al.* [27] constructed a three-qubit state that is biseparable under any bipartition, but nevertheless entangled, refuting a conjecture made in Ref. [33]. Such a phenomenon is possible because the basis used to decompose ρ into separable states for one bipartition may be very different from the basis used for another bipartition, so we cannot simply use a unified basis to separate the state for a tripartition. The existence of such bipartite-separable, but multipartite-entangled states also means that it is not sufficient to just focus on the familiar bipartite entanglement measures (such as negativity or entanglement of formation). Indeed, many multipartite entanglement measures and tests have been proposed to (partially) separate k -separable states with different k [34–45].

Even though our motivation for the current work comes from the many-body physics of anomalous symmetries, some of our results are of interest to multipartite entanglement theory *per se*. In particular, in Sec. III B 2 we present a family of many-body states that are bipartite separable but tripartite entangled, and in Appendix A we study a “good” multipartite entanglement measure of mixed states presented in Ref. [46] that faithfully distinguishes k -partite-separable from entangled states.

B. Strong and weak symmetries

To generalize symmetry anomaly to mixed systems, we first have to define what it means for a mixed state to be symmetric. Following previous works [17–22], we contrast two definitions. The first definition is of *average*, or *weak* symmetry.

Definition 3 (weakly symmetric states). A state ρ is *weakly symmetric* under an (anti)unitary operator U if $U\rho U^\dagger = \rho$.

The symmetry U can be antiunitary, but we treat only the unitary case in this paper.

The definition of weak symmetry comes naturally from the symmetry action of U on pure states, $|\psi\rangle \mapsto U|\psi\rangle$. Furthermore, there is an equivalent condition on purifications $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}_A$: ρ is weakly symmetric under U if, and only if, its purification $|\Psi\rangle$ is symmetric under $U \otimes U_A$, for some unitary U_A acting only on the ancilla space $\mathcal{H}_A$. Moreover, we can choose U_A to be (unitarily equivalent to) the complex conjugate of U . Indeed, one purification of ρ is $|\Psi\rangle = \sqrt{\rho} \otimes \mathbb{1} |\Psi_{00}\rangle$, where $|\Psi_{00}\rangle = \sum_i |ii\rangle \in \mathcal{H} \otimes \mathcal{H}$ is maximally entangled, satisfying $U \otimes U^* |\Psi\rangle = |\Psi\rangle$. One physical interpretation is that ρ may come from an interaction with an environment that is symmetric under the extended symmetry $U \otimes U_A$.

All previous statements generalize accordingly to group representations $U: G \rightarrow U(\mathcal{H})$. In such a case, another

equivalent description comes from Schur’s lemma, which implies ρ is block-diagonal $\rho = \oplus_\alpha \rho^{(\alpha)}$, where each block matrix $\rho^{(\alpha)}$ acts on the multiplicity space of the irreducible representation (irrep) α of U as a multiple of identity between each irrep copy [47].

Of importance to us is the subset of weakly symmetric states for which the ancilla symmetry action U_A can be made trivial, i.e., $U_A \propto \mathbb{1}_A$. If ρ comes from the system interacting with its environment, this condition prohibits charge exchange between the symmetry and its environment. In such case, we say ρ is *strongly symmetric*.

Definition 4 (strongly symmetric state). A state ρ is *strongly symmetric* under a unitary operator U if $U\rho = \lambda\rho$, for some phase $\lambda \in U(1)$ that we call the *charge* of ρ .

In Sec. IV A, we prove that any tripartite-separable mixed state $\sum_i p_i |A_i\rangle |B_i\rangle |C_i\rangle$ is not *strongly* symmetric under an anomalous $(1+1)$ D symmetry. The reason why we restrict this result to strongly symmetric states, instead of weakly symmetric or to another subset entirely, is twofold. First, the weak symmetry condition is too weak, since even the maximally mixed state is weakly symmetric under any symmetry: $U\mathbb{1}U^\dagger \equiv \mathbb{1}$. Second, and most importantly, the following lemma about strongly symmetric states enables us to leverage the anomaly-nonseparability proof from pure states to mixed states.

Lemma 1. All (unnormalized) vectors $|\tilde{\psi}_i\rangle$ in any decomposition of $\rho = \sum_i |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i|$ are symmetric with equal charge if, and only if, ρ is strongly symmetric with the same charge.

Proof. The “only if” statement is trivial. For the “if” part, note that the strong symmetry condition $U\rho = \lambda\rho$ implies any state in the image of ρ is also symmetric with the same charge. Since $\text{Ker}(\rho)^\perp = \text{Im}(\rho)$, we prove that each $|\tilde{\psi}_i\rangle$ is orthogonal to all vectors in the kernel of ρ . Indeed, for any $|\tilde{\phi}\rangle \in \text{Ker}(\rho)$, we have $0 = \langle \tilde{\phi} | \rho | \tilde{\phi} \rangle = \sum_i |\langle \tilde{\phi} | \tilde{\psi}_i \rangle|^2$, implying $\langle \tilde{\phi} | \tilde{\psi}_i \rangle = 0$. Hence, $|\tilde{\psi}_i\rangle \in \text{Ker}(\rho)^\perp = \text{Im}(\rho)$. ■

Therefore, if no *pure* symmetric state is tripartite separable, then by the lemma above, no *strongly* symmetric *mixed* state is either. Note that the lemma above does not assume the states $|\tilde{\psi}_i\rangle$ in a given decomposition of ρ are orthogonal.

A direct corollary of Lemma 1 ties together strong symmetry to trivial symmetry extensions, as alluded earlier.

Corollary 1. Any purification $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}_A$ of a strongly symmetric state ρ , i.e., $U\rho = \lambda\rho$, is symmetric under the trivially extended unitary $U \otimes \mathbb{1}_A$ with same charge λ .

Proof. Let $|\Psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle_A$ be the Schmidt decomposition of $|\Psi\rangle$. Since the vectors $|\tilde{\psi}_i\rangle := \sqrt{p_i} |\psi_i\rangle$ forms a decomposition of ρ , Lemma 1 implies they are symmetric under U , which, in turn, implies $U \otimes \mathbb{1}_A |\Psi\rangle = \lambda |\Psi\rangle$. ■

When $U(g)$ is part of a group representation, $\lambda(g)$ forms a one-dimensional irrep. In such a case, a strongly symmetric ρ is still a mixture of pure states with the same

charge given by λ , whereas a weakly symmetric one can be an ensemble of pure states with different charges.

These definitions are compatible with weakly and strongly symmetric channels [17,18,48–50], which we briefly review here. A weakly symmetric channel commutes with the adjoint action of the symmetry, $\rho \mapsto U\rho U^\dagger$, and a strongly symmetric channel has a collection of Kraus operators K_i , all commuting with U . Naturally, a strongly symmetric channel is also weakly symmetric. A physical motivation for the former is that the strong symmetry condition for Lindbladian evolutions implies each jump operator commutes with the symmetry, which is the algebraic form of requiring symmetric interactions between the system and its environment. As expected, a weakly symmetric state stays weakly symmetric after the application of a weakly symmetric channel, and the same is true for strong symmetry.

C. Anomaly

For a large class of anomalies—specifically, those classified by group cohomology (bosons) [5,14,51] and supercohomology (fermions) [52,53]—the anomaly can be realized in an on-site Hilbert space [i.e., given by Eq. (1)], with the symmetry acting in a non-on-site manner. In this work, we focus on bosonic systems with anomaly classified by group cohomology—for bosonic open systems in space dimension $d < 4$, all nontrivial SPT phases fall into this class [22]. We begin with $(0+1)$ D systems with a unitary G symmetry, which can be thought of as the boundary of a $(1+1)$ D SPT.

The anomaly of the boundary of a $(1+1)$ D SPT system is manifested by the projective action of the symmetry near each edge [54–56]. More precisely, the symmetry representation has the same action on ground states as another representation $U(g) = U_L(g) \otimes U_R(g)$ with support only near the left and right end points of the chain. This restriction to the edge is possible because the bulk SPT system is short-range entangled [51]. Importantly, each restriction $U_{L,R}$ is a *projective* representation:

$$U_{L,R}(g)U_{L,R}(h) = \omega_{L,R}(g, h)U_{L,R}(gh), \quad (3)$$

with $\omega_L(g, h) = \omega_R(g, h)^{-1} \in U(1)$ complex phases that cancel each other so U remains nonprojective. Thus, without loss of generality, we can focus only on the left end point. Associativity of U_L implies

$$\omega_L(g, h)\omega_L(gh, k) = \omega_L(g, hk)\omega_L(h, k), \quad (4)$$

which means $\omega_L: G \times G \rightarrow U(1)$ is a 2-cocycle. Moreover, the freedom to redefine $U_L(g) \mapsto \lambda(g)U_L(g)$ implies we should identify $\omega_L(g, h) \sim \omega_L(g, h)\lambda(gh)\lambda(g)^{-1}\lambda(h)^{-1}$. The equivalence classes formed from this identification are elements of the second cohomology group $H^2[G, U(1)]$. When an edge symmetry U corresponds to a nontrivial

cohomology class, we say it is *anomalous*. The anomaly is a property of the SPT phase itself, since it is invariant under symmetric local unitaries acting on the bulk and how the symmetry is restricted to the edge.

Let us look at a concrete SPT system with exactly solvable edge theory: the 1D cluster chain, with Hamiltonian

$$H_{ZZZ} = - \sum_{i=2}^{L-1} Z_{i-1} X_i Z_{i+1}. \quad (5)$$

This Hamiltonian has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by $X_{\text{even}} := \prod_{k=1}^{\lfloor L/2 \rfloor} X_{2k}$ and $X_{\text{odd}} := \prod_{k=1}^{\lfloor L/2 \rfloor} X_{2k-1}$ (see Fig. 2). H_{ZZZ} can be exactly solved by transforming it under the local unitary transformation $U_{CZ} := \prod_{i=1}^{L-1} CZ_{i,i+1}$, as $U_{CZ}^\dagger H_{ZZZ} U_{CZ} = - \sum_{i=2}^{L-2} X_i$. Importantly, the ground state subspace of H_{ZZZ} is fourfold degenerate, generated by states of the form $U_{CZ}|\phi_L\rangle \otimes |+\dots+\rangle \otimes |\phi_R\rangle$, where $|\phi_L\rangle$ and $|\phi_R\rangle$ are arbitrary one-qubit states at the left and right end points of the chain, respectively, and $|+\rangle$ is the $+1$ eigenstate of X . Since this mapping from the ZZZ Hamiltonian to a trivial quantum paramagnet is a local unitary, then, by definition, the ground states of H_{ZZZ} are short-range entangled in the bulk. However, note that each gate of U_{CZ} is not symmetric under X_{even} and X_{odd} . In fact, no local unitary with symmetric gates could trivialize H_{ZZZ} , as it is topologically protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry above.

Restricting to the ground state subspace, $X_{\text{odd}} \stackrel{\text{GS}}{=} X_1 \otimes X_L$ and $X_{\text{even}} \stackrel{\text{GS}}{=} Z_1 X_2 \otimes X_{L-1} Z_L$ for odd system size L , and for even L , $X_{\text{odd}} \stackrel{\text{GS}}{=} X_1 \otimes X_{L-1} Z_L$ and $X_{\text{even}} \stackrel{\text{GS}}{=} Z_1 X_2 \otimes X_L$, where we use $\otimes$ to indicate the separation between the two end points of the chain. In both cases, we have $U(g) = U_L(g) \otimes U_R(g)$, $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$, with U_L and U_R projective representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$, as their generators anticommute. These representations are nothing but the logical X and Z operators of the effective qubits at each end of the chain.

For $(2+1)$ D SPTs, the boundary anomaly is manifested by the non-on-site character of the symmetry action [6,51]. Generically, the boundary symmetry operator $U(g)$ is a finite-depth local unitary (FDLU), up to exponential tails. Moreover, there is no equivalent on-site representation.

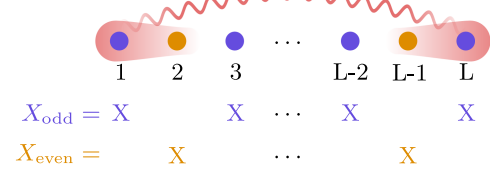


FIG. 2. Cluster chain for odd number of sites L . The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is realized by X_{odd} and X_{even} . When the low-energy system is anomalous and symmetric, the degenerate edge states (in red) are entangled.

One way to see this is by the reduction procedure of Else and Nayak [51], which we summarize below. Since we revisit the same arguments in Sec. IV A, we leave the details to that section.

Given a $(2+1)$ D SPT, we can still restrict the symmetry action to the boundary when the bulk is in the (SRE) ground state. Let $U(g)$ be this boundary symmetry representation, which acts as a finite-depth local unitary. Following Else and Nayak [51], $U(g)$ can be restricted to a unitary $U_M(g)$ with support on a region M of the boundary so that U_M acts as U in the bulk of M . Thus, U_M should still be a linear representation, up to boundary operators $\Omega = \Omega_L \otimes \Omega_R$:

$$U_M(g)U_M(h) = \Omega(g, h)U_M(gh). \quad (6)$$

Similarly to the 0D anomaly, associativity of $U_M(g)U_M(h)U_M(k)$ imposes a consistency relation on Ω , which is also satisfied by Ω_L and Ω_R individually up to a phase $\omega(g, h, k) \in U(1)$:

$$\begin{aligned} \omega(g, h, k)\Omega_L(g, h)\Omega_L(gh, k) \\ = U_M(g)\Omega_L(h, k)U_M(g)^{-1}\Omega_L(g, hk). \end{aligned} \quad (7)$$

One can prove [51] that such ω is a closed 3-cocycle:

$$\omega(g, h, k)\omega^{-1}(gh, k, l)\omega(g, hk, l)\omega^{-1}(g, h, kl)\omega(h, k, l) = 1, \quad (8)$$

or, more compactly, $\delta\omega = 1$. Furthermore, the phase indetermination of $\Omega_{L,R}$ implies we have to identify ω 's that differ by exact 3-cocycles:

$$\omega(g, h, k) \sim \omega(g, h, k)\mu(g, h)\mu^{-1}(gh, k)\mu(h, k)\mu^{-1}(g, hk), \quad (9)$$

or, more compactly, $\omega \sim \omega \cdot \delta\mu$. Thus, to each $(2+1)$ D SPT phase, there is an element $[\omega]$ in the third cohomology group $H^3[G, U(1)]$ that classifies it.

As an example, for $G = \mathbb{Z}_2 = \{1, g\}$, we can canonically choose $\Omega(1, g) = \Omega(g, 1) = 1$, and then the only nontrivial 3-cocycle $\omega(g, g, g)$ is defined by

$$\omega(g, g, g)\Omega_L(g, g) = U_M(g)\Omega_L(g, g)U_M(g)^{-1}. \quad (10)$$

In words, $\omega(g, g, g)$ measures whether $\Omega_L(g, g)$ is charged under g or not.

For higher-dimensional SPT systems, similar reduction procedures can be performed to reduce the problem of classifying d -dimension anomaly to $d = 0$, where it goes back to a projective phase, but now in the $H^{d+2}[G, U(1)]$ cohomology group. We leave the details of the higher-dimensional construction to Sec. IV D, where it is applied to the anomaly-nonseparability connection.

III. PROTOTYPICAL EXAMPLES

Before discussing the anomaly-nonseparability connection for generic anomalous systems, we illustrate it with some concrete examples.

A. $(0+1)$ D: Projective representation

As we saw in Sec. II C, one-dimensional SPT systems have anomalous edge states characterized by the projective nature of the symmetry action. Furthermore, we argue now that the symmetry operator U_L at the left end point cannot have an invariant state $|\psi\rangle \in \mathcal{H}_L$, and equally for U_R . Indeed, if $U_L(g)|\psi\rangle = \lambda(g)|\psi\rangle$ for some $\lambda: G \rightarrow U(1)$, then the projective factor $\omega(g, h) \in U(1)$ in

$$U_L(g)U_L(h) = \omega(g, h)U_L(gh) \quad (11)$$

would equal to $\omega(g, h) = \lambda(g)\lambda(h)/\lambda(gh)$, by acting on Eq. (11) with $|\psi\rangle$. Such ω can be trivialized by redefining $U_L(g) \mapsto \lambda(g)^{-1}U_L(g)$ and $U_R(g) \mapsto \lambda(g)U_R(g)$, contradicting the assumption that they are (nontrivial) projective representations. This implies that any symmetric pure state $|\Psi\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R$ must be entangled, namely, $|\Psi\rangle \neq |\psi_L\rangle \otimes |\psi_R\rangle$. By Lemma 1, we immediately conclude that any strongly symmetric mixed state $\rho \in \mathcal{Q}(\mathcal{H}_L \otimes \mathcal{H}_R)$ must be bipartite nonseparable. This is the simplest case of the anomaly-nonseparability connection.

Coming back to the cluster chain example, it is easy to see that there cannot be a symmetric pure state at one of the edges, since it would be a simultaneous eigenstate of both logical X and Z . Therefore, the only symmetric ground states are the ones where the end points are entangled (see Fig. 2). For the cluster chain, these correspond to the states $U_{CZ}|\Psi_{ab}\rangle_{1L} \otimes_{i=2}^{L-1} |+\rangle_i$, $a, b \in \{0, 1\}$, where $|\Psi_{ab}\rangle = (1/\sqrt{2})(1 \otimes Z^a X^b)(|00\rangle + |11\rangle)$ is a Bell pair state. Equivalently, the only *mixed* state of *one qubit* that is *weakly* symmetric under the anomalous symmetry is the maximally mixed state $\mathbb{1}/2$, which purifies to a Bell state. Since there is only one Bell state in each $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry sector, then this trivially says that any *strongly* symmetric mixed state at the $(0+1)$ D edge of the $(1+1)$ D cluster chain SPT is also bipartite entangled.

Similar statements can be established for more general G . For simplicity, let us assume that the edge Hilbert space transforms as an irreducible projective representation. In this case, by Schur's lemma, the only state ρ weakly symmetric under G [i.e., $\rho U(g) = U(g)\rho$ for all $g \in G$] is the maximally mixed state $\rho \propto \mathbb{1}$, which can be purified into a maximally entangled state between the two edges.

B. $(1+1)$ D: Four-qubit CZX

Moving on to one spatial dimension, we now consider a system with anomalous $\mathbb{Z}_2$ symmetry as our prototypical example. It can appear on the boundary of a $\mathbb{Z}_2$ -protected bosonic topological phase [14, 15]. Concretely, the system

is a simple 1D chain of L qubits with $\mathbb{Z}_2$ symmetry realized by the CZX unitary [14]:

$$U_{\text{CZX}} = \prod_{i=1}^L X_i \prod_{i=1}^L \text{CZ}_{i,i+1}, \quad (12)$$

where $\text{CZ}_{ij} = 1 - 2|1_i 1_j\rangle\langle 1_i 1_j|$ is the controlled-Z gate between qubits i and j with $\{|0\rangle, |1\rangle\}$ the Pauli-Z basis. Since $U_{\text{CZX}}^2 = (-1)^L$, we require an even number of sites. We show that any density matrix ρ that is strongly symmetric under this $\mathbb{Z}_2$ symmetry ($U_{\text{CZX}}\rho = \lambda\rho$) cannot be tripartite separable. The essence of our arguments can already be seen through the four-qubit system— $L = 4$ qubits is the smallest system size for which U_{CZX} is nontrivially distinct from ordinary on-site $\mathbb{Z}_2$. For higher even L , the same arguments and conclusions follow, with suitable modifications [57]. We then show, with explicit examples, that a similar statement on bipartite nonseparability does not hold.

1. Tripartite entanglement

We now prove that there is no mixed state ρ that is strongly symmetric under U_{CZX} and is tripartite separable, that is, there is no ρ satisfying $U_{\text{CZX}}\rho = \lambda\rho$, $\lambda \in U(1)$, and also

$$\rho = \sum_i p_i |A_i\rangle\langle A_i| \otimes |B_i\rangle\langle B_i| \otimes |C_i\rangle\langle C_i|, \quad (13)$$

where we partition the four-qubit chain into three regions, A , B , and C , with $|A_i\rangle \in \mathcal{H}_A$, etc. Without loss of generality, we may assume A contains two adjacent spins, whereas B and C contain just one. See Fig. 3.

First, we use Lemma 1 to reduce the problem from mixed to pure states by noting that $U_{\text{CZX}}\rho = \lambda\rho$ implies $U_{\text{CZX}}|\psi\rangle = \lambda|\psi\rangle$ for all $|\psi\rangle$ in the range of ρ . For the tripartite-separable state of Eq. (13), this means $U_{\text{CZX}}|A_i\rangle|B_i\rangle|C_i\rangle = \lambda|A_i\rangle|B_i\rangle|C_i\rangle$, even if such states are not orthogonal to each other. Let us now focus on one such pure state $|A\rangle|B\rangle|C\rangle$ in the decomposition of ρ and assume by contradiction that it is symmetric under U_{CZX} .

Let α_{ij} , $i, j \in \{0, 1\}$, be a nonzero coefficient in the Z -basis expansion of $|A\rangle = \sum_{i,j \in \{0,1\}} \alpha_{ij} |ij\rangle$. By the CZX

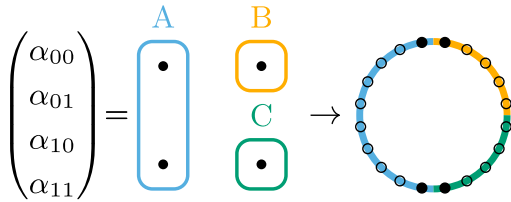


FIG. 3. Four-qubit tripartite-separable state of the CZX model. Their behavior is representative of spins near the boundaries between regions AB and AC of a much larger spin chain.

symmetry, $|\alpha_{\bar{i}\bar{j}}| = |\alpha_{ij}| \neq 0$ (we denote $\bar{k} = 1 - k$). We then have

$${}_A\langle \bar{i}\bar{j} | U_{\text{CZX}} | A \rangle | B \rangle | C \rangle = \alpha_{\bar{i}\bar{j}} \lambda | B \rangle | C \rangle. \quad (14)$$

This means that we can restrict the symmetry U_{CZX} to the BC region via

$$\begin{aligned} U_{BC} &:= {}_A\langle \bar{i}\bar{j} | U_{\text{CZX}} | A \rangle / \alpha_{\bar{i}\bar{j}} \\ &= \frac{\alpha_{ij}}{\alpha_{\bar{i}\bar{j}}} (-1)^{ij} Z_B^i Z_C^j \cdot U_{BC}^{\text{strict}}, \end{aligned} \quad (15)$$

where $U_{BC}^{\text{strict}} = X_B X_C \text{CZ}_{BC}$ is the restriction of symmetry by taking only the operators that act strictly inside region BC . Crucially, the restricted operator U_{BC} is an unitary symmetry for the remaining state in BC :

$$U_{BC} | B \rangle | C \rangle = \lambda | B \rangle | C \rangle. \quad (16)$$

Nevertheless, instead of being a $\mathbb{Z}_2$ -valued operator, U_{BC} satisfies the algebra

$$U_{BC}^2 \propto (U_{BC}^{\text{strict}})^2 \propto Z_B Z_C, \quad (17)$$

where $\propto$ means equal up to some phase factor. This is a simple manifestation of the Else-Nayak mechanism [51]. Since $|B\rangle|C\rangle$ must also be an eigenstate of U_{BC}^2 , it follows that $Z_B |B\rangle = \mu_B |B\rangle$ and $Z_C |C\rangle = \mu_C |C\rangle$ for some $\mu_B, \mu_C = \pm 1$. This immediately contradicts the fact that the operators Z_B and Z_C are charged under the $\mathbb{Z}_2$ symmetry, i.e., $U_{\text{CZX}} Z_{B(C)} = -Z_{B(C)} U_{\text{CZX}}$, as this would imply $\lambda \mu_{B(C)} = -\mu_{B(C)} \lambda = 0$. Thus, our initial assumption that there is a strongly symmetric tripartite-separable state cannot hold.

The above argument can be extended, with minor modifications, to arbitrary even system length L . For a larger segment A , we can write

$$|A\rangle = \sum_{ij} \alpha_{ij} |ij\rangle \otimes |\psi_{ij}\rangle, \quad (18)$$

where $|ij\rangle$ labels the two spins, in the Z basis, at the two boundaries of A (one adjacent to B and the other to C), and $|\psi_{ij}\rangle$ labels all the other spins in A . We can then construct the restricted symmetry action on BC similar to Eq. (15):

$$\begin{aligned} U_{BC} &:= {}_A\langle \psi_{\bar{i}\bar{j}} | \langle \bar{i}\bar{j} | U_{\text{CZX}} | A \rangle / \alpha_{\bar{i}\bar{j}} \\ &\propto Z_b^i Z_c^j \cdot U_{BC}^{\text{strict}}, \end{aligned} \quad (19)$$

where U_{BC}^{strict} is again the restriction of symmetry by taking only the operators that act strictly inside region BC and Z_b (Z_c) is the Pauli-Z operator for the spin at the boundary of B (C) adjacent to A . Crucially, we still have

$$U_{BC}^2 \propto Z_b Z_c, \quad (20)$$

which leads to the conclusion that a symmetric tripartite-separable state does not exist.

2. Bipartite separability

Although there are no tripartite-separable states strongly symmetric under the CZX anomalous symmetry, there is a complete basis of bipartite-separable symmetric pure states on any two connected regions A and $B = \bar{A}$, the complement of A . These are formed from the tensor product of cat states in each of the regions: $|\alpha^{(s)}\rangle \otimes |\beta^{(t)}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, where $|\xi^{(r)}\rangle \equiv (1/\sqrt{2})(|\xi\rangle + r|\bar{\xi}\rangle)$, with ξ a bit string and $r \in U(1)$. Without doing additional calculations, we can already guess that such cat states might form symmetric bipartite-separable states from the proof of tripartite entanglement presented in the last section. This is because one can still follow a similar procedure outline above to find a restricted symmetry action U_A of the state in A satisfying $U_A^2 \propto Z_1 Z_{|A|}$. This implies a long-range connected correlation function $|\langle Z_1 Z_{|A|} \rangle - \langle Z_1 \rangle \langle Z_{|A|} \rangle| > 0$, which is satisfied by the prototypical cat states $|\alpha^{(s)}\rangle$. More explicitly, we find the constraints over α, β, t, s , and λ such that

$$U|\alpha^{(s)}\rangle|\beta^{(t)}\rangle = \lambda|\alpha^{(s)}\rangle|\beta^{(t)}\rangle. \quad (21)$$

After some algebra, we find that the symmetry condition can be satisfied if and only if

$$s^2 = t^2 = (-1)^{N_A+1+\alpha_1+\alpha_{|A|}+\beta_1+\beta_{|B|}}, \quad (22)$$

where N_A is the number of sites of region A and $\alpha_1, \alpha_{|A|}$ ($\beta_1, \beta_{|B|}$) are the values of α (β) at the two boundaries of A (B), where α_1 is adjacent to β_1 and $\alpha_{|A|}$ to $\beta_{|B|}$. Equation (22) dictates whether s and t take values in ± 1 or $\pm i$, depending on the bit strings α and β . The corresponding symmetry eigenvalue is

$$\lambda = st(-1)^{\sum_{i=1}^{|A|-1} \alpha_i \alpha_{i+1} + \sum_{j=1}^{|B|-1} \beta_j \beta_{j+1} + \alpha_1 \beta_1 + \alpha_{|A|} \beta_{|B|}}. \quad (23)$$

Since any pair of bit strings α, β gives a total of four orthogonal symmetry eigenstates by varying the signs of t and s , the collection of all such bipartite-separable states forms a complete basis.

We expect this procedure of finding a symmetric bipartite-separable basis to apply similarly for $\mathbb{Z}_n$ and any finite Abelian symmetry group.

3. Strongly symmetric infinite-temperature state

A natural way to generate strongly symmetric states is to consider a thermal state of a symmetric Hamiltonian H at temperature $T = \beta^{-1}$ in the “canonical ensemble,” i.e., that

includes only states transforming as a particular 1D irrep of the symmetry group:

$$\rho_{T,\lambda} \propto \sum_{|E\rangle \in V_\lambda} e^{-\beta E} |E\rangle \langle E| = P_\lambda e^{-\beta H}. \quad (24)$$

Here, $\lambda: G \rightarrow U(1)$ denotes the 1D irrep, and $V_\lambda \subset \mathcal{H}$ its subspace:

$$V_\lambda = \{|\psi\rangle | U(g)|\psi\rangle = \lambda(g)|\psi\rangle, \forall g \in G\}. \quad (25)$$

and P_λ is the orthogonal projector to V_λ . In particular, the infinite-temperature limit $\lim_{T \rightarrow \infty} \rho_{T,\lambda} = (1/\dim V_\lambda) P_\lambda$ depends only on the symmetry and not on the particular Hamiltonian H .

In the case of the $\mathbb{Z}_2$ CZX symmetry, we have two infinite-temperature states

$$\rho_{\infty,\pm} = \frac{1}{2L} (\mathbb{1} \pm U_{\text{CZX}}), \quad (26)$$

corresponding to the two values the symmetry operator U_{CZX} can take. $\rho_{\infty,\pm}$ is the maximally mixed state within the charge sector $V_\pm \subset \mathcal{H}$. From the previous discussion in Sec. III B 2, we know that for each charge sector $V_\pm$ and any bipartition $\{A, B = \bar{A}\}$, we have a complete basis of bipartite-separable states $\{|\alpha^{(s)}\rangle \otimes |\beta^{(t)}\rangle\}$. This immediately implies that

$$\rho_{\infty,\pm} \propto \sum_{\substack{\alpha^{(s)}, \beta^{(t)} \\ \lambda = \pm}} |\alpha^{(s)}\rangle \langle \alpha^{(s)}|_A \otimes |\beta^{(t)}\rangle \langle \beta^{(t)}|_B, \quad (27)$$

where the summation is constrained only by the eigenvalue condition Eq. (23). Since $\{A, B\}$ is arbitrary, we conclude that $\rho_{\infty,\pm}$ is bipartite separable for all bipartitions. Yet, surprisingly, from our discussion in Sec. III B 1, we know that $\rho_{\infty,\pm}$ is not separable for any tripartition.

To numerically verify that $\rho_{\infty,\pm}$ is tripartite entangled, we employ the multipartite permutation criterion of Ref. [36]. In summary, the criterion says that permuting the indices $(i_1, \dots, i_n, j_1, \dots, j_n)$ —including input i_α and output j_α indices—of an n -partite-separable density matrix $\rho = [\rho_{i_1, \dots, i_n}^{j_1, \dots, j_n}]$ can decrease only its trace norm from $\|\rho\|_1 := \text{Tr}|\rho| = 1$. Conversely, if some index permutation increases the trace norm to a value higher than $\|\rho\|_1 = 1$, then it is nonseparable. For the four-qubit $\rho_{\infty,\pm}$, this happens if one permutes i_B and j_C in $[\rho_{i_A, i_B, i_C}^{j_A, j_B, j_C}]$, where i_R and j_R are the input and output (multi-)indices of region $R \in \{A, B, C\}$, respectively (see Fig. 3). Namely, we prove in Appendix B that the resulting matrix after the permutation $\tilde{\rho} = [\rho_{i_A, j_C, i_C}^{j_A, j_B, i_B}]$ has trace norm $\|\tilde{\rho}\|_1 = \frac{1}{2} + (1/\sqrt{2}) > 1$.

From the results shown in Sec. IV B, we can conclude that not only $\rho_{\infty,\pm}$ is tripartite entangled, but that it cannot

be connected to a tripartite-separable state via a finite-depth local channel. This means that the tripartite entanglement is long-range in nature. Despite this, we show in Appendix C that it can be prepared via a finite-depth local *adaptive* circuit, which includes not only local unitaries and local measurements, but also (nonlocal) classical feedback [58–60]. This adaptive procedure adds insight into the state preparation complexity of $\rho_{\infty,\pm}$. Combined with the fact that many interesting properties of $\rho_{\infty,\pm}$ are already present for a system with just four qubits, the local adaptive procedure shows a practical way to realize such state in a quantum computer via midcircuit measurements and feedback, as has recently been done with long-range entangled states [61,62].

Another interesting property of the states $\rho_{\infty,\pm}$ is that they have the maximally disordered subsystems property [63]. This means that taking the partial trace of even one qubit at any site i completely trivializes the system: $\text{Tr}_i \rho_{\infty,\pm} = \mathbb{1}/2^{L-1}$. In other words, $\rho_{\infty,\pm}$ has only *global* correlations, and as such the state cannot be distinguished from the trivial maximally mixed state $\mathbb{1}/2^L$ by measuring any k -point (k finite) correlation function $\text{Tr}[O(x_1)O(x_2)\dots O(x_k)\rho]$. The maximally disordered subsystems property, however, is not a generic feature of anomalous states, and it is not stable against local operations.

We have seen that strongly symmetric $T = \infty$ anomalous states are great examples of almost featureless states that, at the same time, cannot be trivial due to the anomaly-nonseparability constraint. Next, we generalize the anomaly-nonseparability connection to other symmetries and dimensions.

IV. MULTIPARTITE NONSEPARABILITY: GENERAL RESULTS

In this section, we generalize the discussions in Sec. III. We consider bosonic lattice systems in d spatial dimensions and a global symmetry G implemented by finite-depth local unitaries $U(g \in G)$ with maximum circuit depth $D \sim O(1)$. We further assume that the symmetry implementation has a quantum anomaly characterized by an element in the group cohomology $[\omega] \in H^{d+2}[G, U(1)]$. In one dimension, our central result is a general connection between anomaly and tripartite nonseparability.

Theorem 1. For $d = 1$, if the quantum anomaly $[\omega] \in H^3[G, U(1)]$ is nontrivial, and ρ is strongly symmetric under G in the sense that $\forall g \in G, U(g)\rho \propto \rho$, then

- (1) ρ is tripartite nonseparable, namely,

$$\rho \neq \sum_i p_i |A_i\rangle\langle A_i| \otimes |B_i\rangle\langle B_i| \otimes |C_i\rangle\langle C_i|, \quad (28)$$

as long as $|A|, |B|, |C| \gg D$, the maximum depth of $U(g)$, and A, B, C are all connected;

- (2) the tripartite entanglement of ρ is long-ranged, in the sense that ρ cannot be prepared from a tripartite-separable state (call it ρ_{3s}) via a finite-depth local channel, namely,

$$\rho \neq \mathcal{E}_{\text{FD}}[\rho_{3s}]. \quad (29)$$

We prove the first statement in Sec. IV A and the second statement in Sec. IV B.

A natural generalization of Theorem 1 (and the trivial $d = 0$ example from Sec. III A) is the following conjecture.

Conjecture 1. In d space dimensions, if the quantum anomaly $[\omega] \in H^{d+2}[G, U(1)]$ is nontrivial, and ρ is strongly symmetric under G , then

- (1) ρ is $(d + 2)$ -partite nonseparable for some $(d + 2)$ -partitions;
- (2) the $(d + 2)$ -partite entanglement of ρ is long-ranged:

$$\rho \neq \mathcal{E}_{\text{FD}}[\rho_{(d+2)s}], \quad (30)$$

where $\rho_{(d+2)s}$ is a $(d + 2)$ -partite-separable state.

Although a general proof is unknown, in Sec. IV D, we motivate the above conjecture by proving a weaker version, when the symmetry action U_g is assumed to take some specific form.

A. Tripartite entanglement for $(1 + 1)\text{D}$ systems

Let $U(g)$ be a finite-depth local unitary action of a symmetry group G on a $(1 + 1)\text{D}$ system with the anomaly represented by a cohomology class $[\omega] \in H^3[G, U(1)]$. We now show that if there is a strongly symmetric state ρ that is tripartite separable, then there is no anomaly, i.e., $[\omega] = [1]$. As discussed in Secs. II B and III, since any pure state in the ensemble ρ is also symmetric under G , it suffices to consider just a pure state.

We partition the system (a ring) into three disjoint intervals [see Fig. 4(a)]. In order to reduce the symmetry and extract the anomalous phase ω out of one boundary operator, we impose two requirements on the tripartition. First, each region must be larger than the maximum depth of the quantum circuits defining the symmetry operators $U(g)$. This guarantees we can reduce the symmetry into the subregions. Second, there must exist two regions whose shared boundary is exactly one point. Otherwise, we are not able to extract the anomalous phase, as is clear from the proof below [see Fig. 4(b) for an example] [64]. In $(1 + 1)\text{D}$, one natural choice for which the second condition is valid is when A, B , and C are connected, so we are using this one in what follows.

Under the preceding conditions, suppose there is a tripartite-separable pure state $|\psi\rangle = |A\rangle|B\rangle|C\rangle$ that is symmetric under U :

$$U(g)|A\rangle|B\rangle|C\rangle = \lambda(g)|A\rangle|B\rangle|C\rangle. \quad (31)$$

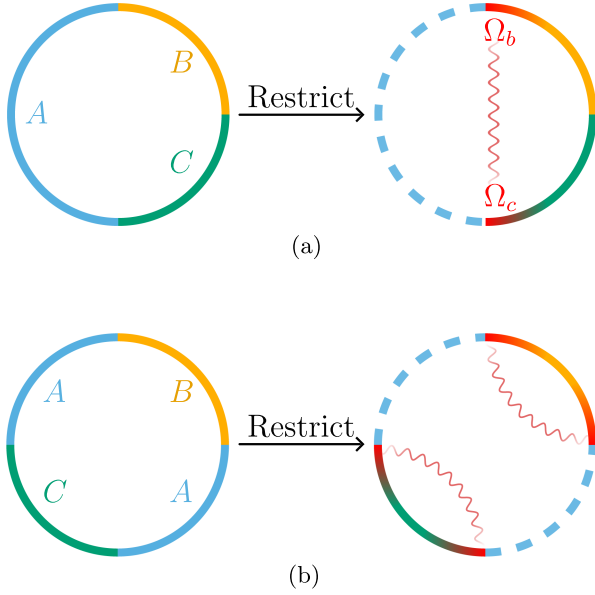


FIG. 4. The anomaly-nonseparability connection on a tripartite ring depends crucially if there is an intersection point between a unique set of regions. (a) If A , B , and C are connected, we can restrict the symmetry to the complement of A and conclude B and C are entangled if the symmetry is anomalous. (b) If A is disconnected, B and C are not necessarily entangled. In this case, two boundary operators appear for each interval in the complement of A . Since the anomaly comes from the phase freedom of only one of them, B and C might not be entangled.

We prove that this leads to a contradiction if $[\omega]$ is nontrivial.

The anomaly of the symmetry is manifest in its reductions to subregions with boundaries. Thus, we now construct one such reduction $U_{BC}(g)$ to region BC that acts in the same way as $U(g)$ away from the boundaries of BC and, importantly, has $|B\rangle|C\rangle$ as an eigenvector.

For now, we omit the dependence of the symmetry on group elements g for simplicity of notation. First, we group the gates of U near the intersection of regions A and B into a separate local unitary u_{AB} , so that $u_{AB}^\dagger U$ has no gate acting in A and B at the same time. We do the same for A and C with u_{AC} (see Fig. 5 for an explicit decomposition). In this way, we can separate U into $U = (u_{AB} \otimes u_{AC})(\tilde{U}_A \otimes \tilde{U}_{BC})$, with

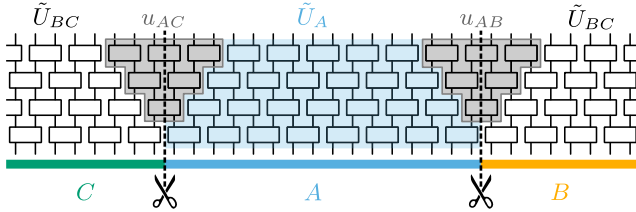


FIG. 5. Decomposition of $U = (u_{AB} \otimes u_{AC})(\tilde{U}_A \otimes \tilde{U}_{BC})$ into unitaries u_{AB} and u_{AC} that straddle the boundaries and unitaries $\tilde{U}_A$ and $\tilde{U}_{BC}$, entirely supported in A and BC , respectively. Lemma 2 enables us to “cut” U into separate pieces acting independently on each region.

$\tilde{U}_A$ ($\tilde{U}_{BC}$) supported on A (BC) and acting as U in the bulk of its support.

For the moment, let us focus on the region AB . Although u_{AB} can entangle states in AB , Eq. (31) suggests that it does not entangle the state $\tilde{U}_A|A\rangle \otimes \tilde{U}_B|B\rangle$, since the final state is still separable with respect to AB after applying all three unitaries. We show in Appendix E this is indeed the case [65], i.e., u_{AB} does not entangle $\tilde{U}_A|A\rangle \otimes \tilde{U}_B|B\rangle$, if the third region C is also included in the argument. Then we can replace u_{AB} by the tensor product of two unitaries, $v_A \otimes v_B$, each one supported either in A or in B . This is guaranteed by the following simple lemma.

Lemma 2. Given pure states $|\psi_{12}\rangle, |\phi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $|\psi_{34}\rangle, |\phi_{34}\rangle \in \mathcal{H}_3 \otimes \mathcal{H}_4$ and an unitary $U_{23} \in U(\mathcal{H}_2 \otimes \mathcal{H}_3)$ satisfying $U_{23}|\psi_{12}\rangle|\psi_{34}\rangle = |\phi_{12}\rangle|\phi_{34}\rangle$, then there exist unitaries $V_2 \in U(\mathcal{H}_2)$ and $V_3 \in U(\mathcal{H}_3)$ satisfying $|\phi_{12}\rangle = V_2|\psi_{12}\rangle$ and $|\phi_{34}\rangle = V_3|\psi_{34}\rangle$. In diagrammatic form,

$$(32)$$

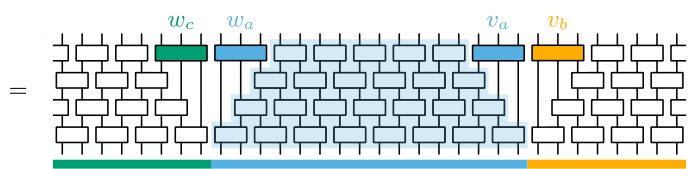
Proof. Tracing out subsystems 2, 3, and 4 gives two pure states reducing to the same density matrix in $\mathcal{H}_1$: $\text{Tr}_2[|\psi_{12}\rangle\langle\psi_{12}|] = \text{Tr}_2[|\phi_{12}\rangle\langle\phi_{12}|]$. Hence, by the unitary equivalence of purifications [66], there exists a unitary V_2 such that $|\phi_{12}\rangle = V_2|\psi_{12}\rangle$. V_3 can be analogously defined by tracing out subsystems 1, 2, and 3. ■

Applying the same reasoning to region AC , we arrive at the similar conclusion that u_{AC} can be replaced by $w_a \otimes w_c$. Finally, we can define $U_{BC} := (v_b \otimes w_c)\tilde{U}_{BC}$ satisfying

$$U_{BC}(g)|B\rangle|C\rangle = \lambda(g)|B\rangle|C\rangle, \quad (33)$$

after tracing out region A from Eq. (31) (see Fig. 5). For the simple CZX example in Sec. III B 1, this restricted unitary U_{BC} can be explicitly constructed via Eq. (19).

The reduced operator $U_{BC}(g)$ is still a finite-depth local unitary, since the supports of $v_b(g)$ and $w_c(g)$ have size of the same order as the depth $D = O(1)$ of the original $U(g)$.



For the same reason, we know $U_{BC}(g)$ acts in the same way as $U(g)$ in the interior of the BC region, $O(D)$ sites away from its boundary points. In particular, this means that U_{BC} is an unitary representation of G up to boundary operators $\Omega(g_1, g_2) = \Omega_b(g_1, g_2) \otimes \Omega_c(g_1, g_2)$, since $U(g)$ is a representation: $U(g_1)U(g_2) = U(g_1g_2)$. More precisely, we have

$$U_{BC}(g_1)U_{BC}(g_2) = \Omega(g_1, g_2)U_{BC}(g_1g_2). \quad (34)$$

Applying this equation to $|B\rangle|C\rangle$ and using Eq. (33), we have

$$\begin{aligned} \Omega_b(g_1, g_2)|B\rangle &= \mu_b(g_1, g_2)|B\rangle, \\ \Omega_c(g_1, g_2)|C\rangle &= \mu_c(g_1, g_2)|C\rangle \end{aligned} \quad (35)$$

with $\mu_b \cdot \mu_c(g_1, g_2) = \delta\lambda(g_1, g_2) \equiv \lambda(g_1)\lambda(g_2)/\lambda(g_1g_2)$, where δ is the coboundary operator of the second group cohomology $H^2[G, U(1)]$.

Equation (35) means that we can assign definite phases to Ω_b and Ω_c , at least when acting on states $|B\rangle$ and $|C\rangle$, respectively. Intuitively, the 3-cocycle ω measures the obstruction to remove phase factors from Ω_b and Ω_c for all states they can act on. Hence, we expect the SPT state is trivial, with $[\omega] = 1$. Indeed, associativity for the boundary operators $\Omega_c(g_1, g_2)$ implies

$$\begin{aligned} \omega(g_1, g_2, g_3)\Omega_c(g_1, g_2)\Omega_c(g_1g_2, g_3) \\ = U_{BC}(g_1)\Omega_c(g_2, g_3)U_{BC}(g_1)^{-1}\Omega_c(g_1, g_2g_3). \end{aligned} \quad (36)$$

Applying this equation to $|B\rangle|C\rangle$ gives $\omega = \delta\mu_c$, a coboundary. Thus, the group cohomology class $[\omega] \in H^3(G, U(1))$ is trivial.

B. Long-range entanglement

In the previous section, we prove that no tripartite-separable state $|A\rangle|B\rangle|C\rangle$ can be symmetric under an anomalous symmetry in $(1+1)D$, no matter how entangled each of the three states are in their internal degrees of freedom. Although this space of tripartite-separable states is very large, it is fragile to any local perturbation on the boundary between any two of the three regions. This stands in contrast with the well-known result that pure symmetric states under an anomalous symmetry are not short-range entangled [14, 51, 67]. In this section, we unify these two results by proving that not only is any tripartite-separable mixed state not symmetric under an anomalous symmetry, but also any other state that can be prepared from it by a finite-depth local channel (FDLC).

More specifically, by finite-depth local channel we mean a completely positive trace-preserving map $\mathcal{E}_{\text{FDLC}}$ that is the restriction of a finite-depth local unitary acting over an enlarged Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A$, where $\mathcal{H}_S$ is the original, “physical” Hilbert space and $\mathcal{H}_A$ the ancilla space. We

enlarge it by adding to each on-site Hilbert space a finite-dimensional auxiliary Hilbert space, geometrically positioned as close to the original physical site as the other neighboring physical sites. Furthermore, the auxiliary degrees of freedom start in some trivial product state, such as $|00\cdots 0\rangle_A$. Following previous works [19, 22, 68–70], we use finite-depth local channels as the extension of finite-depth local unitaries to mixed states.

Suppose, by contradiction, that $\rho = \mathcal{E}_{\text{FDLC}}(\rho_3)$ is a strongly symmetric state under $U(g)$, where $\rho_3 = \sum_i p_i [A_i B_i C_i]$ is tripartite separable, where $[\psi] = |\psi\rangle\langle\psi|$. By definition, there exists a local unitary V acting on an enlarged Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A$ such that $\rho = \sum_i p_i \text{Tr}_E[V|\tilde{A}_i\rangle\langle\tilde{A}_i| |\tilde{B}_i\rangle\langle\tilde{B}_i| |\tilde{C}_i\rangle\langle\tilde{C}_i|]$, where $|\tilde{A}_i\rangle = |A_i\rangle_S |0\rangle_A$ and similarly for $|\tilde{B}_i\rangle$ and $|\tilde{C}_i\rangle$. By virtue of the strong symmetry condition, Lemma 1 and its corollary imply that not only each mixed state in the previous mixture is strongly symmetric, but also that each enlarged pure state $V|\tilde{A}_i\rangle\langle\tilde{A}_i| |\tilde{B}_i\rangle\langle\tilde{B}_i| |\tilde{C}_i\rangle\langle\tilde{C}_i|$ is symmetric under the trivially enlarged symmetry $\tilde{U}(g) = U(g)_S \otimes \mathbb{1}_A$. Equivalently, the state $|\tilde{A}_i\rangle\langle\tilde{A}_i| |\tilde{B}_i\rangle\langle\tilde{B}_i| |\tilde{C}_i\rangle\langle\tilde{C}_i|$ is symmetric under $U'(g) := V^\dagger \tilde{U}(g) V$.

With this conjugated symmetry representation, we can follow the same steps that reached Eq. (36) and identify a trivial group cohomology phase $[\omega'] \in H^3[G, U(1)]$. Even though the symmetry representation $U'(g)$ is different, we expect that this group cohomology class still corresponds to the same bulk SPT phase, since the anomaly should be stable under finite-depth local unitaries. To make this precise, we prove in Appendix D, using techniques developed in Ref. [51], that ω' is in the same cohomology class as ω of the original symmetry, appearing in Eq. (36).

For mixed quantum states, a natural notion of “phases of matter” is based on the two-way connectivity via finite-depth quantum channels [19, 22, 68–70]: Two mixed states ρ_1 and ρ_2 belong to the same phase if, and only if, there are two finite-depth quantum channels $\mathcal{E}$ and $\mathcal{E}'$ such that $\mathcal{E}[\rho_1] = \rho_2$ and $\mathcal{E}'[\rho_2] = \rho_1$. Here, we do not impose any symmetry restriction on $\mathcal{E}$ and $\mathcal{E}'$ (if we did, the phases would be symmetry protected). From this point of view, our result on long-range tripartite entanglement can be rephrased as: Any state ρ that is strongly symmetric under an anomalous symmetry G must belong to a nontrivial phase, in which any state must be tripartite entangled.

Let us now revisit the infinite-temperature example in Sec. III B 1: $\rho_{\infty,+} \propto \mathbb{1} + U_{\text{CZX}}$. From our discussion here, we see that $\rho_{\infty,+}$ is long-range (tripartite) entangled [71]. The long-range entanglement is also “noninvertible,” in the sense that there is no “inverse” state $\tilde{\rho}$ such that $\rho_{\infty,+} \otimes \tilde{\rho}$ can be prepared from a trivial state in finite depth—otherwise, we immediately obtain a finite-depth preparation of $\rho_{\infty,+}$ by tracing out $\tilde{\rho}$. Furthermore, any k -point (k finite) connected correlation function $\langle O(x_1) \dots O(x_k) \rangle_c$ vanishes exponentially. In fact, any region A shares only an “area law” correlation with its complement $\bar{A}$, in the sense that the mutual information $I_{A\bar{A}} = \log 2$. Furthermore, for

any two regions A and B , as long as $B \neq \bar{A}$ we have $I_{AB} = 0$. These features, in the pure state context, are the defining characteristics of intrinsic topological orders, which exist only in two space dimensions or above. From this point of view, states like $\rho_{\infty,+}$ can be viewed as certain type of intrinsic (or noninvertible) topological phases in $(1+1)D$, which are possible only for mixed states.

C. $\rho_{\infty,\pm}$ as an intrinsically mixed quantum phase

We now recall two key properties of $\rho_{\infty,\pm} = (1 \pm U_{CZX})/2^L$:

- (1) $\rho_{\infty,\pm}$ is long-range tripartite entangled, meaning $\rho_{\infty,\pm} \neq \mathcal{E}_{\text{FD}}(\rho_{\text{tri-sep}})$;
- (2) $\rho_{\infty,\pm}$ is bipartite separable for any (connected) bipartition of the system.

It turns out these two properties guarantee that $\rho_{\infty,\pm}$ belongs to an *intrinsically mixed quantum phase*, in the sense that it is not two-way connected to any pure state through FD local channels. To see this, suppose there are two pure states $|\Psi_{1,2}\rangle$ and FD local channels $\mathcal{E}_{1,2}$ such that

$$|\Psi_1\rangle \xrightarrow{\mathcal{E}_1} \rho_{\infty,\pm} \xrightarrow{\mathcal{E}_2} |\Psi_2\rangle. \quad (37)$$

Then, the long-range tripartite entanglement of $\rho_{\infty,\pm}$ guarantees that $|\Psi_1\rangle$ must be long-range entangled. The bipartite separability of $\rho_{\infty,\pm}$, on the other hand, requires $|\Psi_2\rangle$ to be short-range entangled. To see this, notice that for any decomposition $\rho_{\infty,\pm} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, we have $\mathcal{E}_2: |\psi_i\rangle \mapsto |\Psi_2\rangle$. Since we can choose $|\psi_i\rangle$ to be bipartite separable for a interval bipartition $A|\bar{A}$, $|\Psi_2\rangle$ must also be bipartite separable for $A|\bar{A}$, up to a finite-depth local unitary (denoted as $U_{\partial A}$) acting near the boundary ∂A , with width W proportional to the depth of the channel $\mathcal{E}_2$. Since this is true for any bipartition $A|\bar{A}$, we can have $O(L)$ number of such disentangling unitaries $U_{\partial A_i}$, each separated by a finite distance $l \gg W$ from the others. Upon acting with $\prod_i U_{\partial A_i}$, $|\Psi_2\rangle$ becomes a product state with enlarged unit cells of size l —in other words, $|\Psi_2\rangle$ is short-range entangled. Therefore, $|\Psi_1\rangle$ and $|\Psi_2\rangle$ cannot be the same state, meaning $\rho_{\infty,\pm}$ is not two-way connected to any pure state.

We can further consider a more general type of finite-depth local channel: Instead of starting with an ancilla $\mathcal{H}_A$ in a pure product state $|00\dots 0\rangle_A$, we allow initiating the ancilla in a convex sum of short-range-entangled pure states: $\rho_A^{(0)} = \sum_j p_j |\psi_j^{\text{SRE}}\rangle\langle\psi_j^{\text{SRE}}|_A$. We can then define different states to belong to the same quantum phase if they are two-way connected through such generalized finite-depth local channels. This definition of phases is stronger, in the sense that long-range classical correlations, reflected in the ordinary correlation function such as $\text{Tr} \rho O_1 O_2$, become trivializable [68].

It is easy to check that, for the $\rho_{\infty,\pm}$ state, our previous argument can be modified even for such generalized channels [73]. We again conclude that if Eq. (37) holds, then $|\Psi_1\rangle$ must be long-range (tripartite) entangled while $|\Psi_2\rangle$ must be short-range entangled, and there is no pure state that is two-way connected to $\rho_{\infty,\pm}$. The state $\rho_{\infty,\pm}$ is, thus, *intrinsically mixed* in a rather strong sense—to our knowledge, such an intrinsically mixed many-body state has not been discussed before [74].

D. Higher dimensions

We can now generalize the anomaly-nonseparability connection to higher dimensions. The central statement (Conjecture 1) is that a d -dimensional bosonic system with an anomalous strong symmetry G is nonseparable for certain $(d+2)$ partition of the system.

To motivate this conjecture, we follow the Else-Nayak argument [51] to prove a weaker version, in which we assume a specific form of the symmetry actions. Specifically, we assume each symmetry element $g \in G$ to act as

$$U(g) = \sum_{\alpha} e^{i\mathcal{N}(g)[\alpha]} |g\alpha\rangle\langle\alpha|, \quad (38)$$

where $\{|\alpha\rangle = \bigotimes_{x \in \Lambda} |\alpha_x\rangle\}_{\alpha}$ is an on-site basis, $\alpha_x \mapsto g\alpha_x$ is an on-site g -action on classical labels α , and $\mathcal{N}(g)$ is a local functional of α . For each anomaly described by the group cohomology $H^{d+2}[G, U(1)]$, there are certain “fixed-point” models realizing the boundary symmetry action in this specific form [5,51]. The anomalous symmetries described in Sec. III all take the form of Eq. (38).

Now consider a symmetric k -separable state $|A_1\rangle|A_2\rangle \cdots |A_k\rangle =: |A_1\rangle|\bar{A}_1\rangle$, satisfying

$$U(g)|A_1\rangle|\bar{A}_1\rangle = \lambda(g)|A_1\rangle|\bar{A}_1\rangle. \quad (39)$$

We show there is a restricted FDLU symmetry action $U_{\bar{A}_1}$, where $\bar{A}_1 = A_2 \cup \cdots \cup A_k$, that acts like U in the bulk of $\bar{A}_1$ and has $|A_2\rangle \cdots |A_k\rangle$ as an eigenvector. The construction of $U_{\bar{A}_1}$ is similar to that in Sec. III B 1: Choose a product state $|\alpha_{A_1}\rangle$ in the classical on-site basis such that $\langle\alpha_{A_1}|A_1\rangle \neq 0$, and then a restricted local unitary is given by

$$U(g)_{\bar{A}_1} =: \frac{\langle g\alpha_{A_1}|U(g)|A_1\rangle}{\langle\alpha_{A_1}|A_1\rangle} = \sum_{\alpha} e^{i\mathcal{N}_{\bar{A}_1}(g)[\alpha]} |g\alpha\rangle\langle\alpha|, \quad (40)$$

where $|\alpha\rangle$ lives in $\bar{A}_1$ and $\mathcal{N}_{\bar{A}_1}(g)[\alpha] =: \mathcal{N}(g)[(g\alpha_A)\alpha]$ is a local functional supported on $\bar{A}_1$ that acts in the same way as $\mathcal{N}$ in the bulk of $\bar{A}_1$. We have, thus, obtained a local unitary restriction to the subsystem $\bar{A}_1 = A_2 \cup A_3 \cdots \cup A_k$, with the same form as Eq. (38) but the local functional $\mathcal{N}_{\bar{A}_1}$ is modified near the boundary of $\bar{A}_1$.

Importantly, the reduced unitary will be a representation up to boundary operators $\Omega(g, h)$. These, in turn, will also be of the form (38), now with the local functional depending on two group elements, $\mathcal{N}(g, h)$. We can then iterate the above procedure, restricting the symmetry action to fewer regions and lower dimensions—each step producing a local functional $\mathcal{N}$ that is modified from the previous one near the boundaries. This collection of local functionals $\{\mathcal{N}\}$ satisfies the symmetry algebra only up to boundary terms, similar to the 0D and 1D cases discussed earlier (see Ref. [51] for details). The violation of the symmetry algebra on the subsystem boundaries is characterized precisely by the group cohomology $H^{d+2}[G, U(1)]$ [51].

To illustrate the idea more concretely, we consider a $(2+1)$ D anomalous system and a symmetric pure state $|\Psi\rangle = |A\rangle|B\rangle|C\rangle|D\rangle$ that is separable for the four-partition $\{A, B, C, D\}$, as in Fig. 6(a). Following Eq. (40), we can reduce the symmetry to region BCD . Then, we extract its boundary operator $\Omega_{BCD}(g, h)$ and reduce it to CD . Extracting the boundary operator again, we arrive at $\Omega_c(g, h, k)$ [and $\Omega_d(g, h, k)$], supported near the intersection points of A, B , and C (and D), inside C (D). The separability of C and D regions require $\langle\Omega_c\rangle\langle\Omega_d\rangle = \langle\Omega_c\Omega_d\rangle \neq 0$, which, in turn, requires $\Omega_{c,d}$ to carry trivial symmetry charge. Finally, since the four-cocycle $[\omega] \in H^4[G, U(1)]$ is extracted from the symmetry charges of $\Omega_c(g, h, k)$, we conclude that the anomaly $[\omega]$ must be trivial for the symmetric state to be separable for such four-partitions.

Notice that for $d \geq 2$ different partitions may have different topology, and not all $(d+2)$ partitions will result in a boundary intersection consisting of only one point. This is essential for our anomaly-nonseparability proof, and we have already seen partitions in $(1+1)$ D failing to satisfy this condition [see Fig. 4(b)]. In higher dimensions, there are cases with no intersection point at all, such as the partition of the space into parallel “strips” in Fig. 6(b). For such topologies, our argument does not hold, and it is not clear if anomaly still implies nonseparability for them.

Even though our argument is valid only for the specific form of symmetry action Eq. (38), the universal nature of the result in $d \leq 1$ motivates us to conjecture that the anomaly-nonseparability connection holds as long as the symmetry action $U(g)$ is anomalous. The argument in Sec. IV B then implies that the nonseparability is long-ranged, in the sense that a symmetric state cannot be prepared from a $(d+2)$ -separable state using a finite-depth local channel.

V. MIXED STRONG-WEAK ANOMALY

We now consider effects of weak symmetries. If there is only weak symmetry, then the maximally mixed state $\rho \propto \mathbb{1}$ will always be symmetric, since the identity matrix $\mathbb{1}$ commutes with all symmetry elements. This means that weak symmetry alone does not guarantee nontrivial universal structure, which agrees with previous results that weak

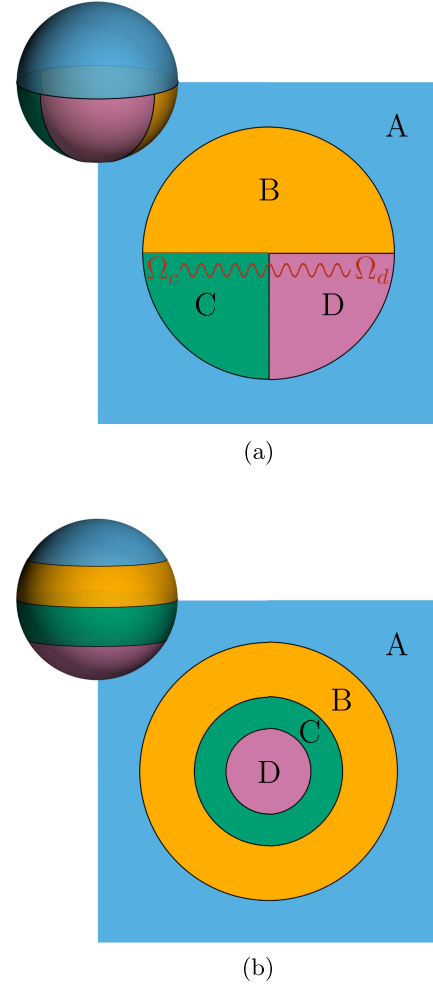


FIG. 6. Distinct four-partitions in $(2+1)$ D, shown on the sphere and the plane after stereographic projection. (a) This partition obstructs 4-partite separability. By reducing the symmetry to the edge of BCD , and then to the points c and d , we reach a contradiction if regions C and D are separable. (b) The anomaly-nonseparability connection presented here cannot be applied to this partition, since it does not have any intersection point to which we can reduce the symmetry.

symmetry alone does not support nontrivial bulk SPT phases [18,19,22]. The next step, then, is to consider systems with both strong and weak symmetries, with mixed anomalies involving both. In the simplest case, a system has a mixed anomaly under $G \times H$ if it is anomalous under a symmetry representation of the product group $G \times H$, but not under one of its factors, say, G , alone. If a subgroup, say, H , is only a weak symmetry, then according to previous results the anomaly remains nontrivial [19,22]. However, our preceding result for nonseparability will no longer apply.

Let us first consider a simple example. Consider a periodic qubit chain (1d) with $\mathbb{Z}_2^3$ symmetry generated by

$$X_{\text{odd}} = \prod_{i \text{ odd}} X_i, \quad X_{\text{even}} = \prod_{i \text{ even}} X_i, \quad U_{CZ} = \prod_{i=1}^L CZ_{i,i+1}. \quad (41)$$

This symmetry action is anomalous [16,76], as can be seen by reducing it to a region $A = \{1, 2, \dots, 2n\}$ and calculating the boundary operators: $U_{CZ}^{(A)} X_{\text{odd}}^{(A)} U_{CZ}^{(A)} = Z_{2n} X_{\text{odd}}^{(A)}$. In this case, the boundary operator Z_{2n} anticommutes not with the symmetry X_{odd} that generated it, but with the other one, X_{even} . This is the signature of a mixed anomaly. In contrast, the boundary operator of the anomalous $\mathbb{Z}_2$ in Sec. III B 1 was charged under the same symmetry U_{BC} that generated it [77]. In this section, we mainly focus on this particular $\mathbb{Z}_2^3$ symmetry for simplicity, even though we expect our discussions to be generalizable to other anomalous symmetries in $1d$.

If a mixed state is strongly symmetric under, say, just the $\mathbb{Z}_2^2$ subgroup, and weakly on the other $\mathbb{Z}_2$, our anomaly-nonseparability result no longer applies. As an example, if we choose the strong $\mathbb{Z}_2^2$ to be generated by X_{even} and X_{odd} , the infinite-temperature states $\rho_\infty = (1/2^L)(\mathbb{1} + s_1 X_{\text{even}})(\mathbb{1} + s_2 X_{\text{odd}})$ are equal mixtures of fully separable states $|\pm \pm \dots \pm\rangle$, with appropriate choice of signs $\pm$ to respect $X_{\text{even}} = s_1$ and $X_{\text{odd}} = s_2$. Hence, as mixed states, they are also fully separable.

Even though mixed anomaly can evade the nonseparability results, recent works on average SPT phases [19,22] suggest that states with strong-weak mixed anomaly must still be nontrivial in some sense. So what is nontrivial about the above ρ_∞ or, more generally, states that respect strong and weak symmetries with mixed anomaly? We now show several ways in which strong-weak anomaly constrains its symmetric states. First, we show that, similar to the pure state case, mixed anomaly serves as an obstruction of “symmetry localization” (Sec. VA). This has two consequences: The obstruction of localization for strong symmetry indicates that the mixed state cannot be symmetrically invertible [22] (Sec. VB); the obstruction of localization for weak symmetry, on the other hand, indicates that the mutual information and conditional mutual information cannot both be exponentially decaying (Sec. VC)—this condition is analogous to being “gapless” for a mixed quantum state [79]. The general lesson is that while the strong-weak mixed anomaly, in general, does not constrain quantum entanglement, it does constrain correlations of mixed states in more subtle ways [80]. Finally, we remind the reader that strong symmetries are also weak, by definition. Thus, all of the following results also apply to states with purely strong anomalous symmetries.

A. Obstruction of symmetry localization

For pure states, the proposition of “symmetry localization” states that if a gapped ground state has a global symmetry $U|\psi\rangle = |\psi\rangle$, then the action of the symmetry restricted to a region A (denoted as U_A) on $|\psi\rangle$ is equivalent to a unitary $V_{\partial A}$ acting only on the boundary of A :

$$U_A |\psi\rangle = V_{\partial A} |\psi\rangle. \quad (42)$$

Here, the boundary region ∂A has a width larger than the correlation length of the state. Symmetry localization is crucial for our understanding of symmetric gapped states, including symmetry-protected and symmetry-enriched topological orders [51,83]. On the other hand, states with spontaneous symmetry breaking do not satisfy symmetry localization.

We now generalize the definition of symmetry localization to mixed states, for both strong and weak symmetries. We show that, in the presence of mixed anomaly, localization of one symmetry inevitably leads to the spontaneous breaking of the other symmetry. First, we start with strong symmetry.

Definition 5 (strong symmetry localization). A strong symmetry U is localized on state ρ if, when restricted on a large region A , U_A acts as a boundary operator $V_{\partial A}$:

$$U_A \cdot \rho = V_{\partial A} \cdot \rho, \quad (43)$$

where

- (1) $V_{\partial A}$ has operator norm (the largest singular value) $\|V_{\partial A}\| = 1$;
- (2) in $1d$, $V_{\partial A}$ is locality preserving. In particular, when ∂A is composed of two disconnected components, $\partial A = L \sqcup R$, we should have $V_{\partial A} = V_L \otimes V_R$.

The norm-1 condition on $V_{\partial A}$ (instead of unitarity) is needed for later purposes and does not seem to bear much physical meaning beyond improving mathematical precision. Furthermore, the locality-preserving condition is not required for dimensions $d \geq 2$ to allow for more exotic states and symmetries that may elude the group cohomology classification outlined in Sec. IV D and localize to “nonlocal” boundary operators $V_{\partial A}$.

Using an argument similar to those in Secs. III B 1 and IV A, we have the following result.

Theorem 2. For a $1d$ state ρ with the anomalous $\mathbb{Z}_2^3$ symmetry Eq. (41), where $K = \mathbb{Z}_2$ is strong and $G = \mathbb{Z}_2^2$ is weak, if the strong symmetry K is localized on ρ , then ρ must break G spontaneously.

One example of the situation described above when K is generated by X_{odd} , and G , by X_{even} and U_{CZ} , is the state $\rho = |+\rangle\langle +|_{\text{odd}} \otimes (\frac{1}{2}[00\dots 0] + \frac{1}{2}[11\dots 1])_{\text{even}}$, where $|\psi\rangle := |\psi\rangle\langle\psi|$, for which $V_{\partial A} = \mathbb{1}$ and X_{even} is spontaneously broken from a weak symmetry to nothing.

We expect this result to hold for general $1d$ symmetries with mixed strong-weak anomaly. But for simplicity we prove only the statement for the special $\mathbb{Z}_2^3$ case.

Proof. Without loss of generality, we choose the strong $\mathbb{Z}_2$ symmetry to be generated by X_{odd} and the weak $\mathbb{Z}_2^2$ symmetry by X_{even} and U_{CZ} [see Eq. (41)]. Let us denote the restricted action of X_{odd} on a segment A as $X_{\text{odd},A}$, and the corresponding boundary operator $V_{\partial A} = V_L V_R$. For convenience, we can always define the boundary so that $X_{\text{odd},A}$ and $V_{L/R}$ commute.

Now consider the effect of the weak symmetry X_{even} . First, we show that, as long as X_{even} is not spontaneously broken, V_L (and V_R) can at most transform as a single irrep of X_{even} under conjugation. Suppose on the contrary $V_{L/R} = V_{L/R,+} + V_{L/R,-}$, where $V_{L/R,+}$ is invariant under conjugation by X_{even} and $V_{L/R,-}$ picks up a minus sign, and none of these operators annihilate ρ . Since $X_{\text{odd},A} \cdot \rho$ commutes with X_{even} , we must have

$$(V_{L,+}V_{R,-} + V_{L,-}V_{R,+})\rho = 0. \quad (44)$$

By assumption, the operators $V_{L/R,\pm}$ do not annihilate ρ individually. Since $V_{L,\pm}$ and $V_{R,\pm}$ act on different regions, their bilinear product should not annihilate ρ either. We can then left multiply the above equation with $V_{L,-}^\dagger V_{R,+}^\dagger$ to obtain

$$\begin{aligned} & \text{Tr}[(V_{L,-}^\dagger V_{L,+})(V_{R,+}^\dagger V_{R,-})\rho] \\ &= -\text{Tr}[(V_{L,-}^\dagger V_{L,-})(V_{R,+}^\dagger V_{R,+})\rho] \\ &= -c < 0, \end{aligned} \quad (45)$$

where c is an $O(1)$ number. The above equation indicates spontaneous breaking of the X_{even} symmetry, with $V_{L,-}^\dagger V_{L,+}$ being the local order parameter. Therefore, in the absence of spontaneous symmetry breaking, $V_{L/R}$ should transform as a single irrep under X_{even} .

We can now consider the effect of the weak $\mathbb{Z}_2$ generated by U_{CZ} :

$$\begin{aligned} \rho &= V_L X_{\text{odd},A} V_R \rho \\ &= V_L X_{\text{odd},A} V_R U_{CZ} \rho U_{CZ}^\dagger \\ &= V'_L Z_L X_{\text{odd},A} Z_R V'_R \rho \\ &= (V'_L Z_L V_L^\dagger)(V'_R Z_R V_R^\dagger) \rho, \end{aligned} \quad (46)$$

where $V'_{L/R} := U_{CZ} V_{L/R} U_{CZ}^\dagger$. $Z_{L,R}$ are the Pauli-Z operators at the boundary of A , transforming nontrivially under X_{even} —the manifestation of the mutual anomaly. Since U_{CZ} commutes with X_{even} , $V'_{L/R}$ carries the same X_{even} irrep as $V_{L/R}$. Therefore, Eq. (46) indicates spontaneous breaking of the X_{even} symmetry, with $V'ZV^\dagger$ being the local order parameter.

We, therefore, conclude that, in the absence of spontaneous breaking of the weak symmetries, the strong symmetry (which has a mixed anomaly with the weak ones) cannot be localized. ■

In Sec. VB, we show that the nonlocalization of strong symmetry indicates that the anomalous state cannot be “symmetrically invertible.”

Now we turn to weak symmetries.

Definition 6 (weak symmetry localization). A weak symmetry W is localized on state ρ if, when restricted

on a large region A , W_A acts as a boundary quantum channel $\mathcal{E}_{\partial A}$:

$$W_A \rho W_A^\dagger = \mathcal{E}_{\partial A}[\rho], \quad (47)$$

where the channel $\mathcal{E}_{\partial A}$ acts nontrivially only near the boundary, and is locality preserving in 1d: $\mathcal{E}_{\partial A} = \mathcal{E}_L \otimes \mathcal{E}_R$ for A an interval with $\partial A = L \sqcup R$.

The strong-weak mixed anomaly also serves as an obstruction of the weak symmetry localization.

Theorem 3. For a 1d state ρ with the anomalous $\mathbb{Z}_2^3$ symmetry Eq. (41), where $K = \mathbb{Z}_2$ is strong and $G = \mathbb{Z}_2^2$ is weak, if the weak symmetry G is localized on ρ , then ρ must break the strong symmetry K spontaneously.

An example of the above when $K = \langle X_{\text{odd}} \rangle$ and $G = \langle X_{\text{even}}, U_{CZ} \rangle$ is the state $\rho = (1/2^L)(\mathbb{1} + X_{\text{odd}})$. When acting on ρ , the boundary channel of X_{even} is $\mathcal{E}_{\partial A}^{\text{even}} = \mathbb{1}$, and the one of U_{CZ} is conjugation by $\prod_{(i,j) \in \partial A} CZ_{ij}$. Indeed, ρ is the prototypical state with a strong symmetry X_{odd} that is spontaneously broken to a weak symmetry [84,85].

Again, we expect this result to hold for general 1d symmetries with mixed strong-weak anomaly but prove the statement only for the special $\mathbb{Z}_2^3$ case.

Proof. Without loss of generality, we choose the strong $\mathbb{Z}_2$ symmetry to be generated by X_{odd} and the weak $\mathbb{Z}_2^2$ symmetry by X_{even} and U_{CZ} [see Eq. (41)]. Consider the restriction of $W := X_{\text{even}}$ on a large segment A (denoted as $X_{\text{even},A}$), which by assumption satisfies Eq. (47) (again for convenience we can choose the boundary region so that $X_{\text{even},A}$ and $\mathcal{E}_{L/R}$ commute). By considering the dilated (or purified) form of Eq. (47) $W_A |\Psi_\rho\rangle = V_{\partial A} |\Psi_\rho\rangle$, we can use the same argument in the proof of Theorem 2 to show that $\mathcal{E}_{L/R}$ can at most create a definite $\mathbb{Z}_2$ charge (+1 or -1) of the strong symmetry X_{odd} . We then consider the effect of the weak U_{CZ} symmetry:

$$\begin{aligned} \rho &= X_{\text{even},A} \mathcal{E}_L \circ \mathcal{E}_R [\rho] X_{\text{even},A} \\ &= X_{\text{even},A} \mathcal{E}_L \circ \mathcal{E}_R [U_{CZ} \rho U_{CZ}^\dagger] X_{\text{even},A} \\ &= X_{\text{even},A} Z_L Z_R \mathcal{E}'_L \circ \mathcal{E}'_R [\rho] X_{\text{even},A} Z_L Z_R \\ &= Z_L Z_R (\mathcal{E}'_L \circ \mathcal{E}_L \circ \mathcal{E}'_R \circ \mathcal{E}_R [\rho]) Z_L Z_R, \end{aligned} \quad (48)$$

where the modified channel

$$\mathcal{E}'_{L/R}[\rho] := U_{CZ}^\dagger \mathcal{E}_{L/R} [U_{CZ} \rho U_{CZ}^\dagger] U_{CZ} \quad (49)$$

creates the same strong X_{odd} charge as $\mathcal{E}_{L/R}$. This means that the local channel $\mathcal{E}'_{L/R} \circ \mathcal{E}_{L/R}$ preserves the strong X_{odd} symmetry, while $Z_{L/R}$ on ∂A is charged under X_{odd} . Equation (48) is, thus, the statement that the strong X_{odd} symmetry is spontaneously broken to a weak symmetry, or nothing [84].

We, therefore, conclude that, in the absence of spontaneous breaking of the strong symmetry, the weak symmetry which has a mixed anomaly with the strong one cannot be localized. ■

In Sec. V C, we show that the nonlocalization of weak symmetry indicates that, in $1d$, anomalous state cannot be “gapped,” which is analogous to the statement in pure states.

B. Strong symmetry localization and symmetric invertibility

First, we review the notion of *symmetric invertibility* from Ref. [22].

Definition 7 (symmetric invertibility). A mixed state on the physical Hilbert space $\rho \in \mathcal{H}_S$ is *symmetrically invertible* if there exists an ancillary mixed states (the “inverse” state) $\rho_I \in \mathcal{H}_A$, and a pair of symmetric finite-depth local channels $\mathcal{E}_T, \mathcal{E}_P: \mathcal{Q}(\mathcal{H}_S \otimes \mathcal{H}_A) \rightarrow \mathcal{Q}(\mathcal{H}_S \otimes \mathcal{H}_A)$ such that

$$\rho \otimes \rho_I \xrightarrow{\mathcal{E}_T} |\psi_{\text{Trivial}}\rangle \xrightarrow{\mathcal{E}_P} \rho \otimes \rho_I, \quad (50)$$

where $|\psi_{\text{Trivial}}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_A$ is a symmetric pure product state.

We now explain the details of the definition above. First, the ancillary Hilbert space $\mathcal{H}_A$ and the enlarged Hilbert space to which the local channels are purified have to be compatible with the geometry of the physical Hilbert space $\mathcal{H}_S$. Here, this means that each on-site Hilbert space of $\mathcal{H}_S$ is individually enlarged. Second, we extend the symmetry action from $\mathcal{H}_S$ to $\mathcal{H}_S \otimes \mathcal{H}_A$ in a faithful manner: For a symmetry element g , we extend the symmetry action to $U_g \rightarrow U_g \otimes U_g^A$, where U_g^A is a finite-depth unitary acting on the ancilla. Third, each gate composing the channels $\mathcal{E}_T$ and $\mathcal{E}_P$ is symmetric under this extended symmetry in accordance to the strong and weak parts of the symmetry as explained in Sec. II B.

For pure states, an invertible state can be viewed as a unique gapped ground state of some local Hamiltonian, which forms the basis of classifying unique gapped ground states using invertible topological quantum field theory [86,87]. For mixed-state SPT, invertibility is also needed to make the bulk topological invariants well defined [22].

In Ref. [22], it was argued that symmetric invertibility guarantees strong symmetry localization. We review this result below.

Theorem 4. For a symmetric invertible state ρ , a strong symmetry g can always be localized (in the sense of Definition 5).

Proof. Consider a dilated form of $\mathcal{E}_T$ in Eq. (50) (with the ancilla a):

$$U_T \rho \otimes \rho_I \otimes |0\rangle\langle 0|_a U_T^\dagger = |\psi_{\text{Trivial}}\rangle\langle \psi_{\text{Trivial}}| \otimes \rho_a. \quad (51)$$

For the trivial product state $|\psi_{\text{Trivial}}\rangle$, the restricted symmetry action $g_A \otimes g_A^A$ acts trivially, so we have

$$g_A \otimes g_A^A \cdot \rho \otimes \rho_I \otimes |0\rangle\langle 0|_a = \tilde{V}_{\partial A} \cdot \rho \otimes \rho_I \otimes |0\rangle\langle 0|_a, \quad (52)$$

where $\tilde{V}_{\partial A} = (g_A g_A^A) U_T^\dagger (g_A g_A^A)^\dagger U_T$ acts nontrivially on the boundary region ∂A because U_T commutes with the strong symmetry in the bulk of A . Tracing out $\mathcal{H}_A \otimes \mathcal{H}_a$, we obtain the desired form of symmetry localization Eq. (43). ■

If we combine the above theorem with Theorem 2, we conclude that a $1d$ mixed state with mixed strong-weak anomaly cannot be symmetrically invertible as long as the weak symmetry is not spontaneously broken. Then, we notice that spontaneous breaking of weak symmetry also leads to noninvertibility, since the existence of $\mathcal{E}_P$ in Eq. (50) requires all connected correlation functions to vanish. We, therefore, conclude the following.

Corollary 2. A $1d$ mixed state ρ with mixed strong-weak anomaly cannot be symmetrically invertible.

The above result covers both mixed strong-weak anomaly and purely strong anomaly. Purely weak symmetries do not have nontrivial anomaly, as discussed in Refs. [18–22]. For example, the maximally mixed state is weakly symmetric under any symmetry [88].

It is natural to expect that our result can be extended to higher dimensions. However, in higher dimensions, we do not have a controlled argument, which has to do with the fact that the Else-Nayak mechanism for ’t Hooft anomaly has not been rigorously developed. In this work, we leave the higher-dimensional version as a conjecture.

Recall that, for pure states, a noninvertible state always comes with some sharp feature, such as spontaneous symmetry breaking (SSB), gapless low-energy spectrum, or topological order. From this perspective, the noninvertibility of the seemingly innocuous mixed state ρ_∞ may come as a surprise. Even though we do not fully understand the landscape of noninvertible mixed states at this point, the simple ρ_∞ state does come with a sharp feature: It spontaneously breaks the strong $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry down to only a weak symmetry, akin to an SSB cat state. Such an unusual pattern of spontaneous symmetry breaking, possible only in mixed quantum states, has been proposed recently in Refs. [22,91] and was studied in more detail in Ref. [84].

C. Weak symmetry localization and gapped Markovian states

First, we define the analogue of “gapped states” in mixed states.

Definition 8 (gapped Markovian states). Consider a disk region A , a buffer region B surrounding A of width w and the rest of the system C , so that $\text{dist}(A, C) \geq w$ (see Fig. 7). We call ρ a *gapped Markovian state* if for any such partitioning of space, we have

- (i) exponentially decaying mutual information (MI) $I(A:C) = O(e^{-w/\xi_{\text{MI}}})$ for some finite correlation length ξ_{MI} ; and

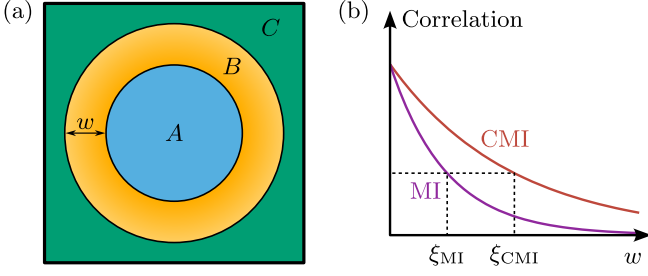


FIG. 7. (a) Partitioning of the space used for Definition 8 of gapped Markovian states and (b) plot of the exponential decay of the MI and CMI with the width w of the buffer region B for such states. Note that we do not have $I(A:C|B) \geq I(A:C)$ nor $\xi_{\text{CMI}} \geq \xi_{\text{MI}}$, in general, and (b) illustrates just one possible scenario.

- (ii) exponentially decaying conditional mutual information (CMI) $I(A:C|B) = O(e^{-w/\xi_{\text{CMI}}})$ for some finite “Markov length” ξ_{CMI} .

Recall that the MI is defined as $I(A:C) \equiv S(A) + S(C) - S(AC)$ (S being the von Neumann entropy), and the CMI is $I(A:C|B) \equiv S(AB) + S(BC) - S(C) - S(ABC)$. For pure states, the MI coincides with CMI, and the gap condition reduces to the usual one that demands correlation functions to decay exponentially, which is indeed satisfied by ground states of gapped Hamiltonians [92,93]. For general mixed states, the MI condition is rather intuitive as it demands correlation functions to decay exponentially. The CMI condition was emphasized recently in Ref. [79] as a desirable property of mixed states with features similar to ground states of gapped Hamiltonians. For our purpose, the most important consequence of the CMI condition is that the state on A can be recovered, using the Petz map, by a channel acting on AB only as long as $w \gg \xi_{\text{CMI}}$ [94], up to an error that decays exponentially with r/ξ_{CMI} in trace distance.

We now present the main result of this subsection.

Theorem 5. For a gapped Markovian state ρ , a weak symmetry g can always be localized (in the sense of Definition 6).

Proof. Consider a disk region R and the state

$$\rho' := W_R \rho W_R^\dagger, \quad (53)$$

where W_R is the weak symmetry restricted to R . We first use the exponentially decaying MI property to argue that ρ' is equal to ρ far from the boundary of R and then use the exponentially decaying CMI to locally recover ρ' from ρ , thus arriving at a localized action of the weak symmetry [see Eq. (47)].

Let ∂R_w be the boundary of R thickened by a width w , and define $R_w^- := R \setminus \partial R_w$ and $R_w^+ := R \cup \partial R_w$ (see Fig. 8). If $w \gg \xi_{\text{MI}}$, then ρ has almost no correlations between regions R_w^- and $(R_w^+)^c$, separated by a distance w [95]. That is,

$$\text{Tr}_{\partial R_w}[\rho] \approx \rho_{R_w^-} \otimes \rho_{(R_w^+)^c}, \quad (54)$$

up to an exponentially small deviation in w/ξ_{MI} .

Now let r_w be the range of the W , defined as the smallest length r such that the support of $W^\dagger X_A W$ is contained in A_r^+ for X_A an arbitrary observable supported in a region A . Then, for an extended width $\tilde{w} > 2r_w + w$, the state $\rho' = W_R \rho W_R^\dagger$ is similarly uncorrelated at large distances:

$$\text{Tr}_{\partial R_w}[\rho'] = \text{Tr}_{\partial R_w \setminus \partial R_w}[\tilde{W}_R \text{Tr}_{\partial R_w}[\rho] \tilde{W}_R^\dagger] \quad (55)$$

$$\stackrel{(54)}{\approx} \rho'_{R_w^-} \otimes \rho_{(R_w^+)^c}, \quad (56)$$

where $\tilde{W}_R$ acts like W_R in R_w^- but trivially in ∂R_w . Such operator can be constructed by removing gates outside the “light cone” coming from R_w^- (see Fig. 8). Moreover, we have

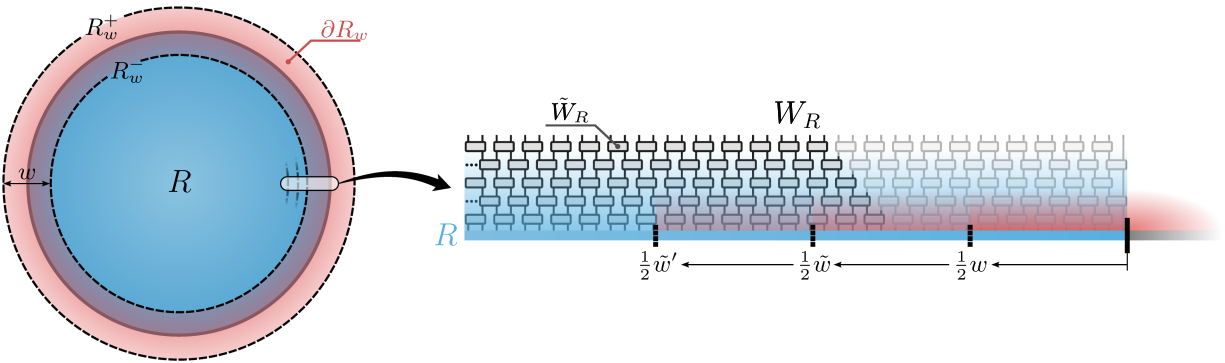


FIG. 8. Partitioning of the space used in the proof of Theorem 5. On the left, a disklike region R is pictured. ∂R is its boundary region with thickness w , from which we define $R_w^- := R \setminus \partial R_w$ and $R_w^+ := R \cup \partial R_w$. On the right, a snapshot of the radial direction around ∂R is depicted. The symmetry FDLU W is restricted to region R by removing all gates acting outside it, resulting in W_R . Importantly, we can remove further gates (the ones more transparent to the right) to arrive at a FDLU $\tilde{W}_R$ that acts like W_R on R_w^- but trivially on ∂R_w .

$$\rho'_{R_{\tilde{w}}} = \text{Tr}_{(R_{\tilde{w}})^c} [W_R \rho W_R^\dagger] = \text{Tr}_{(R_{\tilde{w}})^c} [W \rho W^\dagger] = \rho_{R_{\tilde{w}}}^-, \quad (57)$$

where in the last equality we use that W is a weak symmetry of ρ . Therefore,

$$\text{Tr}_{\partial R_{\tilde{w}}} [\rho'] \approx \rho_{R_{\tilde{w}}}^- \otimes \rho_{(R_{\tilde{w}})^c}^+ \approx \text{Tr}_{\partial R_{\tilde{w}}} [\rho]. \quad (58)$$

Finally, we extend the boundary region of R even more to a width of $\tilde{w}'$ satisfying $\tilde{w}' \gg \xi_{\text{CMI}} + \tilde{w}$. By treating the extension $\partial R_{\tilde{w}'} \setminus \partial R_{\tilde{w}}$ as a buffer region B separating the smaller boundary $A = \partial R_{\tilde{w}}$ from the bulk regions $R_{\tilde{w}'}^- \cup (R_{\tilde{w}'}^+)^c$, the exponentially small CMI $I(A:C|B)$ from the gapped Markovian condition [97] implies the existence of a channel $\mathcal{E}_{B \rightarrow AB}$ from B to $AB = \partial R_{\tilde{w}'}$ such that [94]

$$\rho' \approx \mathcal{E}_{B \rightarrow AB}(\rho'_{BC}) \stackrel{(58)}{\approx} \mathcal{E}_{B \rightarrow AB}(\rho_{BC}), \quad (59)$$

which is of the form of Eq. (47) if we define $\mathcal{E}_{\partial R} := \mathcal{E}_{B \rightarrow AB} \circ \text{Tr}_A$.

In $1d$, we can further guarantee $\mathcal{E}_{\partial R}$ to be locality preserving by applying the steps above in each of the connected components of ∂R separately. For R an interval, for example, we would arrive at $\mathcal{E}_{\partial R} = \mathcal{E}_L \otimes \mathcal{E}_R$, where $\mathcal{E}_L$ and $\mathcal{E}_R$ are supported around the left and right end points of R , respectively. ■

The above theorem combined with Theorem 3 implies that a $1d$ mixed state with mixed strong-weak anomaly cannot be gapped Markovian if the strong symmetry is not spontaneously broken. Moreover, it was shown in Ref. [84] that a state with spontaneously broken strong symmetry cannot have a finite Markov length for all tripartitions ABC and, hence, cannot be gapped Markovian. We therefore conclude the following.

Corollary 3. A $1d$ mixed state ρ with mixed strong-weak anomaly cannot be gapped Markovian (in the sense of Definition 8).

The above statement directly generalizes the familiar result in pure states. To illustrate the necessity of both the MI and CMI conditions, let us consider the $\mathbb{Z}_2^2$ symmetry generated by U_{CZ} and $X := X_{\text{even}} X_{\text{odd}}$. We can take X to be strong and U_{CZ} to be weak, and then the state $\rho_0 \propto \mathbb{1} + X$ is fully symmetric and has zero MI between distant regions A and C , $I(A:C) = 0$. However, the CMI $I(A:C|B)$ does not decay with $|B|$, instead saturating at $\log 2$. This example shows that the CMI can be positive even though the MI is zero. For the contrary, we can take U_{CZ} to be strong and X to be weak. A simple symmetric state would be the classically correlated state

$$\rho \propto |\uparrow\uparrow\cdots\rangle\langle\uparrow\uparrow\cdots| + |\downarrow\downarrow\cdots\rangle\langle\downarrow\downarrow\cdots|. \quad (60)$$

This state has vanishing CMI, but the MI between any two regions is $\log 2 > 0$.

We note that a simpler argument to Corollary 3 can be made if all the symmetries are strong. Indeed, a $1d$ gapped Markovian state ρ can be prepared from a tripartite-separable state using a finite-depth channel [98]. Based on our result on long-range tripartite entanglement of Sec. IV A, we conclude that gapped Markovianity is also forbidden by strong symmetry anomaly in $1d$.

Another consequence of Theorem 3 is that, for a symmetric gapped state, we can define and discuss the notion of SPT by examining the symmetry properties of the boundary channels $\mathcal{E}_{\partial A}$ —this leads naturally to the “decorated domain wall” picture advocated in Refs. [19,22].

In this section, we discuss (the absence of) two mixed-state properties: symmetric invertibility and gapped Markovianity. In $1d$, these two notions are essentially equivalent for pure states—for example, one can use either to define the notion of SPT. An open question is whether these two notions remain equivalent for mixed states. We leave this question to future study.

VI. LIEB-SCHULTZ-MATTIS AND NONSEPARABILITY

In Sec. V, we discuss the general consequences of strong-weak mixed anomaly in terms of symmetry localization, invertible and gapped Markovian states. It turns out that for some special (yet physically interesting) cases we can have stronger results on quantum entanglement. In this section, we discuss spin-1/2 chains (1D) with continuous non-Abelian on-site strong symmetry [SO(3) or O(2)] and weak lattice translation symmetry. For pure states, it is by now well established that there is a mutual ’t Hooft anomaly involving the on-site and translation symmetries [10,13]. This mutual anomaly strongly constrains the long-range behavior of the system, resulting in the celebrated Lieb-Schultz-Mattis (LSM) theorem [7]. Below, we show that the LSM anomaly also puts a strong constraint on mixed states; namely, any mixed state ρ that does not break the weak symmetries (explicitly or spontaneously) must be bipartite nonseparable. Furthermore, this nonseparability is long-ranged: ρ cannot be obtained from a bipartite-separable state ρ_0 through a finite-depth local channel.

Suppose on the contrary that for a bipartition A, B ,

$$\rho = \sum_i p_i |A_i\rangle\langle A_i| \otimes |B_i\rangle\langle B_i|, \quad (61)$$

where $|A_i\rangle$ ($|B_i\rangle$) are states living on the subsystem A (B). Let us consider the Pauli operator Z_x that generates an SO(2) symmetry, with the total SO(2) charge $Q = \sum_x Z_x$. Now consider two points x and y inside the regions A and B , respectively. We have

$$\text{Tr}[\rho Z_x Z_y] = \sum_i p_i \langle A_i | Z_x | A_i \rangle \langle B_i | Z_y | B_i \rangle = 0. \quad (62)$$

In the last equality above, we use the fact that either $O(2)$ or $SO(3)$ contains a π -rotation operation that sends $Z_x \rightarrow -Z_x$ and that we can apply this rotation on A (or B) alone, since each $|A_i\rangle$ is individually symmetric from the strong symmetry of ρ and the on-siteness of the spin flip $\prod_i X_i$.

The condition Eq. (62) is valid only for $x \in A$ and $y \in B$. To include cases in which both $x, y \in A$ or B , we invoke the weak translation symmetry of ρ , implying that $\text{Tr}[\rho Z_x Z_y] = 0$ for all $x \neq y$. The strong symmetry condition also requires $\sum_x Z_x \rho = 0$; namely, ρ contains only states with zero total $SO(2)$ charge. We then have

$$\begin{aligned} 0 &= \text{Tr} \rho \left(\sum_x Z_x \right)^2 \\ &= \text{Tr} \rho \left(\sum_x Z_x^2 + \sum_{x \neq y} Z_x Z_y \right) \\ &= L, \end{aligned} \quad (63)$$

which is a contradiction. So our original proposition Eq. (61)—namely, ρ is bipartite separable—must be false.

Notice that if we have only the Abelian on-site strong symmetry $SO(2)$, then Eq. (62) may not hold, since we could have $\langle A_i | Z_x | A_i \rangle \neq 0$. In this case, we can have a fully separable density matrix $\rho \propto P_{Q=0}$ (the projection operator to the subspace with $Q = \sum_x Z_x = 0$) that clearly does not spontaneously break translation symmetry. This is curiously in accordance with the fact that the LSM constraint no longer corresponds to a standard 't Hooft anomaly once the on-site symmetry is lowered to $SO(2)$ [10,13,99,100].

Back to the $O(2)$ or $SO(3)$ case, we now show that a symmetric state ρ not only is bipartite nonseparable, but also cannot be prepared from any bipartite-separable state $\rho_0 = \sum_i p_i |A_i\rangle\langle A_i| \otimes |B_i\rangle\langle B_i|$ using a finite-depth local channel. In this statement, there is no symmetry requirement on the initial state ρ_0 and the finite-depth channel $\mathcal{E}_{FD}$.

To prove the above statement, suppose on the contrary that $\rho = \mathcal{E}_{FDLC}(\rho_0)$, then consider part of the ensemble $\rho_i := \mathcal{E}_{FDLC}(|A_i\rangle\langle A_i| \otimes |B_i\rangle\langle B_i|)$. The connected correlation function $\langle Z_x Z_y \rangle - \langle Z_x \rangle \langle Z_y \rangle = 0$ on $|A_i\rangle \otimes |B_i\rangle$ for x (y) deep in the region A (B), simply because the two regions are uncorrelated. Now under a finite-depth channel, this connected correlation function should remain zero, or at least exponentially small in $|x - y|$ if the channel is made of gates with exponential tails. But for the state ρ_i , $\langle Z_x \rangle = \langle Z_y \rangle = 0$ by strong symmetry. Therefore, $\langle Z_x Z_y \rangle = 0$ on ρ_i as well as on $\rho = \sum_i p_i \rho_i$, up to exponential tails.

It was proved in Ref. [101] that as long as $\sum_x Z_x \rho = 0$, namely, that $SO(2)$ is a strong symmetry at half filling, then we have either (a) spontaneous breaking of the weak (average) translation symmetry, in the sense that for some local operator $V(x)$, $\text{Tr}[\rho V(x) V(y)] \sim (-1)^{(x-y)}$; or (b) the two-point function $\text{Tr}[\rho Z_x Z_y]$ should decay no faster than

$1/|x - y|^2$. Since $\langle Z_x Z_y \rangle = 0$ (up to exponential tails) clearly violates the second condition, we conclude that states ρ that are bipartite separable up to finite-depth local channels with strong on-site $O(2)$ or $SO(3)$ symmetry must break the weak translation symmetry, either explicitly or spontaneously [102].

We note that the LSM theorem in mixed-state context was also studied recently in Ref. [103], from a very different perspective.

Finally, we notice that if we lower the on-site symmetry to a discrete subgroup, for example, $\mathbb{Z}_2 \times \mathbb{Z}_2$, then our nonseparability result no longer applies, and the situation becomes very similar to the general discussions in Sec. V. Indeed, we can have a symmetric density matrix $\rho \propto (1 + \prod_i X_i)(1 + \prod_i Z_i)$ that is $(L/2)$ -separable under disjoint Bell pairs—it is just *symmetrically noninvertible* and has a *diverging Markov length*.

VII. RELATIONSHIP WITH OTHER WORKS

We have focused on consequences of 't Hooft anomaly for mixed quantum states, in terms of constraints on long-range entanglement (for strong symmetry anomaly) and correlation (for mixed strong-weak anomaly). We now place our results in the broader context of research on mixed-state quantum phases, an area that has recently attracted significant attention.

The notion of 't Hooft anomaly is intimately linked to SPT phases. Recently, SPT phases have been generalized to mixed states [18–22]. While prior works mainly focused on the bulk physics (which have only short-ranged entanglement and correlation), our work is concerned with the boundary physics. We have shown that a nontrivial bulk indeed leads to a nontrivial boundary in terms of long-range correlation and/or entanglement.

Shortly after the first version of our paper appeared on arXiv, two independent works on anomaly constraints for mixed states were posted [81,82]. While a common theme of all these works (including ours) is that mixed states with nontrivial strong-weak 't Hooft anomaly must be nontrivial in some sense (in agreement with previous classification of bulk SPT phases), our work focused on different aspects of mixed-state phases in comparison to others. For strong symmetry anomaly, we focused on multipartite entanglement, which allowed us to identify a class of *intrinsically mixed* quantum phases that are not two-way connected to any pure states (Sec. IV C). For mixed strong-weak anomaly, we first discussed its impact on *symmetric invertibility* and in a subsequent update discussed its impact on more information-theoretic aspects of long-range correlations.

For systems with only weak symmetry, there is no nontrivial anomaly in our definition. The simplest way to see this is to note that the maximally mixed state $\rho \sim \mathbb{1}$, which certainly has no entanglement nor correlation, is always invariant under any weak symmetry. However,

a different notion of “nontriviality” can be formulated in terms of the degeneracies in the eigenvalue spectrum of ρ . References [89,90,103] studied the notion of anomaly and LSM constraints from this point of view, which is different from the view taken in our work.

VIII. DISCUSSION

In this work, we explored several consequences of ’t Hooft anomaly for mixed quantum states. Most notably, we showed that an anomalous strong symmetry requires the mixed state to be multipartite entangled, or nonseparable. Specifically, for a d -dimensional bosonic lattice system with symmetry G and anomaly $\omega \in H^{d+2}[G, U(1)]$, any strongly symmetric mixed state must be $(d+2)$ -partite nonseparable. Furthermore, the nonseparability is long-ranged, in the sense that the symmetric state cannot be obtained from a $(d+2)$ -partite-separable state using a finite-depth local quantum channel. We proved this result for $d \leq 1$ and provided some plausibility argument for $d > 1$. We note that, in a different context, separability properties of bulk mixed-state SPT phases have been studied in Ref. [104]. It remains to understand the precise relation between the bulk and “boundary” perspectives.

From a quantum informational perspective, anomalies can be useful to generate examples of states with interesting patterns of mixed-state entanglement. For example, strongly symmetric infinite-temperature states ρ_∞ can be $(d+2)$ -nonseparable and $(d+1)$ -separable at the same time. Even in $d = 1$, we discovered examples [such as ρ_∞ of the CZX model in Eq. (26)] that are long-ranged entangled in the sense that they cannot be prepared from trivial states using finite-depth local channels; and, at the same time, any proper subsystems are maximally disordered, so that all connected correlation functions behave trivially.

We also discussed mixed anomaly involving both strong and weak symmetries. In general, the mixed strong-weak anomaly no longer constrains quantum entanglement. Instead, such anomaly constrains the long-range correlations and forbids the states from being symmetrically invertible or gapped Markovian. However, for the special—but physically relevant—case of spin-1/2 chain with weak translation and strong $SO(3)$ [or $O(2)$] spin rotation symmetries, we again concluded that any symmetric mixed state must be long-range entangled. It remains to see to what extent we can generalize such statement to other systems.

We end with some open questions.

- (1) In this paper, we have focused on bosonic anomalies of unitary zero-form symmetries. A natural question is whether our results can be extended to include fermions and/or higher-form symmetries. Even for bosonic zero-form symmetries, our result is rigorously established only for $d \leq 1$, and a controlled treatment of higher-dimensional cases is left to future work.
- (2) In this work, we focused on entanglement properties of lattice systems. It is natural to ask whether we can formulate similar results for continuum field theories, where the notion of ’t Hooft anomaly can also be naturally defined.
- (3) The simple example states discussed in Secs. III and V are infinite-temperature states that exhibit strong-to-weak spontaneous symmetry breaking [22,84,91]. For pure states with anomalous symmetry, it is known that if spontaneous symmetry breaking is forbidden, we can have stronger constraints on the states—typically, the state will exhibit either power-law correlation function or intrinsic topological order. Similarly, it is natural to ask: If there is no strong-to-weak spontaneous symmetry breaking, can we make even stronger statements about the mixed quantum states?
- (4) For ground states, anomalous symmetries not only impose constraints on pure states, leading to non-trivial features like long-range entanglement, but also restrict their parent Hamiltonians, which cannot be symmetrically gapped without topological order. Can an analogous statement be established regarding the dynamics of anomalous open systems? In such context, the Lindbladian naturally replaces the role of the Hamiltonian as a generator of dynamics. One interesting direction is to investigate the conditions under which the steady states of Lindbladian evolutions are gapped Markovian or not (Definition 8) and how this relates to properties of the Lindbladian.
- (5) In Sec. VI, we have seen that whether the strong on-site symmetry is continuous or discrete makes a significant difference—the former makes the states nonseparable while the latter does not, even though the ’t Hooft anomaly is nontrivial in both cases. Are non-Abelian continuous symmetries intrinsically special in some sense? Or is this just a special feature of the 1D spin-1/2 chain?
- (6) Last, but perhaps the most important: Can we find some physically observable consequences of the multipartite nonseparability?

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APPENDIX A: A “GOOD” MEASURE OF MULTIPARTITE ENTANGLEMENT

In Sec. II A, we describe how hard it is to decide whether a given multipartite state is separable or not. Even harder is to calculate a good measure of multipartite entanglement, as there is a trade-off between practical computability and having desirable properties, such as being zero if and only if the state is (k -)separable.

Based on the general framework for classifying multipartite entanglement and constructing their measures given in Ref. [46], we present here one good but hard-to-compute mixed-state measure of entanglement with respect to a k partition. First, given a pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and a k partition $\mathcal{P} = \{A_i\}_{i=1}^k$, we define the k -partite entanglement $E_{\mathcal{P}}$ to be (half of) the sum of entanglement entropies for each set in the partition:

$$E_{\mathcal{P}}(|\psi\rangle\langle\psi|) := \frac{1}{2} [S(\rho_{A_1}) + S(\rho_{A_2}) + \cdots + S(\rho_{A_k})]. \quad (\text{A1})$$

Then, we extend $E_{\mathcal{P}}$ to all mixed states by taking its convex roof:

$$E_{\mathcal{P}}(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_{\mathcal{P}}(|\psi_i\rangle\langle\psi_i|). \quad (\text{A2})$$

Defined in this way, $E_{\mathcal{P}}$ enjoy the following properties:

- (i) monotonic under local operations and classical communication,
- (ii) convex under mixtures,
- (iii) zero if, and only if, the state is separable with respect to $\mathcal{P}$, and
- (iv) reduces to entanglement of formation for $k = 2$.

Hence, we can reformulate our main result by stating that a strongly symmetric anomalous state ρ has $E_{\mathcal{P}}(\rho) > 0$ for certain $(d+2)$ partitions $\mathcal{P}$.

Moreover, since the set of all strongly symmetric states is compact, then there exists an anomalous state that achieves a lower bound of multipartite entanglement $E_{U,d,L}^{\min} > 0$ that depends only on the group action $U(g)$, the dimension d , and the number of sites L . As an example for the $d = 1$ CZX symmetry, the bipartite-separable states (see Sec. III B 2), if arranged on the $A|BC$ bipartition, have $\log 2$ tripartite entanglement. Surprisingly, there are symmetric states with lower tripartite entanglement: By randomly sampling symmetric pure states and computing their tripartite entanglement $E_{A|B|C}(\rho) \equiv E_3(\rho)$, we conjecture that, for the CZX symmetry, $E_3^{\min} \approx 0.9 \log 2$ independently of the system size. The candidate states with this value of tripartite entanglement constitute a complete basis of the form

$$|\alpha\beta\gamma^{(s)}\rangle + \lambda_A |\bar{\alpha}\beta\gamma^{(r_A)}\rangle + \lambda_B |\alpha\bar{\beta}\gamma^{(r_B)}\rangle + \lambda_C |\alpha\beta\bar{\gamma}^{(r_C)}\rangle, \quad (\text{A3})$$

where $|\xi^{(r)}\rangle \equiv (1/\sqrt{2})(|\xi\rangle + r|\bar{\xi}\rangle)$, with ξ a bit string and $r \in U(1)$, $\alpha, \beta, \gamma \in \{0, 1\}^{|A|, |B|, |C|}$ bit strings in A , B , and C , respectively, and $\lambda_i, r_j \in U(1)$ phase factors. Requiring symmetry with respect to U_{CZX} with eigenvalue $\mu \in \{\pm 1\}$ constrains

$$\mu = s(-1)^{\sum_i (\alpha\beta\gamma)_i (\alpha\beta\gamma)_{i+1}} \quad (\text{A4})$$

and

$$\begin{aligned} s &= (-1)^{\alpha_1 + \alpha_{|A|} + \beta_1 + \gamma_{|C|} + |A| + 1} r_A \\ &= (-1)^{\beta_1 + \beta_{|B|} + \gamma_1 + \alpha_{|A|} + |B| + 1} r_B \\ &= (-1)^{\gamma_1 + \gamma_{|C|} + \alpha_1 + \beta_{|B|} + |C| + 1} r_C, \end{aligned} \quad (\text{A5})$$

and minimizing their tripartite entanglement E_3 implies

$$s = r_A \lambda_A^2 = r_B \lambda_B^2 = r_C \lambda_C^2, \quad (\text{A6})$$

$$\lambda_A^2 \lambda_B^2 \lambda_C^2 = -1, \quad (\text{A7})$$

$$\text{with } \lambda_i^4 = 1, \quad r_j^2 = 1. \quad (\text{A8})$$

By varying all the parameters respecting the constraints above, one can verify that the states of Eq. (A3) form a orthonormal basis, each having tripartite entanglement $E_3^{\min} = \frac{3}{2}(-p \log p - q \log q) \approx 0.9 \log 2$, for $p = 1 - q = (2 + \sqrt{2})/4$. Furthermore, because they form a basis of symmetric states, then the tripartite entanglement $E_3(\rho_{\infty, \pm})$ of the CZX infinite-temperature states $\rho_{\infty, \pm}$ of Eq. (26) is upper bounded by $E_3^{\min}$ above, being equal if our conjecture that they minimize E_3 is indeed correct.

APPENDIX B: TRIPARTITE ENTANGLEMENT OF INFINITE-TEMPERATURE STATES WITH CZX SYMMETRY VIA PERMUTATION CRITERION

Consider the four-qubit infinite-temperature state $\rho_{\infty} = (1/2^4)(1 + U_{CZX})$. For each Z -basis state $|ij\rangle$ of $\mathcal{H}_A$, ρ_{∞} is block diagonal in the subspace V_{ij} spanned by acting with X_A , X_B , X_C , or products thereof on $|ij\rangle_A |0\rangle_B |0\rangle_C$. For example, the block matrix for the $V_{00} = V_{11}$ subspace is

$$\frac{1}{2^4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{B1})$$

Importantly, the permutation $\rho_{i_A, i_B, i_C}^{j_A, j_B, j_C} \rightarrow \rho_{i_A, j_C, i_C}^{j_A, j_B, i_B}$ keeps this block-diagonal structure, so we can apply it to each subspace V_{ij} separately. For V_{00} , it becomes

$$\frac{1}{2^4} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{B2})$$

whose trace norm is $\frac{1}{2}[\frac{1}{2} + (1/\sqrt{2})]$. The same value is found for $V_{01} = V_{10}$, and since the trace norm of the direct sum of matrices is the sum of the trace norms, we find the trace norm of the state after permutation is $\frac{1}{2} + (1/\sqrt{2}) > 1$. It is also straightforward to generalize this result to the case where A is larger, but B and C are still one-qubit long and adjacent to each other. We expect a similar argument to hold for general A , B , and C that are connected.

APPENDIX C: LOCAL ADAPTIVE PREPARATION OF INFINITE-TEMPERATURE STATES WITH CZX SYMMETRY

Here, we show a finite-depth local adaptive preparation of the infinite-temperature states $\rho_{\infty, \pm} \propto \mathbb{1} \pm U_{\text{CZX}}$, as discussed in Sec. III B 3. By “local adaptive preparation” we mean starting from a product state and applying a quantum circuit involving (geometrically) local unitary gates and local measurements, whose outcomes may influence future parts of the circuit. A familiar example of an adaptive circuit is the syndrome measurement of a stabilizer code followed by error correction conditioned on the syndrome.

We divide the adaptive preparation in three parts.

- (1) *Initial state preparation.*—Start from the $|++\cdots+\rangle$ state in a system of L qubits with periodic boundary condition.
- (2) *ZZ measurement.*—Measure $Z_i Z_{i+1}$ for all $i \in \{1, 2, \dots, L\}$, whose outcome $c_i = \pm 1$ we record for future classical processing. Having knowledge of all results $(c_i)_{i=1}^L$, we can be sure that the post-measurement state is a GHZ-like state in the Z basis of the form $(1/\sqrt{2})(|b\rangle + |\bar{b}\rangle)$, where $b \in \{0, 1\}^L$ is a bit string and $\bar{b}_i = 1 - b_i$ is its complement. The relation between b and $(c_i)_{i=1}^L$ is given by $(-1)^{b_i}(-1)^{b_{i+1}} = (-1)^{\bar{b}_i}(-1)^{\bar{b}_{i+1}} = c_i$. In other words, from $(c_i)_{i=1}^L$ we can find the two bit strings $\{b, \bar{b}\}$ satisfying the equation above.
- (3) *Unitary correction with feedback.*—The postmeasurement GHZ-like state is also an eigenstate of the

CXZ symmetry, with eigenvalue $\lambda = (-1)^{\sum_i b_i b_{i+1}} = (-1)^{\sum_i \bar{b}_i \bar{b}_{i+1}}$. Since we can compute λ based *only* on the ZZ measurement results $(c_i)_{i=1}^L$, we can conditionally apply Z_1 if $\lambda = -1$ to make it an $+1$ eigenstate of the CZX symmetry, or vice versa.

The resulting density matrix coming from the mixture of the final states over all measurement results is exactly equal to $\rho_{\infty, \pm} \propto \mathbb{1} \pm U_{\text{CZX}}$. We note that, even though the non-local nature of the feedback is crucial to establish the necessary global correlations, the quantum part of the circuit is entirely local and finite depth.

APPENDIX D: LONG-RANGE ENTANGLEMENT PROOF

Here, we prove that the anomaly-nonseparability connection remains if a local unitary symmetry representation $U(g)$ is conjugated by another local unitary V . In particular, we prove that the group cocycle ω' coming from the $U'(g) = VU(g)V^\dagger$ representation is in the same group cohomology class as ω , from $U(g)$.

First, we follow the same arguments of Sec. IV A to reach Eq. (36) with U_{BC} , Ω , and ω replaced by U'_{BC} , Ω' , and ω' , respectively:

$$\begin{aligned} & \omega'(g_1, g_2, g_3) \Omega'_c(g_1, g_2) \Omega'_c(g_1 g_2, g_3) \\ &= U'_{BC}(g_1) \Omega'_c(g_2, g_3) U'_{BC}(g_1)^{-1} \Omega'_c(g_1, g_2 g_3). \end{aligned} \quad (\text{D1})$$

Then, we define V_{BC} to be a finite-depth local unitary supported in BC and acting the same as V in the bulk of BC . One can construct such unitary by removing the gates in the “causal past” of the boundary of BC as in Fig. 9. By conjugating Eq. (D1) with V_{BC} , we find that $\tilde{U}_{BC}(g) = V_{BC}^\dagger U'_{BC}(g) V_{BC}$ and $\tilde{\Omega}(g_1, g_2) := V_{BC}^\dagger \Omega'(g_1, g_2) V_{BC}$ satisfy the same relation *with the same cocycle* $\tilde{\omega} = \omega'$. Furthermore, $\tilde{U}_{BC}$ is still supported on BC and acts as $U(g)$ in the bulk of BC . In other words, $\tilde{U}_{BC}$ is but another

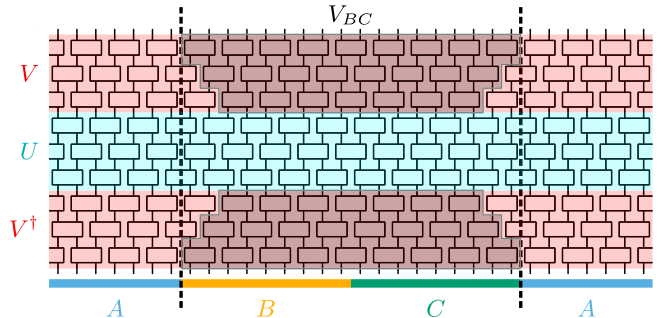


FIG. 9. The original symmetry $U(g)$ conjugated by the (extended) FDLU V . The circuit V_{BC} acts as V in the bulk of BC and is entirely supported on it. By removing V_{BC} from $U'(g) := VU(g)V^\dagger$, we “expose” U in the bulk and, thus, arrive at another unitary restriction to BC with the same nontrivial cohomology class $[\tilde{\omega}] = [\omega']$.

unitary restriction of the original symmetry U to BC . Since it can be shown that two different symmetry restrictions U_{BC} and $\tilde{U}_{BC}$ give rise to 3-cocycles ω and $\tilde{\omega}$ that differ by an exact 3-cocycle [51], then $\omega' = \tilde{\omega}$ is in the same group cohomology class as ω . QED.

APPENDIX E: THE PROOF THAT u_{AB} DOES NOT ENTANGLE A AND B

Following the discussion in Sec. IV A, we assume u_{AB} is supported in a neighborhood near the AB boundary, etc.

Lemma 3. If $u_{AB}u_{BC}u_{AC}|A\rangle|B\rangle|C\rangle = |\tilde{A}\rangle|\tilde{B}\rangle|\tilde{C}\rangle$ as in the diagram below, then u_{AB} does not entangle $|A\rangle$ and $|B\rangle$, and similarly for u_{BC} and u_{AC} :

$$(E1)$$

Proof. We prove the lemma by contradiction. Suppose u_{AB} does entangle A and B , i.e., $u_{AB}|A\rangle|B\rangle \neq |A'\rangle|B'\rangle$ for any $|A'\rangle$ and $|B'\rangle$. Consider dividing A into A_1 and A_2 , and C into C_1 and C_2 , so that A_1 and C_2 are adjacent to the AC boundary, and u_{AC} is fully supported on $A_1 \cup C_2$.

Let us consider the von Neumann entropy S of various regions. In the initial state $|A\rangle|B\rangle|C\rangle$, we have $S_A = 0$, and $S_{A \cup C_2} = S_A + S_{C_2} = S_{C_2}$, the entanglement entropy, or EE, of C_2 in $|C\rangle$. Then, after applying u_{AB} , the EE of A must increase, subsequently leading to an increase in the EE of $A \cup C_2$, as A and C remain separated at this point. Moreover, further applications of u_{BC} and u_{AC} do not change the EE of $A \cup C_2$. On the other hand, we also know that, in the final state, regions A , B , and C become separated again, so

$$S_{C_2}^{\text{final}} = S_{A \cup C_2}^{\text{final}} > S_{A \cup C_2} = S_{C_2}. \quad (E2)$$

Importantly, the inequality above is strict since u_{AB} is assumed to entangle $|A\rangle|B\rangle$.

Similarly, by considering $A_1 \cup C$ we find that $S_{A_1}^{\text{final}} = S_{A_1 \cup C}^{\text{final}}$ does not decrease compared to S_{A_1} . In fact, if we assume u_{BC} entangles $|B\rangle$ and $|C\rangle$, S_{A_1} has to increase, too.

Because A and C are separated in both initial and the final states, we have

$$S_{A_1 \cup C_2}^{\text{final}} = S_{A_1}^{\text{final}} + S_{C_2}^{\text{final}} > S_{A_1} + S_{C_2} = S_{A_1 \cup C_2}. \quad (E3)$$

However, this is impossible, since all the unitaries acting on the initial separated state act either entirely within $A_1 \cup C_2$ or strictly outside it. Hence, they cannot change the entanglement entropy of $A_1 \cup C_2$. We have thus reached a contradiction, and our initial assumption of u_{AB} entangling $|A\rangle$ and $|B\rangle$ must be false. ■

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