

Scaling Laws of Quantum Information Lifetime in Monitored Quantum Dynamics

Bingzhi Zhang^{1,*}, Fangjun Hu², Runzhe Mo¹, Tianyang Chen², Hakan E. Türeci², and Quntao Zhuang^{1,3,†}

¹*Ming Hsieh Department of Electrical and Computer Engineering, University of Southern California, Los Angeles, California 90089, USA*

²*Department of Electrical and Computer Engineering, Princeton University, Princeton, New Jersey 08544, USA*

³*Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089, USA*



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Quantum information is typically fragile under measurements and environmental coupling. Remarkably, we find that its lifetime can scale exponentially with system size when the environment is continuously monitored via midcircuit measurements—regardless of bath size. Starting from a maximally entangled state with a reference, we analytically prove this exponential scaling for typical Haar-random unitaries and confirm it through numerical simulations in both random unitary circuits and chaotic Hamiltonian systems. In the absence of bath monitoring, the lifetime exhibits a markedly different scaling: It grows at most linearly—or remains constant—with system size and decays inversely with the bath size. We further extend our findings numerically to a broad class of initial states. In the intermediate regime of partial monitoring, we identify and prove a two-scale transition, where the quantum mutual information decays logarithmically at microscopic timescales but linearly at macroscopic timescales. We discuss implications for monitored quantum circuits in the weak-measurement limit, quantum algorithms such as quantum diffusion models and quantum reservoir computing, and quantum communication. Finally, we experimentally verify the gap of persisted information on IBM Quantum hardware.

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I. INTRODUCTION

Quantum measurement is a defining feature of quantum mechanics—introducing irreducible randomness and collapsing superpositions in a way fundamentally distinct from unitary evolution. Beyond its foundational role, measurement has emerged as a powerful resource for engineering quantum states and probing the structure of many-body wave functions. In particular, midcircuit measurements offer pathways for understanding thermalization in quantum systems, where subsystem measurements induce an emergent maximally random state ensemble in the complementary system [1,2]. The monitored quantum circuit, a brickwork unitary circuit probabilistically doped with projective measurements, exhibits the measurement-induced phase transition (MIPT): As the measurement probability increases, a system can transition from the

volume-law to area-law entanglement [3,4]. These results have resonated broadly, stimulating research across quantum information science, statistical mechanics, and condensed matter physics.

In the era of quantum science and engineering, a detailed understanding of the role of quantum measurement is essential for realizing quantum error correction [5], a cornerstone of fault-tolerant quantum computation. Experimental advances, particularly in superconducting qubit platforms, have demonstrated midcircuit operations conditioned on measurement outcomes as a potential practical pathway toward scalable quantum computation [6]. Extending the lifetime of encoded quantum information in this setting requires carefully designed codes that preserve coherence in the logical quantum space during syndrome extraction. Midcircuit measurements have also found utilities in quantum algorithms beyond error correction. To begin with, they enable the reuse of physical qubits as fresh ancillae, thereby reducing the physical resources in quantum computation. This approach has recently been demonstrated in the holographic deep thermalization protocol for quantum and classical random number generation [7,8] and discussed for a range of other algorithms [9–16]. In these applications and beyond, the notion of quantum information lifetime plays an important role, and yet its

*Contact author: bingzhiz@usc.edu

†Contact author: qzhuang@usc.edu

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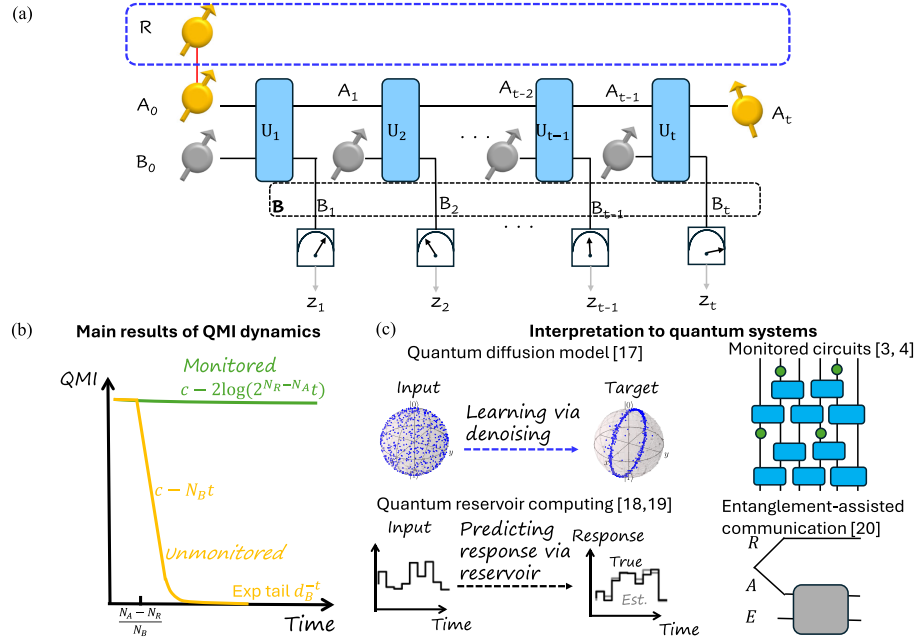


FIG. 1. Schematic plot of quantum dynamics with midcircuit measurements. In (a), initial input A_0 (N_A qubits) goes through interactions with bath via unitaries $U_1, U_2, \dots, U_t$ to produce the final output A_t . Between each unitary, bath qubits B (N_B qubits) are measured and reset to $|0\rangle$'s in most of our discussions, except in Sec. VIII. Equivalently, one can regard the measurement as applied on bath qubits $B_1, B_2, \dots, B_t$, each initialized in $|0\rangle^{\otimes N_B}$. To track the evolution of quantum information, we introduce the reference system R initially maximally entangled with A_0 . The corresponding QMI dynamics is sketched in (b), which decays logarithmically in monitored dynamics but linearly in time with an initial perfect information protection and a late exponential tail with unmonitored bath. (c) highlights connections between the model above and other models and algorithmic frameworks, including QuDDPM [17], QRC [18,19], monitored quantum circuits [3,4], and entanglement-assisted communication [20]. The reference system surrounded by the blue dashed box in (a) indicates that it is not included in the original quantum circuit models except for communication setting.

connection to how midcircuit measurements are performed is unclear, as we detail below.

In generative quantum models such as the quantum denoising diffusion probabilistic model (QuDDPM) [17], of interest in the present work, midcircuit measurement plays a central role. QuDDPM aims to learn a backward denoising process that maps a random state ensemble to the target ensemble, in a step-by-step fashion guided by a sequence of reference ensembles interpolating the target and initial ensembles, which are produced in a controlled forward scrambling process [see Fig. 1(c)]. In each denoising step, midcircuit measurement combined with a parametrized unitary is trained under a cost function. In this regard, midcircuit measurement serves a dual role: It supplies intrinsic quantum randomness for generative learning and acts as a Maxwell's demon, denoising the system by steering transitions between quantum state ensembles. In this setting, the *quantum information lifetime* defined in this work serves as a proxy for the efficiency of state conversion and the rate of learning.

Quantum reservoir computing (QRC) [21,22], a general-purpose quantum machine learning paradigm for processing time-dependent data, is another setting where the scaling behavior of quantum information lifetime is a key design consideration. An implementation of the QRC

algorithm tailored for near-term quantum devices using midcircuit measurements has been recently proposed and investigated [18,19,23–25]. In the noisy intermediate-scale QRC scheme (NISQRC) [19] analyzed here, a subset of qubits, designated as readout qubits, is repeatedly measured and reset as the data are streamed into the circuit, to approximate a target function of past inputs [see Fig. 1(c)]. The remaining qubits, referred to as memory qubits, remain unmeasured. A key quantity of interest is the *memory time*, the ability of the system state to retain information about its past inputs [26,27]. In reservoir computing, memory time plays a subtle role and must be carefully balanced with the nonlinearity the system introduces on its inputs—a trade-off well recognized in classical reservoir computing [28–30]. It is, therefore, important to understand the scaling of memory time—quantum information lifetime—with the measurement rate and system size.

In this work, we derive scaling laws for the quantum information lifetime in dynamical settings with and without bath monitoring enabled by midcircuit measurements. When the system interacts with a periodically monitored bath and the measurement trajectory is recorded, the information lifetime can scale exponentially with the system size (in the context of NISQRC, this corresponds to the size of the memory subsystem); while, without the

monitoring, when the individual measurement trajectory is not recorded, the lifetime scales linearly or is a constant in system size. In the presence of partial monitoring, we uncover a two-scale transition: Quantum mutual information (QMI) decays logarithmically on microscopic scales but linearly on macroscopic scales. We establish our results analytically for typical unitaries using quantum mutual information between the system and a maximally entangled reference. These findings are supported by numerical simulations under Haar-random and random Clifford unitaries and chaotic Hamiltonian dynamics across various initial states. For chaotic Hamiltonians, we identify atypical regimes where residual correlations persist for times that scale exponentially with system size. Our analysis highlights the importance of training in QuDDPM for achieving efficient generative learning. In the context of monitored quantum circuits studied for the entanglement phase transitions, our results demonstrate an exponentially long quantum information lifetime in the weak-measurement limit. We also interpret these results in the context of quantum reservoir computing, suggesting strategies for the design of the memory time in general dynamical systems. In the quantum communication setting, our results have implications for entanglement-assisted protocols, both with and without environment assistance. Finally, we propose a hardware-friendly protocol for verifying mutual information dynamics through experiments on the IBM Quantum hardware.

II. OVERVIEW

As depicted in Fig. 1(a), we study a quantum circuit, where system A interacts unitarily with bath B in a repeated manner. Between successive unitaries, midcircuit measurements are applied on the bath B . The central question we ask in this work is *how does the initial quantum information in A decay through unitary evolution and midcircuit measurements?*

To quantify the quantum information lost, we introduce a reference system R initially correlated with A_0 [see Fig. 1(a)]. We adopt the QMI between R and A_t to quantify the amount of quantum information in the system A_t for $t \geq 0$. The involvement of measurements means that access to measurement outcomes shapes the information dynamics, much like in Wigner’s friend scenario [31]. When the entire measurement trajectory is recorded, we refer to the bath as “monitored,” indicating complete knowledge of the measurement dynamics [Eq. (1)]; conversely, lack of the full information of measurement trajectory decreases our knowledge about the final quantum state. In the extreme case, when the entire measurement trajectory is discarded, we regard the bath as being not monitored [Eq. (2)]. In this work, we begin with an examination of the QMI dynamics under midcircuit measurements with and without bath monitoring and extend to QMI dynamics with partial bath monitoring.

We analyze the QMI dynamics by first deriving analytical scaling expressions with respect to system size, assuming the R - A system starts in a maximally entangled Bell state with $N_R \leq N_A$, in general. As sketched in Fig. 1(b), we prove that, for typical monitored dynamics, the QMI vanishes logarithmically with time steps, leading to a lifetime that scales exponentially with system size, $\tau \sim 2^{N_A}$, and independent of bath size; while, for typical unmonitored dynamics, the QMI initially experiences a perfect information protection plateau thanks to the dynamical quantum error correction (QEC) until $t \sim (N_A - N_R)/N_B$ and then decays linearly with time $2N_R/N_B$ before entering a long-tail residual region, leading to a shorter lifetime with different scaling laws: $\tau \sim (N_A + N_R)/N_B$ or $\tau \sim 1/N_B$ depending on whether one focuses on the initial linear decay or the long-tail residual part. Our results are analytically established for typical dynamics under a maximally entangled initial state of the R - A system and are further corroborated through numerical simulations of a broad class of correlated initial states. When erasure error is introduced as a partially monitored bath, the noisy conditional dynamics forms an interpolation between the monitored and unmonitored limits. We uncover a two-scale transition, with microscopic logarithmic decay accompanied by a macroscopic-scale linear decay.

Besides the typical cases represented by dynamics generated by Haar unitaries, we also examine atypical cases of Hamiltonian dynamics without bath monitoring, which gives rise to a lifetime that scales exponentially.

We also show that QMI dynamics remains unchanged under resetting to a fixed pure state or leaving the qubits in the measured pure state without reset.

Our results have implications for monitored quantum circuits, QuDDPM and QRC algorithms employing partial midcircuit measurements, and entanglement-assisted communication as indicated in Fig. 1(c). Our results show that monitored quantum circuits in the weak-measurement limit support exponentially long quantum information lifetime. In terms of QuDDPM, our results indicate that focusing on a single measurement trajectory is inefficient in learning; It is indeed essential to take QuDDPM’s approach of considering the states under different measurement trajectories. For midcircuit-measurement-based QRC algorithms [18,19,23–25], our findings suggest that optimizing memory time for given resources may require going beyond the marginal distribution of measurements or employing nongeneric Hamiltonian choices.

The paper is organized as follows. In Sec. III, we introduce the framework for information propagation in quantum dynamics with midcircuit measurements and the metric to quantify information lifetime. In Sec. IV, we present results for quantum information dynamics with a monitored bath. In Sec. V, we study the scenario without bath monitoring, including random unitaries and

anomalous Hamiltonian dynamics. In Sec. VI, we investigate the information dynamics with a partially monitored bath. We interpret our results in various scenarios in Sec. VII. In Sec. VIII, we evaluate the effect of reset strategies on the information lifetime. In Sec. IX, we examine the information dynamics from experiments on IBM Quantum devices.

III. QUANTUM SYSTEM AND MUTUAL INFORMATION LIFETIME

A. Quantum circuit setup

We consider a general framework for quantum dynamics with midcircuit measurements, as shown in Fig. 1(a). This framework underlies a broad class of algorithms including recent QRC algorithms [19,32] and the QuDDPM [17]. The quantum system here consists of two subsystems: the data system A and the bath system B comprising N_A and N_B qubits, respectively [33]. For the convenience of description, we also add a subscript t to A and B systems to denote the system at time step t . Starting from the step $t = 0$, the data system A_0 takes an information-encoded quantum state, and the bath B_0 is initialized in a trivial product state, e.g., $|0\rangle_{B_0} \equiv |0\rangle^{\otimes N_B}$. At each time step t , the system undergoes a global unitary evolution U_t , which corresponds to either a chaotic evolution in the case of QRC or a parametrized quantum circuit in QuDDPM. This is followed by a projective measurement on the bath B_t , yielding a single-shot measurement outcome z_t . The resulting postmeasurement state of the data subsystem A is then used as the input for the subsequent time step $t + 1$, where the process of unitary evolution and bath measurement is repeated. We first consider the case in which the bath subsystem B is reset to the trivial product state $|0\rangle_B$ at each time step. In Sec. VIII, we demonstrate that this reset procedure does not affect our main conclusions regarding lifetime scaling.

For simplicity, we take the reference and system to be initially in a pure state $|\Phi\rangle_{RA_0}$ for now. When the bath is fully monitored in each step, in the t th step, the Kraus operator applied on RA joint system corresponding to measurement outcome z_t on bath is $K_t =_{B_t} \langle z_t | (I_R \otimes U_t) | 0 \rangle_{B_{t-1}}$; therefore, the final quantum state of RA_t conditioned on the measurement trajectory $\mathbf{z} = (z_1, \dots, z_t)$ can be expressed as

$$|\psi_{\mathbf{z}}\rangle_{RA_t} = \frac{1}{\sqrt{P_U(\mathbf{z})}} K_t \dots K_2 K_1 |\Phi\rangle_{RA_0}, \quad (1)$$

where $P_U(\mathbf{z})$ is the probability of obtaining the measurement outcome $\mathbf{z}$.

Meanwhile, when the bath is not monitored, we also consider the dynamics of the measurement-unconditioned quantum state. The unconditioned state at step t is the state averaged over all measurement trajectories as

$$\rho_{RA_t} = \sum_{\mathbf{z}} P_U(\mathbf{z}) |\psi_{\mathbf{z}}\rangle \langle \psi_{\mathbf{z}}|_{RA_t}. \quad (2)$$

Equivalently, one can regard the nonmonitored bath being traced out in each time step. In this regard, the input-output relation from A_{t-1} to A_t , $\mathcal{P}_t(\cdot) = \text{tr}_{B_t} [U_t(\cdot \otimes |0\rangle \langle 0|_{B_{t-1}}) U_t^\dagger]$, defines the quantum channel $\mathcal{P}_t$. Therefore, the unconditional joint state can be described by the overall quantum channel

$$\rho_{RA_t} = \mathcal{I} \otimes (\mathcal{P}_t \circ \mathcal{P}_{t-1} \circ \dots \circ \mathcal{P}_1)(\rho_{RA_0}), \quad (3)$$

where $\mathcal{I}$ refers to the identity channel on the reference R .

B. Connecting to quantum models

Here, we connect the quantum circuit dynamics under consideration to several different quantum models. The detailed interpretation can be found in Sec. VII.

The trajectory-conditional state from bath monitoring in Eq. (1) arises in the generative quantum machine learning models such as QuDDPM [17]. As shown in Fig. 1(c), the goal of QuDDPM is to learn to sample a quantum state $|\phi\rangle$ from a certain distribution P_ϕ in a generative manner. The quantum circuit in Fig. 1(a) transports the initial random distribution toward the target distribution represented by the sample. Although the original QuDDPM does not require a reference system R , the dynamical transport of an ensemble of quantum states $\{\phi_k, P_{\phi_k}\}$ can be equivalently represented by the evolution of an entangled state between a reference R and system A , $|\psi\rangle_{RA} = \sum_k \sqrt{P_{\phi_k}} |k\rangle_R |\phi_k\rangle_A$, where $|k\rangle$ is an orthonormal bases to denote the sample index. The induced quantum state ensemble structure change can generally be captured by the change in the quantum correlation between reference R and A_t .

On the other hand, the dynamics of the unconditional state in Eq. (3) provides insight to the current approaches of QRC [19,21,22,32]. QRC aims to learn a complex functional mapping from a sequence of classical data to an output classical data sequence. Reference [19] highlights the importance of balancing memory time and nonlinearity in QRC, using a Volterra-series-based analysis, and argues that the memory time should align with the timescale of the target learning task. In the NISQRC implementation, classical data are encoded into the quantum system via parametric unitaries $U_1, \dots, U_t$ in Fig. 1(a), and the output features are obtained from marginal distributions $P(z_1), \dots, P(z_t)$ of the measurement results (see details in Sec. VII B), with dynamics characterized by Eq. (3). Although the original NISQRC framework does not involve a reference system R , introducing a classical-quantum (CQ) state over RA provides an equivalent description of classical data input into the quantum system A (for details, see Sec. VII B). Therefore, memory time of

classical input can be captured by the correlation between R and A_t , in the unconditional state in Eq. (3).

In the monitored circuit where a brickwork unitary circuit is probabilistically doped with projective measurements, as shown in Fig. 1(c), both trajectory-conditional and unconditional states can be studied. Our model considered in Fig. 1(a) can be regarded as a toy model of monitored circuits, where the entire system is always fully scrambled between each round of subsystem projective measurements on a fixed number of N_B qubits. Because of the fully scrambling unitary, we can fix the N_B measured qubits as the bath and the rest as the system. While the entanglement dynamics is well studied, the loss of quantum information due to measurement is unclear. To capture the quantum information, we can, therefore, introduce the reference system R initially entangled with the data qubits A . In this context, the conditional state in Eq. (1) refers to the resulting state in a particular experiment, while Eq. (2) corresponds to the average state without postselection.

The quantum circuit can also be interpreted in the scenario of entanglement-assisted communication through the channel $\mathcal{P}_t \circ \dots \mathcal{P}_1$ specified in Eq. (3). In this case, the reference system R provides the entanglement assistance in communication, and the QMI quantity directly corresponds to entanglement-assisted communication capacity [20]. The case without environment assistance corresponds to the unmonitored dynamics described in Eq. (3). On the other hand, for monitored dynamics, in each experiment, the receiver will receive not only the conditional state $|\psi_{\mathbf{z}}\rangle_{RA_t}$ in Eq. (1), but also the measurement trajectory $\mathbf{z}$, which is a classical message about the environment bath status; therefore, we can interpret the monitored dynamics as environment-assisted communication.

In all applications, we introduce the reference system R to facilitate the study of the quantum information dynamics. We begin by considering the reference R having the same size as the system A , $N_R = N_A$, and show the scaling of QMI lifetime, a quantity widely examined in the contexts of communication and error correction. We then generalize to the case of a smaller reference $N_R < N_A$ in both monitored and unmonitored dynamics and highlight the resulting distinct behaviors of the QMI dynamics. Note that the case of $N_R > N_A$ is equivalent to $N_R = N_A$, since the maximal initial information is bounded by $2 \min\{N_R, N_A\}$.

C. Quantum mutual information lifetime

Taking into account the measurement-conditioned state $|\psi_{\mathbf{z}}\rangle_{RA_t}$ and unconditioned state ρ_{RA_t} at time step t , we evaluate the information shared by parties R and A_t via QMI, which quantifies the total correlation between R and A_t at time t . Given a measurement trajectory $\mathbf{z}$, the quantum information retained in A_t at time step t is quantified by the *measurement-conditioned QMI* defined as

$$\overline{I(R:A_t|\mathbf{z})} \equiv \mathbb{E}_{\mathbf{z}} I(R:A_t|\mathbf{z}), \quad (4)$$

where $\mathbb{E}_{\mathbf{z}}$ denotes averaging over measurement trajectories according to the outcome probabilities $P_U(\mathbf{z})$ and $I(R:A_t|\mathbf{z})$ is the QMI of the conditional state $|\psi_{\mathbf{z}}\rangle_{RA_t}$ in Eq. (1). Note that the measurement-conditioned QMI is equivalent to the entanglement entropy between R and A_t , as the conditional state $|\psi_{\mathbf{z}}\rangle$ is pure. For the unconditional dynamics, the *measurement-unconditioned QMI*

$$I(R:A_t) \equiv S(\rho_R) + S(\rho_{A_t}) - S(\rho_{RA_t}) \quad (5)$$

is thus evaluated on ρ_{RA_t} in Eq. (3), where ρ_R and ρ_{A_t} are corresponding reduced states of reference and system R and A_t , respectively.

The differing order of QMI evaluation and the ensemble average $\mathbb{E}_{\mathbf{z}}$ in the measurement-conditioned and unconditioned QMI reflects distinct physical scenarios. The measurement-conditioned QMI [Eq. (4)] captures the information retained in a typical conditional state $|\psi_{\mathbf{z}}\rangle$ obtained from the corresponding measurement trajectory. In contrast, the unconditioned QMI is computed from the averaged state ρ_{RA_t} , which contains no information about any measurement outcome [see Eq. (2)]. We further interpret the measurement-conditioned and unconditioned QMI introduced above in the setting of entanglement-assisted communication, as detailed in Sec. VII D.

In both cases of monitored and unmonitored circuits, to quantify the preservation of quantum information, we define the ϵ lifetime τ_{cond} and τ_{uncond} of quantum information by

$$\text{QMI}|_{t=\tau} = \epsilon \times \text{QMI}|_{t=0}, \quad (6)$$

where QMI can be either $\overline{I(R:A_t|\mathbf{z})}$ or $I(R:A_t)$ for conditional and unconditional dynamics and $\epsilon < 1$ is a free constant. Specifically, with the Bell state as the initial state, the quantum communication capability measured by the coherent information $I(R)_{A_t}$ starts with unity and decays to zero at the half-life of QMI (i.e., $\epsilon = 1/2$) in unmonitored dynamics, as detailed in Sec. VII D.

To study the generic properties of quantum dynamics, we model each of the unitary U_k as a fixed unitary sampled from Haar ensemble. This is a good approximation, e.g., when the quantum dynamics is chaotic or a deep random quantum circuit. To characterize the typical case, we consider the Haar ensemble averaged conditional (unconditional) QMI. To validate the asymptotic scaling of QMI dynamics in the large-system limit, we also perform the numerical simulations with random Clifford unitaries. Details of the stabilizer formalism are presented in Appendix A.

To provide a concrete example, we also consider the Hamiltonian dynamics $U = e^{-iH_{IH}}$ generated by an all-to-all coupled Ising Hamiltonian [19] through a relatively long time $t_H \gg 1$:

$$H = \sum_{i<j} J_{ij} \sigma_j^z \sigma_i^z + \sum_{i=1}^L \eta_i^x \sigma_i^x + \sum_{i=1}^L \eta_i^z \sigma_i^z, \quad (7)$$

where the coupling strength $J_{i,j}$, transverse x -field strength η_i^x , and longitudinal z drive η_i^z are randomly chosen, i.e., standard normal distributed (zero mean and unit variance). All unitaries $U_k = U = e^{-iHt_H}$ are identical throughout the various time steps ($t_H = 50$ is chosen throughout the paper). Our numerical results, summarized in Appendix C, indicate that our conclusions remain valid regardless of whether the same unitary is used at each time step or different unitaries are chosen independently. We expect our results to extend to other chaotic Hamiltonians. As an additional illustration, in Appendix D, we analyze a mixed-field Ising model with only nearest-neighbor interactions.

IV. EXPONENTIAL QMI LIFETIME IN MONITORED DYNAMICS

A. Equal reference and system, $N_R = N_A$

With the framework for information propagation in place, we now examine the dynamics of quantum information under midcircuit measurements and resets,

focusing on the measurement-conditioned QMI in Eq. (4) proposed for monitored dynamics. We begin with the maximally entangled state $|\Phi\rangle_{RA_0} \propto \sum_j |j\rangle_R |j\rangle_{A_0}$ to obtain analytical results, and then we numerically extend the analytical results with less-entangled states including perturbed Haar-random states and CQ hybrid states. We define the perturbed Haar-random states as $|\phi^\delta\rangle_{RA_0} \propto |\phi_{\text{Haar}}\rangle_{RA_0} + \delta|\mathbf{0}\rangle_{RA_0}$. Similarly, we define the family of CQ states to be $(\rho^\delta)_{RA_0} \propto \sum |j\rangle\langle j|_R \otimes |u_j^\delta\rangle\langle u_j^\delta|_{A_0}$ with perturbed basis $|u_j^\delta\rangle_{A_0} \propto |u_j\rangle_{A_0} + \delta|\mathbf{0}\rangle_{A_0}$, where $|u_j\rangle$ forms an orthonormal basis. In both cases, δ interpolates between highly correlated ($\delta = 0$) and uncorrelated states ($\delta \gg 1$). In all these cases above, we take the reference and the system having the same size, $N_R = N_A$.

Using Haar-random unitaries to model the chaotic evolution, we have the following theorem, where $d_A = 2^{N_A}$ and $d_B = 2^{N_B}$ are dimensions of system and bath Hilbert spaces (see Appendix F for a proof).

Theorem 1. In quantum dynamics with midcircuit measurements and reset, the expected measurement-conditioned mutual information of an initial Bell state at time t is asymptotically ($d_A \gg 1$) lower bounded by

$$\begin{aligned} \mathbb{E}_{\text{Haar}} \overline{I(R:A_t|\mathbf{z})} &\gtrsim 2(2t+1)N_A + 2tN_B - 2\log_2[(d_A d_B + 1)^t (d_A + 1)^{t+1}] \\ &\quad - 2\log_2 \left[\left(\frac{d_A d_B - 1}{d_A d_B + 1} \right)^t - \left(\frac{d_A - 1}{d_A + 1} \right)^{t+1} \right] + 2 \end{aligned} \quad (8)$$

$$\simeq 2N_A - 2\log_2[(1 - 1/d_B)t + 1]. \quad (9)$$

In the large-bath limit of $d_B \gg 1$, the dynamics of QMI converges to a universal form $\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|\mathbf{z})} \gtrsim 2N_A - 2\log_2(t+1)$. We can estimate the ϵ lifetime for measurement-conditioned QMI as

$$\tau_{\text{cond}} \gtrsim \frac{d_B}{d_B - 1} (2^{(1-\epsilon)N_A} - 1) \sim \Omega[\exp(N_A)], \quad (10)$$

which grows exponentially with the system size N_A but is independent of bath size when $d_B \gg 1$.

Figure 2(a) shows numerical simulations of the measurement-conditioned QMI dynamics in the model where in each step we apply a random unitary sampled from Haar ensemble (dark blue dots), and its decay behavior versus time steps is well characterized by our theoretical lower bound (red dashed line) in Eq. (9) as long as the bound remains positive. The gap between theoretical lower bound and numerical simulation arises from the difference between Rényi-2 and von Neumann entropies, detailed in Appendix I. The relatively small sample fluctuation shown by the error bars further indicates that the Haar average result can represent the dynamics with a typical

implementation of U . Moreover, the numerical results for dynamics of $\overline{I(R:A_t|\mathbf{z})}$ utilizing a fixed Hamiltonian evolution in each step (light blue dots) also closely aligns with Haar results, thus indicating a wide applicability of our theory to different types of quantum dynamics. In the inset in Fig. 2(a), we interpret the QMI dynamics via the lifetime, and similar agreement between the Haar dynamics (blue dots), Hamiltonian dynamics (cyan dots), and the prediction (red dashed line) from Eq. (9) can be found, confirming the exponential scaling of the lifetime with system size N_A .

To verify the logarithmic decay indicated by Eq. (9) in the thermodynamic limit of system size, we also perform numerical simulations of the monitored dynamics using random Clifford unitaries up to $N_R = N_A = 256$ qubits. In Fig. 2(b), we plot the normalized QMI $\overline{I(R:A_t)}/\overline{I(R:A_0)}$ versus a rescaled time of $(1 - 1/d_B)t + 1$ in logarithmic scaling, as suggested by Eq. (9). For stabilizer states, the von Neumann and Rényi-2 entropy coincide, and we find exact agreement between our theoretical lower bound and numerical results. Moreover, the Clifford unitary suggests that Theorem 1 requires at most a unitary 3-design [34].

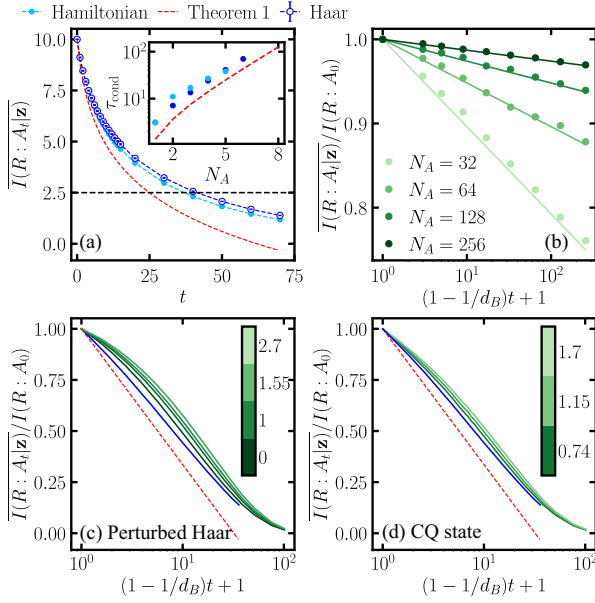


FIG. 2. Measurement-conditioned QMI $\overline{I(R:A_t|z)}$ in monitored dynamics with reset. In (a), we plot numerical simulation results of (von Neumann entropy version) $I(R:A_t|z)$ with Haar-random unitaries (dark blue dots) and fixed Ising Hamiltonian evolution (light blue dots) in a system of $N_R = N_A = 5$, $N_B = 1$ qubits. Error bars represent sample fluctuations of Haar unitary implementations. The red dashed line represents the theoretical asymptotic lower bound of Eq. (9) in Theorem 1. The horizontal black dashed line indicates the threshold of $\overline{I(R:A_t|z)} = \overline{I(R:A_0|z)}/4$. The inset shows the growth of lifetime versus system size N_A of Haar unitary and Hamiltonian evolution. Dots and the red dashed line represent the numerical simulation results and theoretical result, respectively, in Eq. (10) with $\epsilon = 1/4$. In (b), we plot QMI versus a rescaled time in the dynamics with random Clifford unitaries (dots) in a system of $N_R = N_A = 32, 64, 128, 256$ qubits (light to dark) and $N_B = 16$ qubits. The solid lines represent the theory of Eq. (9). In (c) and (d), we plot the decay of normalized QMI in numerical simulations with initial states of perturbed Haar states ϕ_δ and CQ states ρ_δ versus a shifted time separately. Blue lines are Bell-state results with Haar unitary in (a) for reference. The color bar shows the value of δ .

In fact, as shown in Appendix F, we expect that a unitary 2-design already suffices for our theoretical results.

Besides the initial Bell state, we also extend numerical results to the QMI dynamics with various initial states, including the perturbed Haar-random states and CQ states, presented in Figs. 2(c) and 2(d). To compare with Theorem 1, we again plot with the normalized QMI versus a rescaled time $(1 - 1/d_B)t + 1$. As shown in Figs. 2(c) and 2(d), the prediction of Theorem 1 appears linear in such a rescaling (red dashed line). Indeed, the results for various quantum states (green curves) align with the Bell-state result (blue solid line) and demonstrate agreement with the theory prediction (red dashed line).

As the lower bound Eq. (8) [or the simplified version Eq. (9)] becomes negative at extremely late time, we expect

deviations from the lower bound. Indeed, in Figs. 2(c) and 2(d), the curves deviate from the red dashed lower bound curve. In Appendix E, we study this deviation numerically and identify an exponentially decaying tail of QMI. Taking late-time states as the initial states, we show that such a decay still yields a lifetime that is exponential in N_A because of the exponentially suppressed exponent of the tail.

B. Generalization to $N_R < N_A$

We further extend our study of measurement-conditioned QMI dynamics to the regime where the reference is smaller than the system, $N_R < N_A$. Without loss of generality and for analytical convenience, we choose the initial state to consist of N_R Bell pairs shared between reference R and N_R qubits in system A , with the remaining $N_A - N_R$ qubits initialized in $|0\rangle$:

$$|\Phi\rangle_{RA_0} = \frac{1}{\sqrt{d_R}} \sum_{k=0}^{d_R-1} |k\rangle_R |k\rangle_{A_0} \otimes |0\rangle^{\otimes(N_A-N_R)}, \quad (11)$$

where $d_R = 2^{N_R}$ is the Hilbert space dimension of reference. We then have the following theorem for the QMI dynamics (see Appendix F for a proof).

Theorem 2. In quantum dynamics with midcircuit measurements and reset, the expected measurement-conditioned mutual information of an initial state of N_R Bell pairs at time t is asymptotically ($d_R, d_A \gg 1$) lower bounded by

$$\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|z)} \gtrsim 2N_R - 2 \log_2 \left[1 + \frac{d_R t}{d_A} \left(1 - \frac{d_R + 1}{d_R d_B} + \frac{1}{2d_R} \right) \right]. \quad (12)$$

The full version of the lower bound is given in Appendix F. In the limit of $N_R = N_A \gg 1$, it reduces to Eq. (9) in Theorem 1. Moreover, in the large-bath limit of $d_B \gg 1$, the QMI dynamics approaches $\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|z)} \gtrsim 2N_R - 2 \log_2 (1 + d_R t / d_A)$, which exhibits logarithmic decay with rescaled time $(d_R / d_A)t$. Consequently, the ϵ lifetime of the measurement-conditioned QMI is

$$\tau_{\text{cond}} \gtrsim \frac{d_A}{d_R} (d_R^{1-\epsilon} - 1) \sim 2^{N_A - \epsilon N_R}, \quad (13)$$

which still grows exponentially with the system size N_A and is enhanced by a factor of $\sim (d_A / d_R)^\epsilon$ compared with Eq. (10). Another relative timescale is $t \lesssim d_A / d_R^{1-\epsilon}$, over which the QMI remains at least $2(1 - \epsilon)N_R$, indicating near-perfect information protection. Physically, $N_A - N_R$ “idler” qubits act as an auxiliary memory: Although they do not initially encode information, unitary scrambling transfers the encoded information into these degrees of freedom and helps protect it against decoherence from projective

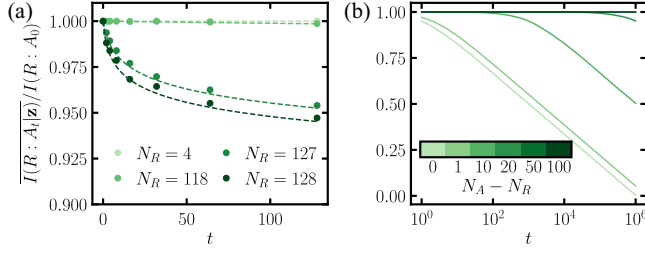


FIG. 3. Normalized measurement-conditioned QMI $\overline{I(R:A_t|z)}$ for $N_R < N_A$ in monitored dynamics with reset. In (a), we plot numerical simulation results with random Clifford unitaries (dots) in a system of $N_A = 128$, $N_B = 16$ qubits and different N_R reference qubits. Dashed line represents the theoretical asymptotic lower bound of Eq. (12) in Theorem 2 for the corresponding N_R . In (b), we plot the asymptotic dynamics of $\overline{I(R:A_t|z)}/I(R:A_0) = 1 - \log_2(d_R t/d_A + 1)/N_R$ with $N_R = 20$ and different N_A (shown by the color bar).

measurements. In particular, for a constant encoded size $N_R \sim \mathcal{O}(1)$, this near-perfect information protection can persist up to $t \sim 2^{N_A}$.

In Fig. 3(a), we verify our theory (dashed lines) with numerical simulations of Clifford circuits (dots) for different reference sizes. For $d_R \ll d_A$ (e.g., $N_R = 4$ and $N_R = 118$), we observe robust information protection over the plotted time window $t \leq 256$. To highlight the dependence on idler size $N_A - N_R$, we plot the universal scaling of normalized conditional QMI in the regime of $d_R, d_B \gg 1$ as $\overline{I(R:A_t|z)}/I(R:A_0) = 1 - \log_2(1 + d_R t/d_A)/N_R$ in Fig. 3(b). For $N_R = 20$, when $N_A - N_R \sim \mathcal{O}(1)$, the QMI shows a logarithmic decay from the initial time $t = 1$, consistent with Eq. (9). As N_A increases, the finite-time dynamics exhibits an increasingly long regime of near-perfect protection (light to dark curves).

V. QMI LIFETIME IN UNMONITORED DYNAMICS

In this section, we study the unmonitored dynamics where the measurement trajectory of bath is not recorded during the dynamics described by Eq. (3).

A. Linear and constant lifetime for typical unitary with $N_R = N_A$

To characterize the decay of quantum information in the unmonitored scenario, we analyze the QMI of the unconditional state defined in Eq. (5). As in the monitored case, we begin with a maximally entangled Bell state of $N_R = N_A$ to facilitate analytical calculations. For computational convenience, we adopt a Rényi version of mutual information $I_2(R:A_t)$ defined by replacing the von Neumann entropy in Eq. (5) with Rényi-2 entropies. Although this extension may not satisfy all property requirements of QMI, we numerically find that it fits well with the dynamics of the actual QMI. Indeed, for special classes of quantum states,

Rényi-2-based QMI well captures quantum correlation [35] and agrees well with von Neumann-based QMI [36], which is also utilized in the experiment of MIPT [37]. With regard to the dynamics of Rényi extended mutual information, we, thus, have the following theorem (see Appendix G for a proof).

Theorem 3. The expected Rényi-2 extended measurement-unconditioned QMI of a Bell initial state in the quantum dynamics of 2-design unitaries with midcircuit measurements and reset at time t is

$$\begin{aligned} \mathbb{E}_{\text{Haar}} I_2(R:A_t) & \simeq \log_2 \left(\frac{(d_A^2 - 1)d_B \left(\frac{(d_A^2 - 1)d_B}{d_A^2 d_B^2 - 1} \right)^t + d_B + 1}{d_A^2 d_B + 1} \right) \\ & - \log_2 \left(\frac{d_A^2 (d_B + 1) - (d_A^2 - 1) \left(\frac{(d_A^2 - 1)d_B}{d_A^2 d_B^2 - 1} \right)^t}{d_A^2 d_B + d_A} \right) + N_A. \end{aligned} \quad (14)$$

In the asymptotic limit $d_A, d_B \gg 1$, when $t \ll 2N_A/N_B$,

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq 2N_A - tN_B, \quad (15)$$

while, for $t \gg 2N_A/N_B$,

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq d_A^2 d_B^{-t}. \quad (16)$$

At early time, Eq. (15) indicates a linear decay of quantum information; while, at late time, Eq. (16) indicates an exponential decay of quantum information.

The late-time exponential decay of QMI in Eq. (16) can be also explained from a quantum channel spectrum approach [19], which holds for general initial states. Here, we focus on the case where the same unitary is applied repeatedly, and, therefore, $\mathcal{P}_k = \mathcal{P}$ for all time steps $1 \leq k \leq t$. Consider the spectrum of the channel defined as the following: $\mathcal{P}\sigma_\alpha = \lambda_\alpha \sigma_\alpha$, where we regard the quantum operators as vectors, e.g., expanded in the Pauli basis. Note that σ_α 's are generally not quantum states. The largest eigenvalue corresponds to the fixed point state of the channel, $\lambda_0 = 1$ and $\sigma_0 = \rho_{\text{fix}}$ satisfying $\mathcal{P}\rho_{\text{fix}} = \rho_{\text{fix}}$. The second-largest eigenvalues (in terms of the norm) generally come in conjugate pairs $\{\lambda_1, \lambda_1^*\}$, with corresponding operator representation of eigenvector $\{\sigma_1, \sigma_1^\dagger\}$ given the real matrix representation of channel $\mathcal{P}$. For the initial state ρ_{RA_0} , we can, in general, decompose it as

$$\rho_{RA_0} = \sum_{i,j} c_{ij} \gamma_i \otimes \sigma_j = \sum_j o_j \otimes \sigma_j, \quad (17)$$

where γ_j forms a basis of operators on reference system R and $o_j = \sum_i c_{ij} \gamma_i$ are operators. Note that o_j 's are

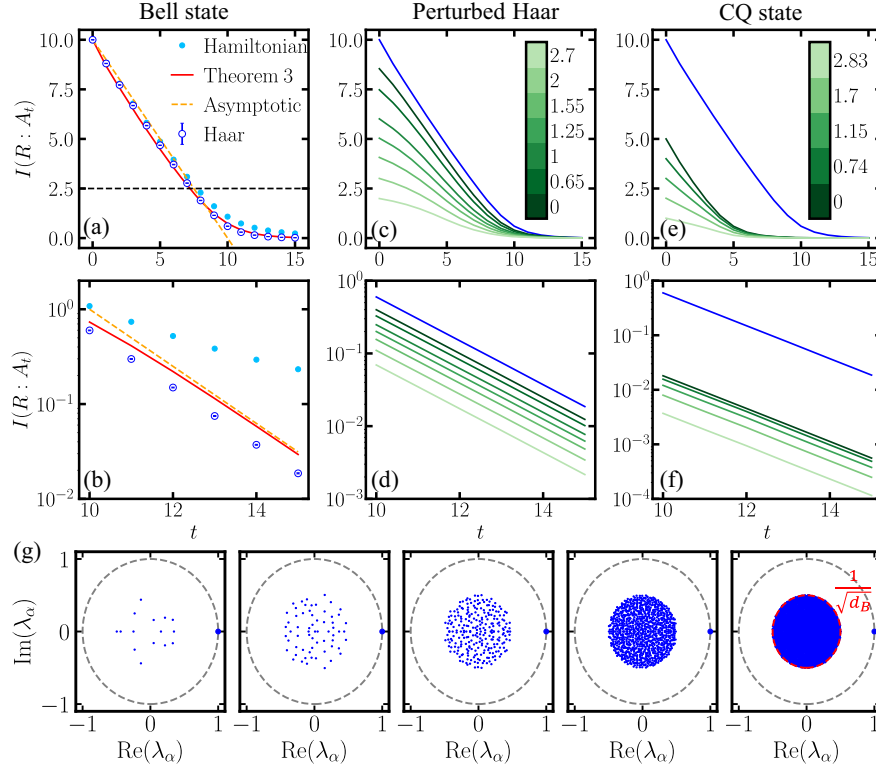


FIG. 4. Measurement-unconditioned QMI $I(R:A_t)$ in unmonitored dynamics with reset. In (a), we plot numerical simulation results of $I(R:A_t)$ of a Bell initial state with random Haar unitary (dark blue dots) and fixed Ising Hamiltonian evolution (light blue dots) in a system of $N_A = 5$, $N_B = 1$ qubits. Error bars represent sample fluctuations of Haar unitary implementations. Red and orange lines represent the exact and asymptotic theory of Eqs. (14) and (15) in Theorem 3. The horizontal black dashed line indicates the threshold of $\epsilon = 1/4$. Late-time dynamics is presented in (b) with the orange line showing the asymptotic theory of Eq. (16) in Theorem 3. In (c) and (e), we plot the numerical simulation of $I(R:A_t)$ under Haar unitary dynamics with initial perturbed Haar states and CQ states separately. Dark to light green lines represent different perturbations δ indicated by the color bar. Blue lines represent the Haar result from (a) for reference. In both cases, we have five reference and data qubits. (d) and (f) are corresponding late-time dynamics, and blue lines is the Haar result from (b). In (g), we show the eigenspectrum of $\mathcal{P}$ implemented by Haar unitary from $N_A = 2$ to 6 (from left to right) with $N_B = 2$ bath qubits. The red dashed circle indicates the radius of $1/\sqrt{d_B}$.

generally not valid quantum states. Applying the channels $\mathcal{P}$ for $t \gg 1$ times following the unmonitored dynamics in Eq. (3), we have

$$\rho_{RA_t} = \sum_j o_j \otimes \lambda_j' \sigma_j \quad (18)$$

$$\simeq o_0 \otimes \rho_{\text{fix}} + \lambda_1^t o_1 \otimes \sigma_1 + \lambda_1^{*t} o_1^\dagger \otimes \sigma_1^\dagger. \quad (19)$$

As QMI is continuous in the quantum states, from Taylor expansion the QMI

$$I(R:A_t) \propto |\lambda_1|^{ct}, \quad (20)$$

where c is a positive integer.

For Haar-random unitary induced channel $\mathcal{P}$, all eigenvalues filling a disk of radius $1/\sqrt{d_B}$ except for λ_0 [38] [see Fig. 4(g)]. Therefore, we have $I(R:A_t) \propto d_B^{-ct}$, which recovers the prediction from Theorem 3.

In Fig. 4(a), we verify the predictions in Theorem 3 for Bell state, finding the close agreement between QMI of numerics (circles) and Rényi-2 extended QMI of theory (lines). The QMI indeed has a linear decay for both the Haar unitary dynamics and the Hamiltonian dynamics, when the environment is not monitored; in the late-time region, we also observe an exponential decay [see Fig. 4(b)]. The Hamiltonian dynamics also shows similar behavior, except that the late-time decay exponent is different from the theory prediction, which we explore in detail in Sec. V C. In Figs. 4(c)–4(f), we further extend the numerical simulation to less correlated initial states, including various perturbed Haar-random states and CQ states. Both the early-time linear decay and late-time exponential decay are observed. To validate our results for larger systems, we perform Clifford circuit simulations and show the results in Fig. 5(a). Up to $N_A = 256$ qubits, the numerics are in exact agreement with the early-time asymptotic prediction in Eq. (15). This agreement again follows from the coincidence of von Neumann and Rényi-2

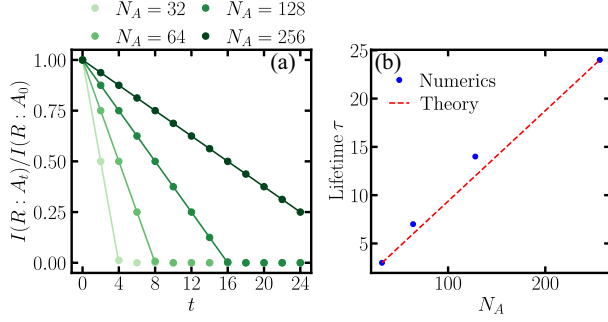


FIG. 5. Measurement-unconditioned QMI $I(R:A_t)$ in unmonitored Clifford dynamics with reset. The initial state is the Bell state with $N_R = N_A$. In (a), we plot the numerical simulations of the normalized QMI $I(R:A_t)/I(R:A_0)$ (dots) in a system of various N_A and $N_B = 16$. The solid lines represent the asymptotic result of Eq. (15) in Theorem 3. In (b), we show the QMI lifetime. Blue dots are numerical results obtained from (a), and the red dashed line is the theoretical prediction of Eq. (21) with $\epsilon = 1/4$.

entropies for stabilizer states. Since stabilizer-state mutual information is quantized, the unconditioned QMI drops linearly to zero and does not exhibit the late-time exponential tail seen for Haar dynamics [Fig. 4(b)].

With Theorem 3 and its generalizations established, we anticipate that the scaling behavior of QMI dynamics can vary depending on the choice of initial state, as explained below.

1. Linear lifetime for typical cases

For typical states of interest, the dynamics is dominated by the linear decay, and we can obtain the unmonitored ϵ lifetime from Eq. (15) as

$$\tau_{\text{uncond}} \simeq 2(1 - \epsilon)N_A/N_B. \quad (21)$$

Interestingly, the lifetime without monitoring the bath agrees with the notion of “thermalization time” in *holographic deep thermalization* [7], which scales $\sim N_A/N_B$ though with a factor of 2 difference due to the Bell initial state. On the contrary, the lifetime of Eq. (10) with bath monitoring reveals a different timescale of information lost toward equilibrium. In Fig. 6(a), we verify the linear QMI lifetime of the Bell initial state for both Haar unitary and Hamiltonian evolution. We further verify the scaling of QMI lifetime in random Clifford dynamics for larger systems in Fig. 5(b). Moreover, through our observations in Figs. 4(c) and 4(e), we expect that the linear lifetime holds for other types of typical states as we considered there.

2. Constant lifetime for residual dynamics

For certain states with extremely low correlation, the late-time dynamics of Eq. (16) dominates. This happens for initial states whose expansion in Eq. (19) becomes precise even at an early time. Since Eq. (16) indicates an

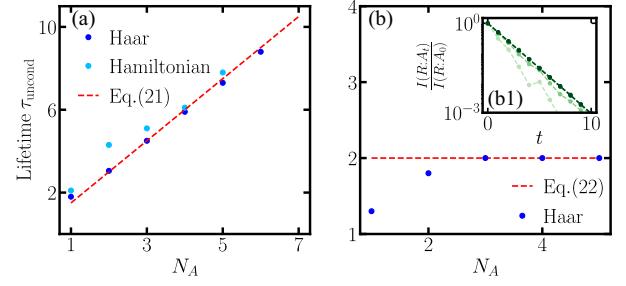


FIG. 6. Lifetime of measurement-unconditioned QMI $I(R:A_t)$ in unmonitored dynamics with reset. In (a), we show QMI lifetime versus system size N_A with Bell initial state for Haar (blue) and Hamiltonian (cyan) unitary implementation. The red dashed line represents the theoretical result in Eq. (21). In (b), we show QMI lifetime for Haar unitary with initial state ρ_{RA_0} to be an unconditional state in dynamics with a Bell initial state at late time. The red dashed line represents the theory in Eq. (22). The inset shows the decay of normalized QMI with $N_A = 1$ to 5 (light to dark). In both cases, we take $\epsilon = 1/4$.

exponential decay at the rate of d_B^{-t} , the lifetime defined in Eq. (6) with time zero shifted to late time leads to

$$\tau_{\text{uncond}} \simeq \log_2(1/\epsilon)/N_B, \quad (22)$$

a constant independent of data system size. In the inset [Fig. 6(b1)], we do see a collapse of exponential QMI decay with increasing system size N_A , when we adopt the initial state from the late-time ($t \gg 2N_A/N_B$) unconditional state [Eq. (2)] evolved from a Bell state $|\Phi\rangle_{RA_0}$. For such weakly correlated states, QMI lifetime does not depend on system size, as shown in Fig. 6(b).

B. Dynamical transition in the generalization to $N_R < N_A$

In this section, we take the initial state to be Eq. (11) and study the measurement-unconditioned QMI dynamics. For analysis convenience, we focus on the Rényi-extended QMI and obtain the following theorem (see Appendix G for a proof).

Theorem 4. In the quantum dynamics of 2-design unitaries with midcircuit measurements and reset, the expected Rényi-2 extended measurement-unconditioned QMI of an initial N_R Bell pair at time t is asymptotically ($d_A, d_B \gg 1$)

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq \log_2(1 + d_R d_A d_B^{-t}) - \log_2 \left[1 + \left(\frac{d_A}{d_R} - 1 \right) d_B^{-t} \right]. \quad (23)$$

The full version is given in Appendix G. From the asymptotic dynamics in Eq. (23), we define two characteristic timescales

$$\tau_{\mp} = (N_A \mp N_R)/N_B, \quad (24)$$

and the unconditional QMI $I_2(R:A_t)$ can be approximated as

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq \begin{cases} 2N_R & t \leq \tau_-, \\ N_A + N_R - N_B t & \tau_- \leq t \leq \tau_+, \\ 2^{N_R + N_A - N_B t} / \log 2 \rightarrow 0 & t \geq \tau_+. \end{cases} \quad (25)$$

This piecewise behavior indicates a dynamical transition of QMI. As a sanity check, setting $N_R = N_A$ in Eq. (25) recovers the asymptotic expansion of Eqs. (15) and (16) in Theorem 3. For finite and nonvanishing timescales $\tau_{\mp}$, we consider the thermodynamic limit of system $N_A \rightarrow \infty$ but finite bath-to-system ratio $N_B/N_A \rightarrow b \in (0, 2)$. Under this assumption, the normalized QMI in Eq. (25) exhibits a dynamical transition. In particular, when the reference N_R is asymptotically small, compared to the system, i.e., $N_R/N_A \rightarrow 0$, the two timescales coincide, $\tau_- = \tau_+ = N_A/N_B =: \tau_c$, and the QMI at the critical time reduces to $I_2(R:A_{\tau_c}) = N_R$.

The nature of this dynamical transition can be understood from a QEC viewpoint. Via an equivalent expansion of the bath at different times, the final state can be viewed as a random QEC code that encodes N_R logical qubits into $N_A + tN_B$ physical qubits (in spacetime). In the noise layer, tN_B qubits undergo fully depolarizing channels. The *quantum Hamming bound* then requires a functioning QEC code to satisfy

$$N_A + tN_B - N_R \geq \log_2 4^{tN_B} = 2tN_B, \quad (26)$$

which leads to the condition of $t \leq (N_A - N_R)/N_B \equiv \tau_-$ in Eq. (25). For $t > \tau_-$, with the increase of erasure errors, the dynamical random code can no longer correct all the errors. In particular, in the limit of $N_A \rightarrow \infty$ with $N_R/N_A \rightarrow 0$, the code undergoes a phase transition from a “good” code to a “bad” code at the critical code rate of $R_c = N_R/(2N_A)$.

In Fig. 7(a), we numerically validate the dynamical transition of unconditioned QMI with $N_R < N_A$ qubits. Using Clifford circuits, the numerical results exactly agree with Eq. (23) in Theorem 4, and, for $t \geq \tau_+$, the QMI vanishes. For a fixed N_A , the time window $t \leq \tau_-$ of near-perfect information protection and correctable errors shrinks as N_R increases (light to dark). We next fix the encoded information to $N_R = 1$ (a single logical qubit) and set $N_B/N_A = 1/8$. As the system size N_A increases (light to dark), the normalized unconditioned QMI dynamics collapse onto a universal scaling form [see Fig. 7(b)], consistent with a dynamical phase transition. In the inset, we further plot the temporal derivative of the normalized QMI and observe a peak at the critical time $t = \tau_c$. Because of

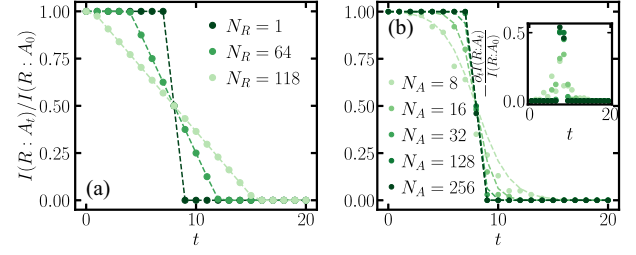


FIG. 7. Normalized measurement-unconditioned QMI $I(R:A_t)/I(R:A_0)$ in unmonitored dynamics with reset and $N_R < N_A$. In (a), we plot numerical simulations with $N_A = 128$, $N_B = 16$ qubits and various reference N_R (dots). In (b), we set $N_R = 1$ and $N_B/N_A = 1/8$ for different system sizes N_A . The dashed lines in both subplots are theoretical results of Eq. (23) in Theorem 4. The inset shows the time derivative of the normalized QMI. The sharpness of transition is limited by the discrete t .

discrete temporal steps in the circuit, the peak only remains finite height rather than diverging.

C. Anomaly in Hamiltonian dynamics: Exponential lifetime from outliers

Figure 4(b) shows that, under the dynamics generated by the Hamiltonian in Eq. (7), the exponent governing the exponential decay of the QMI is significantly smaller than the theoretical prediction given in Theorem 3, indicating a notable deviation from Haar dynamics. In the following, we examine this anomaly in detail.

For the late-time dynamics, Eq. (20), derived using the quantum channel spectrum approach, remains generally applicable. Therefore, we start with an analysis based on the spectrum of the Hamiltonian-induced quantum channel. In Fig. 8(a), we numerically find that, for the Hamiltonian evolution considered, most eigenvalues concentrate within a constant-size disk of size $1/\sqrt{d_B}$ (red dashed circle) for increasing system size N_A , similar to the Haar case in Fig. 4(g). However, there exist isolated eigenvalues located between the bulk of the spectrum and the fixed point eigenvalue $\lambda_0 = 1$. These outliers contribute to the residual information retained in the system and serve as a phenomenological analog, in the channel spectrum, to the eigenstate spectrum observed in the “many-body scar” Hamiltonian [39]. We note that the Hamiltonian in Eq. (7) does not feature a low-entanglement energy tower, which is one of the key hallmarks of many-body scarred systems. As a result, we find that the theoretical memory time, given by $\tau_{\text{eig}} = -1/\log_2(|\lambda_1|)$, deviates from the Haar prediction and instead exhibits exponential growth with N_A , as indicated by the black dashed line in Fig. 8(d).

As an example, we can construct a classically correlated state between the reference and memory system as

$$\rho_{RA_0} = \frac{1}{2} [|0\rangle\langle 0| \otimes \rho_{\text{fix}} + |1\rangle\langle 1| \otimes (\rho_{\text{fix}} + a(\sigma_1 + \sigma_1^\dagger))], \quad (27)$$

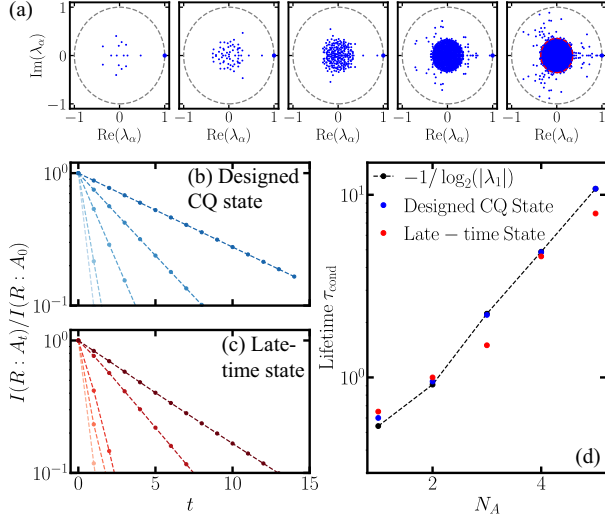


FIG. 8. Measurement-unconditioned QMI in unmonitored Hamiltonian dynamics with reset. In (a), we show the eigen-spectrum of channel $\mathcal{P}_0$ from $N_A = 2$ to 6 (left to right). The red dashed circle indicates the radius of $1/\sqrt{d_B}$. In the left bottom, we plot the decay of normalized QMI in different system sizes with initial state ρ_{RA_0} to be (b) the designed CQ state in Eq. (27) and (c) unconditional state in dynamics with a Bell initial state at late time separately. Colored lines from light to dark represent system size of $N_A = 1$ to 5 . In (d), we show their corresponding lifetime and compare to the memory time τ_{eig} . In all cases, we fix bath size $N_B = 3$.

for a properly chosen a such that $\rho_{\text{fix}} + a(\sigma_1 + \sigma_1^\dagger)$ is a valid quantum state. Here, ρ_{fix} and σ_1 are the eigenvectors of the Hamiltonian-induced quantum channel $\mathcal{P}$. Iterating the dynamics with channel $\mathcal{P}$ in Eq. (3) to time step t , we have

$$\rho_{RA_t} = \frac{1}{2} [|0\rangle\langle 0| \otimes \rho_{\text{fix}} + |1\rangle\langle 1| \otimes (\rho_{\text{fix}} + a\lambda_1^t \sigma_1 + a\lambda_1^{*t} \sigma_1^\dagger)], \quad (28)$$

which is a special case of Eq. (19). Therefore, we can infer the lifetime as in Eq. (20) with $|\lambda_1|$ following exponentially scaling as we have seen. We see that the choice of initial state in Eq. (27) guarantees the dominance of the first nonunity eigenvalue λ_1 in Eq. (19) even at early time. Therefore, the QMI decays exponentially at early time, as confirmed in Fig. 8(b).

Besides the designed CQ state, we numerically evaluate the QMI with initial state from a late-time ($t \gg 2N_A/N_B$) unconditional state evolved from a Bell state under the Hamiltonian dynamics in Fig. 8(c). In contrast to the collapsed behavior in the inset in Fig. 6(b1), QMI decays exponentially at different rates with increasing system size N_A . Figure 8(d) clearly shows an exponential increase in the QMI lifetime, consistent with the scaling of the memory time $\tau_{\text{eig}} = -1/\log_2(|\lambda_1|)$ discussed above.

VI. TRANSITION OF QMI LIFETIME IN PARTIALLY MONITORED DYNAMICS

In this section, we study the dynamics with a tunable monitoring strength. We implement this by applying an erasure channel on $N_E \leq N_B$ qubits in the bath every s time steps [illustrated in Fig. 9(a)]. This is because, when a bath qubit is erased (and to be reset in the next time step), it is equivalent to trace out—the quantum state becomes the average state even conditioned on $\mathbf{z}$. Equivalently, one can consider a fully depolarizing channel instead of erasure. The resulting measurement-conditioned dynamics, therefore, interpolates between fully monitored and unmonitored dynamics, as also discussed in Ref. [40]. In particular, $s = \infty$ reproduces the monitored dynamics of Sec. IV, whereas $s = 1$ with $N_E = N_B$ yields the unmonitored dynamics of Sec. V. For simplicity, we focus on $N_R = N_A$ (the extension to $N_R < N_A$ is analogous). Because the conditional state is also mixed, we work with the Rényi-2 extended conditional QMI. Assuming Haar-random unitaries at each step, we obtain the following theorem where $d_E = 2^{N_E}$ is the Hilbert-space dimension of erased bath qubits (see Appendix H for a proof).

Theorem 5. In the quantum dynamics with midcircuit measurements, reset, and periodic erasure channel on N_E qubits in every s steps, the expected Rényi-2 extended measurement-conditioned QMI of a Bell initial state at time t in the asymptotic limit of $d_A, d_B/d_E \gg 1$ is

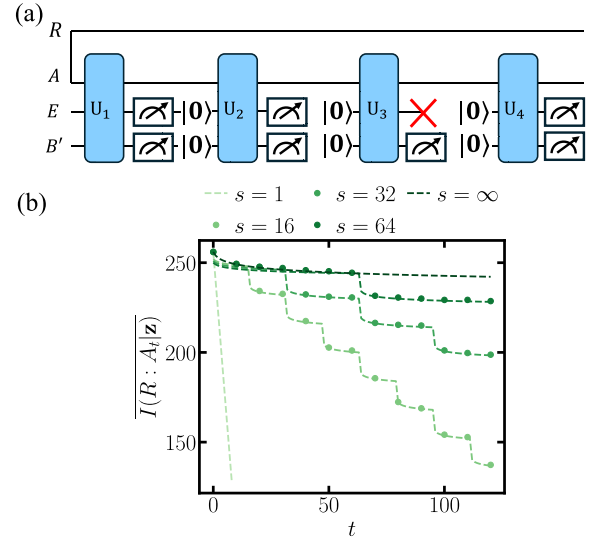


FIG. 9. Measurement-conditioned QMI $\overline{I(R:A_t|\mathbf{z})}$ in partially monitored dynamics with reset. In (a), we show an example of the partially monitored circuit with $s = 3$. The red cross represents the erasure channel. In (b), we plot numerical simulations of Clifford circuits with $N_R = N_A = 128$, $N_{B'} = N_E = 16$ qubits (circles). Dashed lines represent the theoretical prediction of Eq. (30) in Theorem 5.

$$\begin{aligned} & \mathbb{E}_{\text{Haar}} \overline{I_2(R:A_t|\mathbf{z})} \\ & \simeq 2N_A - n_s N_E - \log_2 \left(s \frac{1 - d_E^{-n_s}}{d_E - 1} + t - sn_s + 1 \right) \\ & \quad - \log_2 \left(\frac{s d_E (1 - d_E^{-n_s})}{d_E - 1} + (t - sn_s + 1) d_E^{-n_s} \right). \end{aligned} \quad (29)$$

Here, $n_s = \lfloor t/s \rfloor$ is the number of periods for erasure channels. When $d_E \gg 1$,

$$\begin{aligned} \mathbb{E}_{\text{Haar}} \overline{I_2(R:A_t|\mathbf{z})} & \simeq 2N_A - n_s N_E \\ & \quad - \log_2(1 + t - sn_s) - \log_2 s. \end{aligned} \quad (30)$$

Equation (30) shows that, within each erasure period, the conditional QMI exhibits a logarithmic decay. At each erasure event, the QMI decreases by N_E , leading to a two-scale transition. More explicitly, for $t \in [n_s s, (n_s + 1)s]$ with fixed n_s , $\mathbb{E}_{\text{Haar}} \overline{I_2(R:A_t|\mathbf{z})} \simeq (2N_A - n_s N_E) - \log_2 s - \log_2(1 + t - sn_s)$ presents a logarithmic decay of QMI with t . In contrast, if we probe times $t/s \in \mathbb{N}^+$ at the erasure events, the QMI $\mathbb{E}_{\text{Haar}} \overline{I_2(R:A_t|\mathbf{z})} \simeq 2N_A - \log_2 s - tN_E/s$ reduces to an effective linear decay on macroscopic time-scales. Overall, we observe this two-scale transition in the conditional QMI dynamics. This transition admits a statistical physics interpretation. The QMI can be mapped to the free energy of a permutation-spin model, where each spin takes values of identity or swap. An erasure event imposes an identity boundary condition at that time slice, biasing subsequent spins toward the identity. As a result, logarithmic decay survives only when erasures are rare $t/s \sim o(1)$, whereas, for $t/s \sim \Omega(1)$, the effective behavior is linear.

For the ϵ -QMI lifetime, we estimate the critical number of periods as $n_s^* = [(2 - \epsilon)N_A - \log_2(s)]/N_E$, so that $\tau_{\text{cond}} = sn_s^* = s[(2 - \epsilon)N_A - \log_2(s)]/N_E$. If $s \sim \mathcal{O}(\text{poly}(N_A))$, then $\tau_{\text{cond}} \sim \mathcal{O}(\text{poly}(N_A))/N_E$. In contrast, for $s \sim \exp(N_A)$, the lifetime can grow exponentially, $\tau_{\text{cond}} \sim \exp(N_A)$.

In Fig. 9(b), we present Clifford circuit simulations of the conditional QMI in the partially monitored setting with different s . When $s \rightarrow \infty$, the conditional QMI decays logarithmically in time, consistent with Eq. (9) in Theorem 1. For a fixed time window [e.g., $t \leq 128$ in Fig. 9(b)], decreasing s increases the frequency of erasure events, producing more steplike plateaus (“stairs”) and shortening the logarithmic-decay segments between successive drops. As a result, the dynamics crosses over continuously from a logarithmic to a linear scaling. In the extreme case of $s = 1$, we recover Eq. (15) in Theorem 3.

VII. INTERPRETATION IN QUANTUM CIRCUIT MODELS

Having completed the analysis of QMI lifetime, we now apply the framework introduced in Sec. III to interpret our results within the context of quantum computing models, monitored quantum circuits, and entanglement-assisted communication.

A. Quantum diffusion models

In quantum diffusion models, i.e., QuDDPM [17], we aim to learn the distribution of a target state ensemble and generate new samples following this distribution, referred to as quantum generative learning [see Fig. 1(c)]. In the forward process of QuDDPM, the state samples from the target state ensemble are evolved with controllable scrambling unitaries to approach the Haar-random state ensemble. The forward process stands as a reference dynamical process for the learning in backward process. In the backward learning process, the data and bath system are initially prepared in a random state and a trivial product state. Here, we introduce the reference system to index the states on the data system for analysis convenience. In particular, for the state ensemble $\{|\phi_k\rangle, P_{\phi_k}\}_{k^*}$, it can be equivalently represented as

$$|\psi\rangle_{RA} = \sum_k \sqrt{P_{\phi_k}} |k\rangle_R |\phi_k\rangle_A. \quad (31)$$

In the t th step, the joint system of data and bath evolves through a parametrized unitary $U_t(\theta_t)$ followed by mid-circuit measurements on the bath system. Ideally, we expect the output state ensemble in the t th step to match the corresponding state ensemble in the forward process, and, as the backward denoising process proceeds, the final output state ensemble from backward circuits follows the same distribution of the target state ensemble of interest. As the output state depends on the whole measurement trajectory, the backward learning process refers to the monitored dynamics studied in Sec. IV.

According to our theoretical results in Theorem 1, in the backward denoising process, if we simply apply unitary circuits with randomly initialized parameters and focus on a single measurement trajectory, it at least requires an exponential number of steps to erase the correlation between the reference index system and the states in the data system, which makes the denoising process inefficient and hard for near-term devices and applications. This negative insight, thus, indicates that one should not focus on single-trajectory learning. In addition, training may be able to resolve the issue for the single-trajectory case. Indeed, when we compare the output state ensemble of backward circuits and forward scrambling, minimizing a well-designed loss function can guide us to optimize the parametrized unitary $U_t(\theta_t)$, which stores information for mapping from the input state ensemble from the prior step

in the backward process toward the output ensemble in the corresponding forward step and, therefore, boosts the backward learning process in reducing the number of steps.

B. Quantum reservoir computing

The objective of QRC is to learn a functional that maps a temporal sequence $\mathbf{u} = \{\dots, u_0, u_1, \dots, u_t, \dots\}$ to an output sequence $\mathbf{y}$. In a midcircuit-measurement-based QRC setup [19,25], the system consists of a memory and readout register with N_A and N_B qubits, respectively. The system undergoes a sequence of unitary evolutions $U_t(u_t)$, conditioned on the classical input u_t [see Fig. 10(a)]. In typical implementations of $U_t(u_t)$, the unitary, for instance, can be generated by a parametrized Hamiltonian $H + u_t \sum_{i=1}^L \eta_i^z \sigma_i^z$ with H defined in Eq. (7).

After implementing each unitary, one measures the readout system to obtain results $\{\dots, z_0, z_1, \dots, z_t, \dots\}$, each $z_t \in \{0, 1\}^{N_B}$ is a binary bit string. Through repetitions of experiments, one can obtain the learning features $\mathbf{x}(t)$, a normalized 2^{N_B} -dimensional vector of unconditional measurement statistics of $z_t \in \{0, 1\}^{N_B}$, with each component being the probability of the corresponding measurement outcome z_t . For example, with $N_B = 3$ readout qubits, $x_7(t) = \text{Prob}[z_t = 111]$. The final output y_t is estimated as a weighted sum: $y_t = \mathbf{w} \cdot \mathbf{x}(t)$.

Consider the dynamics that starts with unitary $U_0(u_0)$ encoded with the classical input u_0 , as shown in Fig. 10(b); one can equivalently introduce a reference system R , whose quantum state $|u_0\rangle$ determines the dynamics of the memory-readout system $A_{-1}B_{-1}$ via a measurement: One measures R in the encoding bases $\{|u_0\rangle\}$ and then performs the unitary $U_0(u_0)$ conditioned on the measurement result. Taking a step further in Fig. 10(c),

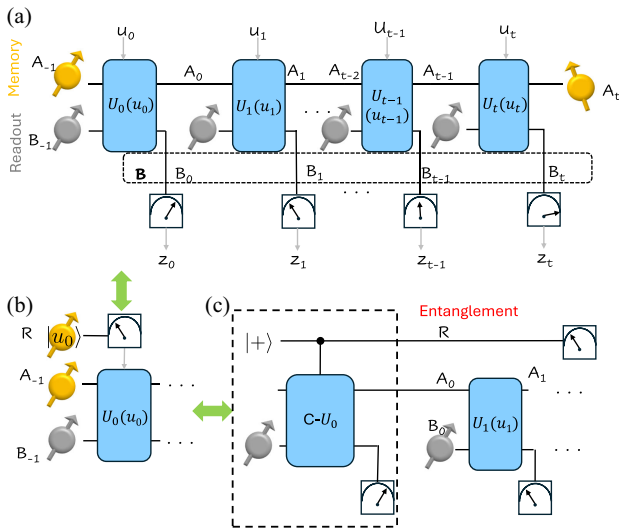


FIG. 10. Reduction from the reservoir computing paradigm with classical inputs $u_0 u_1 \dots u_t$ to the quantum circuit with entangled input RA_0 .

one can consider R in a superposition state $|+\rangle \propto \sum_{u_0} |u_0\rangle$ assuming equal prior probability of u_0 without losing generality, and the measurement-conditioned unitary is realized as a control-unitary gate defined as $C - U_0 = \sum_{u_0} |u_0\rangle\langle u_0|_R \otimes U_0(u_0)$. In this regard, assuming an initial trivial product state, the $RA_{-1}B_{-1}$ system becomes entangled via the control unitary, and, after the first measurement on the readout system, the state RA_0 remains entangled while B_0 is reset to $|0\rangle_{B_0}$, leading to a reduction toward the setup in Fig. 1(a). Since our theoretical analysis assumes typical and sufficiently complex unitary dynamics $U_1, \dots, U_t$, we can neglect the subsequent inputs $u_1, \dots, u_t$ when focusing on the memory of the initial input u_0 , following the same reasoning as in Ref. [19].

In this context, the fading memory effect in reservoir computing can be quantified by the correlation between the initial input u_0 and the readout z_t as a function of the time step t . Since QRC employs marginal measurement statistics to compute $y_t = \mathbf{w} \cdot \mathbf{x}(t)$, all relevant information is contained within the unconditional quantum state described in Eq. (3). Consequently, the correlation between u_0 and z_t can be characterized by the unconditional classical mutual information $I^C(u_0; z_t)$, which is upper bounded by the quantum mutual information in Eq. (5), as dictated by the data processing inequality.

By initializing the system in a Bell state on RA_0 , the linear-in-system-size lifetime of the unconditional QMI dynamics discussed in Sec. VA also implies a corresponding memory time linear in the size of the memory subsystem of QRC, assuming sufficient complex unitary dynamics. For Hamiltonian dynamics, on the other hand, the structure of the Hamiltonian considered supports exponential lifetime in system size, as we recognize in Sec. VC when tailoring the correlation to spectrum structure. However, such a specific weak initial correlation requires the input u_0 to affect the dynamics in a specific form and may not be utilized in a specific learning task. Overall, the general framework to interpret information loss in QRC is also applicable to recent advances in feedback-driven QRC with appropriate modifications [21].

To make use of more information from measurements, we propose to utilize a longer sequence of the memory outcome in multiple consecutive steps instead of a single step. Let us begin with an extreme case, where at each step one adopts the full measurement outcome history $\mathbf{z}_t = (z_1, z_2, \dots, z_t)$ in learning the approximation of the function. Considering the $N_B t$ total number of qubits measured, the measurement statistics of $\mathbf{z}$ defines the feature $\mathbf{x}(t) = \{x_0(t), \dots, x_{2^{N_B t}-1}(t)\}$, i.e., the probability of the measurement outcomes, and output can be estimated by $y_t = \mathbf{w} \cdot \mathbf{x}(t)$. Note that such an extended scheme includes the case of adopting only the distribution of measurement result z_t from the t th step via choosing a weight to reproduce the output only from marginal distribution. Consider the case of $t = 2$ as an example,

$z \in \{00, 01, 10, 11\}$ and $x(2) = \{x_{00}, x_{01}, x_{10}, x_{11}\}$; by limiting the weight to be $w = (w_0, w_1, w_0, w_1)$, the output using the whole $x(2)$ then becomes $y_2 = w_0(x_{00} + x_{10}) + w_1(x_{01} + x_{11})$, which matches the output from marginal distribution at $t = 2$.

Apparently, now y_t is derived from the statistics of the entire trajectory, and the mutual information between y_t and u_0 is upper bounded by

$$I^C(u_0 : y_t) \leq I^C(u_0 : z_t). \quad (32)$$

In this case, memory time is maximal, as the early measurement result z_1 always appears in any y_t as expected. However, we can ask how much additional information ΔI_{t+1}^C the additional measurement z_{t+1} provides about u_0 to provide insight on whether increasing t can assist in the learning of u_0 . Taking the upper bound as an estimation, we have

$$\Delta I_{t+1}^C \equiv I^C(u_0 : \{z_1, \dots, z_t, z_{t+1}\}) - I^C(u_0 : \{z_1, \dots, z_t\}) \quad (33)$$

$$= I^C(u_0 : z_{t+1} | \{z_1, \dots, z_t\}) \equiv I^C(u_0 : z_{t+1} | z_t). \quad (34)$$

In this case, the additional correlation between z_{t+1} and u_0 can be described by the classical conditional mutual information $I^C(u_0 : z_{t+1} | z_t)$, as previous measurement outcomes are taking into account in the learning. Adopting data processing inequality in each trajectory of z_t , we have $I^C(u_0 : z_{t+1} | z_t) \leq I(R : A_t | z_{t-1})$. Accordingly, results in Sec. IV indicate a slow logarithmic decay, and, therefore, the additional information is appreciable for an exponentially long time.

C. Monitored quantum circuits

In the monitored quantum circuit setup, a brickwork-style unitary circuit is applied onto a trivial product initial state, with projective measurements randomly performed on qubits following every layer of unitary gates at a chosen probability q . Recent studies [3,4] have shown that, with the increasing measurement probability q , the bipartite entanglement entropy of the resulting state transits from volume law to area law, known as the MIPT. The model we considered in Fig. 1(a) can be regarded as a toy model of monitored circuits in the weak-measurement limit, where the data and bath system are always fully scrambled between each time of projective measurements. In addition, instead of measuring each qubit with a certain probability, we fix the number of qubits to be measured as N_B among the $N_A + N_B$ qubits. For a typical Haar-random unitary with exponential complexity [41], the equivalent measurement probability $q \sim N_B / (N_A + N_B) \exp(N_A + N_B)$ can be exponentially small; therefore, our model can be regarded as the monitored circuit in the weak-measurement

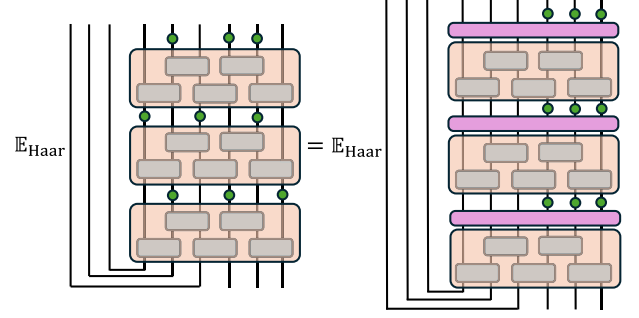


FIG. 11. Equivalence of the monitored quantum circuit in the weak-measurement limit (left) and the monitored circuit in the weak-measurement limit with fixed qubits to be measured (right). Purple rectangles represent permutation operators.

limit and the output state follows volume-law entanglement for any cut within the data system.

With the extension of the monitored circuit, initial information stored in a local system, i.e., the data system, propagates across the whole system via random unitary circuit and gets lost due to the projective measurements. We focus on the information remained in the unmeasured subsystem through the dynamical process of monitored circuit in the weak-measurement limit. To keep track of the initial information in the local subsystem, we introduce the same reference system that is maximally entangled with the data system. In the weak-measurement limit, given that the unitary circuit between every two layers of measurements is sufficient complex (Fig. 11, left), we are allowed to insert an extra layer of permutation operation (purple) to rearrange the position of measured and unmeasured qubits in the system while without changing the ensemble average result (see Fig. 11, right). With this equivalence transformation, the measurement-conditioned QMI $\overline{I(R : A_t | z)}$ defined in Eq. (4) characterizes the typical amount of information remained in the unmeasured system, and, according to our theoretical result of Theorem 1 in Sec. IV, the quantum information initially stored in a local subsystem can persist in the quantum system for exponentially long time through monitored circuits in the weak-measurement limit. However, if we do not keep the record on the measurement outcome trajectory in the experiment, the quantum information can persist in a linearly long time only following Theorem 3 in Sec. V.

D. Entanglement-assisted communication

To close this section, we interpret our results in the entanglement-assisted communication setting. While we focus on classical messages, quantum messages are equivalent at half of the communication rate due to quantum teleportation [20].

Let us begin with the measurement-unconditioned QMI in Eq. (5). As shown in Fig. 12, top, the unmonitored

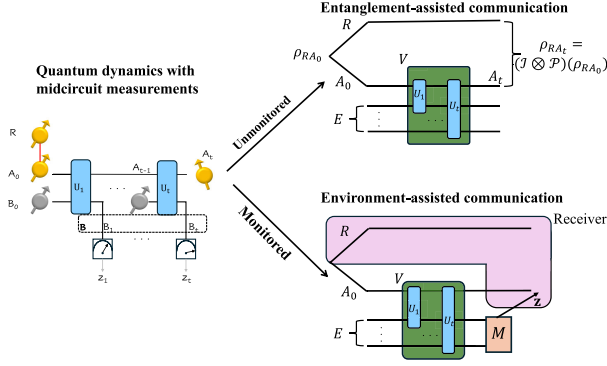


FIG. 12. Connection from quantum dynamics with midcircuit measurements in unmonitored and monitored cases to entanglement-assisted communication without (top) and with (bottom) environment assistance.

dynamics naturally fits into the setup of entanglement-assisted communication [20]. The sender aims to send a classical message via the input A_0 toward the quantum channel $\mathcal{I}_R \otimes \mathcal{P}$ with $\mathcal{P} = \mathcal{P}_t \circ \dots \circ \mathcal{P}_1$. Following the definition of channel $\mathcal{P}$ of Eq. (3), we can represent the channel $\mathcal{P}$ via a unitary $V = \prod_{\ell=1}^t U_\ell$ applied on the sender A_0 and environment $E = B_0 \dots B_{t-1}$, which is a joint system of expanded baths initialized in $|0^{\otimes N_{B^t}}\rangle_E$. A reference system R to purify the input A_0 is sent through the identity channel as the entangled ancilla, which is required by definition of the communication setup and different from the cases discussed above. The entanglement-assisted classical capacity of the channel $\mathcal{P}$ quantifies the maximum bit transmission rate, provided the shared entanglement, is known as [20]

$$C_E(\mathcal{P}) = \max_{\rho_{A_0}} S(\rho_{A_0}) + S[\mathcal{P}(\rho_{A_0})] - S((\mathcal{I} \otimes \mathcal{P})|\Phi\rangle\langle\Phi|_{RA_0}) \quad (35)$$

$$= \max_{\rho_{A_0}} S(\rho_R) + S(\rho_{A_t}) - S(\rho_{RA_t}) \quad (36)$$

$$= \max_{\rho_{A_0}} I(R:A_t), \quad (37)$$

where in the second line we utilize the fact that R purifies A_0 . Therefore, our results on the unmonitored dynamics in Theorem 3 provide an achievable lower bound on the entanglement-assisted capacity, as Bell-state input is one of the possible input states. Similarly, in *noisy superdense coding*, the unconditional QMI $I(R:A_t)$ quantifies the amount of classical bits to be sent provided a shared entangled state ρ_{RA_t} [42].

Next, we connect the measurement-conditioned QMI defined by Eq. (4) in the scenario of environment-assisted classical communication. We still follow the setup of noisy superdense coding, where Alice applied a quantum channel $\mathcal{P}$ to her side and send it to Bob, who owns the other side R

entangled with A_0 and A_t at the end to recover the information sent from Alice, shown by the pink shaded area. However, unlike the earlier unconditional case, now at the output side of the environment, Charlie performs measurement M and transmits the measurement outcome $\mathbf{z}$ to the receiver A_t via classical communication. In each time conditioned on the measurement outcome $\mathbf{z}$, the maximum information Bob can learn about Alice's information is the QMI of state $|\psi_{\mathbf{z}}\rangle_{RA_t}$; therefore, the measurement-conditioned QMI $\overline{I(R:A_t|\mathbf{z})}$ represents an achievable rate of communication *on average case*, and the logarithmic decay of conditional QMI in Theorem 1 indicates a large advantage in communication rate due to the assistance of classical information from environment.

Besides the entanglement-assisted communication scenario, we can further consider a connection to quantum communication. As $S(\rho_R)$ remains a time-dependent constant in our setup [for instance, $S(\rho_R) = N_A$ for Bell-state input], the decay of unconditional QMI, in fact, reveals the dynamics of coherent information

$$I(R:A_t) \equiv S(\rho_{A_t}) - S(\rho_{RA_t}) \quad (38)$$

$$= I(R:A_t) - S(\rho_R). \quad (39)$$

Following our definition of QMI lifetime in Eq. (6), for a given initial input state ρ_{RA_0} , there always exists a threshold above which a positive quantum communication from R to A_t can be realized. The threshold corresponds to $I(R:A_t) = I(R:A_t) - S(\rho_R) = 0$, and, since from Eq. (6) we define $I(R:A_t) = \epsilon I(R:A_0)$, we have the threshold value $\epsilon = S(\rho_R)/I(R:A_0)$. Specifically, for a pure entangled state $|\Phi\rangle_{RA_0}$, we have $\epsilon = 1/2$.

VIII. QMI DYNAMICS WITH AND WITHOUT RESET

Up to this point, our analysis has focused on the quantum dynamics with midcircuit measurements followed by a reset operation, which erases any prior information in the bath and resets it to a trivial pure state—e.g., $|0\rangle^{\otimes N_B}$ —for the next time step. In this section, we compare with the QMI dynamics without the reset operation, where the measurement outcome is retained in the bath qubit and carried into the next time step.

For the measurement-conditioned QMI $\overline{I(R:A_t|\mathbf{z})}$, we prove that the dynamics is unchanged by whether the reset operation is applied (see the proof in Appendix B). When we do not perform any reset operation to erase the measurement outcome $\mathbf{z}$ in the preceding step, the initial state of bath in the following step is $|\mathbf{z}\rangle$. The physical intuition behind this insensitivity is that we can insert an adaptive layer of local Pauli- X gates that transforms the bath state $|\mathbf{z}\rangle$ to the trivial reset state $|0\rangle$. Since local gates do *not* change entanglement, they leave the conditional

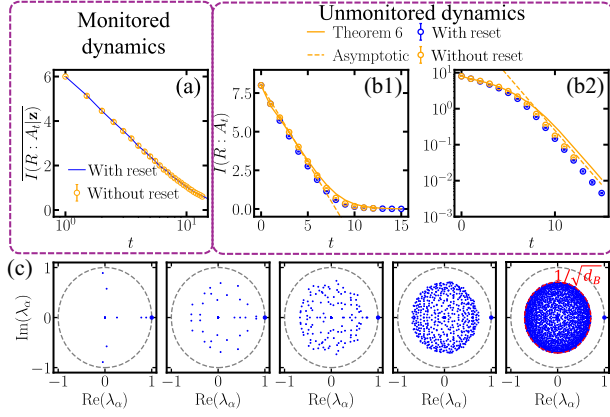


FIG. 13. QMI dynamics in monitored and unmonitored dynamics with and without reset. In (a), we plot decay of measurement-conditioned QMI with reset (blue) and without reset (orange). In (b1) and (b2), we plot numerical simulation of measurement-unconditioned QMI with pure state reset (blue circles) and without pure state reset (orange circles) in early and late time separately. Orange solid and dashed lines show the theory of Eq. (42) in Theorem 6 and asymptotic theory, respectively. Error bars represent sample fluctuations of Haar unitary implementations. The system consists of (a) $N_A = 3$ and (b1), (b2) $N_A = 4$ data qubits and $N_B = 1$ bath qubits. In (c), we show the spectrum of Haar channel without reset of system size from $N_A = 1$ to 5 (left to right) with $N_B = 1$ bath qubits. The red dashed circle indicates the radius of $1/\sqrt{d_B}$.

QMI unchanged. Equivalently, these X gates can be absorbed into the subsequent global unitary layer. We verify the conclusion through numerical simulations for the logarithmic decay in Fig. 13(a) (see verification on the residual dynamics in Appendix E).

For measurement-unconditioned QMI $I(R:A_t)$, if no reset is applied after each bath measurement, the joint reference-system-bath state becomes a classical mixture over measurement-conditioned states. At step t , the resulting unconditioned dynamics can be written as the channel

$$\mathcal{Q}_t(\cdot) = \sum_{z_t} |z_t\rangle\langle z_t|_{B_t} (U_t(\cdot) U_t^\dagger) |z_t\rangle\langle z_t|_{B_t}, \quad (40)$$

which corresponds to a fully dephasing channel on the bath and also considered in Ref. [19]. The corresponding output joint state at step t can be expressed analogously to Eq. (3) as

$$\rho_{RA_t B_t} = \mathcal{I} \otimes (\mathcal{Q}_t \circ \mathcal{Q}_{t-1} \circ \cdots \circ \mathcal{Q}_1)(|\Phi\rangle\langle\Phi|_{RA_0} |0\rangle\langle 0|_{B_0}). \quad (41)$$

For analytical convenience, we consider the Rényi-2 extended unconditional QMI and obtain the following theorem (see Appendix G for a proof).

Theorem 6. In the quantum dynamics of 2-design unitaries with midcircuit measurements and without

reset, the expected Rényi-2 extended measurement-unconditioned QMI of a Bell initial state at time $t \geq 1$ is

$$\begin{aligned} \mathbb{E}_{\text{Haar}} I_2(R:A_t) & \simeq -\log_2 \left(\frac{(d_A^2 - 1)(d_B - 1)}{d_A^2 d_B - 1} \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t + 1 \right) \\ & + \log_2 \left((d_A^2 - 1) \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t + 1 \right). \end{aligned} \quad (42)$$

In the asymptotic limit $d_A, d_B \gg 1$, when $t \ll 2N_A/N_B$, $\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq 2N_A - tN_B$, while, for $t \gg 2N_A/N_B$, $\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq d_A^2 d_B^{-t}$.

Theorem 6 shows that the unconditioned QMI dynamics without reset operation coincides with the reset case of Theorem 3 in the asymptotic limit. Intuitively, because the unitaries are random and strongly scrambling, the measurement-conditioned states obtained without reset are effectively typical, so the bath's initial state has little influence on the resulting QMI. We verify this numerically by comparing the unconditioned dynamics without reset (orange circles) with the reset case (blue circles) in Figs. 13(b1) and 13(b2), finding a strong overlap. Our theoretical prediction of Eq. (42) in Theorem 6 also agrees with the numerical simulations, reproducing the early-time linear decay and the late-time exponential tail. On the other hand, if the bath is reset to a fully mixed state, the unconditioned QMI decays twice faster than the pure state reset case (see Appendix B for details).

We also analyze the channel spectrum of Eq. (40) for interpreting the late-time behavior as illustrated in Sec. V (see Appendix G 3 for some details). In Fig. 13(c), we plot the eigenvalue distribution of the quantum channel $\mathcal{Q}$ for a Haar-random unitary U and also observe that the spectrum is bounded within a disk of radius $1/\sqrt{d_B}$ (red circle) in the large- N_A limit. This behavior is consistent with the phenomenon observed in the unmonitored dynamics with reset [see Fig. 4(g)]. Therefore, we conclude that whether adopting the reset or not does not affect QMI in both monitored and unmonitored dynamics. Finally, we emphasize that, even though the QMI memory time of the unconditioned states is identical in both the reset-present and reset-free cases, this quantity characterizes memory only during the transient regime of QRC. In the more general time-asymptotic regime $t \rightarrow \infty$, the reset-free QRC exhibits no response to any new input u_t , whereas the reset-present QRC retains a nontrivial persistent memory to new inputs. This is a fundamentally crucial difference between reset-present and reset-free settings when applying the unconditioned dynamics model to QRC.

Because of the intrinsic chaotic property of Hamiltonian dynamics considered here, we expect that the reset strategy would lead to similar scaling laws. Meanwhile, the anomaly case of QMI lifetime in Hamiltonian dynamics

without the reset strategy remains unexplored, and we leave it as an interesting open question.

IX. EXPERIMENT OF QMI DYNAMICS ON NEAR-TERM HARDWARE

In this section, we validate the exponential separation in measurement-conditioned and unconditioned QMI dynamics through near-term hardware compatible realizations. With a focus on the superconducting platform of IBM Quantum devices, we implement the brickwork circuit to mimic the implementation of Haar-random unitary in theoretical modeling of Fig. 1(a) with appropriate modifications to reduce circuit compilation cost (see Appendix J). We perform both ideal and noisy simulations of IBM Quantum devices via Qiskit [43]. For the IBM Quantum Torino we targeted at, it has median T_1 as 185.04 μs , median control-Z (CZ) gate error 2.566×10^{-3} , median square-root-X (SX) error 3.262×10^{-4} , and median readout (measurement) error 3.052×10^{-2} . We also perform the noisy simulation of IBM Quantum Sherbrooke and present the results in Appendix J.

In Fig. 14(a), we evaluate the QMI dynamics implemented with the brickwork circuit. For the ideal simulation, both measurement-conditioned (light blue dots) and unconditioned (light green dots) QMI align with the simulation results with Haar unitaries (solid lines), demonstrating its scrambling ability to mimic the dynamics with random unitaries.

The above simulation results on the dynamics of QMI rely on a complete tomography of quantum states. In practice, one could consider the alternatives based on Rényi entropy [37]. However, for Rényi entropy estimation via the randomized measurement, there still exists a statistical error factor of $1/\sqrt{M}$ for applying M sets of measurement unitaries due to central limit theorem, and a large number of shots are still required for each choice of measurement unitaries [44]. To mitigate the resource cost for practical applications, inspired by the cross-entropy benchmark for verifying MIPT [45], we propose a

quantum-to-classical (Q2C) mutual information protocol. Through the same dynamics as in Fig. 1(a), we define the measurement-conditioned Q2C mutual information as

$$\overline{I^{\text{Q2C}}(R:A_t|\mathbf{z})} \equiv \mathbb{E}_{\mathbf{z}} D_{\text{KL}}[P_{U|\mathbf{z}}(z_R, z_{A_t}) || P_{U|\mathbf{z}}(z_R) \otimes P_{U|\mathbf{z}}(z_{A_t})], \quad (43)$$

where $P_{U|\mathbf{z}}(z_R) = |\langle z_R | \psi_{\mathbf{z}} \rangle_{RA_t}|^2$ is the marginal distribution of measurement result on subsystem R and similar for $P_{U|\mathbf{z}}(z_{A_t})$, while $P_U(z_R, z_{A_t}) = |\langle z_R, z_{A_t} | \psi_{\mathbf{z}} \rangle_{RA_t}|^2$ is the corresponding joint distribution of (z_R, z_{A_t}) on the conditional state $|\psi_{\mathbf{z}}\rangle_{RA_t}$ [see definition in Eq. (1)], and D_{KL} is the KL divergence. In the same spirit, we can also define the measurement-unconditioned Q2C mutual information as

$$I^{\text{Q2C}}(R:A_t) \equiv D_{\text{KL}}[P_U(z_R, z_{A_t}) || P_U(z_R) \otimes P_U(z_{A_t})], \quad (44)$$

with marginal and joint distributions measured on the unconditional state ρ_{RA_t} defined in Eq. (2), i.e., $P_U(z_R) = \text{tr}(\rho_{RA_t} |z_R\rangle\langle z_R|)$ and $P_U(z_R, z_{A_t}) = \text{tr}(\rho_{RA_t} |z_R, z_{A_t}\rangle \times \langle z_R, z_{A_t}|_{RA_t})$.

Through the conversion of quantum correlation to state measurement outcome statistics, we need to perform only computational basis measurement in the Q2C protocol without any extra quantum operation, which significantly reduces the sample complexity. The measurement results $\{(z_R, z_{A_t}, \mathbf{z})\}$ with measurement trajectory of bath $\mathbf{z}$ can be further utilized to estimate all statistics involved in Eqs. (43) and (44). This protocol can save more resources in running quantum devices following the spirit of “*measure first, ask questions later*” [46] though at the cost of losing characterization of quantum information partially through the quantum-to-classical conversion. We leave an in-depth analysis of complexity and statistical error as future studies.

In Fig. 14(b), we show the dynamics of Q2C mutual information in the ideal setup. Similar to the behavior of

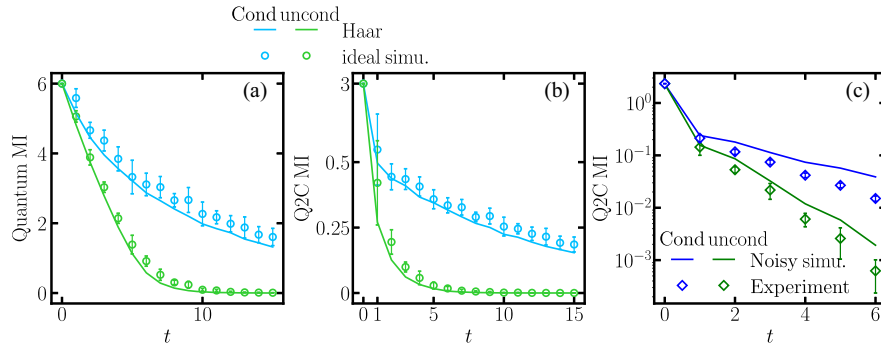


FIG. 14. Dynamics of measurement-conditioned and unconditioned (a) QMI and (b),(c) Q2C MI. Solid lines in (a) and (b) represent Haar simulation results. Light colored dots with error bars represent ideal simulation results using a brickwork circuit. In (c), solid lines and diamonds represent noisy simulation and experimental results on IBM Quantum Torino. Here, we perform simulation on a system of $N_A = 3$, $N_B = 1$ qubits with circuit consisting of $L = 4$ layers in each step. Each data point is averaged over ten randomized circuits.

the QMI dynamics discussed above, the measurement-conditioned Q2C mutual information (light blue circles) also decays logarithmically with steps, while the unconditioned one (light green circles) simply decays linearly, and both of them agree with the corresponding Haar simulation results (solid lines), presenting a large gap there. We further perform experiments on IBM Quantum devices, shown in Fig. 14(c). Despite the effect of the device and shot noise, we still observe a significant gap between the measurement-conditioned Q2C mutual information (blue diamonds) and unconditioned one (green diamonds), which enlarges with time and approaches an order of magnitude at $t = 6$. The noisy simulations (lines) including gate noise and idling decoherence also align with the experimental results, though small deviations remain due to unmodeled noise resources such as crosstalk. Overall, the Q2C protocol provides a practical diagnostic of the exponential separation in mutual information lifetime on near-term devices with reduced experimental overhead.

X. DISCUSSION

In this work, we demonstrate that quantum information can exhibit a lifetime that scales exponentially with system size, even in the presence of interactions with an arbitrary environment, provided the environment (bath) is monitored at each time step. In contrast, when the environment is left unmonitored, we show that the lifetime of quantum information—such as that stored in a Bell state—scales only linearly with system size. In the partially monitored dynamics, we uncover a two-scale transition: The QMI decays logarithmically on microscopic timescales and linearly on macroscopic scales.

In the case of Hamiltonian dynamics, we observe an anomalously long exponential lifetime of classical correlations. This behavior is attributed to Hamiltonian-dependent outliers in the spectrum of the corresponding single-step quantum channel—a phenomenon whose precise origin remains an intriguing open question from the perspective of quantum many-body physics. While we have focused on all-to-all coupled Ising Hamiltonian as an illustrative example in the main text, we also observe same behavior in the mixing-field Ising model with only nearest-neighbor interaction in Appendix D.

To reduce experimental resources, we propose a hardware-efficient Q2C mutual information protocol to diagnose the separation of QMI in monitored and unmonitored dynamics and demonstrate the exponential separation in both ideal simulations and experiments on IBM Quantum devices.

Our QMI framework provides a useful perspective for understanding memory dynamics in midcircuit-measurement-based QRC algorithms. We propose extending memory time by leveraging statistics from sequences of measurement outcomes. This approach holds potential for improved performance, warranting further investigation through detailed numerical and experimental studies.

Our results also offer valuable insights for the field of quantum machine learning, including applications such as quantum diffusion models and quantum reservoir computing. For generative quantum machine learning models such as QuDDPM that incorporate recorded midcircuit measurements, our theoretical result on the exponentially long QMI lifetime reveals a limitation: For single-trajectory learning, efficient learning of the transformation from random initial states to a target ensemble is challenging for chaotic circuits with high representation power. Instead, one should go beyond the states generated under a single measurement trajectory. Additionally, the linear lifetime of mutual information in unmonitored dynamics offers guidance for setting the appropriate temporal window in QRC when learning individual signals. Incorporating statistics from extended sequences of measurement outcomes may further enhance the ability to learn non-Markovian sequences. Finally, our findings provide theoretical support for the security of holographic deep thermalization against entanglement attacks, as identified in Ref. [7].

We point out some future directions. The robustness of the exponential separation in quantum information lifetime against various types of noise is interesting to explore for both theoretical understanding and practical applications on near-term devices. Extending our analysis to different Hamiltonian classes (e.g., chaotic, integrable, many-body-localized, and symmetry-constrained) is an interesting direction to understand how Hamiltonian structure protects information against measurement-induced loss [47]. For practical experimental verification, a statistical analysis of the proposed Q2C mutual information is useful for uncovering both its strengths and limitations. Another experimental direction for demonstrating QMI dynamics is to design a reliable witness of the exponential separation in QMI behavior that avoids postselection, thereby further reducing the required resources. An intriguing question is whether the decay of QMI observed in this work can be linked to local system thermalization and the emergent state design in holographic deep thermalization [7].

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The views, opinions, and findings expressed are solely the authors' and not the U.S. government's.

DATA AVAILABILITY

The data that support the findings of this article are openly available [48].

APPENDIX A: CLIFFORD CIRCUIT SIMULATION WITH STABILIZER FORMALISM

In this section, we discuss the stabilizer formalism for Clifford circuit simulation, with further details presented in Refs. [3,49].

A (mixed) stabilizer state ρ of N qubits can be fully described as

$$\rho = \prod_{i=1}^{|G|} \frac{\mathbf{I} + g_i}{2}, \quad (\text{A1})$$

where mutually commuting Pauli operators $G = \{g_1, \dots, g_{|G|}\}$ formulate a set of generators for the stabilizer group of the state. The size of generator set satisfies $|G| \leq N$ where the equality holds for pure states. For analysis and calculation convenience, we can write the generators $g \in G$ in the *clipped gauge*, such that there are exactly two generators with the left or right end on any single qubit. Furthermore, if the two generators have only left or right ends on a given qubit, then the Pauli operator of this qubit must be different in the two generators. The advantage of *clipped gauge* is that the entanglement in a subregion A can be directly read out from the generators as

$$S_A = |A| - |\{g | \text{supp}(g) \subseteq A\}|, \quad (\text{A2})$$

where $\text{supp}(g)$ is the support of operator g . The mutual information between two neighboring regions A and B can be written out in a similar way. Our numerical simulation of the Clifford circuit is performed with QuantumClifford.jl [50].

APPENDIX B: QMI DYNAMICS WITH AND WITHOUT RESET

In this section, we prove that the dynamics of measurement-conditioned QMI remains the same regardless of reset or not. Similar to Eq. (1), the conditional state at step t without reset strategy is

$$|\psi'_z(U)\rangle_{RA_t} \quad (\text{B1})$$

$$\propto_B \langle z | U_t \dots U_2 U_1 [|\Phi\rangle_{RA_0} \otimes (|0\rangle_{B_0} \otimes_{k=1}^{t-1} |z_k\rangle_{B_k})] \quad (\text{B2})$$

$$\propto_B \langle z | U_t \dots U_2 U_1 [|\Phi\rangle_{RA_0} \otimes (\otimes_{k=1}^{t-1} X^{z_k} |0\rangle_B)] \quad (\text{B3})$$

$$= |\psi_z(U^z)\rangle_{RA_t}, \quad (\text{B4})$$

where $B = B_t \dots B_2 B_1$ is the joint system of bath system in each step and we have defined $U^z = U_t X^{z_{t-1}} U_3 X^{z_2} U_2 X^{z_1} U_1$ with $\{X^{z_k}\}_{k=1}^{t-1}$ applied on the bath system only.

Then, from the definition of measurement-conditioned QMI in Eq. (4),

$$\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|z)'} \quad (\text{B5})$$

$$= \mathbb{E}_{U \sim \text{Haar}} 2 \mathbb{E}_z S(\text{tr}_R(|\psi'_z(U)\rangle\langle\psi'_z(U)|_{RA_t})) \quad (\text{B6})$$

$$= 2 \mathbb{E}_{U \sim \text{Haar}} \sum_z P_{U^z}(z) S(\text{tr}_R(|\psi_z(U^z)\rangle\langle\psi_z(U^z)|_{RA_t})) \quad (\text{B7})$$

$$= 2 \sum_z \mathbb{E}_{U \sim \text{Haar}} P_{U^z}(z) S(\text{tr}_R(|\psi_z(U^z)\rangle\langle\psi_z(U^z)|_{RA_t})) \quad (\text{B8})$$

$$= \mathbb{E}_{\text{Haar}} \overline{I(R:A_t|z)} \quad (\text{B9})$$

equals the original measurement-conditioned QMI with reset strategy.

We further consider the unmonitored dynamics while resetting the bath state to a fully mixed state in each step instead. We expect that it can provide a faster decay, as it is equivalent to a fully depolarizing channel. For the Rényi-2 extended unconditioned QMI, we have the following theorem (see Appendix G for a proof).

Theorem 7. The expected Rényi-2 extended measurement-unconditioned QMI of a Bell initial state in the quantum dynamics of 2-design unitaries with midcircuit measurements and fully mixed state reset at time t is

$$\begin{aligned} \mathbb{E}_{\text{Haar}} I_2(R:A_t) &\simeq \log_2 \left(\left(d_B - \frac{1}{d_A^2} \right) \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A^2} \right) \\ &\quad - \log_2 \left(\frac{d_B - 1}{d_A} \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A} \right) + N_A. \end{aligned} \quad (\text{B10})$$

In the asymptotic limit $d_A, d_B \gg 1$, when $t \ll N_A/N_B + 1/2$,

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq 2N_A + N_B - 2N_B t, \quad (\text{B11})$$

while, for $t \gg N_A/N_B + 1/2$,

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq d_A^2 d_B^{1-2t}. \quad (\text{B12})$$

Compared to Theorem 3, the information dynamics in both early and late time still exhibits a corresponding linear and exponential decay but at a decay rate twice larger. Here, we provide the physical intuition for the different decay rates. In each step of reset, we can purify the fully mixed state by a Bell pair of dimension of d_B^2 , which can be regarded as an expanded bath system with twice the number of qubits. Furthermore, we can also regard the

whole unmonitored dynamics with Bell purification as the Hayden-Preskill protocol [51], although the scrambling unitary over $N_A + N_B t$ qubits is slightly different. According to the decoupling theorem [51], when $N_A < 1/2(N_A + N_B t)$ and, thus, $t > N_A/N_B$, the output side of system is nearly fully decoupled from the reference, which also indicates the decay rate of unconditioned QMI to be $2N_B$.

Accordingly, the ϵ lifetime with fully mixed state reset is estimated as

$$\tau_{\text{uncond}} \simeq \begin{cases} (1 - \epsilon)(N_A/N_B + 1/2), & t \ll N_A/N_B + 1/2, \\ \log_2(1/\epsilon)/2N_B, & t \gg N_A/N_B + 1/2, \end{cases} \quad (\text{B13})$$

which is only approximately half of the corresponding lifetime compared to the one with zero state reset strategy in Eqs. (21) and (22).

Through numerical simulation with Haar unitaries [see Figs. 15(a) and 15(b)], the measurement-unconditioned QMI dynamics with fully mixed state reset (blue triangles) overlaps with our theory prediction in Theorem 7 and, thus, demonstrates the double-speed linear and exponential decay compared to zero state reset strategy (orange line) in early and late stage. Moreover, the eigenspectrum of channel with Haar unitary and fully mixed state reset strategy concentrates

in a disk of radius $1/d_B$ [38], shown in Fig. 15(c), providing support to the intuition and theorem proposed above.

APPENDIX C: IDENTICAL UNITARY OR RANDOM UNITARY

In this section, we provide additional numerical simulations to compare the QMI dynamics and its corresponding lifetime in quantum dynamics with randomly sampled unitary and identical unitaries. To be precise, in the main text, for Haar quantum dynamics, we randomly sample t unitaries $\{U_k\}_{k=1}^t$ from Haar ensemble. Therefore, we regard this as random Haar dynamics; meanwhile, for Hamiltonian-based quantum dynamics, we randomly choose a group of parameters $\{\eta_i^x, \eta_i^z, J_{ij}\}$ and keep it fixed for all $\{U_k\}_{k=1}^t$, which we regard as identical Hamiltonian dynamics. We also consider two additional cases here. For identical Haar quantum dynamics, we randomly sample a single unitary U from Haar ensemble and utilize it across the dynamics for t steps; for random Hamiltonian-based quantum dynamics, we randomly choose t groups of parameters for the Hamiltonian in Eq. (7) and take their evolution for unitaries in each step.

In Fig. 16, we numerically simulate the measurement-conditioned and unconditioned QMI dynamics for the four

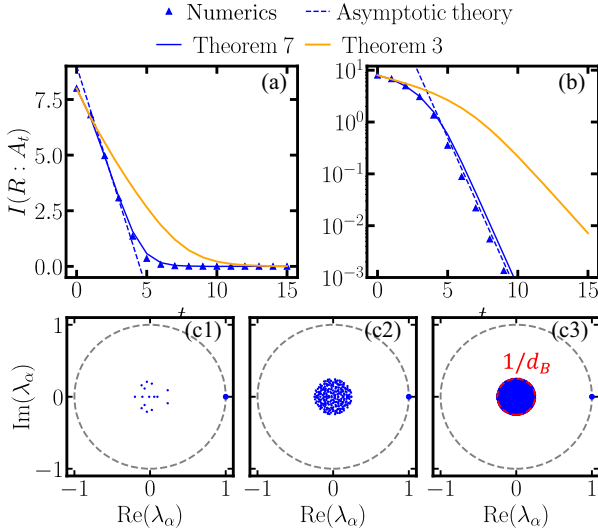


FIG. 15. Measurement-unconditioned QMI in unmonitored dynamics with full-mixed state reset. In (a) and (b), we plot numerical simulation of measurement-unconditioned QMI with fully mixed state reset (blue triangles) in early and late time separately. Blue solid lines show the analytical theory of Eq. (B10) in Theorem 7. Blue dashed lines in (a) and (b) show the corresponding asymptotic theory. The orange solid line represents the theory with pure state reset in Eq. (14). The system consists of $N_A = 4$ data qubits and $N_B = 1$ bath qubits in (a), (b). In (c), we show the spectrum of Haar channel with fully mixed state reset of system size $N_A = 2, 4, 6$ (left to right) with $N_B = 2$ bath qubits. The red dashed circle indicates the radius of $1/d_B$.

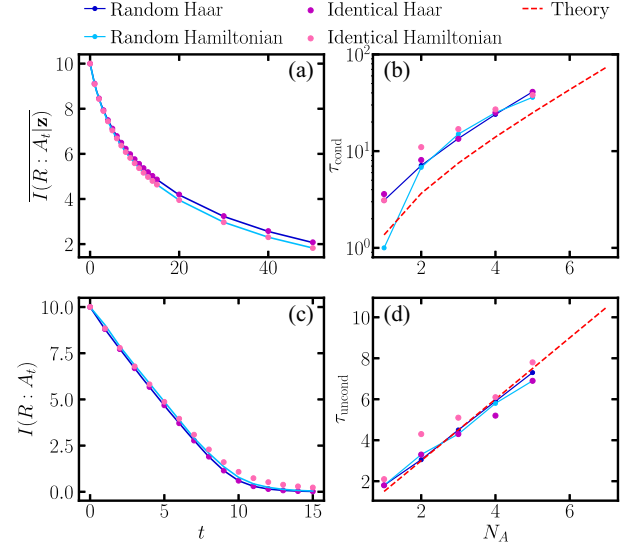


FIG. 16. Comparisons on dynamics of QMI and the corresponding lifetime in quantum dynamics with random unitary and identical unitary. We present measurement-conditioned QMI and measurement-unconditioned QMI in the top and bottom, respectively. Dark and light blue lines represent numerical simulations with randomly sampled Haar unitary and random Hamiltonian evolution separately. Purple and pink dots show results with identical fixed Haar unitary and fixed Hamiltonian evolution separately. The red dashed lines in (b) and (d) represent theory from Eqs. (10) and (21) with $\epsilon = 1/4$. In (a) and (c), we have system of $N_A = 5$ data qubits. In all cases, the bath consists of $N_B = 1$ qubits.

cases above and estimate the corresponding QMI lifetime. The QMI dynamics for identical Haar and Hamiltonian evolution (purple and pink) overlap with the results for random Haar and Hamiltonian (dark and light blue) for a system of $N_A = 5$ data qubits. The QMI lifetime among the four cases also approach each other in the same asymptotic scaling though with deviations at small N_A . In conclusion, given a large system size of N_A , the QMI dynamics with random unitaries or identical unitaries become indistinguishable, making our theories of Theorems 1 and 3 applicable to wide scenarios of quantum dynamics from holographic thermalization to reservoir computing.

APPENDIX D: QMI DYNAMICS WITH NEAREST-NEIGHBOR HAMILTONIAN

In the main text, we consider the Hamiltonian to be an all-to-all coupled Ising Hamiltonian with random parameters. In this section, we turn to another type of Hamiltonian, mixing-field Ising model (MFIM), which

involves only nearest-neighbor couplings:

$$H_{\text{MFIM}} = h_x \sum_i \sigma_i^x + h_y \sum_i \sigma_i^y + \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x. \quad (\text{D1})$$

Under the choice of $h_x, h_y = 0.8090, 0.9045$, this Hamiltonian dynamics is consistent with the eigenstate thermalization hypothesis and is also adopted in the studies of deep thermalization [2]. Here, we adopt this parameter option to numerically validate the corresponding QMI dynamics with and without bath monitoring in Fig. 17. In both monitored and unmonitored dynamics, we find that the QMI dynamics behaves similarly to the all-to-all Ising Hamiltonian considered in Figs. 2(a) and 4(a), and it is well described by our theory in Theorems 1 and 3. Even for the late-time unmonitored dynamics, the unconditioned QMI still decays exponentially though deviating from the theory (inset).

APPENDIX E: RESIDUAL MONITORED DYNAMICS

In this section, we provide numerical evidence on the decay of QMI in time beyond the lifetime τ_{cond} in Eq. (10). As the lower bound Eq. (8) [or the simplified version Eq. (9)] becomes negative at extremely late time, we expect deviations from the lower bound. For the quantum states under consideration, though, these residual dynamics correspond to $t \gg \tau_{\text{cond}}$, which is beyond exponential in system size. Moreover, at this point, the QMI is already infinitesimal and less interesting. Nevertheless, we present some numerical results demonstrating the exponential decay for $t \gg \tau_{\text{cond}}$ in Fig. 18(a). Here, we choose the initial state for different N_A to be the conditional state in Eq. (1) with Bell state $|\Phi\rangle$ and $\mathbf{z} = (\mathbf{0}, \dots, \mathbf{0})$ at $t \gg \tau_{\text{cond}}$. The exponential decay of QMI further indicates that the logarithmic decay of QMI identified in Theorem 1 is *not* universal for *all* states, despite proven for the Bell state and generic for various states considered above.

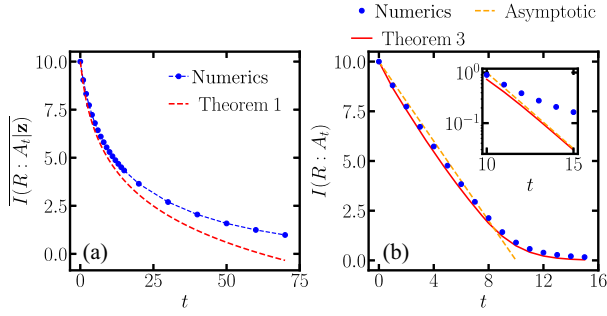


FIG. 17. (a) Measurement-conditioned QMI and (b) measurement-unconditioned QMI dynamics with MFIM Hamiltonian and reset. Blue circles represent numerical simulations of $N_R = N_A = 5$ and $N_B = 1$ qubits. The red dashed line in (a) shows Eq. (9) in Theorem 1. The red solid line and orange dashed line in (b) represent Eqs. (14) and (15), respectively, in Theorem 3. The inset shows the late-time dynamics in logarithmic scaling, and the orange dashed line is Eq. (16).

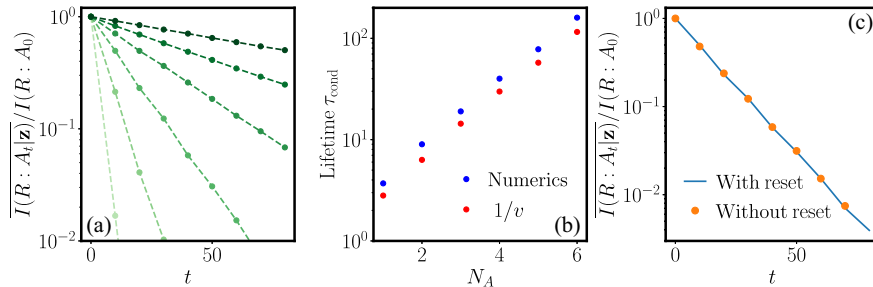


FIG. 18. Measurement-conditioned QMI $\overline{I(R:A_t|\mathbf{z})}$ in monitored residual dynamics. The initial state $|\psi\rangle_{RA_0}$ is chosen to be the late-time conditional state in dynamics with a Bell initial state. In (a), we plot the normalized QMI dynamics for $N_A = 1$ to 6 (light to dark). In (b), blue dots correspond to the estimated lifetime from numerical simulation in (a) for $\epsilon = 1/4$, and red dots represent the scaling of $1/v$, where v is the rate from linear fitting of $\overline{I(R:A_t|\mathbf{z})} \sim e^{-vt}$. In (c), we compare the numerical simulation of QMI dynamics with (blue) and without reset (orange) in a system of $N_A = 3$ qubits. In all cases, we take $N_B = 1$ bath qubits.

Counterexample can be constructed by initializing the quantum state in the state at $t \gg \tau_{\text{cond}}$ in those cases, as demonstrated by the exponential decay starting from time zero in our numerical examples. Despite the exponential decay with time, we find that the lifetime versus the system size N_A is still exponential, as shown in Fig. 18(b) (blue dots). This is because the exponent v in the exponential decay $\overline{I(R:A_t|z)} \sim e^{-vt}$ is vanishing as the system size increases (red dots). Furthermore, we would like to point out that our conclusion on the equivalence between cases with and without reset in measurement-conditioned QMI still holds, as shown via the comparison in Fig. 18(c).

APPENDIX F: DERIVATION FOR MEASUREMENT-CONDITIONED QMI DYNAMICS (THEOREMS 1 AND 2)

In this section, we derive the dynamics of measurement-conditioned QMI. We begin with the lower bound to connect the measurement-conditioned QMI to purity of reduced state:

$$\overline{I(R:A_t|z)} \equiv \mathbb{E}_z[S(R|z) + S(A_t|z) - S(RA_t|z)] \quad (\text{F1})$$

$$= \mathbb{E}_z 2S(A_t|z) \quad (\text{F2})$$

$$\geq \mathbb{E}_z 2S_2(A_t|z) = -2\mathbb{E}_z \log_2(\text{tr}(\rho_{A_t|z}^2)) \quad (\text{F3})$$

$$\geq -2\log_2(\mathbb{E}_z \text{tr}(\rho_{A_t|z}^2)), \quad (\text{F4})$$

where $S(A_t|z)$ is the von Neumann entanglement entropy of the reduced state $\rho_{A_t|z} = \text{tr}_R(|\psi_z\rangle\langle\psi_z|_{RA_t})$ and so as others. The second line comes from the fact that the conditional state $|\psi_z\rangle_{RA_t}$ is a pure state. We utilize the monotonicity in Rényi entropy to obtain inequality (F3) and obtain the last inequality utilizing the concavity of logarithmic function. Utilizing the concavity property again, we can obtain the lower bound for the Haar-averaged measurement-conditioned QMI as

$$\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|z)} \geq -2\log_2(\mathbb{E}_{\text{Haar}} \mathbb{E}_z \text{tr}(\rho_{A_t|z}^2)). \quad (\text{F5})$$

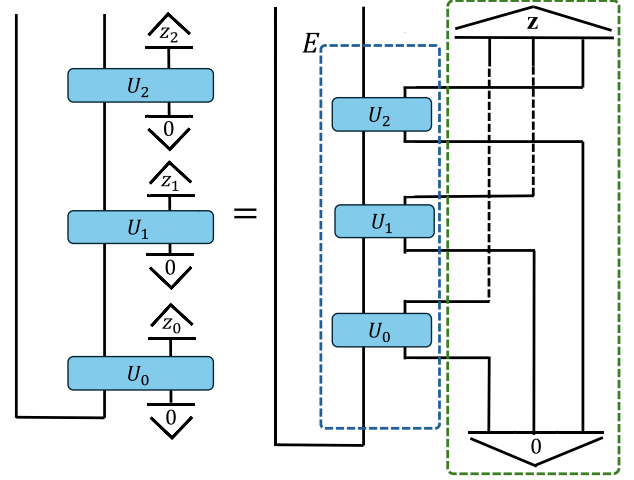


FIG. 19. Equivalent relation on the quantum dynamics with midcircuit measurements and zero state reset via expanding the bath. The linear map E (blue box) is applied on the initial maximally entangled state. The dashed lines indicate that those outputs are directly connected to the measurement results $|z\rangle$. The blue box shows the linear map E , and the green dashed box shows the temporal input $|\phi_z\rangle$. Here, we show an example for $t = 3$.

1. Proof of Theorem 1 for $N_R = N_A$

In this subsection, we begin with the proof for Theorem 1 where the initial state is a maximally entangled state and, thus, $N_R = N_A$. Via the bath equivalent expansion in Fig. 19, we can first write out the linear map E (blue box) as

$$E = \sum_{\mathbf{a}, \mathbf{b}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle | a_t \rangle_{A_t} \times \langle a_0 |_{A_0} \otimes_{k=0}^{t-1} \langle b_{2k+1} | \langle b_{2k} |, \quad (\text{F6})$$

where $\mathbf{a} = (a_0, \dots, a_t)$ and $\mathbf{b} = (b_0, b_1, \dots, b_{2t-2}, b_{2t-1})$ are vector representations of indices. By applying the linear map E on the maximally entangled state $|\Phi\rangle_{RA_0}$, we have

$$E' \equiv (\mathbb{I}_R \otimes E) |\Phi\rangle_{RA_0} = \frac{1}{\sqrt{d_A}} \sum_{\mathbf{a}, \mathbf{b}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle | a_0 \rangle_R | a_t \rangle_{A_t} \otimes_{k=0}^{t-1} \langle b_{2k+1} | \langle b_{2k} |, \quad (\text{F7})$$

which is a linear map from the expanded bath including input and measurement to the output conditional state in system RA_t . The same formalism is also adopted in Refs. [7,52]. From Fig. 19, the temporal input generated from the green dashed box is $|\phi_z\rangle = \otimes_{k=0}^{t-1} |z_t\rangle |0\rangle$, and, therefore, we can obtain the output conditional state and measurement probability as

$$\begin{cases} |\psi_z\rangle_{RA_t} = E' |\phi_z\rangle / \sqrt{P(\mathbf{z})}, \\ P(\mathbf{z}) = \langle \phi_z | E'^{\dagger} E' | \phi_z \rangle. \end{cases} \quad (\text{F8})$$

The corresponding density operator is $|\psi_{\mathbf{z}}\rangle\langle\psi_{\mathbf{z}}|_{RA_t} = P(\mathbf{z})^{-1}E'|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|E'^{\dagger}$, and the reduced state $\rho_{A_t|\mathbf{z}}$ is

$$\rho_{A_t|\mathbf{z}} = P(\mathbf{z})^{-1}\text{tr}_R(E'|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|E'^{\dagger}). \quad (\text{F9})$$

The measured-conditioned purity of $\rho_{A_t|\mathbf{z}}$ is, thus,

$$\bar{\gamma} \equiv \mathbb{E}_{\mathbf{z}}\text{tr}(\rho_{A_t|\mathbf{z}}^2) = \sum_{\mathbf{z}} P(\mathbf{z})\text{tr}(\rho_{A_t|\mathbf{z}}^2) \quad (\text{F10})$$

$$= \sum_{\mathbf{z}} P(\mathbf{z})P(\mathbf{z})^{-2}\text{tr}(\text{tr}_R(E'|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|E'^{\dagger})^2) \quad (\text{F11})$$

$$= \sum_{\mathbf{z}} P(\mathbf{z})^{-1}\text{tr}(E'^{\otimes 2}|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes 2}E'^{\dagger\otimes 2}(\mathbb{I}_R \otimes \mathbb{S}_{A_t})) \quad (\text{F12})$$

$$= \sum_{\mathbf{z}} \langle\phi_{\mathbf{z}}|E'^{\dagger}E'|\phi_{\mathbf{z}}\rangle^{-1}\text{tr}(E'^{\otimes 2}|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes 2}E'^{\dagger\otimes 2}(\mathbb{I}_R \otimes \mathbb{S}_{A_t})), \quad (\text{F13})$$

where $\mathbb{S}_{A_t}$ is the swap operator over two replicas in system A_t . To evaluate it analytically, we introduce the pseudo measurement-conditioned purity from replica trick as

$$\bar{\gamma}^{(m)} = \sum_{\mathbf{z}} \langle\phi_{\mathbf{z}}|E'^{\dagger}E'|\phi_{\mathbf{z}}\rangle^m \text{tr}(E'^{\otimes 2}|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes 2}E'^{\dagger\otimes 2}(\mathbb{I}_R \otimes \mathbb{S}_{A_t})) \quad (\text{F14})$$

$$= \sum_{\mathbf{z}} \text{tr}(E'|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|E'^{\dagger})^m \text{tr}(E'^{\otimes 2}|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes 2}E'^{\dagger\otimes 2}(\mathbb{I}_R \otimes \mathbb{S}_{A_t})) \quad (\text{F15})$$

$$= \sum_{\mathbf{z}} \text{tr}(E'^{\otimes m}|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes m}E'^{\dagger\otimes m}) \text{tr}(E'^{\otimes 2}|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes 2}E'^{\dagger\otimes 2}(\mathbb{I}_R \otimes \mathbb{S}_{A_t})) \quad (\text{F16})$$

$$= \sum_{\mathbf{z}} \text{tr}(E'^{\otimes(m+2)}|\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes(m+2)}E'^{\dagger\otimes(m+2)}(\mathbb{I}_R \otimes (\mathbb{I}_m \otimes \mathbb{S}_{A_t}))) \quad (\text{F17})$$

$$= \text{tr}\left(E'^{\otimes(m+2)}\left(\sum_{\mathbf{z}} |\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes(m+2)}\right)E'^{\dagger\otimes(m+2)}(\mathbb{I}_R \otimes (\mathbb{I}_m \otimes \mathbb{S}_{A_t}))\right), \quad (\text{F18})$$

where in the last step $\mathbb{I}_R$ is changed to the identity operator on $m+2$ replicas and $\mathbb{I}_m \otimes \mathbb{S}_{A_t}$ is the swap operator on A_t with trivial identity on first m replicas. Note that

$$\sum_{\mathbf{z}} |\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes(m+2)} = \sum_{\mathbf{z}} \otimes_{k=0}^{t-1} |z_k\rangle\langle z_k|^{\otimes(m+2)} \otimes |0\rangle\langle 0|^{\otimes(m+2)} \quad (\text{F19})$$

$$= \otimes_{k=0}^{t-1} \left(\sum_{z_k} |z_k\rangle\langle z_k|^{\otimes(m+2)} \otimes |0\rangle\langle 0|^{\otimes(m+2)} \right) \quad (\text{F20})$$

$$\equiv \otimes_{k=0}^{t-1} (D_{m+2} \otimes |0\rangle\langle 0|^{\otimes(m+2)}), \quad (\text{F21})$$

where we define $D_{m+2} = \sum_{\mathbf{z}} |z\rangle\langle z|^{\otimes(m+2)}$. Then, we can evaluate $E'^{\otimes(m+2)}(\sum_{\mathbf{z}} |\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes(m+2)})E'^{\dagger\otimes(m+2)}$ as

$$\begin{aligned} & E'^{\otimes(m+2)}\left(\sum_{\mathbf{z}} |\phi_{\mathbf{z}}\rangle\langle\phi_{\mathbf{z}}|^{\otimes(m+2)}\right)E'^{\dagger\otimes(m+2)} \\ &= d_A^{-(m+2)} \sum_{\alpha\beta} \prod_{\alpha'\beta'}^{t-1} \langle\alpha_{k+1}\beta_{2k+1}|U_k^{\otimes(m+2)}|\alpha_k\beta_{2k}\rangle\langle\alpha'_k\beta'_{2k}|U_k^{\dagger\otimes(m+2)}|\alpha'_{k+1}\beta'_{2k+1}\rangle|\alpha_0\rangle\langle\alpha'_0|_R|\alpha_t\rangle\langle\alpha'_t|_{A_t} \\ &\quad \times \otimes_{k=0}^{t-1} \langle\beta_{2k+1}|D_{m+2}|\beta'_{2k+1}\rangle\langle\beta_{2k}|0^{m+2}\rangle\langle 0^{m+2}|\beta'_{2k}\rangle, \end{aligned} \quad (\text{F22})$$

where $|\alpha_k\rangle = \bigotimes_{j=1}^{m+2} |a_k^{(j)}\rangle$, $|\beta_k\rangle = \bigotimes_{j=1}^{m+2} |b_k^{(j)}\rangle$, $|\alpha'_k\rangle = \bigotimes_{j=1}^{m+2} |a'_k{}^{(j)}\rangle$, and $|\beta'_k\rangle = \bigotimes_{j=1}^{m+2} |b'_k{}^{(j)}\rangle$ are the simplified notation for $m+2$ replicas of basis. $\alpha = (\alpha_0, \dots, \alpha_{t-1})$ and $\beta = (\beta_0, \beta_1, \dots, \beta_{2k-2}, \beta_{2k-1})$ are the vector representations and so as the others α' and β' . Utilizing the facts

$$\langle \beta_{2k+1} | D_{m+2} | \beta'_{2k+1} \rangle = \sum_{z_k} \langle \beta_{2k+1} | z_k^{\otimes(m+2)} \rangle \langle z_k^{\otimes(m+2)} | \beta'_{2k+1} \rangle = \sum_{z_k} \delta_{\beta_{2k+1}, z_k} \delta_{\beta'_{2k+1}, z_k}, \quad (\text{F23})$$

$$\langle \beta_{2k} | 0^{m+2} \rangle \langle 0^{m+2} | \beta'_{2k} \rangle = \delta_{\beta_{2k}, 0} \delta_{\beta'_{2k}, 0}, \quad (\text{F24})$$

where $\delta_{\beta, z} = \prod_{j=1}^{m+2} \delta_{b^{(j)}, z}$ is a simplified notation of Kronecker deltas on each replicas, then we have

$$\begin{aligned} & E'^{\otimes(m+2)} \left(\sum_{\mathbf{z}} |\phi_{\mathbf{z}}\rangle \langle \phi_{\mathbf{z}}|^{\otimes(m+2)} \right) E'^{\dagger \otimes(m+2)} \\ &= d_A^{-(m+2)} \sum_{\alpha, \alpha'} \sum_{\mathbf{z}} \prod_{k=0}^{t-1} \langle \alpha_{k+1} | z_k^{\otimes(m+2)} | U_k^{\otimes(m+2)} | \alpha_k 0^{\otimes(m+2)} \rangle \langle \alpha'_k 0^{\otimes(m+2)} | U_k^{\dagger \otimes(m+2)} | \alpha'_{k+1} z_k^{\otimes(m+2)} \rangle | \alpha_0 \rangle \langle \alpha'_0 |_R | \alpha_t \rangle \langle \alpha'_t |_{A_t} \end{aligned} \quad (\text{F25})$$

$$\begin{aligned} &= d_A^{-(m+2)} \sum_{\alpha, \alpha'} \sum_{\mathbf{z}} \prod_{k=0}^{t-1} \text{tr}(U_k^{\otimes(m+2)} (|\alpha_k\rangle \langle \alpha'_k| \otimes |0\rangle \langle 0|^{\otimes(m+2)}) U_k^{\dagger \otimes(m+2)} (|\alpha'_{k+1}\rangle \langle \alpha_{k+1}| \otimes |z_k\rangle \langle z_k|^{\otimes(m+2)})) | \alpha_0 \rangle \langle \alpha'_0 |_R | \alpha_t \rangle \langle \alpha'_t |_{A_t}. \end{aligned} \quad (\text{F26})$$

The pseudo measurement-conditioned purity [Eq. (F18)] then becomes

$$\bar{\gamma}^{(m)} = \text{tr} \left(E'^{\otimes(m+2)} \left(\sum_{\mathbf{z}} |\phi_{\mathbf{z}}\rangle \langle \phi_{\mathbf{z}}|^{\otimes(m+2)} \right) E'^{\dagger \otimes(m+2)} (\mathbb{I}_R \otimes (\mathbb{I}_m \otimes \mathbb{S}_{A_t})) \right) \quad (\text{F27})$$

$$\begin{aligned} &= d_A^{-(m+2)} \sum_{\alpha, \alpha'} \sum_{\mathbf{z}} \prod_{k=0}^{t-1} \text{tr}(U_k^{\otimes(m+2)} (|\alpha_k\rangle \langle \alpha'_k| \otimes |0\rangle \langle 0|^{\otimes(m+2)}) U_k^{\dagger \otimes(m+2)} (|\alpha'_{k+1}\rangle \langle \alpha_{k+1}| \otimes |z_k\rangle \langle z_k|^{\otimes(m+2)})) \\ &\quad \times \langle \alpha'_0 | \alpha_0 \rangle_R \langle \alpha'_t | \mathbb{I}_m \otimes \mathbb{S}_{A_t} | \alpha_t \rangle_{A_t}, \end{aligned} \quad (\text{F28})$$

$$\begin{aligned} &= d_A^{-(m+2)} \sum_{\alpha, \alpha'} \prod_{k=0}^{t-1} \text{tr}(U_k^{\otimes(m+2)} (|\alpha_k\rangle \langle \alpha'_k| \otimes |0\rangle \langle 0|^{\otimes(m+2)}) U_k^{\dagger \otimes(m+2)} (|\alpha'_{k+1}\rangle \langle \alpha_{k+1}| \otimes D_{m+2})) \\ &\quad \times \prod_{j=1}^{m+2} \delta_{a_0^{(j)}, a'_0{}^{(j)}} \prod_{j=1}^m \delta_{a_t^{(j)}, a'_t{}^{(j)}} \delta_{a_t^{(m+1)}, a_t^{(m+2)}} \delta_{a_t^{(m+2)}, a_t^{(m+1)}}. \end{aligned} \quad (\text{F29})$$

Next, we evaluate Haar ensemble average on the pseudo measurement-conditioned purity $\bar{\gamma}^{(m)}$. For Haar unitary twirling, it is known that

$$\mathbb{E}_{\text{Haar}} [U^{\otimes(m+2)} \otimes U^{* \otimes(m+2)}] = \sum_{\sigma, \pi \in S_{m+2}} \text{Wg}(\sigma^{-1} \pi, m+2) |\hat{\sigma}\rangle \langle \hat{\pi}| \quad (\text{F30})$$

$$= \text{Wg}(e, m+2) \sum_{\pi} |\hat{\pi}\rangle \langle \hat{\pi}| + \sum_{\sigma \neq \pi} \text{Wg}(\sigma^{-1} \pi, m+2) |\hat{\sigma}\rangle \langle \hat{\pi}|, \quad (\text{F31})$$

where S_{m+2} is the permutation group of $m+2$ replicas and $\hat{\pi}$ is the operator representation of permutation π . $\text{Wg}(\sigma^{-1} \pi, m+2) = (d^{|\sigma^{-1} \pi|})^{-1}$ is the Weingarten coefficient with $|\sigma^{-1} \pi|$ the number of cycles of permutation $\sigma^{-1} \pi$. Here, e is the identity.

We start from the unitary twirling on the last step boundary condition:

$$\begin{aligned} \mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes(m+2)} \otimes U_{t-1}^{*\otimes(m+2)}]|\tau_2\rangle_A|D_{m+2}\rangle_B &= \text{Wg}(e, m+2) \sum_{\pi} |\hat{\pi}\rangle_{AB} (\hat{\pi}|\tau_2\rangle_A) (\hat{\pi}|D_{m+2}\rangle_B) \\ &+ \sum_{\sigma \neq \pi} \text{Wg}(\sigma^{-1}\pi, m+2) |\hat{\sigma}\rangle_{AB} (\hat{\sigma}|\tau_2\rangle_A) (\hat{\sigma}|D_{m+2}\rangle_B). \end{aligned} \quad (\text{F32})$$

In the following, we evaluate the two terms above separately. For the first term, note that $(\hat{\pi}|D_{m+2}\rangle_B = \text{tr}(\hat{\pi}^\dagger \sum_z |z\rangle\langle z|^{\otimes(m+2)}) = \sum_z \langle z|^{\otimes(m+2)} |\hat{\pi}^\dagger|z^{\otimes(m+2)}\rangle = d_B$; we then have

$$\text{Wg}(e, m+2) \sum_{\pi} |\hat{\pi}\rangle_{AB} (\hat{\pi}|\tau_2\rangle_A) (\hat{\pi}|D_{m+2}\rangle_B) = \text{Wg}(e, m+2) d_B \sum_{\pi} d_A^{|\pi^\dagger \tau_2|} |\hat{\pi}\rangle_{AB}, \quad (\text{F33})$$

and, further connecting $|D_{m+2}\rangle(0|_B$ on the bath system, we have

$$|D_{m+2}\rangle(0|_B \text{Wg}(e, m+2) \sum_{\pi} |\hat{\pi}\rangle_{AB} (\hat{\pi}|\tau_2\rangle_A) (\hat{\pi}|D_{m+2}\rangle_B) = \text{Wg}(e, m+2) d_B \sum_{\pi} d_A^{|\pi^\dagger \tau_2|} |\hat{\pi}\rangle_A |D_{m+2}\rangle_B \quad (\text{F34})$$

$$\simeq \text{Wg}(e, m+2) d_B (d_A^{m+2} |\tau_2\rangle_A + d_A^{m+1} |e\rangle_A) |D_{m+2}\rangle_B, \quad (\text{F35})$$

where we keep the leading-order terms $\pi = \tau_2, \mathbb{I}$ in the last step given large system size $d_A \gg 1$. Similarly, for the second term in Eq. (F32), we have

$$\begin{aligned} |D_{m+2}\rangle(0|_B \sum_{\sigma \neq \pi} \text{Wg}(\sigma^{-1}\pi, m+2) |\hat{\sigma}\rangle_{AB} (\hat{\sigma}|\tau_2\rangle_A) (\hat{\sigma}|D_{m+2}\rangle_B) &\simeq d_B [\text{Wg}(e\tau_2, m+2) d_A^{m+2} |e\rangle_A |D_{m+2}\rangle_B \\ &+ \text{Wg}(\tau_2^{-1}e, m+2) d_A^{m+1} |\tau_2\rangle_A |D_{m+2}\rangle_B] \end{aligned} \quad (\text{F36})$$

$$= \text{Wg}(e\tau_2, m+2) d_A^{m+1} d_B (d_A |e\rangle_A + |\tau_2\rangle_A) |D_{m+2}\rangle_B, \quad (\text{F37})$$

where in the second line we still focus on the leading order $(\sigma, \pi) = (\tau_2, e), (e, \tau_2)$. Combining these two terms together, we have

$$\begin{aligned} |D_{m+2}\rangle(0|_B \mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes(m+2)} \otimes U_{t-1}^{*\otimes(m+2)}]|\tau_2\rangle_A|D_{m+2}\rangle_B \\ \simeq \text{Wg}(e, m+2) d_A^{m+1} d_B (d_A |\tau_2\rangle_A + |e\rangle_A) |D_{m+2}\rangle_B + \text{Wg}(e\tau_2, m+2) d_A^{m+1} d_B (d_A |e\rangle_A + |\tau_2\rangle_A) |D_{m+2}\rangle_B \end{aligned} \quad (\text{F38})$$

$$\simeq \frac{d_A^{m+1} d_B}{d_A^{m+2} d_B^{m+2}} (d_A |\tau_2\rangle_A + |e\rangle_A) |D_{m+2}\rangle_B - \frac{d_A^{m+1} d_B}{d_A^{m+3} d_B^{m+3}} (d_A |e\rangle_A + |\tau_2\rangle_A) |D_{m+2}\rangle_B \quad (\text{F39})$$

$$= d_B^{-(m+1)} (1 - d_A^{-2} d_B^{-1}) |\tau_2\rangle_A |D_{m+2}\rangle_B + d_A^{-1} d_B^{-(m+1)} (1 - d_B^{-1}) |e\rangle_A |D_{m+2}\rangle_B, \quad (\text{F40})$$

where in the second line we approximate it by $\text{Wg}(e, m+2) \simeq d_A^{-(m+2)} d_B^{-(m+2)}$ and $\text{Wg}(e\tau_2, m+2) \simeq -d_A^{-(m+3)} d_B^{-(m+3)}$ given $d_A \gg 1$. Let us move to the next twirling of U_{t-2} and bath operation $|D_{m+2}\rangle(0|_B$. For the boundary condition $|e\rangle_A |D_{m+2}\rangle_B$, we have

$$|D_{m+2}\rangle(0|_B \mathbb{E}_{\text{Haar}}[U_{t-2}^{\otimes(m+2)} \otimes U_{t-2}^{*\otimes(m+2)}]|e\rangle_A|D_{m+2}\rangle_B$$

$$= |D_{m+2}\rangle(0|_B \sum_{\sigma, \pi} \text{Wg}(\sigma^{-1}\pi, m+2)|\hat{\sigma}\rangle_{AB}(\hat{\pi}|e\rangle_A(\hat{\pi}|D_{m+2}\rangle_B) \quad (\text{F41})$$

$$= d_B \sum_{\sigma, \pi} \text{Wg}(\sigma^{-1}\pi, m+2) d_A^{|\pi^\dagger|} |\hat{\sigma}\rangle_A |D_{m+2}\rangle_B \quad (\text{F42})$$

$$= d_B \left[\text{Wg}(e, m+2) \sum_{\pi} d_A^{|\pi^\dagger|} |\hat{\pi}\rangle_A + \sum_{\sigma \neq \pi} \text{Wg}(\sigma^{-1}\pi, m+2) d_A^{|\pi^\dagger|} |\hat{\sigma}\rangle_A \right] |D_{m+2}\rangle_B \quad (\text{F43})$$

$$\simeq d_B [\text{Wg}(e, m+2)(d_A^{m+2}|e\rangle_A + d_A^{m+1}|\tau_2\rangle_A) + \text{Wg}(\tau_2^{-1}e, m+2)d_A^{m+2}|\tau_2\rangle_A + \text{Wg}(e\tau_2, m+2)d_A^{m+1}|e\rangle_A] |D_{m+2}\rangle_B \quad (\text{F44})$$

$$= d_A^{m+1} d_B [(\text{Wg}(e, m+2) + \text{Wg}(\tau_2^{-1}e, m+2)d_A)|\tau_2\rangle_A + (\text{Wg}(e, m+2)d_A + \text{Wg}(e\tau_2, m+2))|e\rangle_A] |D_{m+2}\rangle_B \quad (\text{F45})$$

$$\simeq d_A^{m+1} d_B \left[\left(d_A^{-(m+2)} d_B^{-(m+2)} - d_A^{-(m+3)} d_B^{-(m+3)} d_A \right) |\tau_2\rangle_A \left(d_A^{-(m+2)} d_B^{-(m+2)} d_A - d_A^{-(m+3)} d_B^{-(m+3)} \right) |e\rangle_A \right] |D_{m+2}\rangle_B \quad (\text{F46})$$

$$= d_A^{-1} d_B^{-(m+1)} (1 - d_B^{-1}) |\tau_2\rangle_A |D_{m+2}\rangle_B + d_B^{-(m+1)} (1 - d_A^{-2} d_B^{-1}) |e\rangle_A |D_{m+2}\rangle_B, \quad (\text{F47})$$

where we make the same approximations as we utilized above. In summary, we have the following linear transformation on boundary conditions in every step from Eqs. (F40) and (F47):

$$\begin{cases} |\tau_2\rangle_A |D_{n+2}\rangle_B \rightarrow q_\tau |\tau_2\rangle_A |D_{m+2}\rangle_B + q_e |e\rangle_A |D_{m+2}\rangle_B, \\ |e\rangle_A |D_{n+2}\rangle_B \rightarrow q_e |\tau_2\rangle_A |D_{m+2}\rangle_B + q_\tau |e\rangle_A |D_{m+2}\rangle_B, \end{cases} \quad (\text{F48})$$

where $q_\tau = d_B^{-(m+1)} (1 - d_A^{-2} d_B^{-1})$ and $q_e = d_A^{-1} d_B^{-(m+1)} (1 - d_B^{-1})$ are two coefficients. We can, thus, write the linear transformation in the form of a two-by-two matrix

$$Q = \begin{pmatrix} q_\tau & q_e \\ q_e & q_\tau \end{pmatrix}, \quad (\text{F49})$$

with bases $|\tau_2\rangle_A |D_{m+2}\rangle_B \rightarrow (1, 0)^T$ and $|e\rangle_A |D_{m+2}\rangle_B \rightarrow (0, 1)^T$. Via diagonalization, we can write out Q^t explicitly as

$$Q^t = \frac{1}{2} \begin{pmatrix} \nu_+^t + \nu_-^t & \nu_+^t - \nu_-^t \\ \nu_+^t - \nu_-^t & \nu_+^t + \nu_-^t \end{pmatrix}, \quad (\text{F50})$$

where $\nu_\pm = q_\tau \pm q_e$ are eigenvalues of Q . Considering the initial boundary condition at step t in Fig. 20, we obtain the boundary condition at step 0 as

$$Q^t(1, 0)^T \rightarrow \frac{1}{2} [(q_\tau + q_e)^t + (q_\tau - q_e)^t] |\tau_2\rangle_A |D_{m+2}\rangle_B + \frac{1}{2} [(q_\tau + q_e)^t - (q_\tau - q_e)^t] |e\rangle_A |D_{m+2}\rangle_B, \quad (\text{F51})$$

and, thus, we have the dynamical solution for pseudo measurement-conditioned purity:

$$\mathbb{E}_{\text{Haar}} \bar{\gamma}^{(m)} \simeq d_A^{-(m+2)} (e|_A (0|_B \left(\frac{1}{2} [(q_\tau + q_e)^t + (q_\tau - q_e)^t] |\tau_2\rangle_A |D_{m+2}\rangle_B + \frac{1}{2} [(q_\tau + q_e)^t - (q_\tau - q_e)^t] |e\rangle_A |D_{m+2}\rangle_B \right)) \quad (\text{F52})$$

$$= \frac{1}{2} d_A^{-(m+2)} [(q_\tau + q_e)^t + (q_\tau - q_e)^t] d_A^{m+1} + [(q_\tau + q_e)^t - (q_\tau - q_e)^t] d_A^{m+2} \quad (\text{F53})$$

$$= \frac{1}{2d_A} [(1 + d_A)(q_\tau + q_e)^t + (1 - d_A)(q_\tau - q_e)^t] \quad (\text{F54})$$

$$= \frac{1}{2d_A} \left[(1 + d_A) \left(d_A^{-2} d_B^{-(m+2)} (d_A + 1)(d_A d_B - 1) \right)^t + (1 - d_A) \left(d_A^{-2} d_B^{-(m+2)} (d_A - 1)(d_A d_B + 1) \right)^t \right] \quad (\text{F55})$$

$$= \frac{1}{2d_A} d_A^{-2t} d_B^{-(m+2)t} [(d_A + 1)^{t+1} (d_A d_B - 1)^t - (d_A - 1)^{t+1} (d_A d_B + 1)^t], \quad (\text{F56})$$

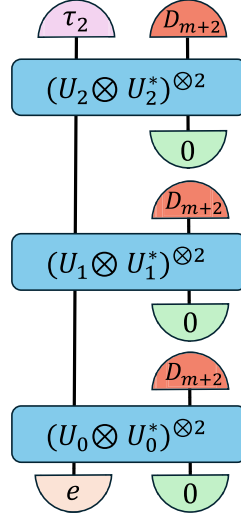


FIG. 20. Tensor network representation of Haar-averaged pseudo measurement-conditioned purity of $\rho_{A_t|z}$ with time steps $t = 3$. Here, $\tau_2 = \mathbb{I}_m \otimes \mathbb{S}$ is the swap operator on the last two replicas.

where we utilize the definition of q_τ and q_e defined above in the second-to-last line. Via taking the limit $m \rightarrow -1$, we solve the Haar-averaged dynamical solution for the measurement-conditioned purity of $\rho_{A_t|z}$ as

$$\mathbb{E}_{\text{Haar}} \mathbb{E}_{\mathbf{z}} \text{tr}(\rho_{A_t|z}^2) = \mathbb{E}_{\text{Haar}} \bar{\gamma} = \lim_{m \rightarrow -1} \mathbb{E}_{\text{Haar}} \bar{\gamma}^{(m)} \quad (\text{F57})$$

$$\simeq \frac{1}{2} d_A^{-(2t+1)} d_B^{-t} [(d_A + 1)^{t+1} (d_A d_B - 1)^t - (d_A - 1)^{t+1} (d_A d_B + 1)^t]. \quad (\text{F58})$$

Finally, we have the asymptotic lower bound for Haar-averaged measurement-conditioned QMI from Eq. (8):

$$\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|\mathbf{z})} \geq -2 \log_2 (\mathbb{E}_{\text{Haar}} \mathbb{E}_{\mathbf{z}} \text{tr}(\rho_{A_t|z}^2)) \quad (\text{F59})$$

$$\gtrsim -2 \log_2 \left(\frac{1}{2} d_A^{-(2t+1)} d_B^{-t} [(d_A + 1)^{t+1} (d_A d_B - 1)^t - (d_A - 1)^{t+1} (d_A d_B + 1)^t] \right) \quad (\text{F60})$$

$$= 2 + 2(2t + 1)N_A + 2tN_B - 2 \log_2 [(d_A + 1)^{t+1} (d_A d_B - 1)^t - (d_A - 1)^{t+1} (d_A d_B + 1)^t] \quad (\text{F61})$$

$$= 2 + 2(2t + 1)N_A + 2tN_B - 2 \log_2 [(d_A d_B + 1)^t (d_A + 1)^{t+1}] - 2 \log_2 \left[\left(\frac{d_A d_B - 1}{d_A d_B + 1} \right)^t - \left(\frac{d_A - 1}{d_A + 1} \right)^{t+1} \right], \quad (\text{F62})$$

which is the solution of Eq. (8) of Theorem 1 in the main text.

At the end of this section, we perform an asymptotic analysis on the scaling of above result in the limit of large system size $d_A \gg 1$:

$$\begin{aligned} & \mathbb{E}_{\text{Haar}} \overline{I(R:A_t|\mathbf{z})} \\ &= 2 + 2(2t + 1)N_A + 2tN_B - 2t \log_2 (d_A d_B + 1) - 2(t + 1) \log_2 (d_A + 1) - 2 \log_2 \left[\left(1 - \frac{2}{d_A d_B + 1} \right)^t - \left(1 - \frac{2}{d_A + 1} \right)^{t+1} \right] \end{aligned} \quad (\text{F63})$$

$$\simeq 2 + 2(2t + 1)N_A + 2tN_B - 2t(N_A + N_B) - 2(t + 1)N_A - 2 \log_2 \left[1 - \frac{2t}{d_A d_B + 1} - \left(1 - \frac{2(t + 1)}{d_A + 1} \right) \right] \quad (\text{F64})$$

$$= 2 - 2\log_2 \left(\frac{2(t+1)}{d_A + 1} - \frac{2t}{d_A d_B + 1} \right) \quad (\text{F65})$$

$$\simeq 2 - 2\log_2 \left(\frac{2(t+1)}{d_A} - \frac{2t}{d_A d_B} \right) \quad (\text{F66})$$

$$= 2 - 2\log_2 \left(\frac{2}{d_A} (t+1 - t/d_B) \right) \quad (\text{F67})$$

$$= 2N_A - \log_2 ((1 - 1/d_B)t + 1). \quad (\text{F68})$$

2. Proof of Theorem 2 for $N_R < N_A$

In this subsection, we provide the proof of Theorem 2, where $N_R < N_A$ and, thus, there are only N_R Bell pairs shared between reference and the system. The derivation of the measurement-conditioned QMI is similar to Theorem 1 presented above.

The initial state on reference R and system A is

$$|\Phi\rangle_{RA_0} = \frac{1}{\sqrt{d_R}} \sum_{i=0}^{d_R-1} |i\rangle_R |i\rangle_{A_1} |0\rangle_{A_2}, \quad (\text{F69})$$

where $|A_1| = N_R$ for qubits in the system entangled with the reference while A_2 stands for the idling qubits. The measurement-conditioned state is

$$\rho_{A|z} = \text{tr}_{RB}(U|\Phi\rangle\langle\Phi|_{RA_0}|0\rangle\langle 0|_B U^\dagger |z\rangle\langle z|_B) / p_z \equiv \tilde{\rho}_{RA|z} / p_z, \quad (\text{F70})$$

where $p_z = \text{tr}(\rho_{A|z})$ is the corresponding probability. Here, U represents the whole circuit. The measurement-conditioned purity is

$$\bar{\gamma} = \sum_{\mathbf{z}} p_{\mathbf{z}} \text{tr}(\rho_{A|z}^2) = \sum_{\mathbf{z}} p_{\mathbf{z}}^{-1} \text{tr}(\tilde{\rho}_{A|z}^2). \quad (\text{F71})$$

We again introduce the pseudo purity as

$$\bar{\gamma}^{(m)} \equiv \sum_{\mathbf{z}} p_{\mathbf{z}}^m \text{tr}(\tilde{\rho}_{A|z}^2) \quad (\text{F72})$$

$$= \sum_{\mathbf{z}} \text{tr} \left(U^{\otimes m} |\Phi\rangle\langle\Phi|_{RA}^{\otimes m} |0\rangle\langle 0|_B^{\otimes m} U^{\dagger \otimes m} \mathbf{I}_R^{\otimes m} \mathbf{I}_A^{\otimes m} |z\rangle\langle z|_B^{\otimes m} \right) \text{tr} \left(U^{\otimes 2} |\Phi\rangle\langle\Phi|_{RA}^{\otimes 2} |0\rangle\langle 0|_B^{\otimes 2} U^{\dagger \otimes 2} \mathbf{I}_R^{\otimes 2} \tau_A |z\rangle\langle z|_B^{\otimes 2} \right) \quad (\text{F73})$$

$$= \sum_{\mathbf{z}} \text{tr} \left(U^{\otimes(m+2)} |\Phi\rangle\langle\Phi|_{RA}^{\otimes(m+2)} |0\rangle\langle 0|_B^{\otimes(m+2)} U^{\dagger \otimes(m+2)} \mathbf{I}_R^{\otimes(m+2)} \mathbf{I}_A^{\otimes m} \tau_A |z\rangle\langle z|_B^{\otimes(m+2)} \right) \quad (\text{F74})$$

$$= d_R^{-(m+2)} \text{tr} \left(U^{\otimes(m+2)} \mathbf{I}_{A_1}^{\otimes(m+2)} |0\rangle\langle 0|_{A_2}^{\otimes(m+2)} |0\rangle\langle 0|_B^{\otimes(m+2)} U^{\dagger \otimes(m+2)} \mathbf{I}_A^{\otimes m} \tau_A \sum_{\mathbf{z}} |z\rangle\langle z|_B^{\otimes(m+2)} \right). \quad (\text{F75})$$

Compared to the expression in Eq. (F29), we notice that the only difference is the initial boundary condition as $\mathbf{I}_A^{\otimes(m+2)} \rightarrow \mathbf{I}_{A_1}^{\otimes(m+2)} \otimes |0\rangle\langle 0|_{A_2}^{\otimes(m+2)}$, while the twirling evolution and final boundary condition remain the same as expected. Therefore, we can still utilize the result in Eq. (F51), and the dynamical solution for the pseudo measurement-conditioned purity becomes

$$\mathbb{E}_{\text{Haar}} \bar{\gamma}^{(m)} \simeq d_R^{-(m+2)} (e|_{A_1} (0|_{A_2} (0|_B \left(\frac{1}{2} [(q_\tau + q_e)^t + (q_\tau - q_e)^t] |\tau_2)_A |D_{m+2})_B + \frac{1}{2} [(q_\tau + q_e)^t - (q_\tau - q_e)^t] |e)_A |D_{m+2})_B \right) \quad (\text{F76})$$

$$= \frac{1}{2} d_R^{-(m+2)} \left([(q_\tau + q_e)^t + (q_\tau - q_e)^t] d_R^{m+1} + [(q_\tau + q_e)^t - (q_\tau - q_e)^t] d_R^{m+2} \right) \quad (\text{F77})$$

$$= \frac{1}{2d_R} [(q_\tau + q_e)^t (1 + d_R) + (q_\tau - q_e)^t (1 - d_R)] \quad (\text{F78})$$

$$= \frac{1}{2d_R} \left[(1 + d_R) \left(d_A^{-2} d_B^{-(m+2)} (d_A + 1) (d_A d_B - 1) \right)^t + (1 - d_R) \left(d_A^{-2} d_B^{-(m+2)} (d_A - 1) (d_A d_B + 1) \right)^t \right] \quad (\text{F79})$$

$$= \frac{1}{2d_R} d_A^{-2t} d_B^{-(m+2)t} \left[(d_A + 1)^t (d_A d_B - 1)^t (1 + d_R) - (d_A - 1)^t (d_A d_B + 1)^t (d_R - 1) \right]. \quad (\text{F80})$$

Taking the replica limit of $m \rightarrow -1$, we have

$$\mathbb{E}_{\text{Haar}} \bar{\gamma} = \frac{1}{2d_R} d_A^{-2t} d_B^{-t} \left[(d_A + 1)^t (d_A d_B - 1)^t (1 + d_R) - (d_A - 1)^t (d_A d_B + 1)^t (d_R - 1) \right], \quad (\text{F81})$$

and the asymptotic lower bound for the Haar-averaged measurement-conditioned QMI from Eq. (8):

$$\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|\mathbf{z})} \geq -2\log_2(\mathbb{E}_{\text{Haar}} \bar{\gamma}) \quad (\text{F82})$$

$$\gtrsim -2\log_2 \left(\frac{1}{2d_R} d_A^{-2t} d_B^{-t} \left[(d_A + 1)^t (d_A d_B - 1)^t (1 + d_R) - (d_A - 1)^t (d_A d_B + 1)^t (d_R - 1) \right] \right) \quad (\text{F83})$$

$$= 2 + 2N_R + 4tN_A + 2tN_B - 2\log_2 \left[(d_A + 1)^t (d_A d_B - 1)^t (1 + d_R) - (d_A - 1)^t (d_A d_B + 1)^t (d_R - 1) \right] \quad (\text{F84})$$

$$= 2 + 2N_R + 4tN_A + 2tN_B - 2\log_2 [(d_A + 1)^t (d_A d_B + 1)^t (d_R + 1)] - 2\log_2 \left[\left(\frac{d_A d_B - 1}{d_A d_B + 1} \right)^t - \left(\frac{d_A - 1}{d_A + 1} \right)^t \left(\frac{d_R - 1}{d_R + 1} \right) \right]. \quad (\text{F85})$$

To simplify the results, we perform an asymptotic analysis in the limit of $d_A, d_R \gg 1$, and then we have

$$\mathbb{E}_{\text{Haar}} \overline{I(R:A_t|\mathbf{z})} = 2 + 2N_R + 4tN_A + 2tN_B - 2\log_2 [(d_A + 1)^t (d_A d_B + 1)^t (d_R + 1)] - 2\log_2 \left[\left(1 - \frac{2}{d_A d_B + 1} \right)^t - \left(1 - \frac{2}{d_A + 1} \right)^t \left(1 - \frac{2}{d_R + 1} \right) \right] \quad (\text{F86})$$

$$\simeq 2 + 2N_R + 4tN_A + 2tN_B - 2tN_A - 2t(N_A + N_B) - 2N_R - 2\log_2 \left[\left(1 - \frac{2t}{d_A d_B + 1} \right) - \left(1 - \frac{2t}{d_A + 1} \right) \left(1 - \frac{2}{d_R + 1} \right) \right] \quad (\text{F87})$$

$$= 2 - 2\log_2 \left[\frac{2}{d_R + 1} + \frac{2t}{d_A} \left(1 - \frac{2}{d_R + 1} - \frac{1}{d_B} \right) \right] \quad (\text{F88})$$

$$= 2\log_2(d_R + 1) - 2\log_2 \left[1 + \frac{d_R t}{d_A} \left(1 - \frac{d_R + 1}{d_R d_B} + \frac{1}{2d_R} \right) \right] \quad (\text{F89})$$

$$\simeq 2N_R - 2\log_2 \left[1 + \frac{d_R t}{d_A} \left(1 - \frac{d_R + 1}{d_R d_B} + \frac{1}{2d_R} \right) \right]. \quad (\text{F90})$$

APPENDIX G: DERIVATION FOR MEASUREMENT-UNCONDITIONED QMI (THEOREMS 3, 4, 6, AND 7)

In this section, we provide the proof on the dynamics for measurement-unconditioned QMI with pure and mixed state reset strategy. As we point out in the main text, for computation convenience, we focus on the Rényi-extended QMI defined by

$$I_2(R:A_t) = S_2(\rho_R) + S_2(\rho_{A_t}) - S_2(\rho_{RA_t}), \quad (\text{G1})$$

where $S_2(\cdot)$ is the Rényi-2 entropy.

In the following, Theorem 3 is proven in Appendix G 1, Theorem 4 is proven in Appendix G 2, Theorem 6 is proven in Appendix G 3, and Theorem 7 is proven in Appendix G 4.

1. Pure-state reset with $N_R = N_A$ (Theorem 3)

We begin with the setup of $N_R = N_A$. For pure-state reset strategy, without losing generality, we simply assume that, in every step, the bath system is reset to the trivial product state $|0\rangle$. From the equivalence relation in Fig. 21, we can write out the linear map E as

$$E = \sum_{\mathbf{a}, \mathbf{b}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{k+1} | U_k | a_k 0 \rangle | a_t \rangle_{A_t} \otimes_{k=0}^{t-1} | b_{k+1} \rangle_{B_{k+1}} \langle a_0 |_{A_0}, \quad (\text{G2})$$

where $\mathbf{a} = (a_0, \dots, a_t)$ and $\mathbf{b} = (b_1, \dots, b_t)$ are vector representations of the corresponding indices on data and bath system. Given the initial maximally entangled state $|\Phi\rangle_{RA_0} = (1/\sqrt{d_A}) \sum_j |j\rangle_R |j\rangle_{A_0}$, then we have the output state on $RA_t B$ to be

$$|\psi\rangle_{RA_t B} = (\mathbb{I}_R \otimes E) |\Phi\rangle_{RA_0} \quad (\text{G3})$$

$$= \sum_{\mathbf{a}, \mathbf{b}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{k+1} | U_k | a_k 0 \rangle | a_t \rangle_{A_t} \otimes_{k=0}^{t-1} | b_{k+1} \rangle_{B_{k+1}} (\mathbb{I}_R \otimes \langle a_0 |_{A_0}) |\Phi\rangle_{RA_0} \quad (\text{G4})$$

$$= \frac{1}{\sqrt{d_A}} \sum_{\mathbf{a}, \mathbf{b}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{k+1} | U_k | a_k 0 \rangle | a_0 \rangle_R | a_t \rangle_{A_t} \otimes_{k=0}^{t-1} | b_{k+1} \rangle_{B_{k+1}}. \quad (\text{G5})$$

With one more step, we can write out the corresponding density operator of $|\psi\rangle_{RA_t B}$ as

$$\rho_{RA_t B} = \frac{1}{d_A} \sum_{\substack{\mathbf{a}, \mathbf{a}' \\ \mathbf{b}, \mathbf{b}'}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{k+1} | U_k | a'_k 0 \rangle \langle a'_k 0 | U_k^\dagger | a'_{k+1} b'_{k+1} \rangle | a_0 \rangle \langle a'_0 |_R | a_t \rangle \langle a'_t |_{A_t} \otimes_{k=0}^{t-1} | b_{k+1} \rangle \langle b'_{k+1} |_{B_{k+1}}. \quad (\text{G6})$$

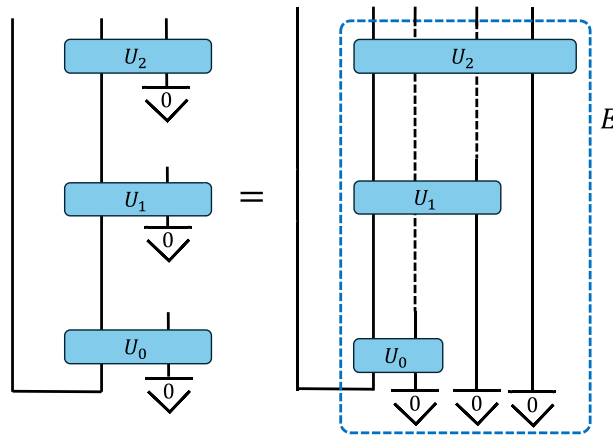


FIG. 21. Equivalent relation on the quantum dynamics with midcircuit measurements and $|0\rangle_B$ reset via expanding the bath. The linear map E (blue box) is applied on the initial maximally entangled state. The dashed lines indicate that those outputs do not interact with the unitaries to be passed through. Here, we show an example for $t = 3$.

Recall the definition of Rényi-extended QMI in Eq. (G1), and now we evaluate the corresponding reduced states and their Rényi-2 entropy. We begin with ρ_R :

$$\rho_R = \text{tr}_{A,B}(\rho_{RA,B}) \quad (\text{G7})$$

$$= \frac{1}{d_A} \sum_{\substack{a,a' \\ b,b'}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{k+1} | U_k | a_k 0 \rangle \langle a'_k 0 | U_k^\dagger | a'_{k+1} b'_{k+1} \rangle | a_0 \rangle \langle a'_0 |_R \delta_{a_t, a'_t} \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}} \quad (\text{G8})$$

$$= \frac{1}{d_A} \sum_{\substack{a,a' \\ b,b'}} \prod_{k=0}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes |0\rangle\langle 0|) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes |b'_{k+1}\rangle\langle b_{k+1}|)) | a_0 \rangle \langle a'_0 |_R \delta_{a_t, a'_t} \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}}. \quad (\text{G9})$$

The purity of ρ_R then becomes

$$\begin{aligned} \text{tr}(\rho_R^2) &= \frac{1}{d_A^2} \sum_{\substack{a,a' \\ b,b'}} \prod_{k=0}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes |0\rangle\langle 0|) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes |b'_{k+1}\rangle\langle b_{k+1}|)) \delta_{a_t, a'_t} \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}} \\ &\quad \times \sum_{\substack{\alpha,\alpha' \\ \beta,\beta'}} \prod_{k=0}^{t-1} \text{tr}(U_k(|\alpha_k\rangle\langle \alpha'_k| \otimes |0\rangle\langle 0|) U_k^\dagger(|\alpha'_{k+1}\rangle\langle \alpha_{k+1}| \otimes |\beta'_{k+1}\rangle\langle \beta_{k+1}|)) \delta_{\alpha_t, \alpha'_t} \prod_{k=0}^{t-1} \delta_{\beta_{k+1}, \beta'_{k+1}} \delta_{a_0, \alpha'_0} \delta_{a'_0, \alpha_0} \end{aligned} \quad (\text{G10})$$

$$\begin{aligned} &= \frac{1}{d_A^2} \sum_{\substack{a,a' \\ b,b'}} \sum_{\substack{\alpha,\alpha' \\ \beta,\beta'}} \prod_{k=0}^{t-1} \text{tr}(U_k^{\otimes 2}(|a_k \alpha_k\rangle\langle a'_k \alpha'_k| \otimes |0\rangle\langle 0|^{\otimes 2}) U_k^{\dagger \otimes 2}(|a'_{k+1} \alpha'_{k+1}\rangle\langle a_{k+1} \alpha_{k+1}| \otimes |b'_{k+1} \beta'_{k+1}\rangle\langle b_{k+1} \beta_{k+1}|)) \\ &\quad \times \delta_{a_0, \alpha'_0} \delta_{a'_0, \alpha_0} \delta_{a_t, \alpha'_t} \delta_{\alpha_t, a'_t} \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}} \delta_{\beta_{k+1}, \beta'_{k+1}}, \end{aligned} \quad (\text{G11})$$

which can be equivalently expressed in the diagram shown in Fig. 22(a) with a choice of $t = 3$.

For Haar ensemble average of unitary twirling over two replicas, we have

$$\mathbb{E}_{\text{Haar}}[U^{\otimes 2} \otimes U^{*\otimes 2}] = \frac{1}{d^2 - 1} \left[|e\rangle\langle e| - \frac{1}{d} |e\rangle\langle \tau| + |\tau\rangle\langle \tau| - \frac{1}{d} |\tau\rangle\langle e| \right], \quad (\text{G12})$$

where $|e\rangle$ and $|\tau\rangle$ are the vectorized operator for identity and transposition between two replicas, respectively, following the convention of Refs. [7,52]. The inner product of any two operator, thus, can be conveniently represented as $\text{tr}(A^\dagger B) = (A|B)$. The unitary twirling on the step t becomes

$$\mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes 2} \otimes U_{t-1}^{*\otimes 2}] |e\rangle_A |e\rangle_B = \frac{1}{d_A^2 d_B^2 - 1} [d_A^2 d_B^2 |e\rangle_{AB} - |e\rangle_{AB} + d_A d_B |\tau\rangle_{AB} - d_A d_B |\tau\rangle_{AB}] \quad (\text{G13})$$

$$= |e\rangle_{AB}, \quad (\text{G14})$$

where we utilize $(e|e)_A = d_A^2$ and $(\tau|e)_A = d_A$ in the second line. Via the resetting with $(0|_B$ and replacing it to $|e\rangle_B$ in the $t - 1$ step, we reobtain the initial boundary condition $|e\rangle_A |e\rangle_B$. Therefore, the Haar-averaged purity becomes

$$\mathbb{E}_{\text{Haar}} \text{tr}(\rho_R^2) = \frac{1}{d_A^2} (\tau|_A (0|_B |e\rangle_A |e\rangle_B) = \frac{1}{d_A}. \quad (\text{G15})$$

Next, we move to the reduced state ρ_{A_t} . Following Eq. (G6), we have

$$\rho_{A_t} = \text{tr}_{RB}(\rho_{RA,B}) \quad (\text{G16})$$

$$= \frac{1}{d_A} \sum_{\substack{a,a' \\ b,b'}} \prod_{k=0}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes |0\rangle\langle 0|) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes |b'_{k+1}\rangle\langle b_{k+1}|)) \delta_{a_0, a'_0} |a_t\rangle\langle a'_t|_{A_t} \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}}, \quad (\text{G17})$$

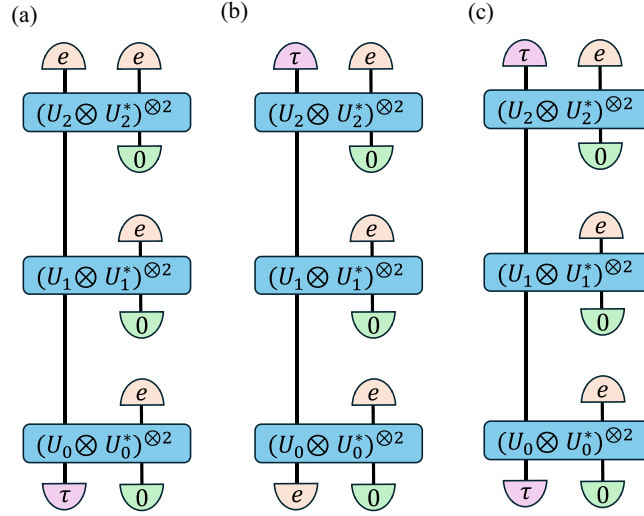


FIG. 22. Tensor network representation of Haar-averaged purity for reduced states (a) ρ_R , (b) ρ_{A_t} , and (c) ρ_{RA_t} with zero-state reset and time steps $t = 3$.

and the purity becomes

$$\begin{aligned} \text{tr}(\rho_{A_t}^2) &= \frac{1}{d_A^2} \sum_{\substack{a,a' \\ b,b'}} \sum_{\substack{a',a' \\ \beta',\beta'}} \prod_{k=0}^{t-1} \text{tr} \left(U_k^{\otimes 2} (|a_k \alpha_k\rangle \langle a'_k \alpha'_k| \otimes |0\rangle \langle 0|^{\otimes 2}) U_k^{\dagger \otimes 2} (|a'_{k+1} \alpha'_{k+1}\rangle \langle a_{k+1} \alpha_{k+1}| \otimes |b'_{k+1} \beta'_{k+1}\rangle \langle b_{k+1} \beta_{k+1}|) \right) \\ &\quad \times \delta_{a_0, a'_0} \delta_{\alpha_0, \alpha'_0} \delta_{a_t, a'_t} \delta_{\alpha_t, \alpha'_t} \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}} \delta_{\beta_{k+1}, \beta'_{k+1}}, \end{aligned} \quad (\text{G18})$$

which is equivalently expressed in the diagram shown in Fig. 22(b). The Haar ensemble average from step t is

$$\mathbb{E}_{\text{Haar}} [U_{t-1}^{\otimes 2} \otimes U_{t-1}^{* \otimes 2}] |\tau\rangle_A |e\rangle_B = \frac{1}{d_A^2 d_B^2 - 1} \left[d_A d_B^2 |e\rangle - \frac{d_A^2 d_B}{d_A d_B} |e\rangle + d_A^2 d_B |\tau\rangle - \frac{d_A d_B^2}{d_A d_B} |\tau\rangle \right] \quad (\text{G19})$$

$$= \frac{d_A(d_B^2 - 1)}{d_A^2 d_B^2 - 1} |e\rangle_{AB} + \frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} |\tau\rangle_{AB}, \quad (\text{G20})$$

where in the second line we utilize the fact $(\tau|\tau)_A = d_A^2$. Catenating with $|e\rangle(0|_B$ results in

$$|e\rangle(0|_B \mathbb{E}_{\text{Haar}} [U_{t-1}^{\otimes 2} \otimes U_{t-1}^{* \otimes 2}] |\tau\rangle_A |e\rangle_B = \frac{d_A(d_B^2 - 1)}{d_A^2 d_B^2 - 1} |e\rangle_A |e\rangle_B + \frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} |\tau\rangle_A |e\rangle_B. \quad (\text{G21})$$

We can find that the second term shares the same boundary condition $|\tau\rangle_A |e\rangle_B$ as the step t we have evaluated, and from the calculation in Eq. (G14) we know that the boundary $|e\rangle_A |e\rangle_B$ remains the same through unitary twirling. Therefore, we can separately evaluate the contribution from $|\tau\rangle_A |e\rangle_B$ and $|e\rangle_A |e\rangle_B$ as follows.

(i) $|\tau\rangle_A |e\rangle_B$:

$$\frac{1}{d_A^2} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t (e|\tau)_A (0|e)_B = \frac{1}{d_A^2} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t. \quad (\text{G22})$$

(ii) $|e\rangle_A|e\rangle_B$:

$$\frac{1}{d_A^2} \frac{d_A(d_B^2 - 1)}{d_A^2 d_B^2 - 1} \sum_{k=0}^{t-1} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^k (e|e)_A (0|e)_B = \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right). \quad (\text{G23})$$

Combining the above two terms together, we obtain the dynamical solution of Haar-averaged purity for ρ_{A_t} :

$$\mathbb{E}_{\text{Haar}} \text{tr}(\rho_{A_t}^2) = \frac{1}{d_A} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t + \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right) \quad (\text{G24})$$

$$= \frac{d_A(1 + d_B)}{1 + d_A^2 d_B} + \frac{1 - d_A^2}{d_A(1 + d_A^2 d_B)} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t. \quad (\text{G25})$$

Finally, we move to the reduced state ρ_{RA_t} :

$$\rho_{RA_t} = \text{tr}_B(\rho_{RA_t B}) \quad (\text{G26})$$

$$= \frac{1}{d_A} \sum_{\substack{a,a' \\ b,b'}} \prod_{k=0}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes |0\rangle\langle 0|) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes |b'_{k+1}\rangle\langle b_{k+1}|)) (|a_0\rangle\langle a'_0|_R \otimes |a_t\rangle\langle a'_t|_{A_t}) \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}}, \quad (\text{G27})$$

and the purity becomes

$$\begin{aligned} \text{tr}(\rho_{RA_t}^2) &= \frac{1}{d_A^2} \sum_{\substack{a,a' \\ b,b'}} \sum_{\substack{a',a'' \\ \beta,\beta'}} \prod_{k=0}^{t-1} \text{tr} \left(U_k^{\otimes 2}(|a_k \alpha_k\rangle\langle a'_k \alpha'_k| \otimes |0\rangle\langle 0|^{\otimes 2}) U_k^{\dagger \otimes 2}(|a'_{k+1} \alpha'_{k+1}\rangle\langle a_{k+1} \alpha_{k+1}| \otimes |b'_{k+1} \beta'_{k+1}\rangle\langle b_{k+1} \beta_{k+1}|) \right) \\ &\quad \times \delta_{a_0, \alpha'_0} \delta_{a'_0, \alpha_0} \delta_{a_t, \alpha'_t} \delta_{a'_t, \alpha_t} \prod_{k=0}^{t-1} \delta_{b_{k+1}, b'_{k+1}} \delta_{\beta_{k+1}, \beta'_{k+1}}, \end{aligned} \quad (\text{G28})$$

which is equivalently expressed in the diagram shown in Fig. 22(c). Since it shares the same boundary condition as in Fig. 22(b), we thus can directly write out the contributions from $|\tau\rangle_A|e\rangle_B$ and $|e\rangle_A|e\rangle_B$ as

(i) $|\tau\rangle_A|e\rangle_B$:

$$\frac{1}{d_A^2} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t (\tau|\tau)_A (0|e)_B = \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t; \quad (\text{G29})$$

(ii) $|e\rangle_A|e\rangle_B$:

$$\frac{1}{d_A^2} \frac{d_A(d_B^2 - 1)}{d_A^2 d_B^2 - 1} \sum_{k=0}^{t-1} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^k (\tau|e)_A (0|e)_B = \frac{d_B + 1}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right). \quad (\text{G30})$$

The dynamical solution of Haar-averaged purity for ρ_{RA_t} becomes

$$\mathbb{E}_{\text{Haar}} \text{tr}(\rho_{RA_t}^2) = \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t + \frac{d_B + 1}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right) \quad (\text{G31})$$

$$= \frac{1 + d_B}{1 + d_A^2 d_B} + \frac{d_B(d_A^2 - 1)}{1 + d_A^2 d_B} \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t. \quad (\text{G32})$$

To evaluate the Haar-averaged of measurement-unconditioned Rényi-extended QMI in Eq. (5), we switch the order of ensemble average and logarithmic, which does not change the final result as long as the fluctuation of entropy is small.

Combining the results in Eqs. (G15), (G25), and (G32), we have

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq -\log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_R^2) - \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{A_t}^2) + \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{RA_t}^2) \quad (\text{G33})$$

$$\begin{aligned} &= N_A - \log_2 \left[\frac{d_A(1+d_B)}{1+d_A^2 d_B} + \frac{1-d_A^2}{d_A(1+d_A^2 d_B)} \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2-1} \right)^t \right] \\ &\quad + \log_2 \left[\frac{1+d_B}{1+d_A^2 d_B} + \frac{d_B(d_A^2-1)}{1+d_A^2 d_B} \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2-1} \right)^t \right], \end{aligned} \quad (\text{G34})$$

which is the result of Eq. (14) in the main text. Assuming large system size $d_A, d_B \gg 1$, we can perform asymptotic analysis:

$$\begin{aligned} \mathbb{E}_{\text{Haar}} I_2(R:A_t) &= N_A - \log_2 \left[\frac{d_A^2(1+d_B) + (1-d_A^2) \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2-1} \right)^t}{d_A(1+d_A^2 d_B)} \right] + \log_2 \left[\frac{1+d_B + d_B(d_A^2-1) \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2-1} \right)^t}{1+d_A^2 d_B} \right] \\ &\simeq N_A - \log_2 \left(\frac{1}{d_A} - \frac{1}{d_A d_B d_B^t} \right) + \log_2 \left(\frac{1}{d_A^2} + \frac{1}{d_B^t} \right) \end{aligned} \quad (\text{G35})$$

$$= N_A - \log_2 \frac{1}{d_A} \left(1 - \frac{1}{d_B^{t+1}} \right) + \log_2 \left(\frac{1}{d_A^2} + \frac{1}{d_B^t} \right) \quad (\text{G36})$$

$$\simeq 2N_A - d_B^{-(t+1)} + \log_2(d_A^{-2} + d_B^{-t}). \quad (\text{G37})$$

For early time $t \ll 2N_A/N_B$, we can further approximate the QMI as

$$\begin{aligned} \mathbb{E}_{\text{Haar}} I_2(R:A_t) &\simeq 2N_A - d_B^{-(t+1)} + \log_2(d_A^{-2} + d_B^{-t}) \\ &= 2N_A - d_B^{-(t+1)} + \log_2 d_B^{-t} (1 + d_A^{-2} d_B^t) \end{aligned} \quad (\text{G38})$$

$$\simeq 2N_A - d_B^{-(t+1)} - tN_B + d_A^{-2} d_B^t \quad (\text{G39})$$

$$\simeq 2N_A - tN_B + \mathcal{O}(d_B^t/d_A^2), \quad (\text{G40})$$

which recovers the linear scaling in the main text. On the other hand, for late time $t \gg 2N_A/N_B$, with a similar approximation method, we have

$$\begin{aligned} \mathbb{E}_{\text{Haar}} I_2(R:A_t) &\simeq 2N_A - d_B^{-(t+1)} + \log_2(d_A^{-2} + d_B^{-t}) \\ &= 2N_A - d_B^{-(t+1)} + \log_2 d_A^{-2} (1 + d_A^2 d_B^{-t}) \end{aligned} \quad (\text{G41})$$

$$\simeq 2N_A - d_B^{-(t+1)} - 2N_A + d_A^2 d_B^{-t} \quad (\text{G42})$$

$$= d_A^2 d_B^{-t} + \mathcal{O}(d_B^{-(t+1)}). \quad (\text{G43})$$

2. Pure-state reset with $N_R < N_A$ (Theorem 4)

In this part, we consider a smaller reference of $N_R < N_A$ and still adopt pure reset strategy. The proof is similar to Appendix G 1 shown above.

The initial state on reference R and system A is still written as Eq. (11), and the measurement-unconditioned state is

$$\rho_{RA} = \text{tr}_B(U|\Phi\rangle\langle\Phi|_{RA_0}|0\rangle\langle 0|_B U^\dagger), \quad (\text{G44})$$

where U is still the whole circuit unitary. Similarly, we can also write out the unconditioned state only for reference and system separately. The purity for the reduced state on reference only is

$$\text{tr}(\rho_R^2) = \text{tr}(U^{\otimes 2}|\Phi\rangle\langle\Phi|_{RA_0}^{\otimes 2}|0\rangle\langle 0|_B^{\otimes 2}U^{\dagger\otimes 2}\tau_R\mathbf{I}_A^{\otimes 2}\mathbf{I}_B^{\otimes 2}) \quad (\text{G45})$$

$$= d_R^{-2}\text{tr}(U^2\tau_{A_1}|0\rangle\langle 0|_{A_2}^{\otimes 2}|0\rangle\langle 0|_B^{\otimes 2}U^{\dagger\otimes 2}\mathbf{I}_A^{\otimes 2}\mathbf{I}_B^{\otimes 2}), \quad (\text{G46})$$

which is same as Eq. (G11) except for the initial boundary condition. Therefore, the Haar-averaged purity becomes

$$\mathbb{E}_{\text{Haar}}\text{tr}(\rho_R^2) = \frac{1}{d_R^2}(\tau|_{A_1}(0|_{A_2}(0|_B|e)_A|e)_B = \frac{1}{d_R}. \quad (\text{G47})$$

For the purity of reduced state on system, we have

$$\text{tr}(\rho_A^2) = \text{tr}(U^{\otimes 2}|\Phi\rangle\langle\Phi|_{RA_0}^{\otimes 2}|0\rangle\langle 0|_B^{\otimes 2}U^{\dagger\otimes 2}\mathbf{I}_R^{\otimes 2}\tau_A\mathbf{I}_B^{\otimes 2}) \quad (\text{G48})$$

$$= d_R^{-2}\text{tr}(U^2\mathbf{I}_{A_1}^{\otimes 2}|0\rangle\langle 0|_{A_2}^{\otimes 2}|0\rangle\langle 0|_B^{\otimes 2}U^{\dagger\otimes 2}\tau_A\mathbf{I}_B^{\otimes 2}), \quad (\text{G49})$$

and utilizing the twirling dynamics illustrated in Eqs. (G20) and (G14), the Haar-averaged purity becomes

$$\mathbb{E}_{\text{Haar}}\text{tr}(\rho_A^2) = \frac{1}{d_R^2} \left[\left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t (e|\tau)_{A_1} + \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right) (e|e)_{A_1} \right] \quad (\text{G50})$$

$$= \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \frac{1}{d_R} + \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right) \quad (\text{G51})$$

$$= \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} + \left(\frac{1}{d_R} - \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} \right) \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t. \quad (\text{G52})$$

Finally, we consider the purity of ρ_{RA} , which can be written as

$$\text{tr}(\rho_{RA}^2) = \text{tr}(U^{\otimes 2}|\Phi\rangle\langle\Phi|_{RA_0}^{\otimes 2}|0\rangle\langle 0|_B^{\otimes 2}U^{\dagger\otimes 2}\tau_R\tau_A\mathbf{I}_B^{\otimes 2}) \quad (\text{G53})$$

$$= d_R^{-2}\text{tr}(U^2\tau_{A_1}|0\rangle\langle 0|_{A_2}^{\otimes 2}|0\rangle\langle 0|_B^{\otimes 2}U^{\dagger\otimes 2}\tau_A\mathbf{I}_B^{\otimes 2}), \quad (\text{G54})$$

and we can utilize the result in Eq. (G52) above to derive

$$\mathbb{E}_{\text{Haar}}\text{tr}(\rho_{RA}^2) = \frac{1}{d_R^2} \left[\left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t (\tau|\tau)_{A_1} + \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right) (\tau|e)_{A_1} \right] \quad (\text{G55})$$

$$= \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t + \frac{d_A(d_B + 1)}{d_A^2 d_B + 1} \left(1 - \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right) \frac{1}{d_R} \quad (\text{G56})$$

$$= \frac{d_A(d_B + 1)}{d_R(d_A^2 d_B + 1)} + \left(1 - \frac{d_A(d_B + 1)}{d_R(d_A^2 d_B + 1)} \right) \left(\frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t. \quad (\text{G57})$$

Therefore, combing Eqs. (G47), (G52), and (G57), we have the Haar-averaged Rényi-2 extended mutual information as

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq -\log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_R^2) - \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_A^2) + \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{RA}^2) \quad (\text{G58})$$

$$\begin{aligned} &= N_R - \log_2 \left[\frac{d_A(d_B+1)}{d_A^2 d_B + 1} + \left(\frac{1}{d_R} - \frac{d_A(d_B+1)}{d_A^2 d_B + 1} \right) \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2 - 1} \right)^t \right] \\ &\quad + \log_2 \left[\frac{d_A(d_B+1)}{d_R(d_A^2 d_B + 1)} + \left(1 - \frac{d_A(d_B+1)}{d_R(d_A^2 d_B + 1)} \right) \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2 - 1} \right)^t \right]. \end{aligned} \quad (\text{G59})$$

In the following, we simplify the expression in the asymptotic limit of $d_A, d_B \gg 1$, and Eq. (G59) can be reduced to

$$\begin{aligned} \mathbb{E}_{\text{Haar}} I_2(R:A_t) &= N_R - \log_2 \left[\frac{d_A(d_B+1)}{d_A^2 d_B + 1} + \left(\frac{1}{d_R} - \frac{d_A(d_B+1)}{d_A^2 d_B + 1} \right) \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2 - 1} \right)^t \right] \\ &\quad + \log_2 \left[\frac{d_A(d_B+1)}{d_R(d_A^2 d_B + 1)} + \left(1 - \frac{d_A(d_B+1)}{d_R(d_A^2 d_B + 1)} \right) \left(\frac{d_B(d_A^2-1)}{d_A^2 d_B^2 - 1} \right)^t \right] \end{aligned} \quad (\text{G60})$$

$$\simeq N_R - \log_2 \left[\frac{1}{d_A} + \left(\frac{1}{d_R} - \frac{1}{d_A} \right) \frac{1}{d_B^t} \right] + \log_2 \left[\frac{1}{d_R d_A} + \left(1 - \frac{1}{d_R d_A} \right) \frac{1}{d_B^t} \right] \quad (\text{G61})$$

$$= N_R + N_A - \log_2 \left[1 + \left(\frac{d_A}{d_R} - 1 \right) d_B^{-t} \right] - (N_R + N_A) + \log_2 [1 + (d_R d_A - 1) d_B^{-t}] \quad (\text{G62})$$

$$= \log_2 (1 + d_R d_A d_B^{-t}) - \log_2 \left[1 + \left(\frac{d_A}{d_R} - 1 \right) d_B^{-t} \right], \quad (\text{G63})$$

which is the theoretical result of Eq. (23) in Theorem 4 in the main text.

Therefore, through the two logarithmic terms above, we can identify two characteristic timescales $\tau_{\mp} = (N_A \mp N_R)/N_B$. When $t \leq \tau_-$, Eq. (G63) becomes

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq \log_2 (d_R d_A d_B^{-t}) - \log_2 (d_A/d_R d_B^{-t}) = 2N_R; \quad (\text{G64})$$

meanwhile, for $\tau_- \leq t \leq \tau_+$, we have

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq \log_2 (d_R d_A d_B^{-t}) - \frac{(d_A/d_R - 1) d_B^{-t}}{\log 2} \simeq N_R + N_A - N_B t; \quad (\text{G65})$$

and, finally, for $t \geq \tau_+$, we have

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq \frac{1}{\log 2} (d_R d_A d_B^{-t} - (d_A/d_R - 1) d_B^{-t}) \simeq \frac{d_R d_A d_B^{-t}}{\log 2} \rightarrow 0. \quad (\text{G66})$$

Therefore, we complete the dynamical transition in Eq. (25) in the main text up to a normalization constant.

3. Dynamics without reset (Theorem 6)

In this part, we prove the dynamics of measurement-unconditioned QMI with bath measured but without reset. The conditioned state $|\psi_{\mathbf{z}}\rangle$ in the dynamics without reset can be written out as

$$|\psi_{\mathbf{z}}\rangle = {}_B \langle \mathbf{z} | (U|\Phi\rangle_{RA} X^{\tilde{\mathbf{z}}} |0\rangle_B) / \sqrt{p_{\mathbf{z}}} \quad (\text{G67})$$

$$= {}_B \langle \mathbf{z} | (U|\Phi\rangle_{RA} |\tilde{\mathbf{z}}\rangle_B) / \sqrt{p_{\mathbf{z}}} \equiv |\tilde{\psi}_{\mathbf{z}}\rangle / \sqrt{p_{\mathbf{z}}}, \quad (\text{G68})$$

where $p_{\mathbf{z}} = \langle \tilde{\psi}_{\mathbf{z}} | \tilde{\psi}_{\mathbf{z}} \rangle$ and $\tilde{\mathbf{z}} = (0, z_1, \dots, z_{t-1})$. The mixed state of interest in the unconditioned dynamics is

$$\rho_{RA_i} = \sum_{\mathbf{z}} p_{\mathbf{z}} |\psi_{\mathbf{z}}\rangle \langle \psi_{\mathbf{z}}| = \sum_{\mathbf{z}} |\tilde{\psi}_{\mathbf{z}}\rangle \langle \tilde{\psi}_{\mathbf{z}}|. \quad (\text{G69})$$

In the following, we derive the Rényi-2 extended mutual information of ρ_{RA_i} . We begin with the purity of reduced state on reference:

$$\text{tr}(\rho_R^2) = \text{tr}(\rho_R^{\otimes 2} \tau_R) \quad (\text{G70})$$

$$= \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(|\tilde{\psi}_{\mathbf{z}}\rangle \langle \tilde{\psi}_{\mathbf{z}}| \otimes |\tilde{\psi}_{\mathbf{z}'}\rangle \langle \tilde{\psi}_{\mathbf{z}'}| \tau_R \mathbf{I}_A^{\otimes 2}) \quad (\text{G71})$$

$$= \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(U^{\otimes 2} |\Phi\rangle \langle \Phi|_{RA}^{\otimes 2} |\tilde{\mathbf{z}}\rangle \langle \tilde{\mathbf{z}}|_B \otimes |\tilde{\mathbf{z}}'\rangle \langle \tilde{\mathbf{z}}'|_B U^{\dagger \otimes 2} \tau_R \mathbf{I}_A^{\otimes 2} |\mathbf{z}\rangle \langle \mathbf{z}|_B \otimes |\mathbf{z}'\rangle \langle \mathbf{z}'|_B) \quad (\text{G72})$$

$$= d_A^{-2} \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(U^{\otimes 2} \tau_A |\tilde{\mathbf{z}}\rangle \langle \tilde{\mathbf{z}}|_B \otimes |\tilde{\mathbf{z}}'\rangle \langle \tilde{\mathbf{z}}'|_B U^{\dagger \otimes 2} \mathbf{I}_A^{\otimes 2} |\mathbf{z}\rangle \langle \mathbf{z}|_B \otimes |\mathbf{z}'\rangle \langle \mathbf{z}'|_B). \quad (\text{G73})$$

We still start the Haar unitary twirling from U_t as

$$\sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B \mathbb{E}_{\text{Haar}} [U_t^{\otimes 2} \otimes U_t^{* \otimes 2}] |e\rangle_A |D_1^{\otimes 2}\rangle_B \quad (\text{G74})$$

$$= \frac{1}{d^2 - 1} \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B \left[|e\rangle \langle e| + |\tau\rangle \langle \tau| - \frac{1}{d} |e\rangle \langle \tau| - \frac{1}{d} |\tau\rangle \langle e| \right] |e\rangle_A |D_1^{\otimes 2}\rangle_B \quad (\text{G75})$$

$$= \frac{1}{d^2 - 1} \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B \left[|e\rangle d_A^2 d_B^2 + |\tau\rangle d_A d_B - \frac{d_A d_B}{d} |e\rangle - \frac{d_A^2 d_B^2}{d} |\tau\rangle \right] \quad (\text{G76})$$

$$= |e\rangle_A \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B |e\rangle_B \quad (\text{G77})$$

$$= |e\rangle_A |D_1^{\otimes 2}\rangle_B, \quad (\text{G78})$$

where we denote $D_1 = \sum_{\mathbf{z}} |\mathbf{z}\rangle \langle \mathbf{z}|$ following the convention and we can immediately see that $(e|D_1^{\otimes 2})_B = d_B^2$ and $(\tau|D_1^{\otimes 2})_B = d_B$. Therefore, we directly have the purity of reference reduced state as

$$\begin{aligned} \mathbb{E}_{\text{Haar}} \text{tr}(\rho_R^2) &= d_A^{-2} \mathbb{E}_{\text{Haar}} \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(U^{\otimes 2} \tau_A |\tilde{\mathbf{z}}\rangle \langle \tilde{\mathbf{z}}|_B \otimes |\tilde{\mathbf{z}}'\rangle \langle \tilde{\mathbf{z}}'|_B U^{\dagger \otimes 2} \mathbf{I}_A^{\otimes 2} |\mathbf{z}\rangle \langle \mathbf{z}|_B \otimes |\mathbf{z}'\rangle \langle \mathbf{z}'|_B) \\ &= d_A^{-2} (\tau|e)_A (0|D_1^{\otimes 2})_B = \frac{1}{d_A}. \end{aligned} \quad (\text{G79})$$

For the purity of reduced state on system A, we have

$$\text{tr}(\rho_A^2) = \text{tr}(\rho_A^{\otimes 2} \tau_A) \quad (\text{G80})$$

$$= \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(|\tilde{\psi}_{\mathbf{z}}\rangle \langle \tilde{\psi}_{\mathbf{z}}| \otimes |\tilde{\psi}_{\mathbf{z}'}\rangle \langle \tilde{\psi}_{\mathbf{z}'}| \mathbf{I}_R^{\otimes 2} \tau_A) \quad (\text{G81})$$

$$= \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(U^{\otimes 2} |\Phi\rangle \langle \Phi|_{RA}^{\otimes 2} |\tilde{\mathbf{z}}\rangle \langle \tilde{\mathbf{z}}|_B \otimes |\tilde{\mathbf{z}}'\rangle \langle \tilde{\mathbf{z}}'|_B U^{\dagger \otimes 2} \mathbf{I}_R^{\otimes 2} \tau_A |\mathbf{z}\rangle \langle \mathbf{z}|_B \otimes |\mathbf{z}'\rangle \langle \mathbf{z}'|_B) \quad (\text{G82})$$

$$= d_A^{-2} \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(U^{\otimes 2} \mathbf{I}_A^{\otimes 2} |\tilde{\mathbf{z}}\rangle \langle \tilde{\mathbf{z}}|_B \otimes |\tilde{\mathbf{z}}'\rangle \langle \tilde{\mathbf{z}}'|_B U^{\dagger \otimes 2} \tau_A |\mathbf{z}\rangle \langle \mathbf{z}|_B \otimes |\mathbf{z}'\rangle \langle \mathbf{z}'|_B). \quad (\text{G83})$$

For Haar twirling of U_t , we first have

$$\begin{aligned} & \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B \mathbb{E}_{\text{Haar}}[U_t^{\otimes 2} \otimes U_t^{*\otimes 2}] |\tau\rangle_A |D_1^{\otimes 2}\rangle_B \\ &= \frac{1}{d^2 - 1} \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B \left[|e\rangle \langle e| + |\tau\rangle \langle \tau| - \frac{1}{d} |e\rangle \langle \tau| - \frac{1}{d} |\tau\rangle \langle e| \right] |\tau\rangle_A |D_1^{\otimes 2}\rangle_B \end{aligned} \quad (\text{G84})$$

$$= \frac{1}{d^2 - 1} \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B \left[|e\rangle d_A d_B^2 + |\tau\rangle d_A^2 d_B - \frac{d_A^2 d_B}{d} |e\rangle - \frac{d_A d_B^2}{d} |\tau\rangle \langle e| \right] \quad (\text{G85})$$

$$= \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B \left[\frac{d_A}{d^2 - 1} (d_B^2 - 1) |e\rangle + \frac{d_B}{d^2 - 1} (d_A^2 - 1) |\tau\rangle \right] \quad (\text{G86})$$

$$= \frac{d_A}{d^2 - 1} (d_B^2 - 1) |e\rangle_A |D_1^{\otimes 2}\rangle_B + \frac{d_B}{d^2 - 1} (d_A^2 - 1) |\tau\rangle_A |D_2\rangle_B, \quad (\text{G87})$$

where $D_2 = \sum_z |z\rangle \langle z|^{\otimes 2}$. It leads to two boundary conditions, where the first one of $|e\rangle_A |D_1^{\otimes 2}\rangle_B$ remains invariant under twirling while the second one $|\tau\rangle_A |D_2\rangle_B$ is a new boundary condition to evaluate:

$$\begin{aligned} & \sum_{z_{t-2}, z'_{t-2}} |z_{t-2}, z'_{t-2}\rangle \langle z_{t-2}, z'_{t-2}|_B \mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes 2} \otimes U_{t-1}^{*\otimes 2}] |\tau\rangle_A |D_2\rangle_B \\ &= \frac{1}{d^2 - 1} \sum_{z_{t-2}, z'_{t-2}} |z_{t-2}, z'_{t-2}\rangle \langle z_{t-2}, z'_{t-2}|_B \left[|e\rangle \langle e| + |\tau\rangle \langle \tau| - \frac{1}{d} |e\rangle \langle \tau| - \frac{1}{d} |\tau\rangle \langle e| \right] |\tau\rangle_A |D_2\rangle_B \end{aligned} \quad (\text{G88})$$

$$= \frac{d_B}{d^2 - 1} \sum_{z_{t-2}, z'_{t-2}} |z_{t-2}, z'_{t-2}\rangle \langle z_{t-2}, z'_{t-2}|_B \left[|e\rangle d_A + |\tau\rangle d_A^2 - \frac{d_A^2}{d} |e\rangle - \frac{d_A}{d} |\tau\rangle \right] \quad (\text{G89})$$

$$= \frac{1}{d^2 - 1} \sum_{z_{t-1}, z'_{t-1}} |z_{t-1}, z'_{t-1}\rangle \langle z_{t-1}, z'_{t-1}|_B [d_A (d_B - 1) |e\rangle + (d_A^2 d_B - 1) |\tau\rangle] \quad (\text{G90})$$

$$= \frac{d_A}{d^2 - 1} (d_B - 1) |e\rangle_A |D_1^{\otimes 2}\rangle_B + \frac{d_A^2 d_B - 1}{d^2 - 1} |\tau\rangle_A |D_2\rangle_B, \quad (\text{G91})$$

where we utilize $(\pi|D_2)_B = d_B$ in the third line for any permutation π . We can now still utilize a two-dimensional linear system to model the twirling dynamics with $|\tau\rangle_A |D_2\rangle_B \rightarrow (1, 0)^T$ and $|e\rangle_A |D_1^{\otimes 2}\rangle_B \rightarrow (0, 1)^T$. The transfer matrix is

$$M = \frac{1}{d^2 - 1} \begin{pmatrix} d_A^2 d_B - 1 & 0 \\ d_A (d_B - 1) & d^2 - 1 \end{pmatrix}, \quad (\text{G92})$$

and the purity for $t \geq 1$ then becomes

$$\mathbb{E}_{\text{Haar}} \text{tr}(\rho_A^2) = d_A^{-2} \langle e|_A \langle 0|_B M^{t-1} \left(\frac{d_B (d_A^2 - 1)}{d^2 - 1}, \frac{d_A (d_B^2 - 1)}{d^2 - 1} \right)^T \quad (\text{G93})$$

$$= d_A^{-2} \left[\frac{(d_A^2 - 1) d_B \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B - 1} \right)^t}{d_A^2 d_B - 1} d_A + \frac{1 - \frac{(d_A^2 - 1) \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B - 1} \right)^t}{d_A^2 d_B - 1}}{d_A} d_A^2 \right] \quad (\text{G94})$$

$$= \frac{(d_A^2 - 1) (d_B - 1)}{d_A (d_A^2 d_B - 1)} \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B - 1} \right)^t + \frac{1}{d_A}. \quad (\text{G95})$$

Similarly, the purity of the joint state of reference and system is

$$\text{tr}(\rho_{RA}^2) = \text{tr}(\rho_{RA}^{\otimes 2} \tau_R \tau_A) \quad (\text{G96})$$

$$= \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(|\tilde{\psi}_{\mathbf{z}}\rangle \langle \tilde{\psi}_{\mathbf{z}}| \otimes |\tilde{\psi}_{\mathbf{z}'}\rangle \langle \tilde{\psi}_{\mathbf{z}'}| \tau_R \tau_A) \quad (\text{G97})$$

$$= \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(U^{\otimes 2} |\Phi\rangle \langle \Phi|_{RA}^{\otimes 2} |\tilde{\mathbf{z}}\rangle \langle \tilde{\mathbf{z}}|_B \otimes |\tilde{\mathbf{z}}'\rangle \langle \tilde{\mathbf{z}}'|_B U^{\dagger \otimes 2} \tau_R \tau_A |\mathbf{z}\rangle \langle \mathbf{z}|_B \otimes |\mathbf{z}'\rangle \langle \mathbf{z}'|_B) \quad (\text{G98})$$

$$= d_A^{-2} \sum_{\mathbf{z}, \mathbf{z}'} \text{tr}(U^{\otimes 2} \tau_A |\tilde{\mathbf{z}}\rangle \langle \tilde{\mathbf{z}}|_B \otimes |\tilde{\mathbf{z}}'\rangle \langle \tilde{\mathbf{z}}'|_B U^{\dagger \otimes 2} \tau_A |\mathbf{z}\rangle \langle \mathbf{z}|_B \otimes |\mathbf{z}'\rangle \langle \mathbf{z}'|_B). \quad (\text{G99})$$

The Haar-averaged purity is, thus,

$$\mathbb{E}_{\text{Haar}} \text{tr}(\rho_{RA}^2) = d_A^{-2} (\tau|_A \langle 0|_B M^{t-1} \left(\frac{d_B(d_A^2 - 1)}{d^2 - 1}, \frac{d_A(d_B^2 - 1)}{d^2 - 1} \right)^T \quad (\text{G100})$$

$$= d_A^{-2} \left[\frac{(d_A^2 - 1) d_B \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t}{d_A^2 d_B - 1} d_A^2 + \frac{1 - \frac{(d_A^2 - 1) \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t}{d_A^2 d_B - 1}}{d_A} d_A \right] \quad (\text{G101})$$

$$= \frac{d_A^2 - 1}{d_A^2} \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A^2}. \quad (\text{G102})$$

Combining Eqs. (G79), (G95), and (G102), we have the Haar-averaged Rényi-2 extended QMI as

$$\mathbb{E}_{\text{Haar}} I_2(R : A_t) \simeq -\log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_R^2) - \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_A^2) + \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{RA}^2) \quad (\text{G103})$$

$$= -\log_2 \left(\frac{1}{d_A} \right) - \log_2 \left(\frac{(d_A^2 - 1)(d_B - 1)}{d_A(d_A^2 d_B - 1)} \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A} \right) + \log_2 \left(\frac{d_A^2 - 1}{d_A^2} \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A^2} \right) \quad (\text{G104})$$

$$= \log_2 \left((d_A^2 - 1) \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t + 1 \right) - \log_2 \left(\frac{(d_A^2 - 1)(d_B - 1)}{d_A^2 d_B - 1} \left(\frac{d_A^2 d_B - 1}{d_A^2 d_B^2 - 1} \right)^t + 1 \right). \quad (\text{G105})$$

In the asymptotic limit of $d_A, d_B \gg 1$, we can further reduce it to

$$\mathbb{E}_{\text{Haar}} I_2(R : A_t) = \log_2(d_A^2 d_B^{-t} + 1) - \log_2(d_B^{-t} + 1) \quad (\text{G106})$$

$$\simeq 2N_A - d_B^{-t} + \log_2(d_B^{-t} + d_A^{-2}). \quad (\text{G107})$$

For early time $t \ll 2N_A/N_B$, the QMI becomes

$$\mathbb{E}_{\text{Haar}} I_2(R : A_t) \simeq 2N_A - d_B^{-t} + \log_2 d_B^{-t} (1 + d_A^{-2} d_B^t) \quad (\text{G108})$$

$$= 2N_A - d_B^{-t} - tN_B + d_A^{-2} d_B^t \quad (\text{G109})$$

$$= 2N_A - tN_B + \mathcal{O}(d_B^t/d_A^2). \quad (\text{G110})$$

On the other hand, for late time of $t \gg 2N_A/N_B$, we have

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq 2N_A - d_B^{-t} + \log_2 d_A^{-2} (d_A^2 d_B^{-t} + 1) \quad (\text{G111})$$

$$= d_A^2 d_B^{-t} + \mathcal{O}(d_B^{-t}). \quad (\text{G112})$$

We complete the proof of Theorem 6.

We end this section by providing an alternative way of describing the channel of Eq. (40) adopted in the QRC literature [19]. As detailed in Eq. (10) of Supplementary Note 2 in Ref. [19], the quantum channel can be described by the superoperator

$$\mathcal{Q}_t(\cdot) = \mathcal{M}[U_t(\cdot)U_t^\dagger]. \quad (\text{G113})$$

Here, we introduce the *measurement-induced decoherence superoperator* $\mathcal{M}(\cdot) = (\mathbf{1}_A \otimes \mathbf{I}_B) \odot (\cdot)$ to represent the

full decoherence of the subsystem B [19], where $\mathbf{1}_A \in \mathbb{R}^{d_A \times d_A}$ is an all-one square matrix on subsystem A , $\mathbf{I}_B \in \mathbb{R}^{d_B \times d_B}$ is the identity matrix on subsystem B , and the Hadamard product $\odot$ is the entrywise product between two matrices.

4. Fully mixed state reset (Theorem 7)

In this part, we focus on a different reset strategy—the bath system is initialized with a fully mixed state $\omega = \mathbf{I}_B/d_B$ starting from the second step.

Similar to the derivation in pure-state reset strategy, we first write out the linear map E (shown by the blue box in Fig. 23) as

$$E = \sum_{\mathbf{a}, \mathbf{b}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle | a_t \rangle_{A_t} \langle a_0 |_{A_0} \otimes_{k=0}^{t-1} | b_{2k+1} \rangle \langle b_{2k} |, \quad (\text{G114})$$

where different from above we define the vector representation $\mathbf{b} = (b_0, b_1, \dots, b_{2t-2}, b_{2t-1})$. Via applying the linear map E on maximally entangled state $|\Phi\rangle_{RA_0}$ and the expanded bath system (shown in Fig. 23), we have the output state over the whole system as

$$\rho_{RA,B} = (\mathbb{I}_R \otimes E)(|\Phi\rangle\langle\Phi|_{RA_0} (\otimes_{k=0}^{t-1} \omega) \otimes |0\rangle\langle 0|) (\mathbb{I}_R \otimes E)^\dagger \quad (\text{G115})$$

$$\begin{aligned} &= \frac{1}{d_A} \sum_{\substack{\mathbf{a}, \mathbf{b}, \\ \mathbf{a}', \mathbf{b}'}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle \langle a'_k b'_{2k} | U_k^\dagger | a'_{k+1} b'_{2k+1} \rangle | a_0 \rangle \langle a'_0 |_R | a_t \rangle \\ &\quad \times \langle a'_t |_{A_t} (\otimes_{k=0}^{t-1} | b_{2k+1} \rangle \langle b'_{2k+1} |) \langle b_0 |_0 \langle 0 | b'_0 \rangle \otimes_{k=1}^{t-1} \langle b_{2k} | \omega | b'_{2k} \rangle \end{aligned} \quad (\text{G116})$$

$$\begin{aligned} &= \frac{1}{d_A} \frac{1}{d_B^{t-1}} \sum_{\substack{\mathbf{a}, \mathbf{b}, \\ \mathbf{a}', \mathbf{b}'}} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle \langle a'_k b'_{2k} | U_k^\dagger | a'_{k+1} b'_{2k+1} \rangle | a_0 \rangle \langle a'_0 |_R | a_t \rangle \\ &\quad \times \langle a'_t |_{A_t} (\otimes_{k=0}^{t-1} | b_{2k+1} \rangle \langle b'_{2k+1} |) \delta_{b_0,0} \delta_{b'_0,0} \prod_{k=1}^{t-1} \delta_{b_{2k}, b'_{2k}}. \end{aligned} \quad (\text{G117})$$

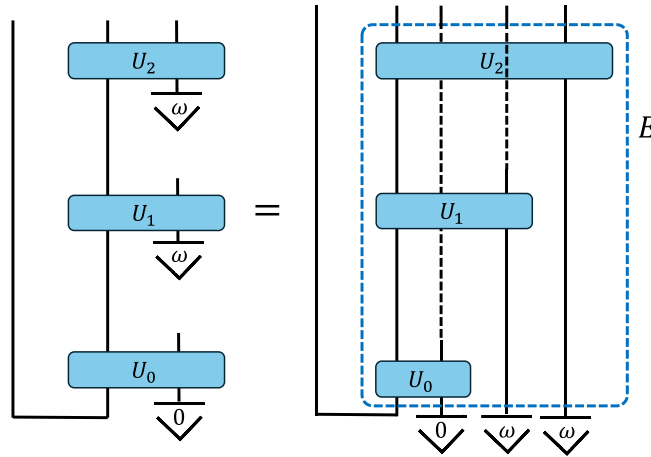


FIG. 23. Equivalent relation on the quantum dynamics with midcircuit measurements and diagonal-mixed-state reset via expanding the bath. Here, $\omega = \mathbf{I}_B/d_B$ represents the fully mixed state. The linear map E (blue box) is applied on the initial maximally entangled state. The dashed lines indicate that those outputs do not interact with the unitaries to be passed through. Here, we show an example for $t = 3$.

For the reduced state ρ_R , we have

$$\rho_R = \text{tr}_{A,B}(\rho_{RA,B}) \quad (\text{G118})$$

$$= \frac{1}{d_A} \frac{1}{d_B^{t-1}} \sum_{a,b'} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle \langle a'_k b'_{2k} | U_k^\dagger | a'_{k+1} b'_{2k+1} \rangle | a_0 \rangle \langle a'_0 |_R \delta_{a,a'_t} \prod_{k=0}^{t-1} \delta_{b_{2k+1}, b'_{2k+1}} \delta_{b_0,0} \delta_{b'_0,0} \prod_{k=1}^{t-1} \delta_{b_{2k}, b'_{2k}} \quad (\text{G119})$$

$$= \frac{1}{d_A} \frac{1}{d_B^{t-1}} \sum_{a,a'} \prod_{k=1}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes \mathbf{I}) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|a_0\rangle\langle a'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|a'_1\rangle\langle a_1| \otimes \mathbf{I})) |a_0\rangle\langle a'_0| \delta_{a,a'_t}. \quad (\text{G120})$$

The purity of ρ_R then can be evaluated as

$$\begin{aligned} \text{tr}(\rho_R^2) &= \frac{1}{d_A^2} \frac{1}{d_B^{2t-2}} \sum_{a,a'} \prod_{k=1}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes \mathbf{I}) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|a_0\rangle\langle a'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|a'_1\rangle\langle a_1| \otimes \mathbf{I})) \delta_{a,a'_t} \\ &\quad \times \sum_{\alpha,\alpha'} \prod_{k=1}^{t-1} \text{tr}(U_k(|\alpha_k\rangle\langle \alpha'_k| \otimes \mathbf{I}) U_k^\dagger(|\alpha'_{k+1}\rangle\langle \alpha_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|\alpha_0\rangle\langle \alpha'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|\alpha'_1\rangle\langle \alpha_1| \otimes \mathbf{I})) \delta_{\alpha,\alpha'_t} \delta_{a_0,\alpha'_0} \delta_{a'_0,\alpha_0} \end{aligned} \quad (\text{G121})$$

$$\begin{aligned} &= \frac{1}{d_A^2} \frac{1}{d_B^{2t-2}} \sum_{a,a'} \prod_{k=1}^{t-1} \text{tr}\left(U_k^{\otimes 2}(|a_k \alpha_k\rangle\langle a'_k \alpha'_k| \otimes \mathbf{I}^{\otimes 2}) U_k^{\dagger \otimes 2}(|a'_{k+1} \alpha'_{k+1}\rangle\langle a_{k+1} \alpha_{k+1}| \otimes \mathbf{I}^{\otimes 2})\right) \\ &\quad \times \text{tr}\left(U_0^{\otimes 2}(|a_0 \alpha_0\rangle\langle a'_0 \alpha'_0| \otimes |0\rangle\langle 0|^{\otimes 2}) U_0^{\dagger \otimes 2}(|a'_1 \alpha'_1\rangle\langle a_1 \alpha_1| \otimes \mathbf{I}^{\otimes 2})\right) \delta_{a,a'_t} \delta_{\alpha,\alpha'_t} \delta_{a_0,\alpha'_0} \delta_{a'_0,\alpha_0}, \end{aligned} \quad (\text{G122})$$

which is represented in tensor network in Fig. 24(a).

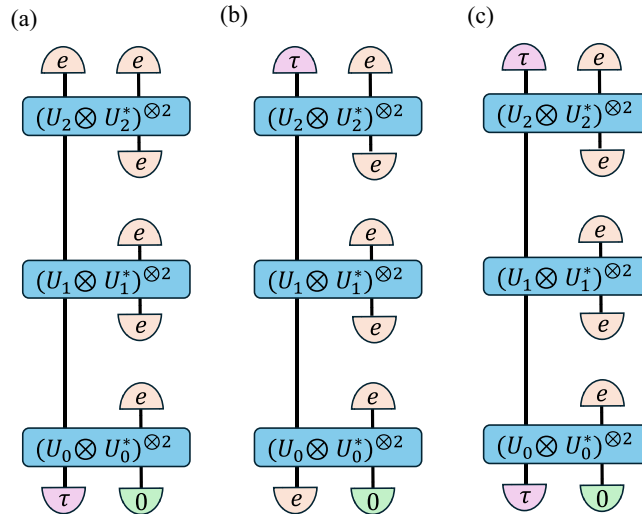


FIG. 24. Tensor network representation of Haar-averaged purity for reduced states (a) ρ_R , (b) ρ_{A_t} , and (c) ρ_{RA_t} , with fully mixed state reset and time steps $t = 3$.

Recall that $\mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes 2} \otimes U_{t-1}^{*\otimes 2}]|e\rangle_A|e\rangle_B = |e\rangle_{AB}$ from Eq. (G14), and the following concatenation with $|e\rangle(e|_B$ results in

$$|e\rangle(e|_B \mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes 2} \otimes U_{t-1}^{*\otimes 2}]|e\rangle_A|e\rangle_B = d_B^2 |e\rangle_A|e\rangle_B. \quad (\text{G123})$$

As the boundary condition is exactly reproduced in the following step with an extra coefficient d_B^2 , then we can directly obtain the solution of purity as

$$\mathbb{E}_{\text{Haar}}\text{tr}(\rho_R^2) = \frac{1}{d_A^2} \frac{1}{d_B^{2t-2}} d_B^{2(t-1)} (\tau|_A(0|_B|e\rangle_A|e\rangle_B) = \frac{1}{d_A}. \quad (\text{G124})$$

Next, we move to the reduced state ρ_{A_t} :

$$\rho_{A_t} = \text{tr}_{RB}(\rho_{RA,B}) \quad (\text{G125})$$

$$= \frac{1}{d_A} \frac{1}{d_B^{t-1}} \sum_{a,b'} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle \langle a'_k b'_{2k} | U_k^\dagger | a'_{k+1} b'_{2k+1} \rangle \delta_{a_0, a'_0} |a_t\rangle \langle a'_t|_{A_t} \prod_{k=0}^{t-1} \delta_{b_{2k+1}, b'_{2k+1}} \delta_{b_0, 0} \delta_{b'_{0,0}} \prod_{k=1}^{t-1} \delta_{b_{2k}, b'_{2k}} \quad (\text{G126})$$

$$= \frac{1}{d_A} \frac{1}{d_B^{t-1}} \sum_{a,a'} \prod_{k=1}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes \mathbf{I}) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|a_0\rangle\langle a'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|a'_1\rangle\langle a_1| \otimes \mathbf{I})) \delta_{a_0, a'_0} |a_t\rangle \langle a'_t|_{A_t}, \quad (\text{G127})$$

and the purity is

$$\begin{aligned} \text{tr}(\rho_{A_t}^2) &= \frac{1}{d_A^2} \frac{1}{d_B^{2t-2}} \sum_{a,a'} \prod_{k=1}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes \mathbf{I}) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|a_0\rangle\langle a'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|a'_1\rangle\langle a_1| \otimes \mathbf{I})) \delta_{a_0, a'_0} \\ &\quad \times \sum_{\alpha, \alpha'} \prod_{k=1}^{t-1} \text{tr}(U_k(|\alpha_k\rangle\langle \alpha'_k| \otimes \mathbf{I}) U_k^\dagger(|\alpha'_{k+1}\rangle\langle \alpha_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|\alpha_0\rangle\langle \alpha'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|\alpha'_1\rangle\langle \alpha_1| \otimes \mathbf{I})) \delta_{\alpha_0, \alpha'_0} \delta_{a_t, \alpha'_t} \delta_{a'_t, \alpha_t}, \end{aligned} \quad (\text{G128})$$

$$\begin{aligned} &= \frac{1}{d_A^2} \frac{1}{d_B^{2t-2}} \sum_{\alpha, \alpha'} \prod_{k=1}^{t-1} \text{tr}\left(U_k^{\otimes 2}(|\alpha_k \alpha_k\rangle\langle \alpha'_k \alpha'_k| \otimes \mathbf{I}^{\otimes 2}) U_k^{\dagger \otimes 2}(|\alpha'_{k+1} \alpha'_{k+1}\rangle\langle \alpha_{k+1} \alpha_{k+1}| \otimes \mathbf{I}^{\otimes 2})\right) \\ &\quad \times \text{tr}\left(U_0^{\otimes 2}(|\alpha_0 \alpha_0\rangle\langle \alpha'_0 \alpha'_0| \otimes |0\rangle\langle 0|^{\otimes 2}) U_0^{\dagger \otimes 2}(|\alpha'_1 \alpha'_1\rangle\langle \alpha_1 \alpha_1| \otimes \mathbf{I}^{\otimes 2})\right) \delta_{\alpha_0, \alpha'_0} \delta_{\alpha_0, \alpha'_0} \delta_{a_t, \alpha'_t} \delta_{a'_t, \alpha_t}, \end{aligned} \quad (\text{G129})$$

which is shown in Fig. 24(b).

Recall from Eq. (G20) $\mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes 2} \otimes U_{t-1}^{*\otimes 2}]|\tau\rangle_A|e\rangle_B = [d_A(d_B^2 - 1)/d_A^2 d_B^2 - 1]|\tau\rangle_{AB} + [d_B(d_A^2 - 1)/d_A^2 d_B^2 - 1]|\tau\rangle_{AB}$; we next apply $|e\rangle(e|_B$ and have

$$|e\rangle(e|_B \mathbb{E}_{\text{Haar}}[U_{t-1}^{\otimes 2} \otimes U_{t-1}^{*\otimes 2}]|\tau\rangle_A|e\rangle_B = \frac{d_A d_B^2 (d_B^2 - 1)}{d_A^2 d_B^2 - 1} |\tau\rangle_A|e\rangle_B + \frac{d_B^2 (d_A^2 - 1)}{d_A^2 d_B^2 - 1} |\tau\rangle_A|e\rangle_B. \quad (\text{G130})$$

Moving downward in Fig. 24(b) up to U_1 , we have the contribution from the two boundary conditions above as

- (i) $|\tau\rangle_A|e\rangle_B$: $[d_B^2(d_A^2 - 1)/d_A^2 d_B^2 - 1]^{t-1} |\tau\rangle_A|e\rangle_B$;
- (ii) $|e\rangle_A|e\rangle_B$: $(d_A d_B^2 (d_B^2 - 1)/d_A^2 d_B^2 - 1) \sum_{k=0}^{t-2} (d_B^2 (d_A^2 - 1)/d_A^2 d_B^2 - 1)^k d_B^{2(t-2-k)} |e\rangle_A|e\rangle_B$.

After finishing the twirling over U_0 , we have the dynamical solution for purity of ρ_{A_t} :

$$\begin{aligned} \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{A_t}^2) &= \frac{1}{d_A^2} \frac{1}{d_B^{2(t-1)}} \left[\left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^{t-1} \frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} d_A + \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^{t-1} \frac{d_A(d_B^2 - 1)}{d_A^2 d_B^2 - 1} d_A^2 \right. \\ &\quad \left. + \frac{d_A d_B^2(d_B^2 - 1)}{d_A^2 d_B^2 - 1} \sum_{k=0}^{t-2} \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^k d_B^{2(t-2-k)} d_A^2 \right] \end{aligned} \quad (\text{G131})$$

$$\begin{aligned} &= \frac{1}{d_A^2} \frac{1}{d_B^{2(t-1)}} \left[\frac{d_A}{d_B} \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t + \frac{d_A^3(d_B^2 - 1)}{(d_A^2 - 1)d_B^2} \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right. \\ &\quad \left. + \frac{d_A d_B^{2(t-1)}}{d_A^2 - 1} \left(d_A^2 - 1 - (d_A^2 d_B^2 - 1) \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t \right) \right] \end{aligned} \quad (\text{G132})$$

$$= \frac{d_B}{d_A} \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{d_A(d_B^2 - 1)}{d_A^2 - 1} \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A(d_A^2 - 1)} \left(d_A^2 - 1 - (d_A^2 d_B^2 - 1) \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t \right) \quad (\text{G133})$$

$$= \frac{d_B - 1}{d_A} \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A}. \quad (\text{G134})$$

Last, the reduced state ρ_{RA_t} is

$$\rho_{RA_t} = \text{tr}_B(\rho_{RA_t B}) \quad (\text{G135})$$

$$\begin{aligned} &= \frac{1}{d_A} \frac{1}{d_B^{t-1}} \sum_{a, b'} \prod_{k=0}^{t-1} \langle a_{k+1} b_{2k+1} | U_k | a_k b_{2k} \rangle \langle a'_k b'_{2k} | U_k^\dagger | a'_{k+1} b'_{2k+1} \rangle | a_0 \rangle \langle a'_0 |_R | a_t \rangle \langle a'_t |_{A_t} \prod_{k=0}^{t-1} \delta_{b_{2k+1}, b'_{2k+1}} \delta_{b_0, 0} \delta_{b'_0, 0} \prod_{k=1}^{t-1} \delta_{b_{2k}, b'_{2k}} \end{aligned} \quad (\text{G136})$$

$$\begin{aligned} &= \frac{1}{d_A} \frac{1}{d_B^{t-1}} \sum_{a, a'} \prod_{k=1}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes \mathbf{I}) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|a_0\rangle\langle a'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|a'_1\rangle\langle a_1| \\ &\quad \times \langle a_1| \otimes \mathbf{I})) | a_0 \rangle \langle a'_0 |_R | a_t \rangle \langle a'_t |_{A_t}, \end{aligned} \quad (\text{G137})$$

and the purity is

$$\begin{aligned} \text{tr}(\rho_{RA_t}^2) &= \frac{1}{d_A^2} \frac{1}{d_B^{2(t-2)}} \sum_{a, a'} \prod_{k=1}^{t-1} \text{tr}(U_k(|a_k\rangle\langle a'_k| \otimes \mathbf{I}) U_k^\dagger(|a'_{k+1}\rangle\langle a_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|a_0\rangle\langle a'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|a'_1\rangle\langle a_1| \otimes \mathbf{I})) \delta_{a_0, a'_0} \delta_{a'_0, a_0} \\ &\quad \times \sum_{\alpha, \alpha'} \prod_{k=1}^{t-1} \text{tr}(U_k(|\alpha_k\rangle\langle \alpha'_k| \otimes \mathbf{I}) U_k^\dagger(|\alpha'_{k+1}\rangle\langle \alpha_{k+1}| \otimes \mathbf{I})) \text{tr}(U_0(|\alpha_0\rangle\langle \alpha'_0| \otimes |0\rangle\langle 0|) U_0^\dagger(|\alpha'_1\rangle\langle \alpha_1| \otimes \mathbf{I})) \delta_{a_t, \alpha'_t} \delta_{a'_t, \alpha_t} \end{aligned} \quad (\text{G138})$$

$$\begin{aligned} &= \frac{1}{d_A^2} \frac{1}{d_B^{2(t-2)}} \sum_{a, a'} \prod_{k=1}^{t-1} \text{tr} \left(U_k^{\otimes 2}(|a_k \alpha_k\rangle\langle a'_k \alpha'_k| \otimes \mathbf{I}^{\otimes 2}) U_k^{\dagger \otimes 2}(|a'_{k+1} \alpha'_{k+1}\rangle\langle a_{k+1} \alpha_{k+1}| \otimes \mathbf{I}^{\otimes 2}) \right) \\ &\quad \times \text{tr} \left(U_0^{\otimes 2}(|a_0 \alpha_0\rangle\langle a'_0 \alpha'_0| \otimes |0\rangle\langle 0|^{\otimes 2}) U_0^{\dagger \otimes 2}(|a'_1 \alpha'_1\rangle\langle a_1 \alpha_1| \otimes \mathbf{I}^{\otimes 2}) \right) \delta_{a_0, \alpha'_0} \delta_{a'_0, \alpha_0} \delta_{a_t, \alpha'_t} \delta_{a'_t, \alpha_t}, \end{aligned} \quad (\text{G139})$$

which is represented as tensor network as in Fig. 24(c). Since it shares the same boundary condition as in Fig. 24(b), the contribution toward U_1 remains the same as in $\text{tr}(\rho_{A_t}^2)$. The dynamical solution for the purity of ρ_{RA_t} is

$$\begin{aligned} \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{RA_t}^2) &= \frac{1}{d_A^2} \frac{1}{d_B^{2(t-1)}} \left[\left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^{t-1} \frac{d_B(d_A^2 - 1)}{d_A^2 d_B^2 - 1} d_A^2 + \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^{t-1} \frac{d_A(d_B^2 - 1)}{d_A^2 d_B^2 - 1} d_A \right. \\ &\quad \left. + \frac{d_A(d_B^2 - 1)}{d_A^2 d_B^2 - 1} d_B^{2(t-1)} \sum_{k=0}^{t-2} \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^k d_B^{-2k} d_A \right] \end{aligned} \quad (\text{G140})$$

$$\begin{aligned} &= \frac{1}{d_A^2} \frac{1}{d_B^{2(t-1)}} \left[\frac{d_A^2}{d_B} \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t + \frac{d_A^2(d_B^2 - 1)}{(d_A^2 - 1)d_B^2} \left(\frac{d_B^2(d_A^2 - 1)}{d_A^2 d_B^2 - 1} \right)^t \right. \\ &\quad \left. + \frac{d_B^{2(t-1)}}{d_A^2 - 1} \left(d_A^2 - 1 - (d_A^2 d_B^2 - 1) \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t \right) \right] \end{aligned} \quad (\text{G141})$$

$$= \left(d_B - \frac{1}{d_A^2} \right) \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A^2}. \quad (\text{G142})$$

We again switch the order of ensemble average and logarithmic and combine the results in Eqs. (G124), (G134), and (G142):

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq -\log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_R^2) - \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{A_t}^2) + \log_2 \mathbb{E}_{\text{Haar}} \text{tr}(\rho_{RA_t}^2) \quad (\text{G143})$$

$$= N_A - \log_2 \left[\frac{d_B - 1}{d_A} \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A} \right] + \log_2 \left[\left(d_B - \frac{1}{d_A^2} \right) \left(\frac{d_A^2 - 1}{d_A^2 d_B^2 - 1} \right)^t + \frac{1}{d_A^2} \right], \quad (\text{G144})$$

which is the result of Eq. (B10) in the main text. In the large limit of system $d_A, d_B \gg 1$, we perform the asymptotic analysis as

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq N_A - \log_2 \left(\frac{1}{d_A} d_B^{1-2t} + \frac{1}{d_A} \right) + \log_2 \left(d_B^{1-2t} + \frac{1}{d_A^2} \right) \quad (\text{G145})$$

$$\simeq 2N_A - d_B^{1-2t} + \log_2(d_A^{-2} + d_B^{1-2t}). \quad (\text{G146})$$

In the early time $t \ll N_A/N_B + 1/2$, we can further approximate it as

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq 2N_A - d_B^{1-2t} + \log_2 d_B^{1-2t} (d_A^{-2} d_B^{2t-1} + 1) \quad (\text{G147})$$

$$\simeq 2N_A - d_B^{1-2t} + (1 - 2t)N_B + d_A^{-2} d_B^{2t-1} \quad (\text{G148})$$

$$\simeq 2N_A + N_B - 2tN_B + \mathcal{O}(d_B^{1-2t}). \quad (\text{G149})$$

For the late time $t \gg N_A/N_B + 1/2$, we can further approximate it as

$$\mathbb{E}_{\text{Haar}} I_2(R:A_t) \simeq 2N_A - d_B^{1-2t} + \log_2 d_A^{-2} (1 + d_A^2 d_B^{1-2t}) \quad (\text{G150})$$

$$\simeq 2N_A - d_B^{1-2t} - 2N_A + d_A^2 d_B^{1-2t} \quad (\text{G151})$$

$$\simeq d_A^2 d_B^{1-2t}. \quad (\text{G152})$$

APPENDIX H: DERIVATION OF MEASUREMENT-CONDITIONED QMI IN PARTIALLY MONITORED DYNAMICS (THEOREM 5)

In this section, we derive the dynamics of the measurement-conditioned QMI in a monitored dynamics with periodically doped erasure channels. Because of the existence of erasure channels, the conditioned state is a mixed state, in general, and, thus, we consider the Rényi-2 extended measurement-conditioned QMI for analysis convenience as

$$\overline{I_2(R:A_t|z)} = \mathbb{E}_z I_2(R:A_t|z) = \mathbb{E}_z [S_2(R|z) + S_2(A_t|z) - S_2(RA_t|z)] \quad (\text{H1})$$

$$\simeq -\log_2(\mathbb{E}_z \text{tr}(\rho_{R|z}^2)) - \log_2(\mathbb{E}_z \text{tr}(\rho_{A_t|z}^2)) + \log_2(\mathbb{E}_z \text{tr}(\rho_{RA_t|z}^2)), \quad (\text{H2})$$

where the approximation can be regarded as annealed approximation to the quenched quantity. Here, we still assume that the initial state of reference and system is a maximally entangled state with $N_R = N_A$.

For notation convenience, we denote B' to represent the bath qubits that are not erased in an arbitrary step. The conditioned state on the reference and system at step t is

$$\rho_{RA_t|z} = \text{tr}_B(U|\Phi\rangle\langle\Phi|_{RA}|0\rangle\langle 0|_B U^\dagger \mathbf{I}_E|z\rangle\langle z|_{B'})/p_z \equiv \tilde{\rho}_{RA_t|z}/p_z, \quad (\text{H3})$$

where $p_z = \text{tr}(\tilde{\rho}_{RA_t|z})$ is the probability of measurement trajectory z . Here, B , E , and B' represent the expanded bath system over all past temporal steps. The reduced state of $\rho_{R|z}$ and $\rho_{A_t|z}$ can be written out similarly. Now we evaluate the conditioned purity of $\rho_{R|z}$, which is

$$\bar{\gamma}_R = \sum_z p_z \text{tr} \rho_{R|z}^2 \quad (\text{H4})$$

$$= \sum_z p_z^{-1} \text{tr}(\tilde{\rho}_{R|z}^2), \quad (\text{H5})$$

and, thus, we utilize the replica trick to defined the pseudo conditioned purity as

$$\bar{\gamma}_R^{(m)} \equiv \sum_z p_z^m \text{tr}(\tilde{\rho}_{R|z}^2) \quad (\text{H6})$$

$$= \sum_z \text{tr} \left(U^{\otimes(m+2)} |\Phi\rangle\langle\Phi|_{RA}^{\otimes(m+2)} |0\rangle\langle 0|_B^{\otimes(m+2)} U^{\dagger \otimes(m+2)} \mathbf{I}_R^{\otimes m} \tau_R \mathbf{I}_A^{\otimes(m+2)} \mathbf{I}_E^{\otimes(m+2)} |z\rangle\langle z|_{B'}^{\otimes(m+2)} \right) \quad (\text{H7})$$

$$= d_A^{-(m+2)} \text{tr} \left(U^{\otimes(m+2)} \mathbf{I}_A^{\otimes m} \tau_A |0\rangle\langle 0|_B^{\otimes(m+2)} U^{\dagger \otimes(m+2)} \mathbf{I}_A^{\otimes(m+2)} \mathbf{I}_E^{\otimes(m+2)} \sum_z |z\rangle\langle z|_{B'}^{\otimes(m+2)} \right). \quad (\text{H8})$$

Therefore, there exist two types of boundary condition for bath in the unitary twirling, $|D_{m+2}\rangle_B$ and $|e\rangle_E |D_{m+2}\rangle_{B'}$, which correspond to the case without and with erasure channels, respectively.

For $|D_{m+2}\rangle_B$, we have derived the twirling results in Appendix F, and the transition can be efficiently represented by the linear transformation matrix Q [see also in Eq. (F49)] as

$$Q = \begin{pmatrix} q_\tau & q_e \\ q_e & q_\tau \end{pmatrix} \simeq d_B^{-(m+1)} \begin{pmatrix} 1 & d_A^{-1} \\ d_A^{-1} & 1 \end{pmatrix}, \quad (\text{H9})$$

where the approximation holds when $d_A, d_B \gg 1$. We remind the readers that here we take the mapping of $|\tau_2\rangle_A \rightarrow (1, 0)^T$ and $|e\rangle_A \rightarrow (0, 1)^T$ omitting the operator on bath.

On the other hand, for $|e\rangle_E |D_{m+2}\rangle_{B'}$, when the system boundary condition is $|e\rangle_A$, we have

$$\begin{aligned} & (0|_{EB'} \mathbb{E}_{\text{Haar}} [U^{\otimes(m+2)} \otimes U^{*\otimes(m+2)}] |e\rangle_A |e\rangle_E |D_{m+2}\rangle_{B'}) \\ &= \sum_{\sigma, \pi \in S_{m+2}} \text{Wg}(\sigma^{-1} \pi, m+2) |\sigma\rangle_A (\pi|e\rangle_A) (\pi|e\rangle_E) (\pi|D_{m+2}\rangle_{B'}) \end{aligned} \quad (\text{H10})$$

$$= d_{B'} \sum_{\sigma, \pi} \text{Wg}(\sigma^{-1} \pi, m+2) |\sigma\rangle_A d_{AE}^{|\pi|} \quad (\text{H11})$$

$$\simeq d_{B'} [\text{Wg}(e, m+2) (|e\rangle_A d_{AE}^{m+2} + |\tau_2\rangle_A d_{AE}^{m+1}) + \text{Wg}(\tau_2, m+2) (|e\rangle_A d_{AE}^{m+1} + |\tau_2\rangle_A d_{AE}^{m+2})] \quad (\text{H12})$$

$$\simeq d_{B'} [d_{AB}^{-(m+2)} (|e\rangle_A d_{AE}^{m+2} + |\tau_2\rangle_A d_{AE}^{m+1}) - d_{AB}^{-(m+3)} (|e\rangle_A d_{AE}^{m+1} + |\tau_2\rangle_A d_{AE}^{m+2})] \quad (\text{H13})$$

$$= d_{B'}^{-(m+1)} (1 - d_{AE}^{-2} d_{B'}^{-1}) |e\rangle_A + d_{B'}^{-(m+1)} d_{AE}^{-1} (1 - d_{B'}^{-1}) |\tau_2\rangle_A. \quad (\text{H14})$$

Meanwhile, for the system boundary condition to be $|\tau_2\rangle_A$, we have

$$\begin{aligned} & (0|_{EB'} \mathbb{E}_{\text{Haar}}[U^{\otimes(m+2)} \otimes U^{*\otimes(m+2)}]|\tau_2\rangle_A|e\rangle_E|D_{m+2}\rangle_{B'} \\ &= \sum_{\sigma, \pi \in S_{m+2}} \text{Wg}(\sigma^{-1}\pi, m+2)|\sigma\rangle_A(\pi|\tau_2\rangle_A)(\pi|e\rangle_E)(\pi|D_{m+2}\rangle_{B'}) \end{aligned} \quad (\text{H15})$$

$$= d_{B'} \sum_{\sigma, \pi} \text{Wg}(\sigma^{-1}\pi, m+2)|\sigma\rangle_A d_A^{|\pi^{-1}\tau_2|} d_E^{|\pi|} \quad (\text{H16})$$

$$\simeq d_{B'} [\text{Wg}(e, m+2)(|e\rangle_A d_A^{m+1} d_E^{m+2} + |\tau_2\rangle_A d_A^{m+2} d_E^{m+1}) + \text{Wg}(\tau_2, m+2)(|e\rangle_A d_A^{m+2} d_E^{m+1} + |\tau_2\rangle_A d_A^{m+1} d_E^{m+2})] \quad (\text{H17})$$

$$\simeq d_{B'} [d_{AB}^{-(m+2)}(|e\rangle_A d_A^{m+1} d_E^{m+2} + |\tau_2\rangle_A d_A^{m+2} d_E^{m+1}) - d_{AB}^{-(m+3)}(|e\rangle_A d_A^{m+2} d_E^{m+1} + |\tau_2\rangle_A d_A^{m+1} d_E^{m+2})] \quad (\text{H18})$$

$$= d_{B'}^{-(m+1)} d_A^{-1} (1 - d_{B'}^{-1} d_E^{-2}) |e\rangle_A + d_{B'}^{-(m+1)} d_E^{-1} (1 - d_A^{-2} d_{B'}^{-1}) |\tau_2\rangle_A. \quad (\text{H19})$$

Now we can summarize the transformation by

$$Q' = d_{B'}^{-(m+1)} \begin{pmatrix} d_E^{-1} (1 - d_A^{-2} d_{B'}^{-1}) & d_{AE}^{-1} (1 - d_{B'}^{-1}) \\ d_A^{-1} (1 - d_E^{-2} d_{B'}^{-1}) & 1 - d_{AE}^{-2} d_{B'}^{-1} \end{pmatrix} \simeq d_{B'}^{-(m+1)} \begin{pmatrix} d_E^{-1} & d_A^{-1} d_E^{-1} \\ d_A^{-1} & 1 \end{pmatrix}, \quad (\text{H20})$$

where the approximation holds when $d_A, d_{B'} \gg 1$.

With the representation of Q and Q' , we can, in general, write out the whole dynamics in permutation basis as

$$(0|_B \mathbb{E}_{\text{Haar}}[U^{\otimes(m+2)} \otimes U^{*\otimes(m+2)}]|e\rangle_E|D_{m+2}\rangle_{B'}) \rightarrow Q^{t-sn_s} (Q' Q^{s-1})^{n_s}, \quad (\text{H21})$$

where $n_s = \lfloor t/s \rfloor$ counts the number of erasure channel periods. Overall, one can apply this linear transformation to different boundary conditions and obtain the corresponding purities of interest, which lead to the Rényi-2 extended measurement-conditioned QMI in Eq. (H2).

The exact analytical formula for the QMI is challenging due to the fact that the matrix Q' is not symmetric. Therefore, to obtain analytical insight, we consider the asymptotic limit of $d_A, d_{B'} \gg 1$ and, thus, focus on only the asymptotic form of Q and Q' in Eqs. (H9) and (H20). We further notice that the asymptotic form of Q' is directly connected to Q by another diagonal matrix multiplication as

$$Q' = \text{diag}(d_E^{-1}, 1) Q \equiv \Lambda Q, \quad (\text{H22})$$

and $Q = d_B^{-(m+1)} (\mathbf{I} + d_A^{-1} \sigma_x)$, where σ_x is the Pauli- X matrix. The above transformation in Eq. (H21) can be reduced to

$$Q^{t-sn_s} (Q' Q^{s-1})^{n_s} = Q^{t-sn_s} (\Lambda Q^s)^{n_s} \quad (\text{H23})$$

$$= d_B^{-(m+1)(t-sn_s)} \left(\mathbf{I} + \frac{\sigma_x}{d_A} \right)^{t-sn_s} \left[\Lambda \left(d_B^{-(m+1)s} \left(\mathbf{I} + \frac{\sigma_x}{d_A} \right)^s \right) \right]^{n_s} \quad (\text{H24})$$

$$= d_B^{-(m+1)(t-sn_s)} \left(\mathbf{I} + (t-sn_s) \frac{\sigma_x}{d_A} \right) d_B^{-(m+1)sn_s} \left(\Lambda + \frac{s}{d_A} \Lambda \sigma_x \right)^{n_s} + \mathcal{O}\left(\frac{1}{d_A^2}\right) \quad (\text{H25})$$

$$= d_B^{-(m+1)t} \left(\mathbf{I} + (t-sn_s) \frac{\sigma_x}{d_A} \right) \left(\Lambda + \frac{s}{d_A} \Lambda \sigma_x \right)^{n_s}. \quad (\text{H26})$$

For $(\Lambda + (s/d_A) \Lambda \sigma_x)^{n_s}$, we have

$$\left(\Lambda + \frac{s}{d_A} \Lambda \sigma_x\right)^{n_s} = \Lambda^{n_s} + \frac{s}{d_A} \sum_{i=0}^{n_s-1} \Lambda^{n_s-1-i} \Lambda \sigma_x \Lambda^i + \mathcal{O}\left(\frac{1}{d_A^2}\right) \quad (\text{H27})$$

$$= \Lambda^{n_s} + \frac{s}{d_A} \sum_{i=0}^{n_s-1} \Lambda^{n_s-i} \sigma_x \Lambda^i \quad (\text{H28})$$

$$= \Lambda^{n_s} + \frac{s}{d_A} \sum_{i=0}^{n_s-1} \begin{pmatrix} 0 & d_E^{-(n_s-i)} \\ d_E^{-i} & 0 \end{pmatrix} \quad (\text{H29})$$

$$= \Lambda^{n_s} + \frac{s}{d_A} \begin{pmatrix} 0 & (1 - d_E^{-n_s})/(d_E - 1) \\ d_E(1 - d_E^{-n_s})/(d_E - 1) & 0 \end{pmatrix} \equiv \Lambda^{n_s} + \frac{s}{d_A} C. \quad (\text{H30})$$

Therefore, the original $Q^{t-sn_s}(Q'Q^{s-1})^{n_s}$ above becomes

$$Q^{t-sn_s}(Q'Q^{s-1})^{n_s} = d_B^{-(m+1)t} \left(\mathbf{I} + (t - sn_s) \frac{\sigma_x}{d_A} \right) \left(\Lambda + \frac{s}{d_A} \Lambda \sigma_x \right)^{n_s} \quad (\text{H31})$$

$$= d_B^{-(m+1)t} \left(\mathbf{I} + (t - sn_s) \frac{\sigma_x}{d_A} \right) \left(\Lambda^{n_s} + \frac{s}{d_A} C \right) \quad (\text{H32})$$

$$= d_B^{-(m+1)t} \left[\Lambda^{n_s} + \frac{s}{d_A} C + \frac{t - sn_s}{d_A} \sigma_x \Lambda^{n_s} + \mathcal{O}\left(\frac{1}{d_A^2}\right) \right] \quad (\text{H33})$$

$$= d_B^{-(m+1)t} \begin{pmatrix} d_E^{-n_s} & \frac{1}{d_A} \left(s \frac{1 - d_E^{-n_s}}{d_E - 1} + t - sn_s \right) \\ \frac{1}{d_A} \left(\frac{s d_E(1 - d_E^{-n_s})}{d_E - 1} + (t - sn_s) d_E^{-n_s} \right) & 1 \end{pmatrix} \equiv \mathcal{T}. \quad (\text{H34})$$

With the global transformation matrix $\mathcal{T}$, we can turn to evaluate all the conditioned pseudopurities of interest. For pseudopurity of $\rho_{R|z}$, we have

$$\mathbb{E}_{\text{Haar}} \tilde{\gamma}_R^{(m)} = d_A^{-(m+2)} (\tau_2 | \mathcal{T}(0, 1)^T \quad (\text{H35})$$

$$= d_A^{-(m+2)} d_B^{-(m+1)t} (\tau_2 |_A \left[\frac{1}{d_A} \left(s \frac{1 - d_E^{-n_s}}{d_E - 1} + t - sn_s \right) |\tau_2)_A + |e)_A \right] \quad (\text{H36})$$

$$= d_A^{-(m+2)} d_B^{-(m+1)t} \left[\frac{1}{d_A} \left(s \frac{1 - d_E^{-n_s}}{d_E - 1} + t - sn_s \right) d_A^{m+2} + d_A^{m+1} \right] \quad (\text{H37})$$

$$= \frac{d_B^{-(m+1)t}}{d_A} \left(s \frac{1 - d_E^{-n_s}}{d_E - 1} + t - sn_s + 1 \right). \quad (\text{H38})$$

Similarly, for pseudopurities of $\rho_{A|z}$ and $\rho_{RA|z}$, we have

$$\mathbb{E}_{\text{Haar}} \tilde{\gamma}_A^{(m)} = d_A^{-(m+2)} (e |_A \mathcal{T}(1, 0)^T \quad (\text{H39})$$

$$= d_A^{-(m+2)} d_B^{-(m+1)t} (e |_A \left[d_E^{-n_s} |\tau_2)_A + \frac{1}{d_A} \left(\frac{s d_E(1 - d_E^{-n_s})}{d_E - 1} + (t - sn_s) d_E^{-n_s} \right) |e)_A \right] \quad (\text{H40})$$

$$= d_A^{-(m+2)} d_B^{-(m+1)t} \left[d_E^{-n_s} d_A^{m+1} + \frac{1}{d_A} \left(\frac{s d_E(1 - d_E^{-n_s})}{d_E - 1} + (t - sn_s) d_E^{-n_s} \right) d_A^{m+2} \right] \quad (\text{H41})$$

$$= \frac{d_B^{-(m+1)t}}{d_A} \left(\frac{s d_E(1 - d_E^{-n_s})}{d_E - 1} + (t - sn_s + 1) d_E^{-n_s} \right) \quad (\text{H42})$$

and

$$\mathbb{E}_{\text{Haar}} \bar{\gamma}_{RA}^{(m)} = d_A^{-(m+2)} (\tau_2|_A \mathcal{T}(1,0))^T \quad (\text{H43})$$

$$= d_A^{-(m+2)} d_B^{-(m+1)t} (\tau_2|_A \left[d_E^{-n_s} | \tau_2 \right)_A + \frac{1}{d_A} \left(\frac{s d_E (1 - d_E^{-n_s})}{d_E - 1} + (t - s n_s) d_E^{-n_s} \right) | e \rangle_A \right] \quad (\text{H44})$$

$$= d_A^{-(m+2)} d_B^{-(m+1)t} \left[d_E^{-n_s} d_A^{m+2} + \frac{1}{d_A} \left(\frac{s d_E (1 - d_E^{-n_s})}{d_E - 1} + (t - s n_s) d_E^{-n_s} \right) d_A^{m+1} \right] \quad (\text{H45})$$

$$= d_B^{-(m+1)t} \left[d_E^{-n_s} + \frac{1}{d_A^2} \left(\frac{s d_E (1 - d_E^{-n_s})}{d_E - 1} + (t - s n_s) d_E^{-n_s} \right) \right] \quad (\text{H46})$$

$$= d_B^{-(m+1)t} d_E^{-n_s} + \mathcal{O}\left(\frac{1}{d_A^2}\right). \quad (\text{H47})$$

By taking the replica limit of $m \rightarrow -1$ and utilizing the definition in Eq. (H2), we have the full expression for the Rényi-2 extended conditioned QMI as

$$\mathbb{E}_{\text{Haar}} \overline{I_2(R:A_t|\mathbf{z})} = -\log_2 \bar{\gamma}_R - \log_2 \bar{\gamma}_A + \log_2 \bar{\gamma}_{RA} \quad (\text{H48})$$

$$= -\log_2 \left[\frac{1}{d_A} \left(s \frac{1 - d_E^{-n_s}}{d_E - 1} + t - s n_s + 1 \right) \right] - \log_2 \left[\frac{1}{d_A} \left(\frac{s d_E (1 - d_E^{-n_s})}{d_E - 1} + (t - s n_s + 1) d_E^{-n_s} \right) \right] + \log_2 d_E^{-n_s} \quad (\text{H49})$$

$$= 2N_A - n_s N_E - \log_2 \left(s \frac{1 - d_E^{-n_s}}{d_E - 1} + t - s n_s + 1 \right) - \log_2 \left(\frac{s d_E (1 - d_E^{-n_s})}{d_E - 1} + (t - s n_s + 1) d_E^{-n_s} \right) \quad (\text{H50})$$

$$\simeq 2N_A - n_s N_E - \log_2 [s(1 - d_E^{-n_s}) d_E^{-1} + t - s n_s + 1] - \log_2 [s(1 - d_E^{-n_s}) + (t - s n_s + 1) d_E^{-n_s}] \quad (\text{H51})$$

$$\simeq 2N_A - n_s N_E - \log_2 (1 + t - s n_s) - \log_2 s, \quad (\text{H52})$$

where the second-to-last line holds when $d_E \gg 1$. The last line is a further simplification for analysis convenience and leads to Eq. (30) of Theorem 5 in the main text.

APPENDIX I: ADDITIONAL NUMERICAL DETAILS

In this section, we provide additional numerical simulation results to support the theorems and estimation of lifetime in the main text. We simulate the dynamics of QMI with TensorCircuit [53].

In Figs. 25(a) and 25(b), we show the decay of normalized measurement-conditioned QMI versus a rescaled time $(1 - 1/d_B)t + 1$ in various data systems N_A with either randomly sampled or identical unitary from Haar ensemble. With increasing N_A , we clearly see a logarithmic decay and, thus, can estimate the corresponding QMI lifetime via a threshold (i.e., $\epsilon = 1/4$). Similar observations are also found for Hamiltonian evolution (shown in Fig. 26). In main text, we show only the asymptotic lower bound of Eq. (F68) (red); here, we

compare it to the complete form of Eq. (F62) (green) in Fig. 25(c). In fact, in the most valid range of decay, the two bounds overlap and align with the numerical results. Moreover, through a comparison to the numerical simulation of lower bound Eq. (F5) in derivation, we find that in the early stage our theoretical results of either Eq. (F62) or (F68) agree well, and the deviation to the simulation of QMI (blue) is due to the difference between von Neumann entropy and its Rényi counterpart.

For measurement-unconditioned QMI, as shown in Fig. 27, we still see that, with increasing system size N_A , the dynamics of QMI behaves similarly as a linear decay for all possible four unitary settings under consideration. Therefore, we can estimate the corresponding QMI lifetime from numerical simulation of dynamics, and we present the results in the main text and Appendix C.

In Figs. 28(b1)–28(b5), we show the normalized bipartite entanglement entropy $S_{\text{half}}/(N_A + N_B)$ of eigenstates of the Ising Hamiltonian in Eq. (7) with different N_A and $N_B = 2$. Since we find only the thermalized state

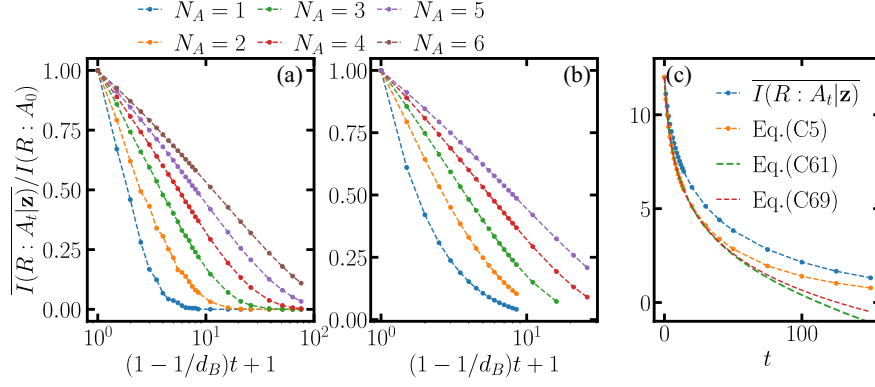


FIG. 25. Dynamics of normalized measurement-conditioned QMI $\overline{I(R:A_t|z)}$ with midcircuit measurements and reset. In (a) and (b), we plot QMI with different data systems N_A and same bath $N_B = 1$ with randomly sampled unitary and identical unitary from Haar ensemble through different steps. In (c), dots are numerical simulation results of $\overline{I(R:A_t|z)}$ with $N_A = 5$ [the same as purple in (a)]. The orange dashed line represents numerical results of the lower bound in Eq. (F5). Green and red dashed lines show the theoretical lower bounds Eqs. (F62) and (F68), respectively.

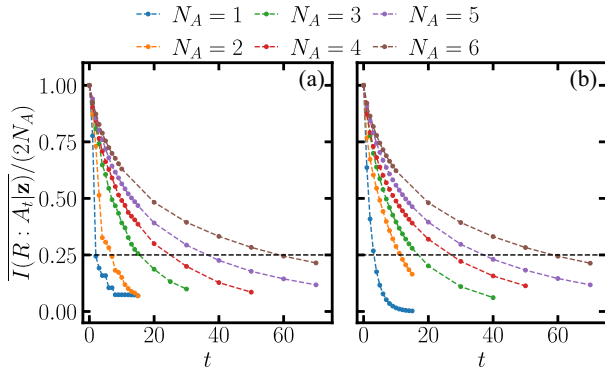


FIG. 26. Dynamics of normalized measurement-conditioned QMI $\overline{I(R:A_t|z)}$ with midcircuit measurements and reset. In (a) and (b), we plot QMI with different data systems N_A and same bath $N_B = 1$ with Hamiltonian evolution with randomly sampled coefficients and identical Hamiltonian evolution.

entanglement spectrums, there does not exist a “many-body scar” signature in the Hamiltonian considered here. Moreover, we evaluate the projection overlap between each eigenstate $|E_j\rangle$ and representative eigenstates of quantum channel $\mathcal{P}_0$ appended with $|0\rangle\langle 0|_B$ on bath. For a fair comparison, we normalize the norm of overlap and plot it versus normalized eigenenergies E_j for different N_A in Figs. 28(c1)–28(c5). For the fixed state (blue) and second-largest eigenstate (green), the largest overlaps with Hamiltonian eigenstates lie at the boundaries, corresponding to eigenstates with nearly zero bipartite entanglement.

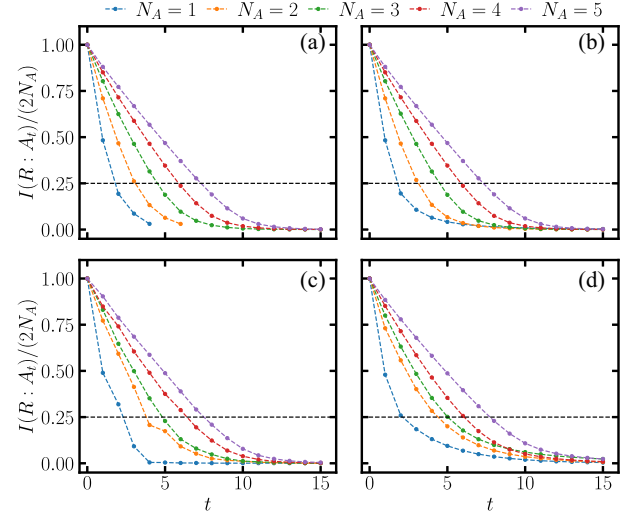


FIG. 27. Dynamics of normalized measurement-unconditioned QMI $I(R:A_t)$ with midcircuit measurements and reset. We plot QMI with different data systems N_A and same bath $N_B = 1$ with (a) randomly sampled Haar unitary, (b) identical Haar unitary, (c) Hamiltonian evolution with random coefficients, and (d) identical Hamiltonian evolution.

In contrast, for σ_1 (orange), the operator with medium eigenvalue norm $|\lambda_{\text{mid}}|$ from spectrum $\mathcal{P}_0$, the major contribution on overlap comes from thermalized Hamiltonian eigenstates instead of the boundary parts. For visualization convenience, we mark the corresponding eigenvalues of ρ_{fix} , σ_1 , and σ_{mid} in Figs. 28(a1)–28(a5).

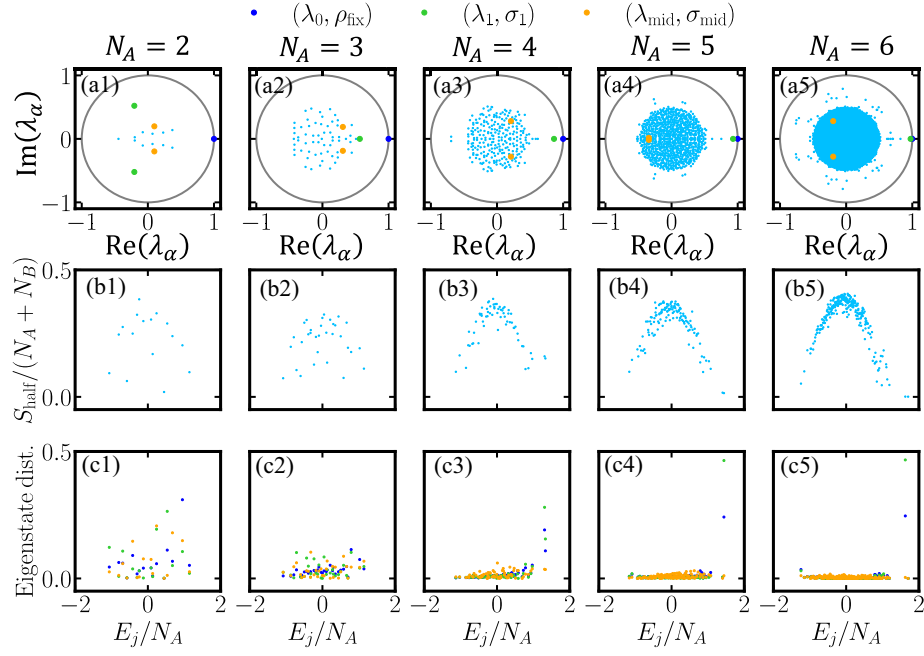


FIG. 28. (a1)–(a5) Eigenspectrum of $\mathcal{P}_0$. Blue, green, and orange dots indicate the eigenstates ρ_{fix} , σ_1 , and σ_{mid} , respectively, to be considered. (b1)–(b5) Normalized bipartite entanglement entropy of Hamiltonian H eigenstates versus normalized eigenenergies. (c1)–(c5) Normalized norm of overlap between eigenstates of $\mathcal{P}_0$ and eigenstates of Hamiltonian versus normalized eigenenergies. In all cases, we have bath $N_B = 2$.

APPENDIX J: DETAILS OF NOISY SIMULATION AND EXPERIMENT

In this section, we provide details of the noisy simulation and experiments of mutual information dynamics on the IBM Quantum devices.

For the experiment performed on IBM Quantum Torino presented in Fig. 14 in the main text, we implement the brickwork circuit to mimic the Haar-random unitaries considered in the theoretical studies. In a system of N qubits, the unitary circuit is defined as

$$U = \prod_{\ell=1}^L \left(\prod_{i=1}^{\lfloor (N-1)/2 \rfloor} \text{CZ}_{2i,2i+1} \right) \left(\prod_{i=1}^{\lfloor N/2 \rfloor} \text{CZ}_{2i-1,2i} \right) \times \left(\bigotimes_{i=1}^N e^{-i\phi_i^{(\ell)} Y_i} e^{-i\theta_i^{(\ell)} X_i} \right), \quad (\text{J1})$$

where $\phi_i^{(\ell)}$ and $\theta_i^{(\ell)}$ are random variables. X_i and Y_i are Pauli- X and $-Y$ operators, respectively, nontrivially supported on the i th qubit, and $\text{CZ}_{i,i+1} = |0\rangle\langle 0|_i \otimes \mathbf{I}_{i+1} + |1\rangle\langle 1|_i \otimes Z_{i+1}$ is the control- Z gate applied on nearest qubits $i, i+1$.

Additionally, we also consider the numerical simulation utilizing a variant of fast scrambling ansatz (FSA) [54] for IBM Quantum Sherbrooke. Compared to the noisy simulation of IBM Quantum Torino in the main text, the noisy simulation here does not consider the device shot noise, measurement error, and idling decoherence. The circuit ansatz is

$$U_{\text{simu}} = \prod_{\ell=1}^L \left(\prod_{i<j} \text{ECR}_{ij} \right) \left(\bigotimes_{i=1}^N e^{-i\phi_i^{(\ell)} Y_i} e^{-i\theta_i^{(\ell)} X_i} \right), \quad (\text{J2})$$

where $\text{ECR}_{ij} \equiv (1/\sqrt{2})(\mathbf{I}_i \otimes X_j - X_i \otimes Y_j)$ with X_i and Y_j denoting the Pauli- X and $-Y$ operator defined nontrivially on the i th and j th qubit. The echoed cross-resonance (ECR) gate is indeed equivalent to the CNOT gate up to some single-qubit rotations.

In Fig. 29, we show the decay of conditional state purity in our noisy simulation. Because of the increasing number

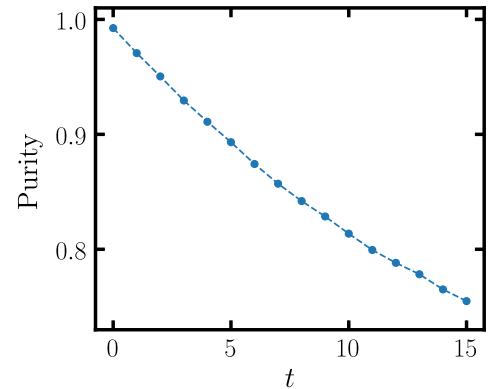


FIG. 29. Purity of postmeasurement conditional state at different time steps in noisy simulation. Here, we perform simulation on a system of $N_A = 3$, $N_B = 1$ qubits with FSA consisting of $L = 2$ layers and 1024 shots for midcircuit measurements on the bath system. Each dot is averaged over ten randomized circuits.

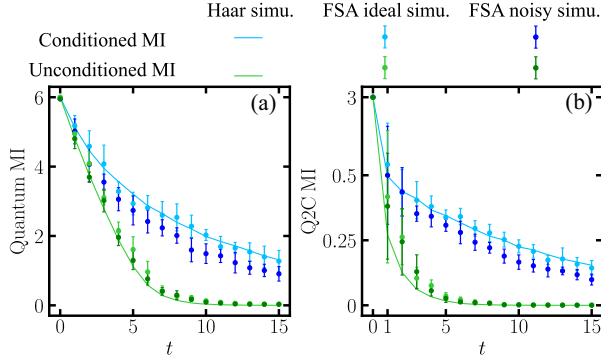


FIG. 30. Dynamics of measurement-conditioned and unconditioned (a) QMI and (b) Q2C MI. Solid lines represent Haar simulation results. Dark and light colored dots with error bars represent ideal and noisy simulation results separately with FSA for IBM Quantum Sherbrooke. Here, we perform simulation on a system of $N_A = 3$, $N_B = 1$ qubits with FSA consisting of $L = 2$ layers and 1024 shots for midcircuit measurements on the bath system. Each dot is averaged over ten randomized circuits.

of gates in the noisy circuit, the purity of the postmeasurement conditional state deviates from the ideal unity and undergoes an approximately linear decay. Within the 15 steps we simulated, the purity of conditional state remains $\gtrsim 0.75$, indicating the coherence of qubits remained in our circuit of simulation.

For IBM Quantum Sherbrooke we targeted at in Fig. 30, it has median T_1 as $275.73 \mu\text{s}$, median ECR gate error 7.726×10^{-3} , median SX error 2.300×10^{-4} , median ECR gate length $5.333 \times 10^{-1} \mu\text{s}$, and median readout (measurement) length $1.216 \mu\text{s}$; thus, our simulation of 15 steps is within the coherence time of qubits, which is also supported by the conditional state purity $\gtrsim 0.75$ (see Fig. 29). Compared to the noisy simulation results presented in the main text, here the shot noise and idling decoherence are not considered in the noisy simulation.

In Fig. 30(a), we evaluate the QMI dynamics implemented with FSA. For the ideal simulation, both measurement-conditioned (light blue dots) and unconditioned (light green dots) QMI align with the simulation results with Haar unitaries (solid lines), demonstrating its scrambling ability to mimic the dynamics with random unitaries. Next, we adopt the noisy circuit model (excluding measurement error) from Qiskit and simulate the QMI dynamics. In the presence of hardware noises, both conditioned (dark blue) and unconditioned (dark green) QMI show the same scaling of dynamics though with a small deviation from the ideal results. We, thus, expect that the exponential separation of QMI lifetime in conditional versus unconditional setup can be robust against circuit noise, suggesting a practical advantage on current-generation hardware.

In Fig. 30(b), we simulate the Q2C mutual information dynamics with FSA. Similar to the behavior of the QMI dynamics discussed above, the measurement-conditioned

Q2C mutual information (blue) also decays logarithmically with steps, while the unconditioned one (green) simply decays linearly, and both of them present a small deviation in the noisy simulation.

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