Covariant Path Integrals for Quantum Fields Backreacting on Classical Space-Time

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We introduce configuration space path integrals for quantum fields interacting with classical fields. We show that this can be done consistently by proving that the dynamics are completely positive directly, without resorting to master equation methods. These path integrals allow one to readily impose space-time symmetries, including Lorentz invariance or diffeomorphism invariance. They generalize and combine the Feynman-Vernon path integral of open quantum systems and the stochastic path integral of classical stochastic dynamics while respecting symmetry principles. We introduce a path integral formulation of general relativity where the space-time metric is treated classically. The theory is a candidate for a fundamental theory that reconciles general relativity with quantum mechanics. The theory is manifestly covariant, and may be inequivalent to the theory derived using master equation methods. We prove that entanglement cannot be created via the classical field, reinforcing proposals to test the quantum nature of gravity via entanglement generation.

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I. INTRODUCTION

Effective theories are ubiquitous in physics: From particle physics to classical statistical mechanics, we often make approximations to an underlying physical theory in order to simplify the dynamical description at hand. Broadly, there are two approaches to constructing effective field theories [1]. In the Wilsonian approach [1,2], one starts with a high-energy theory and asks how the effective low-energy description changes as high momentum modes are integrated out. The second approach involves modifying the theory by hand in order to isolate the desired degrees of freedom while still trying to keep phenomenological accuracy. This usually involves modifying the high-energy description so that the effective description is simpler and easier to use.

Often, we are interested in the effective description of a system where one part behaves classically and the other quantum mechanically, so the system should be described by an effective theory of combined classical-quantum (CQ) dynamics. We do this when we model a quantum measurement since we treat the measurement device as being effectively classical. Of particular interest is where the gravitational field is treated as classical while matter remains quantum since we currently have no quantum theory of gravity. Here, we are often interested in the regime where the quantum fields backreact onto the geometry, for example, when studying vacuum fluctuations in inflationary cosmology or black hole evaporation. Since gravity is a field theory, the correct description of quantum matter backreacting on classical space-time should be described by an effective field theory where the matter degrees of freedom are quantum mechanical, and the gravitational field is treated as being effectively classical.

Consistent classical-quantum master equations, such as the examples introduced in Refs. [3,4], have been used to study the interaction between classical and quantum systems from a master equation perspective. Consistent CQ master equations have been studied in a variety of different contexts [5–9], including gravity [8,10–14], and can be shown to be completely positive, trace preserving (CPTP), and preserve the split between classical and quantum degrees of freedom. CPTP dynamics is required in order to preserve the statistical properties of the density matrix since probabilities are positive and sum to one. The most general form of CPTP classical-quantum dynamics has also been derived [8,15], and takes an analogous form to that of the GSKL or Lindblad equation [16,17] in open quantum systems and the rate equation in stochastic classical dynamics [18].

Importantly, the resulting dynamics do not suffer from the same problems as the standard semiclassical equations [19,20]. There, the backreaction on the classical degrees of freedom is sourced by an expectation value of the quantum state and is known to be inconsistent, inducing a breakdown of either operational no-signaling, the Born rule, or composition of quantum systems under the tensor product [12,21–24]. Instead, the CQ dynamics we consider here is the most general form of dynamics consistent with the state space of quantum mechanics and is both quantitatively and qualitatively different from the standard semiclassical equations [25].

However, it is not known how to frame classical-quantum dynamics in a manifestly covariant framework, which places serious restrictions on any dynamics with an effectively classical field. The problem with the master equation picture is twofold. Firstly, from a practical point of view, field theories are generally better suited to path integral methods. Secondly, in a master equation picture, it is difficult to impose symmetries directly on the master equation. Indeed, writing down master equations for classical-quantum fields directly, without knowing whether or not they are covariant or uphold space-time symmetries, seems to go against much of the principles of modern physics, where one starts with actions based on symmetry principles. If one took the position that there is a fundamentally classical field, such as the gravitational field, it is also not obvious how one could couple it to the standard model while simultaneously ensuring symmetry principles and renormalizability are upheld.

In another paper [26], we give an in-depth analysis of classical-quantum master equations and their associated path integrals, the results of which motivated much of the present work. This work introduces a fully covariant path integral framework to study classical fields interacting with quantum ones. We prove the dynamics are completely positive directly from the path integral perspective, and we do not resort to master equation methods. This is especially important since, in general, it is only possible to go from a master equation picture to a path integral picture when the master equation is less than quadratic in classical or quantum momenta [26]. We find a family of CQ path integrals that are generated by an action, so it is easy to write down theories with space-time symmetries, including gauge symmetries. We do not study the renormalization properties of the dynamics explicitly; however, we have found that the pure gravity theory presented here is formally renormalizable [27].

The path integral we study is a generalization of the Feynman path integral for quantum systems and the stochastic path integral used to study classical stochastic processes [28,29] (see Table I in Appendix A for a comparison and simple examples). The path integrals used in the context of open quantum systems have a Schwinger-Keldish or Feynman-Vernon (FV) form [30], while the classical stochastic actions used to study stochastic processes are known as Onsagar-Machlup functions [28], and are also studied using the Martin-Siggia-Rose (MSR) formalism [31]. Our action combines both the classical and quantum forms and includes an interacting term between the classical and quantum fields. When there is no backreaction on the classical field, the path integral reduces to standard quantum theory with an action that depends on a classical variable. When there is backreaction on the classical field, the path integral includes a summation over all classical configurations and gives rise to a natural, and in fact, necessary [14] mechanism for decoherence. Though the quantum state decoheres, the path integrals preserve purity on the quantum system so that quantum states are mapped to quantum states. There is no loss of quantum information: This is a feature of CQ dynamics under certain natural conditions, which can lead to pure quantum state trajectories conditioned on the trajectory of the classical degree of freedom [9,25]. As discussed in Refs. [3,8,9,25] such dynamics do not require the Born rule to explain the state update rule and probabilistic outcomes of measurements if the classical field is taken as fundamental. Unlike spontaneous collapse models [32–38], the classical variable is itself taken to be dynamical.

Our results have consequences for any theory with a degree of freedom that behaves classically, whether effective or fundamental. With this in mind, we provide a possible template for studying CQ field theories, and we introduce a class of classical-quantum actions which can be used to construct theories with a sensible classical limit. The corresponding path integral can be understood in terms of summing over all classical and quantum paths, where the classical paths deviating too much from their semiclassical configuration are suppressed by a coupling D_0, which also governs the strength of the quantum decoherence. We give an explicit Lorentz invariant model of a classical field coupled to a quantum field. We show how to use perturbation theory to compute correlation functions in Appendix D. This can then be used to compute vacuum expectation values which place experimental constraints on the theory. We also include a discussion of normalization techniques in Appendix E. Since we do not have a full theory of quantum gravity, of particular importance is the construction of theory of quantum matter backreacting on classical space-time, and we discuss the application of our work to the gravitational setting, giving an example of a CQ theory of general relativity. We have since shown that this path integral gives the correct classical limiting behavior in that it suppresses fluctuations away from the Newtonian potential, its relativistic scalar generalization, and gravitational wave solutions of Einstein's equations. In particular, we have since shown in Ref. [27] that the two-point function of the scalar gravitational mode, which includes the Newtonian potential, is positive semidefinite. The two-point function of the tensor mode, corresponding to gravitational waves, has also been shown to be positive semidefinite [39]. The remaining vector sector is positive semidefinite on shell, though the off-shell tachyonic modes are not yet fully understood. We show that the classical gravitational field cannot generate entanglement, demonstrating that this provides a witness for a quantum theory of gravity [40,41]. This resolves a question which has seen considerable debate [42–49]. The theory can be considered as a fundamental theory which is an alternative to quantum gravity. There is also a regime where it may be an effective description of a fully quantum theory of gravity, after taking the classical-quantum limit as outlined in Ref. [50].

II. CLASSICAL-QUANTUM DYNAMICS

We first introduce the general formalism used to describe a classical degree of freedom coupled to a quantum one, and we denote a generic classical degree of freedom by z. For example, it could be a classically treated position variable $z = q$ or a point in phase space $z = (q, p)$. When one considers a hybrid system, the natural set of states to consider are hybrid classical-quantum states. Formally, a classical-quantum state associates to each classical variable an unnormalized density matrix $\varrho(z,t) = p(z,t)\sigma(z,t)$ such that $\mathrm{Tr}_{\mathcal{H}}\varrho(z) = p(z,t) \geq 0$ is a normalized probability distribution over the classical degrees of freedom and $\int dz\varrho(z,t)$ is a normalized density operator on a Hilbert space $\mathcal{H}$. Intuitively, $p(z,t)$ can be understood as the probability density of being in the phase space point z and $\sigma(z,t)$ as the *normalized* quantum state one would have given the classical state z occurs.

Classical-quantum dynamics can then be understood as the set of linear dynamics which maps CQ states to CQ states. Linearity is required in order to maintain a probabilistic interpretation of the density matrix. The dynamics must be completely positive (CP) since we require that states be mapped to other states even when the dynamics act on half an entangled quantum state. In analogy with Kraus theorem for quantum operations, the most general form of *CP* dynamics mapping CQ states onto themselves is described by [8,15]

$$\varrho(z, t+\delta t) = \int dz' \Lambda^{\mu\nu}(z|z', \delta t) L_\mu \varrho(z', t) L_\nu^\dagger, \quad (1)$$

where the $\Lambda^{\mu\nu}(z|z', \delta t)$ defines a positive matrix measure in z, z'. In Eq. (1), the operators L_μ are an arbitrary set of operators on the Hilbert space, and normalization of probabilities requires

$$\int dz \Lambda^{\mu\nu}(z|z', \delta t) L_\nu^\dagger L_\mu = \mathbb{I}. \quad (2)$$

When the dynamics are Markovian, completely positive CQ master equations can be derived from Eq. (1) and have been studied in Refs. [5,6,8,8–10,13,14]. However, it is well known that it is only possible to go from a master equation approach to a position space path integral approach when the master equation is at most quadratic in momenta, or else one cannot perform the Gaussian path integral exactly. Therefore any method of proving consistent dynamics without master equation methods is useful. We shall here work *entirely* within a path integral framework without resorting to master equation methods, and we shall prove complete positivity of the dynamics directly from the path integral approach.

The path integral tells us how the components of the density matrix evolve. Including a classical variable q, the path integral should tell us how to evolve the components of a classical-quantum state,

$$\varrho(q,t) = \int d\phi^+ d\phi^- \varrho(q, t, \phi^+, \phi^-) |\phi^+\rangle\langle\phi^-|, \quad (3)$$

where ϕ represents a continuous quantum degree of freedom and $\varrho(q, t, \phi^+, \phi^-) = \langle\phi^+|\rho(q,t)|\phi^-\rangle$ are the components of the CQ state. Writing Eq. (3) out explicitly, generically, a path integral will take the form

$$\begin{aligned}&\rho(q, \phi^+, \phi^- t_f) \\ &= \int \mathcal{D}q\mathcal{D}\phi^+\mathcal{D}\phi^- \mathcal{N} e^{\mathcal{I}[q,\phi^+,\phi^-,t_i,t_f]} \rho(q_i, \phi_i^+, \phi_i^-, t_i). \end{aligned} \quad (4)$$

In Eq. (4) it is implicitly understood that boundary conditions are to be imposed at t_f, and we have included a normalization factor $\mathcal{N}$ which can depend on the point in configuration space at each instance [51]. In the purely quantum case, one has $\mathcal{I}[\phi^+, \phi^-, t_i, t_f] = iS[\phi^+, t_i, t_f] - iS[\phi^-, t_i, t_f]$ and the path integral is doubled since we are considering density matrices so we must sum over all bra and ket paths. We here consider a general classical-quantum path integral, but the reader may benefit from the simpler examples and comparisons with other path integrals found in Table I.

When the action contains higher derivatives, we can also include additional initial conditions on the time derivatives of the fields in Eq. (4) [52].

III. MAIN RESULT: A COMPLETELY POSITIVE NORM PRESERVING PATH INTEGRAL

Having introduced the classical-quantum formalism, let us now state and prove our main result: Any time-local classical-quantum path integral with action of the form

$$\begin{aligned}\mathcal{I}[\phi^-, \phi^+, q, t_i, t_f] = &\mathcal{I}_{\mathrm{CQ}}[q, \phi^+, t_i, t_f] \\ &+ \mathcal{I}^*_{\mathrm{CQ}}[q, \phi^-, t_i, t_f] - \mathcal{I}_C[q, t_i, t_f] \\ &+ \int_{t_i}^{t_f} dt \sum_\gamma c^\gamma(q,t)(L_\gamma[\phi^+]L^*_\gamma[\phi^-])\end{aligned} \quad (5)$$

defines completely positive CQ dynamics when the terms in Eq. (5) have the following properties: $L_\gamma[\phi^\pm]$ can be any functional of the bra and ket variables, $c^\gamma \geq 0$, $\mathcal{I}_C \geq 0$ for all field configurations, and the real part of $\mathcal{I}_{\rm CQ}$ is nonpositive. We implicitly assume that c^γ is chosen so that the path integral converges.

In the field-theoretic case, the final term of Eq. (5) is replaced by

$$\int_{t_i}^{t_f} dx \sum_\gamma c^\gamma(q,x)\{L_\gamma[\phi^+](x) L_\gamma^*[\phi^-](x)\}, \tag{6}$$

and the resulting path integral in Eq. (5) will be *CP* so long as $c^\gamma(q,x)$ is positive.

In Eq. (5), $\mathcal{I}_{\rm CQ}$ determines the CQ interaction on each of the ket and bra paths and $\mathcal{I}_C[q,t_i,t_f]$ is a purely classical action which takes real values. Since the path integral is written in the field basis, $\mathcal{I}_{\rm CQ}$, $\mathcal{I}_C$, and L_γ are all number-valued functionals of the field configurations, not operators. The above requirements on positive definiteness have been imposed in order for the path integral to be convergent. This condition also arises when studying path integrals associated to CQ master equations [26]. For example, one can take the classical action $\mathcal{I}_C$ to be the action associated to the path integral of the Fokker-Planck equation (14) [28,53,54] which must be positive (semi) definite in order for the path integral to converge. The final term of Eq. (5) contains cross terms between the bra and ket branches ϕ^+, ϕ^- which sends pure states to mixed states and corresponds to including additional noise in the dynamics. It takes the form of a Kraus map acting on the CQ state, which is what ensures complete positivity, and allows one to include classical-quantum Feynman-Vernon [55,56] terms into the action.

If all the $c^\gamma = 0$, the ϕ^+ and ϕ^- integrals factorize in Eq. (5), the path integral preserves the purity of the quantum state conditioned on the classical trajectory. This can be seen from the fact that the absence of $\phi^+\phi^-$ couplings means that the bra field evolves independently of the ket field. If the system starts at q in the pure state $|\phi\rangle\langle\phi|$, then the first term in Eq. (5) evolves $|\phi\rangle$ into $|\phi(t)\rangle$, while the second term evolves $\langle\phi|$ into $\langle\phi\left(t\right)|$, so the final density matrix remains pure: $|\phi(t)\rangle\langle\phi(t)|$. In this case, the absence of cross terms in the action, despite the requirement of Lindblad terms in the hybrid master equation [14], is a remarkable consequence of saturating the decoherence vs diffusion trade-off [26].

To see this more clearly, it is useful to split $\mathcal{I}_{\rm CQ}$ into its real and imaginary components $\mathcal{I}_{\rm CQ} = \mathcal{R}_{\rm CQ} + i\mathcal{S}_Q$. Then Eq. (5) (with $c_n = 0$) reads

$$\mathcal{I}^\pm = \mathcal{R}^+_{\rm CQ} + \mathcal{R}^-_{\rm CQ} + i(\mathcal{S}^+_{\rm CQ} - \mathcal{S}^-_{\rm CQ}) - \mathcal{I}_C, \tag{7}$$

and we are able to get some intuition for each term. Heuristically expanding the actions, or more properly their Lagrangians, in terms of their field dependence $\mathcal{S}_{\rm CQ} \sim \sum_m a_m(q)s_m(\phi)$ and $\mathcal{R}_{\rm CQ} \sim \sum_m b_m(q) r_m(\phi)$, we see that

$$\begin{aligned} \mathcal{S}^+_{\rm CQ} - \mathcal{S}^-_{\rm CQ} &\sim \sum_m a_m(q)[s_m(\phi^+) - s_m(\phi^-)], \\ \mathcal{R}^+_{\rm CQ} + \mathcal{R}^-_{\rm CQ} &\sim \sum_m b_m(q)[r_m(\phi^+) + r_m(\phi^-)]. \end{aligned} \tag{8}$$

Hence, the imaginary part of the integral is associated with things like coherence, which depend on the difference between the ket and bra components of the density matrix, while the real part of the action depends on the sum of the left and right components on the density matrix, which are things like its expectation value. Moreover, conditioned on a classical trajectory $\bar{q}(t)$—which can be represented by inserting a delta function $\delta(q(t) - \bar{q}(t))$ into the classical part of the path integral—we see that the evolution of the quantum state factorizes between the $\phi^\pm$ integrals and hence keeps pure quantum states pure. We shall largely focus on this case; it can be shown that any CQ dynamics which does not preserve the purity of the quantum state conditioned on the classical degree of freedom can be embedded into a larger classical space where the quantum state remains pure, in a CQ version of purification [25].

The backreaction of the quantum system on the classical system is contained in the real components of the CQ action $\mathcal{R}^\pm_{\rm CQ}$. Indeed, when $\mathcal{R}^\pm_{\rm CQ} = 0$, the path integral in Eq. (7) reduces to the standard quantum path integral for the density matrix but also includes a classical variable which can undergo its own autonomous dynamics due to the inclusion of the classical action $\mathcal{I}_C$. However, whenever there is backreaction, Eq. (5) *necessarily* describes nondeterministic evolution: This was proved generally in Ref. [14] using master equation methods.

To prove that the dynamics described by Eq. (5) gives rise to consistent CQ dynamics, we must first show that it leads to completely positive dynamics preserving the positivity of the CQ state. Recall that positivity of the CQ state means that for any Hilbert space vector $|v(q)\rangle$ we have $\mathrm{Tr}[|v(q)\rangle\langle v(q)|\varrho(q)] \geq 0$. In components, positivity of the CQ state requires that for any vector $|v(q)\rangle$ with components $v(\phi,q) = \langle\phi|v(q)\rangle$ we have

$$\int d\phi^+ d\phi^- v(\phi^+,q)^* \varrho(\phi^+,\phi^-,q) v(\phi^-,q) \geq 0. \tag{9}$$

A CQ dynamics Λ is said to be positive if it preserves the positivity of CQ states and completely positive if $\mathbb{I}\otimes\Lambda$ is positive when we act with the identity on any larger system.

Since we assume the dynamics are time local, we can perform a short-time expansion of the path integral. For the action in Eq. (5), in Appendix B we show that the path integral integrand always factorizes into the form

$$\begin{aligned} &[e^{\mathcal{I}_{\rm CQ}[\phi^+,q]} e^{\mathcal{I}_{\rm CQ}[\phi^-,q]^*} + \delta t \sum_\gamma c^\gamma (L^+_\gamma e^{\mathcal{I}_{\rm CQ}[\phi^+,q]}) \\ &\quad \times (L^-_\gamma e^{\mathcal{I}_{\rm CQ}[\phi^-,q]})^*] \mathrm{e}^{-\mathcal{I}_C[q]} + \cdots. \end{aligned} \tag{10}$$

Because Eq. (10) factorizes between $\pm$ branches, it takes the form of a Kraus map and is therefore manifestly completely positive. It is important to note that because of the exponentials, Eq. (10) is always strictly positive, meaning that we do not encounter zero norm states. Instead, the problem of negative norm states and ghosts is mapped to the problem of convergence of the path integral [52].

The path integral defined by Eq. (5) is completely positive, and the other requirement is that it be norm preserving. For time-local dynamics, it is always possible to normalize a CP map in a linear manner to arrive at a CP norm preserving dynamics. Specifically, any time-local CP CQ map of the form

$$\frac{\partial \varrho(z)}{\partial t} = \int dz' W^{\mu\nu}(z|z') L_\mu \varrho(z') L_\nu^\dagger \tag{11}$$

can be normalized by subtracting $\frac{1}{2}\int dz' W^{\mu\nu}(z'|z) \times \{L_\nu^\dagger L_\mu, \varrho(z)\}_+$ to yield CP norm preserving dynamics [8].

With this in mind, for time-local dynamics, Eq. (5) can always be normalized, and taking this into account we can include a normalization factor in the CQ path integral. In practice, and for the path integrals we introduce in this work, normalization of the path integral is accounted for through the classical and quantum kinetic terms in the action [57] (see Appendix E).

IV. COMPARISON TO CLASSICAL PATH INTEGRALS

The path integral action we introduce in Eq. (5) is general. Therefore, it is useful to find CQ actions that give rise to dynamics that have a sensible physical interpretation. To gain some intuition for the classical part of path integral, we can consider the Fokker-Plank equation for a classical probability density $p(z)$,

$$\frac{\partial p(z)}{\partial t} = -\frac{\partial}{\partial z_i}[D_{1,i}(z)p(z)] + \frac{1}{2}\frac{\partial^2}{\partial z_i \partial z_j}[D_{2,ij}(z)p(z)], \tag{12}$$

where $z = (q_1, p_1, \ldots, q_n, p_n)$ for an n dimensional system [58].

In Eq. (12) the coefficient $D_{1,i}$ characterizes the amount of *drift* in the system, and is equal to the evolution of the expectation value of z, $\partial_t \langle z_i \rangle$. If D_{1,q_i} also depends on p_i, then it contributes a friction term. The matrix $D_{2,ij}$ characterizes the amount of *diffusion* in the system and is equal to $\partial_t \langle z_i z_j \rangle$. The corresponding path integral is given by [28,29,53,54]

$$p(z, t_f) = \int \mathcal{D}z \mathcal{N} e^{-\mathcal{I}_C(z,t_i,t_f)} p(z_i, t_i), \tag{13}$$

where

$$\mathcal{I}_C(z, t_i, t_f) = \frac{1}{2}\int_{t_i}^{t_f} dt \left[\frac{dz_i}{dt} - D_{1,i}(z)\right] D_{2,ij}^{-1} \left[\frac{dz_j}{dt} - D_{1,j}(z)\right]. \tag{14}$$

When the matrix D_2 does not depend on z, the normalization is easily computed since the integral is a Gaussian. In this case it does not depend on configuration space and can be taken outside the path integral. The path integral has a natural interpretation in terms of suppressing classical paths which deviate from their expected drift D_1 by an amount that depends on the inverse of the diffusion coefficient D_2^{-1}. If D_2 is z dependent, Eq. (14) can also contain an anomalous contribution [26], but we shall not include it here since Eq. (14) still defines positive classical dynamics.

The simplest nontrivial case is where one diffuses only in momenta. In this case, $\dot{q}_i = (p_i/m_i)$ and the momentum integral acts to enforce a delta function over $\delta(p_i - m_i \dot{q}_i)$. Integrating out the momentum variables, the result is a path integral over only the configuration space variables q_i with action

$$\mathcal{I}_C(q, t_i, t_f) = \frac{1}{2}\int_{t_i}^{t_f} dt \left[m_i \frac{d^2 q_i}{dt^2} - D_{1,i}(q)\right] D_{2,ij}^{-1} \times \left[m_j \frac{d^2 q_j}{dt^2} - D_{1,j}(q)\right], \tag{15}$$

from which we see that the path integral acts to suppress paths away from their expected equations of motion with the amount depending on D_2.

Taking the expected classical equation of motion to itself be generated by an action S_C, the action in Eq. (15) can be rewritten as

$$\mathcal{I}(q, t_i, t_f) = \frac{1}{2}\int_{t_i}^{t_f} dt \frac{\delta S_C}{\delta q_i} D_{2,ij}^{-1} \frac{\delta S_C}{\delta q_j}. \tag{16}$$

Since S_C itself appears in the path integral action $\mathcal{I}(q, t_i, t_f)$, we shall henceforth refer to S_C as the classical *protoaction*. It is important to note that, in general, one can, and generally should, include non-Lagrangian friction terms in the path integral, represented by a more general drift coefficient, as in Eq. (15). We use the term protoaction because one can think of it as an action which generates the equation of motions, which then appear in the CQ action itself as "*equations of motions squared*." This acts to suppress paths which deviate too far from the equations of motion. One typically takes the protoaction to be the action coming from the quantum field theory coupled to the classical degrees of freedom. While this is a useful method for constructing theories which have the desired fully classical limit, we expect that a fundamental theory should be derivable by considering all possible terms which satisfy the requirements of symmetry and complete positivity.

V. NATURAL CLASS OF CQ DYNAMICS

The purely classical action in Eq. (16) generalizes to the combined classical-quantum case. A natural class of theories we find are those derivable from a classical-quantum protoaction $W_{\mathrm{CQ}}[q,\phi]$:

$$\mathcal{I}[\phi^-,\phi^+,q,t_i,t_f] = \int_{t_i}^{t_f} dx\left[i\mathcal{L}_Q^+(q,x) - i\mathcal{L}_Q^-(q,x) - \frac{1}{2}\frac{\delta\Delta W_{\mathrm{CQ}}}{\delta q_i(x)}D_{0,ij}(q,x)\frac{\delta\Delta W_{\mathrm{CQ}}}{\delta q_j(x)} - \frac{1}{2}\frac{\delta\bar{W}_{\mathrm{CQ}}}{\delta q_i(x)}D^{-1}_{2,ij}(q,x)\frac{\delta\bar{W}_{\mathrm{CQ}}}{\delta q_j(x)}\right], \tag{17}$$

where we take $D_0(q,x), D_2(q,x)$ to be symmetric, positive semidefinite real matricies [14] and we impose the matrix restriction $4D_0 \succeq D_2^{-1}$ to ensure the action takes the form of Eq. (5), and hence is completely positive. We show this explicitly in Appendix C. One can further show that Eq. (5) is normalized so long as the CQ protoaction contains classical kinetic terms and $\mathcal{L}_Q$ contains quantum kinetic terms—precise normalization conditions are summarized through Eqs. (E10) and (E11) in Appendix E. In Eq. (17) $W_{\mathrm{CQ}}[q,\phi]$ is a real classical-quantum protoaction which generates the dynamics, and we have introduced the notation $\bar{W}_{\mathrm{CQ}} = \frac{1}{2}(W_{\mathrm{CQ}}[q,\phi^+] + W_{\mathrm{CQ}}[q,\phi^-])$ for the $\pm$ averaged protoaction and $\Delta W_{\mathrm{CQ}} = W_{\mathrm{CQ}}[q,\phi^-] - W_{\mathrm{CQ}}[q,\phi^+]$ for the difference in the protoaction along the $\pm$ branches. The term containing $\bar{W}_{\mathrm{CQ}}$ acts to suppress the classical paths which deviate too far from the classical equations of motion, but sourced by the average of the bra and ket quantum fields, while the ΔW_{CQ} terms decohere the quantum system in a way which preserves complete positivity.

$\mathcal{L}_Q^{\pm}(q,x)$ denotes the purely quantum evolution, which can be any quantum field theory Lagrangian density, but should include the coupling terms from W_{CQ} that depend on both q and $\phi^{\pm}$. In this way, $\mathcal{L}_Q^{\pm}(q,x)$ generates the evolution of the quantum fields including the action of the classical fields on the quantum fields, while W_{CQ} generates the backreaction of the quantum fields on the classical fields. We will typically take $W_{\mathrm{CQ}}[q,\phi^{\pm}] = \mathcal{L}_Q^{\pm}(q,x)$ so that the backreaction of quantum fields on q has the same generator as the action of q on the quantum fields, as is the case with two quantum or two classical systems. Just as in the classical case, one can also add friction terms to Eq. (17) though we shall not do this in the present work.

For simplicity, we here deal with theories with "ultra-local" correlation kernels, meaning the noise kernel is proportional to $\delta^{(3)}(x-y)$, but we also expect our results to extend to the case where D_0, D_2 are positive semidefinite matrix kernels $D_0(x,y), D_2(x,y)$ which have some range [14].

When $4D_0 = D_2^{-1}$, the path integral preserves purity on the quantum system, as shown in Ref. [25] using master equation methods. This form of action is motivated by the study of path integrals [26] for CQ master equations whose backreaction is generated by a Hamiltonian [8,10,25], as well as the purely classical path integral in Eq. (16). Written in the form of Eq. (17), we see that the action of D_2 is to suppress paths that deviate from the $\pm$ averaged Euler-Lagrange equations, which themselves follow from varying the bra-ket averaged protoaction $\bar{W}_{\mathrm{CQ}}$, while the effect of the D_0 term is to decohere the quantum system. The decoherence diffusion trade-off $4D_0 \succeq D_2^{-1}$ [14], required for the dynamics to be CP, means that if coherence is maintained for a long time, then there is necessarily lots of diffusion in the classical system away from its most likely path, with the amount depending on both D_0 and the strength of the coupling which enters in W_{CQ}.

VI. LORENTZ INVARIANT CLASSICAL-QUANTUM DYNAMICS

Lorentz invariant or covariant pure Linbladians have been studied in Refs. [6,56,59]. As a simple example of a classical-quantum Lorentz invariant model, we can consider a classical field $q(x)$ coupled to a quantum field $\phi(x)$ with a manifestly Lorentz invariant protoaction:

$$W_{\mathrm{CQ}} = \int d^4x\left[-\frac{1}{2}\partial_\mu q\partial^\mu q - \frac{1}{2}m_q^2 q^2 - \frac{\lambda}{2}q^2\phi^2\right]. \tag{18}$$

In this case, assuming $4D_0 = D_2^{-1}$, we find the expressions for the CQ coupling terms:

$$\frac{\delta\Delta W_{\mathrm{CQ}}}{\delta q}D_0\frac{\delta\Delta W_{\mathrm{CQ}}}{\delta q} = \lambda^2 D_0 q^2[(\phi^+)^2 - (\phi^-)^2]^2, \tag{19}$$

$$\frac{\delta\bar{W}_{\mathrm{CQ}}}{\delta q}D_2^{-1}\frac{\delta\bar{W}_{\mathrm{CQ}}}{\delta q} = 4D_0\left(\partial^\mu\partial_\mu q - m_q^2 q - \lambda\frac{1}{2}q[(\phi^+)^2 + (\phi^-)^2\right)]^2. \tag{20}$$

We see that Eq. (19) acts to decohere the quantum system into the $|\phi\rangle$ basis by suppressing configurations away from $\phi^+ = \phi^-$ by an amount proportional to $D_0\lambda^2$, where λ characterizes the backreaction on the quantum system. On the other hand, Eq. (20) acts to suppress configurations away from their semiclassical equations of motion—found from varying $\delta\bar{W}_{\mathrm{CQ}}/\delta q$—by an amount also proportional to D_0. Note that this does not depend on the coupling strength so that in the regime where the backreaction is small, one can maintain coherence without deviating too much from the expected classical equations of motion.

This can be used to evaluate CQ path integrals by working perturbatively in the backreaction coupling. We show how this can be done in Appendix D.

Here, when the decoherence vs diffusion trade-off is saturated, the $\phi^+\phi^-$ coupling terms cancel. This is the term which corresponds to the *jump-jump* term in Lindbladian evolution. This makes the total action relatively simple, namely,

$$\begin{aligned}\mathcal{I}[\phi^-,\phi^+,q,t_i,t_f] &= \int_{t_i}^{t_f} dx\left[i\mathcal{L}_Q^+(q,x) - i\mathcal{L}_Q^-(q,x) - \frac{1}{2}\frac{\delta\Delta W_{\rm CQ}}{\delta q(x)}D_0\frac{\delta\Delta W_{\rm CQ}}{\delta q(x)} - \frac{1}{2}\frac{\delta\bar{W}_{\rm CQ}}{\delta q}D_2^{-1}(q,x)\frac{\delta\bar{W}_{\rm CQ}}{\delta q(x)}\right]\\ &= \int_{t_i}^{t_f} dx\left[i\mathcal{L}_Q^+(q,x) - i\mathcal{L}_Q^-(q,x) - \frac{1}{4D_2}(\Box q - q\lambda(\phi^+)^2)^2 - \frac{1}{4D_2}(\Box q - q\lambda(\phi^-)^2)^2\right]\end{aligned} \tag{21}$$

in the $m=0$ case. Note that $i\mathcal{L}_Q^\pm(q,x)$ includes the coupling $\pm i(\lambda/2)q^2\phi^\pm$. Here we see that if D_2 is large, then paths away from the deterministic solutions $\Box q - q\lambda(\phi^\pm)^2 = 0$ will no longer be suppressed, while if D_2 is small (i.e., D_0 is large), the decoherence term will be enhanced. This is the essence of the decoherence vs diffusion trade-off.

VII. DIFFEOMORPHISM INVARIANT CQ GRAVITY

Let us now comment on some of the consequences of classical-quantum theories of gravity. The goal is to try to construct a theory of covariant classical-quantum dynamics that approximates Einstein's equations. We will find that we can construct a theory which is manifestly diffeomorphism invariant. We believe it is likely to be different from the theory presented in Ref. [8] using master equation methods. In particular, we do not yet know if Ref. [8] is diffeomorphism invariant owing to the constraint algebra being unsolved [13]. Nor do we know whether the two theories are equivalent, since one can generally only derive the path integral from the master equation, when the dynamics is at most quadratic in the momenta, which is not the case here.

Since in Eq. (17) the paths away from $[\delta/(\delta q_i)] \times (\bar{W}_{\rm CQ}[q,\phi^\pm])$ are exponentially suppressed by an amount depending on D_2^{-1}, the most likely path will be those for which

$$\frac{\delta}{\delta q_i}(\bar{W}_{\rm CQ}[q,\phi^\pm]) \approx 0. \tag{22}$$

To get a theory that agrees with Einstein's gravity on average, we could therefore try to take $W_{\rm CQ}[g,\phi]$ to be the sum of the Einstein Hilbert action $S_{\rm EH}[g] = [1/(16\pi G_N)] \times \int\sqrt{g}R$, and a matter action $S_m[g,\phi]$ including a cosmological constant. The path integral is over physically distinct geometries, which we could enforce via the Fadeev-Popov procedure. In the case where $W_{\rm CQ} = S_{\rm EH} + S_m$, we have

$$\frac{\delta}{\delta g_{\mu\nu}}(W_{\rm CQ}[g,\phi]) = -\frac{\sqrt{-g}}{16\pi G_N}(G^{\mu\nu} + \Lambda g^{\mu\nu} - 8\pi G_N T^{\mu\nu}). \tag{23}$$

Thus, paths would be exponentially suppressed away from (a $\pm$ branch average of) Einstein's equations. Explicitly, taking the classical degree of freedom to be $g_{\mu\nu}$, the decoherence part of the CQ interaction in Eq. (17) is given by

$$\frac{\delta\Delta W_{\rm CQ}}{\delta g_{\mu\nu}}D_{0,\mu\nu\rho\sigma}\frac{\delta\Delta W_{\rm CQ}}{\delta g_{\rho\sigma}} = \det(-g)\frac{1}{4}(T^{\mu\nu+} - T^{\mu\nu-})D_{0,\mu\nu\rho\sigma}(T^{\rho\sigma+} - T^{\rho\sigma-}), \tag{24}$$

while assuming $4D_0 = D_2^{-1}$ the diffusion part takes the form

$$\frac{\delta\bar{W}_{\rm CQ}}{\delta g_{\mu\nu}}D^{-1}_{2,\mu\nu\rho\sigma}\frac{\delta\bar{W}_{\rm CQ}}{\delta g_{\rho\sigma}} = \frac{1}{64\pi^2G_N^2}\det(-g)\times(G^{\mu\nu} + \bar{\Lambda}g^{\mu\nu} - 8\pi G_N\bar{T}^{\mu\nu})D_{0,\mu\nu\rho\sigma}(G^{\rho\sigma} + \bar{\Lambda}g^{\rho\sigma} - 8\pi G_N\bar{T}^{\rho\sigma}). \tag{25}$$

The action can be put into MSR [31] form by introducing auxiliary fields $X^{\mu\nu}$ via a Hubbard-Stratonovich transformation, such that integrating them out recovers the quadratic diffusion term.

Because the decoherence-diffusion trade-off is saturated, the dynamics take the form of Eq. (5); thus, the dynamics are completely positive, and the quantum state of the fields remains pure conditioned on the metric. The interaction is fully characterized by the tensor density $D_{0,\mu\nu\rho\sigma}$. There are two possible demands one could make on this tensor. The first would be to require that it be a positive semidefinite matrix in the sense that $v^{\mu\nu}D_{0,\mu\nu\rho\sigma}v^{\rho\sigma} \geq 0$ for any matrix $v^{\rho\sigma}$. This would ensure that the dynamics are completely positive and normalizable on any initial state, and classical paths which are close to Einstein's equations are more probable. Constructing diffeomorphism invariant classical-quantum theories of gravity then amounts to trying to find a tensor $D_0^{\mu\nu\rho\sigma}$ which gives rise to a path integral which defines completely positive dynamics.

To meet this demand, the simplest thing one can try is to take $D_{0,\mu\nu\rho\sigma} = D_0 g^{-1/2} g_{\mu\nu} g_{\rho\sigma}$, in which case one finds a diffeomorphism invariant CQ theory of gravity in which paths deviating from the *trace* of Einstein's equations are suppressed. Moreover, according to Eq. (24) the quantum state decoheres into eigenstates of the trace of the stress-energy tensor. In the Newtonian limit, where the trace of the stress-energy tensor is dominated by its mass term, this acts to decohere the quantum state into mass eigenstates. This is related to the amplification mechanism used in spontaneous collapse models [32–38], but here the decoherence mechanism is non-Markovian and arises as a consequence of treating the gravitational field classically and imposing diffeomorphism invariance on the CQ action. Furthermore, although the quantum state decoheres, it remains pure if we condition on the classical trajectory.

This is sufficient to demonstrate that a diffeomorphism invariant CQ theory of gravity is possible. The challenge in constructing a complete theory is to obtain the transverse parts of the Einstein equation, which are the constraints, while still ensuring the path integral over classical geometries remains negative definite so that the path integral converges.

A general form for the diffusion matrix is to take it to be proportional to the generalized Wheeler-deWitt metric in $3+1$ dimensions,

$$D_{0,\mu\nu\rho\sigma} = \frac{1}{8D_2}(-g)^{-1/2}(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2\beta g_{\mu\nu}g_{\rho\sigma}), \tag{26}$$

where D_2 is here the dimensionless coupling constant of the theory. It is related to the diffusion coefficient of Ref. [14], which was given units of $\mathrm{kg}^2\,\mathrm{s}\,\mathrm{m}^{-3}$, by absorbing a factor of G_N^2/c^3. Since we saturate the decoherence-diffusion trade-off $4D_0 = D_2^{-1}$, the decoherence coupling has dimensions $D_0 \sim G_N^2/c^3$, and the theory is characterized by the single dimensionless coupling D_2. While $D_{0,\mu\nu\rho\sigma}$ is not positive semidefinite, we only require it to be positive semidefinite on physical, local degrees of freedom. We have since shown this for the scalar mode [27] and tensor mode [39], whose two-point functions are positive semidefinite kernels. The vector sector is positive semidefinite on shell, though the off-shell tachyonic modes are not yet fully understood [27,51]. Furthermore, normalization of the dynamics is tenable in part because the negative eigenvalues correspond to non-dynamical components of the path integral.

One could conceive of other geometric terms; for example, one could consider various powers of the determinant of $G_{\mu\nu}$ and $T_{\mu\nu}$. On the other hand, if one were to choose a $D_{0,\mu\nu\rho\sigma}$, which was not purely geometric, it either introduces a preferred background or must be made dynamical. The former is more suggestive of an effective theory in which one obtains a classical metric by adding in decoherence or tracing out degrees of freedom in some reference frame. In the latter case, one should add terms proportional to $D_{0,\mu\nu\rho\sigma}g^{\mu\nu}g^{\rho\sigma}$ and $D_{0,\mu\nu\rho\sigma}g^{\mu\sigma}g^{\nu\rho}$ into the classical part of the action. One then needs to ensure that such terms are not in conflict with experiment.

It is also possible to consider a $D_{0,\mu\nu\rho\sigma}(x, x')$ which is a positive-definite kernel in space-time coordinates x, x' in which case one has stochastic processes which are correlated in space-time, and the CQ interaction terms take the form

$$-\frac{1}{2}\int d^4x d^4x' \frac{\delta \Delta W_{\mathrm{CQ}}}{\delta g_{\mu\nu}(x)} D_{0,\mu\nu\rho\sigma}(x, x') \frac{\delta \Delta W_{\mathrm{CQ}}}{\delta g_{\mu\nu}(x')}$$
$$-\frac{1}{2}\int dx dx' \frac{\delta \bar{W}_{\mathrm{CQ}}}{\delta g_{\mu\nu}(x)} D^{-1}_{2,\mu\nu\rho\sigma}(x, x') \frac{\delta \bar{W}_{\mathrm{CQ}}}{\delta g_{\mu\nu}(x')}. \tag{27}$$

Since these noise kernels have spatial correlations, Lorentz invariance implies that they must also have temporal correlations and so they represent non-Markovian dynamics. This is suggestive of an effective CQ theory rather than fundamental ones since non-Markovianity implies the existence of a hidden memory. However, such a theory may have an advantage in terms of suppressing heating and diffusion.

VIII. NO MEDIATION OF ENTANGLEMENT BY CLASSICAL FIELDS

The proposed experiments of Refs. [40,41] are based on the fact that two systems interacting via the gravitational field will not become entangled if the gravitational field is classical. The argument was that local operations and classical communication cannot generate entanglement; a classical interaction can be thought of as a form of classical communication. This provides an experimental basis to test the quantum nature of space-time [40]. However, in the nonrelativistic limit, the interaction between two small masses is dominated by the Newtonian interaction which directly couples the position of the masses and is nondynamical (being a constraint of general relativity). Thus there has been considerable debate on whether a classical Newtonian interaction can generate entanglement [42–44,47–49].

We will now see that any CQ theory which is local in the quantum fields, and action of the form of Eq. (5), cannot generate entanglement via the classical field. It is noteworthy that we do not need to assume that the classical interaction is local. For ease of presentation, we will first take all the $c^\gamma = 0$ (it is straightforward to show that if they are nonzero, they just produce additional local decoherence which cannot generate entanglement). The first key step is one we have already discussed. If $c^\gamma = 0$, then an action of the form of Eq. (5) preserves the purity of the quantum state conditioned on the trajectory Φ_t of the classical field Φ. In particular, the action of the CQ dynamics on the quantum state for a given trajectory Φ_t is proportional to a single operator $L(\Phi_t)$. For any initial pure quantum state $|\Psi_i\rangle$ of the matter distribution and initial configuration Φ_i of the Newtonian potential, we write the initial CQ density matrix entries as $\rho(\Phi_i, m_i^+, m_i^-, t_i) = \langle m^+|\Psi_i\rangle\langle\Psi_i|m^-\rangle\delta(\Phi - \Phi_i)$ and final state given by

$$\begin{aligned}\rho(\Phi_f, m_f^+, m_f^-, t_f) &= \int \mathcal{D}\Phi_t \mathcal{D}m^+ \mathcal{D}m^- \mathcal{N} e^{\mathcal{I}[\Phi, m^+, m^-, t_i, t_f]} \rho(\Phi_i, m_i^+, m_i^-, t_i) \\ &= \int \mathcal{D}\Phi_t \mathcal{D}m^+ \mathcal{D}m^- \mathcal{N} e^{-I_C[\Phi]} e^{\mathcal{I}_{\mathrm{CQ}}[\Phi, m^+, t_i, t_f]} \langle m^+|\Psi_i\rangle\langle\Psi_i|m^-\rangle\delta(\Phi - \Phi_i) e^{\mathcal{I}_{\mathrm{CQ}}[\Phi, m^-, t_i, t_f]} \\ &= \int \mathcal{D}\Phi_t \langle m^+|L(\Phi_t)|\Psi_i\rangle\langle\Psi_i|L^\dagger(\Phi_t)|m^-\rangle\delta(\Phi - \Phi_f). \end{aligned} \tag{28}$$

This is a special case of Eq. (1), with the Kraus operators $L(\Phi_t)$ labeled by a continuous index, given by the trajectories of the classical field. The total evolution is thus an incoherent sum (an integral) over all possible transitions of the quantum state.

We next want to make more precise the requirement that the degree of freedom $m(x)$ that we want to entangle interacts primarily via the classical field. This is required in any experiment since the quantum fields can generate entanglement and need to either be screened or otherwise made small. We will do so by demanding that the quantum interaction is *localized*. For two regions A and B, and for any fixed Φ, we say a CQ action is localized if it is of the form

$$\mathcal{I}_{\mathrm{CQ}}[\Phi, m^\pm, t_i, t_f] \approx \mathcal{I}_A[\Phi, m_A^\pm t_i, t_f] + \mathcal{I}_B[\Phi, m_B^\pm, t_i, t_f], \tag{29}$$

with $\mathcal{I}_A[\Phi, m_A^\pm, t_i, t_f]$ depending only on the matter field $m(x)$ in region A and $\mathcal{I}_B[\Phi, m_B^\pm, t_i, t_f]$ depending only on the matter field $m(x)$ in region B. In other words, the quantum fields do not couple region A with region B. If such a condition is met, the operators corresponding to $\mathcal{I}_A$ and $\mathcal{I}_B$ act on separate Hilbert space factors, and commute, allowing the exponential operators to be separated $e^{\mathcal{I}_{\mathrm{CQ}}} \approx e^{\mathcal{I}_A} e^{\mathcal{I}_B}$, and thus we can see from Eq. (28) that this implies $L(\Phi_t) \approx L_A(\Phi_t) \otimes L_B(\Phi_t)$. This enables us to write the dynamics as the map:

$$\varrho(\Phi_f, t_f) = \int \mathcal{D}\Phi_t L_A(\Phi_t) \otimes L_B(\Phi_t) \varrho(\Phi_i, t_i) L_A^\dagger(\Phi_t) \otimes L_B^\dagger(\Phi_t). \tag{30}$$

Such a map cannot create entanglement between region A and B, since acting it on a product state $|\Psi_A\rangle \otimes |\psi_B\rangle$ produces a statistical mixture of product states, which is a separable state. The path integral approach therefore provides a direct proof that local CQ dynamics cannot generate entanglement.

Let us now discuss which CQ models satisfy the localized assumption of Eq. (29). It requires (i) that the Lindbladian have a sufficiently local noise kernel and (ii) that any quantum interactions between the relevant degrees of freedom in A and B be negligible or screened with some electrostatic shielding. These are both necessary condition—Lindbladians with nonlocal noise kernels can generate entanglement [60], including those which implement Diosi-Penrose decoherence, or the Tilloy-Diosi model [12,49]. Since these nonlocal noise kernels are suggestive of an effective CQ theory rather than a fundamental one, as discussed in relation to the kernels of Eq. (27), we should not be surprised that they can create entanglement. This is also natural since they can be viewed as an interaction with an entangled environment [12]. We have already discussed why requirement (ii) is needed. Quantum interactions between A and B such as electromagnetism needs to be made negligible because these can create entanglement.

The local models discussed here satisfy the locality condition of Eq. (29) whenever (ii) is satisfied. As a specific example, we consider the Newtonian limit of the gravity path integral from the previous section. We consider two mass densities $m_A(x)$ and $m_B(x)$ in each of the regions, interacting via a classical Newtonian potential Φ. The Newtonian limit of our gravity action is given by [61]

$$\begin{aligned}\mathcal{I}_{\mathrm{CQ}}[\Phi, m^+, m^-, t_i, t_f] = \int_{t_i}^{t_f} d^4x \bigg[& i(\mathcal{L}_Q[m^+] - \mathcal{V}_I[\Phi, m^+] - \mathcal{L}_Q[m^-] + \mathcal{V}_I[\Phi, m^-]) \\ & - \frac{1}{8\tilde{D}_2}(m^+(x) - m^-(x))^2 - \frac{1}{2\tilde{D}_2}\left(\frac{\nabla^2\Phi}{4\pi G} - \bar{m}(x)\right)^2 \bigg], \end{aligned} \tag{31}$$

where $\bar{m}(x)=\frac{1}{2}(m^+(x)+m^-(x))$ and $\mathcal{V}_I[\Phi,m^\pm]\approx-2\Phi m^\pm$, coming from expanding the $\sqrt{-g}$ in the matter action. $\mathcal{L}_Q[m^\pm]$ is the matter action in Minkowski space. We can see from this action that Φ satisfies Poisson's equation, sourced by both the bra and ket mass density, with the size of deviations from this controlled by $\tilde{D}_2$.

Let us first consider the Lindbladian term. It is ultralocal, so this part of the action taken in isolation can be split into a part which acts in region A and another part which acts in region B:

$$\int d^4x(m^+(x)-m^-(x))^2\approx\int d^4x(m_A^+(x)-m_A^-(x))^2 + \int d^4x(m_B^+(x)-m_B^-(x))^2. \tag{32}$$

By way of comparison, the correlated Diosi-Penrose noise kernel $\int d^3xd^3y(1/|x-y|)(m^+(x)-m^-(x))(m^+(y)-m^-(y))$ cannot be split into a sum of an A part and a B part, unless $m_A(x)m_B(y)/|x-y|$ is negligible.

The remaining part of the matter action are the bra and ket actions $\int d^4x\{\pm i\mathcal{L}_Q[m^\pm]\mp 2i\Phi m^\pm-[1/(2\tilde{D}_2)]\times[(\nabla^2\Phi)/(4\pi G)]m^\pm\}$. The last two terms do not have any temporal or spatial derivatives of $m^\pm(x)$, so the question of whether the action has the form of Eq. (29) hinges on the form of $\mathcal{L}_Q[m^\pm]$. If gravity is the dominant force, the matter action is dominated by the rest mass density $\mathcal{L}_Q[m^\pm]=m^\pm(x)$ and we can neglect kinetic terms. This action for the matter field is local in the sense of Eq. (29), and thus cannot generate entanglement between separated regions. On the other hand, if we have other interactions between region A and B, then Eq. (29) need not be satisfied, and these other interactions can create entanglement. For example, if we have a scalar field with $\mathcal{L}_Q[\phi^\pm]=\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2\phi^2$, then the $(\partial\phi)^2$ term means that Eq. (29) does not hold. An easy way to check this is to add a source for the quantum field restricted to A and B, $J_{A\pm}(x)\phi^\pm(x)+J_{B\pm}(x)\phi^\pm(x)$, into the action and perform the Gaussian integral over ϕ, which induces couplings $J_A(x)J_B(y)/|x-y|$ into the effective action.

By way of comparison, if we treated the Newtonian potential as a quantum field, with action

$$\begin{aligned}\mathcal{I}_Q&[\Phi^+,\Phi^-,m^+,m^-,t_i,t_f]\\ &=\int_{t_i}^{t_f}d^4xi\bigg[\mathcal{L}_Q[m^+]-\mathcal{V}_I[\Phi^+,m^+]-\mathcal{L}_Q[m^-]\\ &\quad+\mathcal{V}_I[\Phi^-,m^-]+\frac{1}{4\pi G}(\nabla\Phi^+)^2-\frac{1}{4\pi G}(\nabla\Phi^-)^2\bigg],\end{aligned} \tag{33}$$

then we would not be able to write the evolution in terms of product Kraus operators with an index given in terms of a single classical field trajectory Φ_t as in Eq. (30). The fact that we can write the evolution as a convex mixture of products plus the short range of the quantum interaction and Lindbladian is what prohibits the generation of entanglement. The classical interaction itself need not be local. In contrast, given the fully quantum action of Eq. (33) and source terms $m_A^\pm(x)\Phi(x)+m_B^\pm(x)\Phi^\pm(x)$, one can perform the Gaussian integral over the bra and ket Newtonian potentials Φ^+ and Φ^- to get an interaction which clearly can create entanglement between regions A and B as one has terms $\pm iG[m_A^\pm(x)m_B^\pm(y)/|x-y|]$ in the reduced action [46].

IX. DISCUSSION

In this work, we have introduced a general path integral for classical-quantum dynamics, given by Eq. (5), which opens up the way to study classical degrees of freedom coupled to quantum ones via path integral methods. This provides an approach to study covariant theories of classical fields coupled to quantum ones. We have given an explicit example of a Lorentz invariant CQ theory and applied it to classical-quantum theories of gravity.

In particular, we have arrived at a diffeomorphism invariant theory of CQ general relativity—summarized by Eqs. (24) and (25)—which acts to suppress paths that deviate from the trace of Einstein's equations, while simultaneously decohering the quantum system according to the trace of the stress-energy tensor. This provides a first example of diffeomorphism invariant classical-quantum dynamics and, more generally, is a first example of diffeomorphism invariant collapse dynamics [32–38], where the loss of coherence is a derived consequence of the interaction of a quantum system with a classical dynamical variable. We have also proposed a diffeomorphism invariant theory that reproduces all of Einstein's equations as a limiting case. Since $D_{2,\mu\nu\rho\sigma}$ is not positive semidefinite, we have not proven that the dynamics is normalizable, and suppresses paths away from Einstein's equation, but this has since been shown for the scalar and tensor sectors in Refs. [39,51].

We have here given a general construction by which one can write down CQ path integrals that uphold space-time and gauge symmetries. It would be worthwhile to explore this further with concrete examples. For classical-quantum gauge theories, which could be useful in an effective theory of light-matter interactions when there is classical back-reaction, the killing form provides a natural choice for D_0 since for a compact lie group, the killing form is positive semidefinite [62].

The theory and formalism presented here have a number of applications. First, we have shown in Refs. [61,63] that if we first reduce the phase space the nonrelativistic weak field limit of the gravitational path integral reproduces the weak field limit of the theory derived using the master equation method of [8] and that deriving using a measurement and feedback approach [12]. However, there is an important difference. While the nonrelativistic but local theories of Refs. [12,61] are ruled out by experiment via the decoherence-diffusion trade-off, due to having an IR divergence [14], we find that the relativistic theory presented here is not [27]. We have further found that the full theory has a different weak-field limit, where the dynamical degrees of freedom are stochastic gravitational waves and the spacial curvature, while the Newtonian potential is nondynamical [39].

We note that since the propagator scaled like $1/p^4$, the theory could be renormalizable. We have since shown that the pure gravity path integral presented here is formally renormalizable without having tachyons or ghosts [27]. A full proof of renormalizability would require showing that the pole prescription which results in the theory being renormalizable also retains the property that it is completely positive. This was shown for the scalar modes, and has since been extended to the tensor modes [39]. Though effective theories can be nonrenormalizable, the renormalizability of CQ dynamics in the gravitational degrees of freedom has important foundational consequences since the prime motivation for believing that gravity may not be a quantum field is that it reflects the curvature of space-time. If that is the case, then the description of gravity in terms of the metric ought to be a fundamental description which should not break down at some energy scale. In contrast, perturbative quantum gravity is not renormalizable in $3+1$ dimensions, so it is unclear if the geometrical description of gravity would hold in the quantum theory.

For the matter degrees of freedom, the gravitational action described by Eqs. (24) and (25) is not power counting renormalizable due to the terms which are quadratic in the stress-energy tensor, though, as noted in Refs. [56,64], one must be careful with power counting renormalization when considering the density matrix path integral. The additional higher-order terms which are of most interest are terms such as the mass density squared, which for the scalar field go as $(\phi^{\pm})^4$. These serve to decohere the quantum system into mass distributions. On the other hand, the nonrenormalizable terms with negative mass dimension such as $(\partial\phi^{\pm})^4$ act to decohere the state into kinetic energy distributions, and at low energy are suppressed by higher derivatives. From both an effective field theory and in terms of backreacting on the gravitational field, such terms are not relevant at low energy.

From an observational point of view, perhaps the most significant application of the path integral introduced here is that it enables the calculation of the two-point function of the gravitational field. This has since been undertaken in Refs. [27,39,65]. The two-point function for a "mod-squared retarded" pole prescription is found in Ref. [65] and shown to be appropriate for the stochastic Klein-Gordon equation. Understanding the relationship between the Newtonian potential and this scalar mode requires a deeper understanding of coordinate freedoms and diffeomorphism invariance. This provides a path toward strong experimental constraints on the theory, through the decoherence vs diffusion trade-off [14]. Precision acceleration experiments set an upper bound on D_2, while the trade-off implies that interference experiments put a lower bound on D_2, thus constraining the dimensionless coupling constant of the theory from both sides, and possibly falsifying the theory.

We have here approached CQ dynamics from a bottom-up approach: Starting from the description of a system in terms of classical and quantum variables, we have written down a description for the dynamics which leads to consistent evolution. It would be interesting to arrive at classical-quantum theories from a top-down approach. That is, starting from a quantum-quantum system, we should be able to arrive at an effective classical-quantum description. We have also shown that there is a parameter range of the classical-quantum dynamics presented here, which arises from a "classical-quantum limit" of two quantum systems [50]. This occurs via a decoherence mechanism on one of the systems and is closely related to the quantum to classical transition [66–69]. Such an approach would be useful as an effective theory of semiclassical gravitational physics when backreaction is involved, for example, in inflationary cosmology or during black-hole evaporation. As a first step, one would like to extend the result of Ref. [50] to the field-theoretic case, whose natural setting is the path integral formulation described here. Since the pure gravity path integral is renormalizable, one might hope that the effectively classical-quantum theory retains this feature. Thus although perturbative quantum gravity in $3+1$ dimensions might not be renormalizable, it might be in the limit that the system becomes classical. Aside from CQ gravity models, the covariant path integral has subsequently been used to develop an alternative effective theory of wave function collapse by coupling a classical scalar field to perturbative quantum gravity [57]. This could also be related to quantum gravity subject to a decohering environment.

We have presented a simple model of Lorentz invariant classical-quantum field theory. In Appendix D we initiate the study of two interacting Lorentz invariant scalar fields, one classical, one quantum, by demonstrating the use of perturbation theory to compute the partition function, as well as methods for computing the normalization in Appendix E. A slightly simpler model in which the classical-quantum interaction is linear in the classical field $q\phi$ has since been presented in Ref. [27]. Recently, Carney and Matsumura computed the scattering cross section of this model [70] and confirmed both the Lorentz invariance of the result and the non-Markovian nature of the dynamics when the classical field is integrated out. They also found that if one treats planets as point particles, and assuming Markovian dynamics, that there are order one corrections to the Rutherford result. This appears to be related to the fact that diffusion in the classical system can induce secondary decoherence in the quantum system [71], which can result in anomalous heating [72,73]. This secondary effect and suggestions for suppressing it were discussed in Ref. [8], including adding higher-order terms in the classical field as are found in general relativity, as well as friction terms, or field dependent diffusion coefficients. More generally, it is hoped that the methods presented here provide a sufficient template to explore this and other issues in a variety of models which respect space-time and other symmetries.

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APPENDIX A: COMPARISON OF CLASSICAL, QUANTUM, AND CLASSICAL-QUANTUM PATH INTEGRALS

We summarize the classical, quantum, and classical–quantum path integral formulations discussed in this work in Table I. The table collects the corresponding path integral representations, effective actions, and complete positivity conditions, and highlights the role of decoherence and diffusion terms in each case.

TABLE I. Classical, quantum, and classical-quantum path integrals. The classical-quantum path integral generalizes the Feynman-Vernon (FV) path integral of open quantum systems and the stochastic path integral of classical systems.

(a)

	Classical stochastic
Path integral	$p(q,p,t_f)=\int \mathcal{D}q\mathcal{D}p e^{iS_C[q,p]}\delta(\dot q-\frac{\partial H}{\partial p})p(q,p,t_i)$
Action	$iS_C=-\int_{t_i}^{t_f} dt\frac{1}{2}(\frac{\partial H}{\partial q}+\dot p)D_2^{-1}(\frac{\partial H}{\partial q}+\dot p)$
CP condition	D_2^{-1} a positive (semidefinite) matrix, $D_2^{-1}\succeq 0$

The path integral for continuous, stochastic phase space classical dynamics [28,29,53,54]. One sums over all classical configurations (q,p) with a weighting according to the difference between the classical path and its expected force $-\frac{\partial H}{\partial q}$, by an amount characterized by the diffusion matrix D_2. In the case where the force is determined by a Lagrangian L_c, the action S_C describes suppression of paths away from the Euler-Lagrange equations, $iS_C=-\int_{t_i}^{t_f} dt\frac{1}{2}(\frac{\delta L_c}{\delta q_i})(D_2^{-1})^{ij}(\frac{\delta L_c}{\delta q_j})$, by an amount determined by the diffusion coefficient D_2. The most general form of classical path integral can be found in Refs. [26,53,54]. A simple example is Brownian motion, which in configuration space has an action $iS_C=-\frac{1}{2D_2}\int dt(\ddot q)^2$ and $p(q,\dot q,t_f)=\int \mathcal{D}e^{-(1/2D_2)\int dt(\ddot q)^2}p(q,\dot q,t_i)$, which suppresses paths that deviate from $\ddot q=0$. The more general form is discussed in connection to Eq. (14), and the example of Brownian motion is discussed in Feynman and Hibbs [74].

(b)

	Quantum
Path integral	$\rho(\phi^\pm,t_f)=\int \mathcal{D}\phi^\pm e^{iS[\phi^+]-iS[\phi^-]+iS_{\rm FV}[\phi^+,\phi^-]}\rho(\phi^\pm,t_i)$
Action	$S[\phi]=\int_{t_i}^{t_f} dt(\frac{1}{2}\dot\phi^2+V(\phi)), iS_{\rm FV}=\int_{t_i}^{t_f} dt(D_0^{\alpha\beta}L_\alpha^+L_\beta^{*-}-\frac{1}{2}D_0^{\alpha\beta}(L_\beta^{*-}L_\alpha^-+L_\beta^{*+}L_\alpha^+))$
CP condition	$D_0^{\alpha\beta}$ a positive (semidefinite) matrix, $D_0\succeq 0$.

(Table continued)

TABLE I. *(Continued)*

The path integral for a general autonomous quantum system, here taken to be ϕ. The quantum path integral is doubled since it includes a path integral over both the bra and ket components of the density matrix, here represented using the $\pm$ notation. In the absence of the Feynman Vernon term $S_{\rm FV}$ [55], the path integral represents a quantum system evolving unitarily with an action $S[\phi]$. When the Feynman Vernon action $S_{\rm FV}$ is included, the path integral describes the path integral for dynamics undergoing Lindbladian evolution [16,17] with Lindblad operators $L_\alpha(\phi)$. Because of the $\pm$ cross terms, the path integral no longer preserves the purity of the quantum state, and there will generally be decoherence by an amount determined by D_0. As an example, taking $L^\pm = \phi^\pm(x)$ a local field, and $D_0^{\alpha\beta} = D_0$ results in a Feynman-Vernon term $iS_{\rm FV} = -\frac{1}{2} D_0 \int_{t_i}^{t_f} dt dx(\phi^- - \phi^+)^2$ which decoheres the state in the $\phi(x)$ basis, since off-diagonal terms in the density matrix, where $\phi^+(x)$ is different from $\phi^-(x)$, are suppressed.

(c)

	Classical-quantum
Path integral	$\rho(q,p,\phi^\pm,t_f) = \int \mathcal{D}q\mathcal{D}p\mathcal{D}\phi^\pm e^{iS_C[q,p]+iS[\phi^+]-iS[\phi^-]+iS_{\rm FV}[\phi^\pm]+iS_{\rm CQ}[q,p,\phi^\pm]}\delta(\dot{q}-\frac{p}{m})\rho(q,p,\phi^\pm,t_i)$
Action	$iS_C[z] + iS_{\rm CQ}[z,\phi^\pm] = -\frac{1}{2}\int_{t_i}^{t_f} dt D_2^{-1}\left(\frac{\partial H_c}{\partial q} + \frac{1}{2}\frac{\partial V_I[q,\phi^+]}{\partial q} + \frac{1}{2}\frac{\partial V_I[q,\phi^-]}{\partial q} + \dot{p}\right)^2$
CP condition	$D_0 \succeq 0, D_2 \succeq 0$ and $4D_2 \succeq D_0^{-1}$

The phase space path integral for continuous, autonomous classical-quantum dynamics. The path integral is a sum over all classical paths of the variables z, as well as a sum over the doubled quantum degrees of freedom $\phi^\pm$. The action contains the purely quantum term from the quantum path integral in section (b) above, but also includes the term $iS_C + iS_{\rm CQ}$. This suppresses paths away from the averaged drift, which is sourced by both purely classical terms described by the Hamiltonian H_c and the backreaction of the quantum systems on the classical ones, described by a classical-quantum interaction potential V_I. The most general form of classical-quantum path integral can be found in Ref. [26]. In order for the dynamics to be completely positive, the decoherence-diffusion trade-off $4D_2 \succeq D_0^{-1}$ must be satisfied [14,15], where D_0^{-1} is the generalized inverse of D_0, which must be positive semidefinite. When the trade-off is saturated, the path integral preserves the purity of the quantum state, conditioned on the classical degree of freedom [25]. A simple example is a Stern-Gerlach device, where a classical particle in phase space interacts with its own spin σ_z along the direction of a magnetic field with strength D_1 via potential $V_I = -D_1 q \sigma_z$. The interacting term in the action is then $iS_C + iS_{\rm CQ} = -\frac{1}{2D_2}\int dt(\dot{p} - \frac{D_1}{2}\sigma_z^+ - \frac{D_1}{2}\sigma_z^-)^2$ so that the particle receives a force given by the average of the bra and ket spins, with fluctuations which are suppressed with increasing D_2. Complete positivity then requires a decoherence term given by $S_{\rm FV} = -\frac{D_1^2}{8D_2}(\sigma_z^+ - \sigma_z^-)^2$, if we saturate the condition $4D_2 \succeq D_0^{-1}$. This term suppresses superpositions of spin in the direction aligned with the magnetic field.

APPENDIX B: PROOF OF POSITIVITY

In this appendix, we prove the statement made in the main body that Eq. (10) defines completely positive CQ dynamics. To see this in detail, we can perform a short-time expansion of the full path integral, which we can always do since we assume the dynamics are time local.

Let us first consider the case where the quantum state remains pure, so that $c^\gamma = 0$ in Eq. (10). Defining $t_f = t_0 + K\delta t$, $t_i = t_0 + i\delta t$, and discretizing the path integral into steps of size δt, we have that

$$\varrho_{i+1} = \int d\phi_i^+ d\phi_i^- dq_i (e^{\mathcal{I}^+_{i+1,i}})(e^{\mathcal{I}^-_{i+1,i}})^* e^{-\mathcal{I}_{C,i+1,i}}\varrho_i, \tag{B1}$$

where we use the shorthand $\varrho_i = \varrho(\phi_i^+, \phi_i^-, q_i, t_i)$, $\mathcal{I}_{i+1,i} = \mathcal{I}[\phi_{i+1}, \phi_i, q_{i+1}, q_i]$ and $\mathcal{I}_{C,i+1,i} = \mathcal{I}_C[q_{i+1}, q_i]$.

More generally, we can allow for the case where the action contains higher time derivatives, in which case we have $\mathcal{I}[\phi_{i+k_q}, \ldots, \phi_i, q_{i+k_c}, \ldots, q_i]$ and $\mathcal{I}_C[q_{i+k_c}, q_i]$, with $k_c, k_q \geq 2$. In order to retain the usual composition law for the path integral, we must also let the state be described by increasingly higher derivative terms $\varrho\{\phi_i^\pm, \ldots, [(d^{k_q-1}\phi_i^\pm)/(dt^{k_q-1})], q_i, [(d^{k_c-1}\phi_i^\pm)/(dt^{k_c-1})\}$ [52]. The final state then imposes boundary conditions on the components of the action, which contain higher derivative terms so that Eq. (B1) is still well defined.

With this in mind, we can take the trace with respect to an arbitrary vector $|v(q)\rangle$, and for complete positivity, we need to show

$$\int d\phi_{i+1}^+ d\phi_{i+1}^- v_{i+1}^{+*} \varrho_{i+1} v_{i+1}^- \geq 0. \tag{B2}$$

Denoting $\tilde{v}^+_{i+1,i} = e^{\mathcal{I}^+_{i+1,i}} v^{+*}_{i+1}$, then inserting Eq. (B1) into Eq. (B2), we have

$$\int d\phi_{i+1}^+ d\phi_{i+1}^- d\phi_i^+ d\phi_i^- dq_i \tilde{v}^+_{i+1,i} \tilde{v}^{-*}_{i+1,i} e^{-\mathcal{I}_{C,i+1,i}} \varrho_i. \tag{B3}$$

Because the integral factorizes into $\pm$ conjugates, Eq. (B3) will always be positive. To see this explicitly, we first perform the $\phi_i^\pm$ integrals to obtain

$$c_{i+1} = \int d\phi_i^+ d\phi_i^- \tilde{v}^+_{i+1,i} \tilde{v}^{-*}_{i+1,i} \varrho \geq 0, \tag{B4}$$

where we have used the positivity of the state CQ $\varrho(\phi_i^+, \phi_i^-, q_i, t_i)$. What remains is the integral

$$\int d\phi_{i+1}^+ d\phi_{i+1}^- dq_i e^{-\mathcal{I}_{C,i+1,i}} c_{i+1} \geq 0, \tag{B5}$$

which is positive since both c_{i+1} and the exponential are both positive. In Eq. (B5) there is still a free q_{i+1} variable which corresponds to the fact that positivity of the CQ state demands that the CQ dynamics keep quantum states positive conditioned on the classical degrees of freedom. We thus see that the state after applying the time-evolved state will also be positive. Hence the dynamics are positive. When we consider the dynamics as part of a larger system, we apply the identity map on the larger system, and the dynamics still factorize in this way—we perform a delta function path integral on the auxiliary system, so Eq. (5) defines completely positive dynamics.

In the more general case, we can have nonzero $c^\gamma[q,x]$, and there is information loss since the dynamics can send pure states to mixed states. In this case, the only thing that changes is the definition of $\tilde{v}^+_{i+1,i}$ in Eq. (B2). In particular, in the general case, we must also expand out the terms involving c^γ_n in the action of Eq. (5),

$$e^{\delta t\sum_\gamma c^\gamma_{i+1,i}(L^+_\gamma)_{i+1,i}(L^-_\gamma)^*_{i+1,i}} = 1+\delta t\sum_\gamma c^\gamma_{i+1,i}(L^+_\gamma)_{i+1,i}(L^-_\gamma)^*_{i+1,i}+O(\delta t^2), \tag{B6}$$

where $c^\gamma_{i+1,i}=c^\gamma[q_{i+1},q_i]$, $L^+_{i+1,i}=L^+[\phi^+_{i+1},\phi^+_i]$ and similarly for the $^-$ branch.

With this in mind, the integrand of the path integral in Eq. (5) factorizes according to Eq. (B3), and the steps to prove complete positivity are exactly the same but with

$$\tilde{v}^{\gamma+}_{i+1,i}=[1+\sqrt{\delta t c^\gamma_{i+1,i}}(L^+_\gamma)_{i+1,i}]e^{\mathcal{I}^+_{i+1,i}}v^{+*}_{i+1}, \tag{B7}$$

from which the complete positivity of the dynamics follows from the same arguments outlined in Eqs. (B3) and (B4), where we now also sum over γ. Note though that we need only work to first order in δt, had we included them, the higher-order δt terms also factorize in the same way.

In the field-theoretic case, the total CQ action is

$$\begin{aligned}\mathcal{I}[\phi^-,\phi^+,q,t_i,t_f] &= \mathcal{I}_{\mathrm{CQ}}[q,\phi^+,t_i,t_f]+\mathcal{I}^*_{\mathrm{CQ}}[q,\phi^-,t_i,t_f]-\mathcal{I}_C[q,t_i,t_f]\\ &\quad+\int_{t_i}^{t_f}dx\sum_\gamma c^\gamma(q,t,x)[L_\gamma[\phi^+](x)L^*_\gamma[\phi^-](x)],\end{aligned} \tag{B8}$$

and we can repeat the argument for complete positivity, which again follows from the factorization of the path integral integrand. In this case, complete positivity follows from the fact that

$$\int\mathcal{D}\phi^+_{i+1}\mathcal{D}\phi^-_{i+1}\mathcal{D}\phi^+_i\mathcal{D}\phi^-_i\mathcal{D}z_i\sum_\gamma\int d\vec{x}c^\gamma_{i+1,i}(x)(v^+_{i+1,i}(L^+_\gamma)_{i+1,i}(x)e^{\mathcal{I}^+_{i+1,i}})(L^-_\gamma)_{i+1,i}(x)v^-_{i+1,i}e^{\mathcal{I}^-_{i+1,i}})^*e^{-\mathcal{I}_{C,i+1,i}}\varrho_i \tag{B9}$$

is positive when $c^\gamma\geq 0$ and ρ_i is a positive density matrix.

APPENDIX C: SHOWING THE NATURAL CLASS OF CQ DYNAMICS IS CP

In this appendix, we show that the dynamics defined by Eq. (17) takes the form of Eq. (5) and is hence completely positive. Since the purely quantum Lagrangian terms appearing in Eq. (5) are manifestly CP, we shall focus on the CQ interaction term,

$$-\frac{1}{2}\int dx\Delta X^i(q,x)D_{0,ij}(q,x)\Delta X^j(x)-\frac{1}{2}\int dx\bar{X}^i(q,x)D^{-1}_{2,ij}(q,x)\bar{X}^j(q,x), \tag{C1}$$

where we use the shorthand notation $X^i(q,x)=[\delta W_{\mathrm{CQ}}/\delta q_i(x)]$, which we assume is Hermitian since it is generated by a real protoaction W_{CQ}. For ease of presentation, we will here suppress any potential q, x dependence from X^i, D_0, D_2, but these can be added back in.

Expanding Eq. (C1), we can group terms according to D_0, D_2^{-1} as

$$-\frac{1}{8}\int dxd\vec{y}(4D_{0,ij}+D^{-1}_{2,ij})[(X^+)^i(X^+)^j+(X^-)^i(X^-)^j]+\frac{1}{8}\int dx(4D_{0,ij}-D^{-1}_{2,ij})[(X^+)^i(X^-)^j+(X^-)^i(X^+)^j]. \tag{C2}$$

We see that the first line in Eq. (C2) is of the form $I^+_{\mathrm{CQ}}+(I^-_{\mathrm{CQ}})^*$ and so adheres to the form in Eq. (5). If the trade-off is saturated, this completes the proof that Eq. (17) takes the form of Eq. (5). When it is not saturated, we can write

$4D_0 - D_2^{-1} = c \succeq 0$, where $c_{ij}(q,x)$ is a real, symmetric positive semidefinite matrix. We can then expand the second line of Eq. (C2) as

$$\frac{1}{8}\int dx c_{ij}(q,x)[(X^+)^i(X^-)^j + (X^-)^i(X^+)^j], \tag{C3}$$

which, after diagonalizing c_{ij}, takes the form of Eq.(B8), and hence defines CP dynamics whenever the condition $4D_0 - D_2^{-1} = c\succeq 0$ is satisfied.

When the trade-off $4D_0 - D_2^{-1} = c\succeq 0$ is saturated, i.e., when $c = 0$, we can reproduce the full action for the gravity theory of Eq. (17),

$$\begin{aligned}\mathcal{I}[\phi^-,\phi^+,g_{\mu\nu}] = \int dx\Bigg[& i\mathcal{L}^+_{KG} - i\mathcal{L}^-_{KG} - \frac{\det(-g)}{8}(T^{\mu\nu+} - T^{\mu\nu-})D_{0,\mu\nu\rho\sigma}(T^{\rho\sigma+} - T^{\rho\sigma-}) \\ & - \frac{\det(-g)}{128\pi^2}\left(G^{\mu\nu} - \frac{1}{2}[8\pi(T^{\mu\nu})^+ + 8\pi(T^{\mu\nu})^-]\right)D_{0,\mu\nu\rho\sigma}[g]\left(G^{\rho\sigma} - \frac{1}{2}[8\pi(T^{\rho\sigma})^+ + 8\pi(T^{\rho\sigma})^-]\right)\Bigg],\end{aligned} \tag{C4}$$

which we have now verified takes the form of Eq. (5).

APPENDIX D: PERTURBATIVE METHODS OF CALCULATING CORRELATION FUNCTIONS

In this appendix, we study a simple model of CQ interaction to illustrate how one can use standard perturbative methods to calculate classical-quantum correlation functions via CQ Feynman diagrams. In the main body, we considered the path integral which constructs a CQ state at a time t_f from a CQ state at time t_i. In computing correlation functions of classical-quantum observables, the final state is not important, and so we can perform an integral over all final states of the field q_f at t_f to arrive at the partition function,

$$Z = \int dq_f\, \mathrm{Tr}\varrho(q_f, t_f), \tag{D1}$$

which for the configuration space path integral takes the form

$$Z = \int \mathcal{N}\mathcal{D}\phi^-\mathcal{D}\phi^+\mathcal{D}q e^{\mathcal{I}(\phi^-,\phi^+,q,t_i,t_f)}\varrho(q_i,\phi_i^+,\phi_i^-,t_i), \tag{D2}$$

where now there are no final boundary conditions imposed on the path integral.

Formally, we can calculate correlation functions by inserting sources J^+, J^-, J_q for the respective fields ϕ^+, ϕ^-, q into the path integral, and taking functional derivatives with respect to the sources. This is similar to how one computes correlation functions in quantum field theory, and as in that context, if the full action is covariant, then the computed quantities must be as well.

The partition function of interest is therefore

$$Z[J^+,J^-,J_q] = \int \mathcal{N}\mathcal{D}\phi^-\mathcal{D}\phi^+\mathcal{D}q e^{\mathcal{I}(\phi^-,\phi^+,q,t_i,t_f) - iJ_+\phi^+ + iJ^-\phi^- - J_q q}\varrho(q_i,\phi_i^+,\phi_i^-,t_i). \tag{D3}$$

In general, the form of the path integral depends on the initial CQ state $\varrho(q_i,\phi_i^+,\phi_i^-,t_i)$, and any calculation of correlation function must be performed on a case by case basis depending on the initial state.

However, often we are interested in stationary states, and we would like to obtain information on correlation functions over arbitrary long times by taking the limit $t_i \to -\infty, t_f \to \infty$. In open systems, as well as when calculating scattering amplitudes, it is often assumed that the initial state in the infinite past does not affect the stationary state of the system so that there is a complete loss of memory of the initial state [75]. Under this assumption, it is possible to ignore the boundary term containing the initial CQ state $\varrho(q_i,\phi_i^+,\phi_i^-,t_i)$, and we arrive at the partition function:

$$Z[J^+,J^-,J_q] = \int \mathcal{N}\mathcal{D}\phi^-\mathcal{D}\phi^+\mathcal{D}q e^{\mathcal{I}(\phi^-,\phi^+,q,-\infty,\infty) - iJ_+\phi^+ + iJ^-\phi^- - J_q q}. \tag{D4}$$

Using Eq. (D4), we can then use standard perturbation methods for computing correlation functions in CQ theories.

As a simple example, consider the CQ theory of Eq. (18), but in zero spatial dimensions, so that the protoaction is given by

$$W_{\mathrm{CQ}} = \int dt \left[\frac{1}{2}\dot{q}^2 - \frac{m_q^2 q^2}{2} - \frac{\lambda q^2 \phi^2}{2} \right], \tag{D5}$$

and the pure quantum action is given by $S_Q = \int dt \{ \frac{1}{2}\dot{\phi}^2 - [(m_\phi^2 \phi^2)/2] \}$. Assuming the decoherence diffusion trade-off is saturated, we arrive at the total action via the procedure outlined in the Sec. V on the natural class of CQ dynamics [analogous to Eqs. (19) and (20)]:

$$\mathcal{I}[\phi^\pm, q] = \int dt \left[-\frac{i}{\hbar}\frac{\dot{\phi}^{+2} + m_\phi^2(\phi^+)^2}{2} + \frac{i}{\hbar}\frac{\dot{\phi}^{-2} + m_\phi^2(\phi^-)^2}{2} - \frac{1}{4D_2}(\ddot{q} + m_q^2 q + \lambda q(\phi^+)^2)^2 - \frac{1}{4D_2}(\ddot{q} + m_q^2 q + \lambda q(\phi^-)^2)^2 \right]. \tag{D6}$$

Expanding the squared terms, the interactions contain a $(q^2\phi^2)$ four-vertex and a $(q^2\phi^4)$ six-vertex; both will show up explicitly in the Feynman rules below. We see from Eq. (D6) that D_2 in an interacting CQ theory plays exactly the same role as $\hbar$ in an interacting quantum theory, in the sense that to compute correlation functions, we can work perturbatively in D_2. Note that the double limit $D_2 \to 0, D_2^{-1}\lambda \to 0$ defines a deterministic quantum theory with no classical backreaction.

For simplicity, the perturbative expansion below is carried out in the zero-dimensional reduction of this model, in which the time integral and all derivative terms are dropped. The propagators then reduce to numbers and the Feynman rules take their simplest form.

We define the free theory as the action independent of any CQ backreaction:

$$I_{\mathrm{free}} = iS^+ - iS^- - I_C = -i\frac{m_\phi^2(\phi^+)^2}{2\hbar} + i\frac{m_\phi^2(\phi^-)^2}{2\hbar} - \frac{1}{2D_2}q^2 m_q^4. \tag{D7}$$

Inserting sources, we find the partition function of the free theory,

$$Z_{\mathrm{free}}[J_+, J_-, J_q] = \int d\phi^\pm dq e^{I_{\mathrm{free}} - (i/\hbar)J_+\phi^+ + (i/\hbar)J_-\phi^- - (1/D_2)J_q q}, \tag{D8}$$

which can be performed exactly by performing each Gaussian integral individually:

$$Z_{\mathrm{free}}[J_+, J_-, J_q] = \left(\int d\phi^+ e^{-i[m_\phi^2(\phi^+)^2/2\hbar] - iJ_+\phi^+} \right) \left(\int d\phi^- e^{+i[m_\phi^2(\phi^-)^2/2\hbar] + iJ_-\phi^-} \right) \left(\int dq e^{(-1/2D_2)q^2 m_q^4 - J_q q} \right). \tag{D9}$$

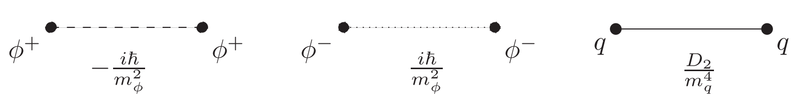
Equation (D9) is evaluated as

$$\begin{aligned} Z_{\mathrm{free}}[J_+, J_-, J_q] &= \left(\frac{-2\pi i\hbar}{m_\phi^2} \right) e^{iJ_+^2/2\hbar m_\phi^2} \left(\frac{2\pi i\hbar}{m_\phi^2} \right) e^{-iJ_-^2/2\hbar m_\phi^2} \left(\frac{\pi D_2}{m_q^4} \right) e^{J_q^2/2D_2 m_q^4} \\ &\coloneqq Z_0 e^{iJ_+^2/2\hbar m_\phi^2} e^{-iJ_-^2/2\hbar m_\phi^2} e^{J_q^2/2D_2 m_q^4}, \end{aligned} \tag{D10}$$

where we have defined Z_0 as the interaction-free partition function without sources. From Eq. (D9) we can read out the propagators for the free theory as we would in standard quantum theory—each one can be found by taking two functional derivatives of the corresponding source, while keeping track of the coefficients of $i/\hbar$ and $1/D_2$ from Eq. (D8). This gives

$$\langle \phi^+\phi^+ \rangle = -\frac{i\hbar}{m_\phi^2}, \qquad \langle \phi^-\phi^- \rangle = \frac{i\hbar}{m_\phi^2}, \qquad \langle qq \rangle = \frac{D_2}{m_q^4}, \tag{D11}$$

and we can represent each of the propagators by the following Feynman diagrams



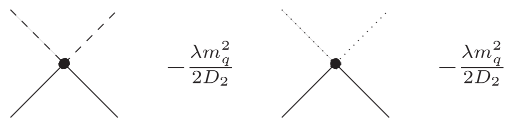
(D12)

The full partition function with the CQ interaction turned on then takes the form

$$Z[J_+, J_-, J_q] = \langle e^{\mathcal{I}_{\mathrm{CQ}}} \rangle = \langle e^{-(1/2D_2)((1/2)\lambda^2 q^2 [(\phi^+)^4 + (\phi^-)^4] + \lambda m_q^2 q^2 [(\phi^+)^2 + (\phi^-)^2])} \rangle, \tag{D13}$$

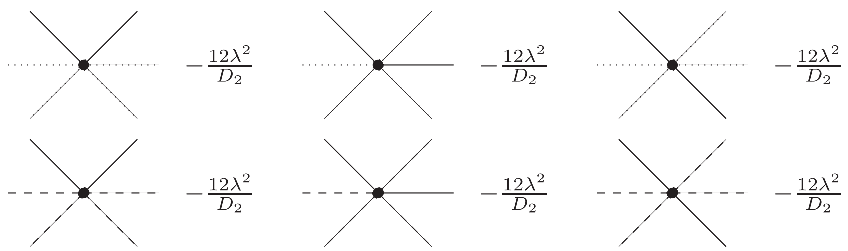
and we can perform an asymptotic expansion of the CQ interaction in terms of D_2 to arrive at the usual Feynman rules for computing correlation functions. Specifically, for terms in the action like $\lambda_{nml}\phi_+^n \phi_-^m q^l$, that is, n copies of ϕ^+, m copies of ϕ^-, and l copies of q, the corresponding Feynman rule assigns to a single vertex with those $n+m+l$ legs the factor $\lambda_{nml} n! m! l!$ to each topologically distinct diagram. The extra factorials are the usual symmetry factors: They count all distinct ways of attaching the identical external legs to that vertex, so you do not have to divide by additional combinatorial numbers when summing over diagrams.

As an example, the CQ interaction term $q^2(\phi^\pm)^2$ in Eq. (D6) has two quartic vertices with strength $-[(\lambda m_q^2)/(2D_2)]$ and vertex factor $-[(2\lambda m_q^2)/(D_2)]$ (i.e., $\times 2! \cdot 2!$), and can be represented by the diagrams



(D14)

We also have the sextic $q^2(\phi^\pm)^4$ interaction with vertex value $-[(\lambda^2 4!2!)/(4D_2)]$ which is assigned to each of the following diagrams:



(D15)

Because the purely quantum part of the CQ action is similar to the Schwinger-Keldish path integral, it is often convenient to change basis to the combinations $\bar{\phi} = \frac{1}{2}(\phi^+ + \phi^-)$ and $\Delta\phi = \phi^+ - \phi^-$. These are typically called the "classical" field and "quantum" field, respectively—terminology that will no doubt be confusing if used in the present context. In this basis, the causal structure of propagators is explicit and only vertices with an odd number of quantum legs contribute to connected correlation functions, exactly as in the standard Keldysh technique. Readers who prefer that basis can translate the foregoing formulas straightforwardly.

APPENDIX E: ENSURING THE CQ PATH INTEGRAL IS NORMALIZED

In this appendix, we show that the CQ action defined by Eq. (17) is normalized so long as it contains appropriate classical and quantum kinetic terms. To see the problem of normalization of CQ path integrals in more detail, we will review how the normalization of quantum states occurs in Lindbladian path integrals with a Feynman-Vernon action [55], and how probabilities are conserved in higher-derivative classical path integrals. Let us first consider higher-order classical path integrals. We refer the reader to Ref. [26] for a complete derivation of normalized CQ path integrals from master equations.

1. Normalization of higher-derivative classical path integrals

When considering a classical path integral that contains higher derivatives, we should treat $q, \dot{q}$ as independent variables. This is outlined in detail in Ref. [52]. To that end, we will show how the normalization of the path integral,

$$p(q_f, \dot{q}_f, t_f) = \int^B \mathcal{D}q e^{-\int_{t_i}^{t_f} dt [\ddot{q} - f(\dot{q}, q)]^2} p(q_i, t_i), \tag{E1}$$

occurs. In Eq. (E1), note that the boundary conditions are given by $B = \{q(t_f) = q_f, \dot{q}(t_f) = \dot{q}_f\}$, which involve both q and $\dot{q}$.

To check normalization, we consider Eq. (E1) for small δt, with $t_n = \delta t + t_{n-1}$:

$$p(q_{n+1},q_{n+2},t_{n+1}) = \int dq_n e^{-\delta t\{[(q_{n+2}-2q_n+q_{n+1})/\delta t]-f(q_{n+1},q_n)\}^2} p(q_n,q_{n+1},t_n). \tag{E2}$$

The norm of the probability distribution is found by performing the integral over the final variables q_{n+1}, q_{n+2}:

$$\int dq_n dq_{n+1} dq_{n+1} e^{-\delta t\{[(q_{n+2}-2q_n+q_{n+1})/\delta t]-f(q_{n+1},q_n)\}^2} p(q_n,q_{n+1},t_n). \tag{E3}$$

Equation (E3) defines a standard Gaussian integral over the q_{n+2} coordinate. Hence, the q_{n+2} integral eats the action up to a Gaussian normalization factor that we can calculate exactly, and we are left with

$$1 = \int dq_n dq_{n+1} \mathcal{N} p(q_n,q_{n+1},t_n), \tag{E4}$$

so we can simply absorb $\mathcal{N}$ into the measure, and the path integral will be normalized. If we were to include a q dependent diffusion coefficient $D_2(q,\dot{q})$ in Eq. (E1), then the Gaussian integral will be q dependent, and this will need to be included in the measure for $\mathcal{D}q$ [26]. The important point is that the higher-derivative terms in the classical path integral are standard Gaussian integrals if we consider q and $\dot{q}$ as independent variables. Hence, by including kinetic terms in the classical part of the action, we expect that the classical contribution to the path integral defined by Eq. (17) can be normalized to give conserved probabilities. We show this explicitly in Appendix E 3.

2. Normalization of Feynman-Vernon path integrals

Let us now consider a Feynman-Vernon quantum path integral with a decoherence term. Consider first the path integral for a quantum state σ,

$$\sigma(\phi_f^+,\phi_f^-,t_f) = \int^B \mathcal{D}\phi^+\mathcal{D}\phi^- e^{\int_{t_i}^{t_f} dt i[\dot{\phi}_+{}^2+V(\phi_+)]-i[\dot{\phi}_-{}^2+V(\phi_-)]-(D_0/2)[L(\phi_+)-L(\phi_-)]^2} \sigma(\phi_i^+,\phi_i^-,t_i), \tag{E5}$$

where B imposes the final state boundary conditions on the bra and ket fields, and $L(\phi)$ is an arbitrary operator of ϕ but not of its derivatives.

For Eq. (E5), it will prove insightful to show how the kinetic term enforces the normalization of the quantum state. To that end, consider the short-time version of Eq. (E6):

$$\begin{aligned}\sigma(\phi_{n+1}^+,\phi_{n+1}^-,t_f) = \int d\phi_n^+ d\phi_n^- & e^{\delta t\{i[(\phi_{n+1}^+-\phi_n^+)/\delta t]^2+iV(\phi_n^+)-i[(\phi_{n+1}^- -\phi_n^-)/\delta t]^2-iV(\phi_n^-)\}} \\ & \times e^{-\delta t(D_0/2)[L(\phi_n^+)-L(\phi_n^-)]^2}\sigma(\phi_n^+,\phi_n^-,t_n).\end{aligned} \tag{E6}$$

The trace of the quantum state is found by matching the $\phi_{n+1}^+ = \phi_{n+1}^- = \phi$ fields and integrating over ϕ:

$$\int d\phi_n^+ d\phi_n^- d\phi e^{(i/\delta t)\phi(\phi_n^+-\phi_n^-)} e^{i\delta t[V(\phi_n^+)-iV(\phi_n^-)]} e^{-\delta t(D_0/2)[L(\phi_n^+)-L(\phi_n^-)]^2}\sigma(\phi_n^+,\phi_n^-,t_n). \tag{E7}$$

Performing the integration over ϕ gives rise to a delta function $\delta(\phi_n^+-\phi_n^-)$. Hence, the quantum state is normalized to constant factors that can be absorbed.

However, had we included higher-order kinetic terms in the decoherence sector, we would not have found this normalization. In particular, if the decoherence term was instead

$$\int_{t_i}^{t_f} dt \frac{1}{2} D_0(\dot{\phi}_+^2 - \dot{\phi}_-^2)^2, \tag{E8}$$

then the delta function integral is not imposed, and the state is no longer normalized to constant factors.

As such, for the path integral to be normalized with higher-derivative decoherence terms, one needs to also add higher-derivative kinetic terms in the action. In this case, the action

$$S = \int_{t_i}^{t_f} dt i[\dot{\phi}_+{}^2 + \ddot{\phi}_+{}^2 - V(\phi_-)] - i[\dot{\phi}_-{}^2 + \ddot{\phi}_-{}^2 - V(\phi_-)] - \frac{1}{2}D_0(\dot{\phi}_+^2 - \dot{\phi}_-^2)^2 \tag{E9}$$

is normalized up to constant factors by the same argument, so long as we treat ϕ and $\dot{\phi}$ as independent variables to be specified in the quantum state; this is also argued for independent reasons in Ref. [52]. To see this, one does the short-time expansion, treating ϕ and $\dot{\phi}$ as independent variables as in the higher-derivative classical path integral. Computing the trace then sets $\phi^{\pm}_{n+2}$ equal to each other, as well as the setting the $\phi^{\pm}_{n+1}$ fields equal to be the same. The ϕ_{n+2} integral then enforces a delta function over $\delta(\phi^+_n - \phi^-_n)$, which kills the decoherence term and means that the path integral is normalized up to constant factors.

3. Normalization of CQ path integrals

In this section, we show that any CQ path integral with action

$$I[q,\phi^+,\phi^-] = \int dt i\dot{\phi}^2_+ + iV(\phi^+) - i\dot{\phi}^2_- - iV(\phi^-) - \frac{D_0(q,\dot{q},\phi^+)}{2}[\ddot{q} + f(q,\dot{q},\phi^+)]^2 - \frac{D_0(q,\dot{q},\phi^-)}{2}[\ddot{q} + f(q,\dot{q},\phi^-)]^2 \tag{E10}$$

is normalized up to constant factors when $D_0 > 0$. In the case where D_0 has a functional dependence on the fields, one must make sure to also include a factor of $\sqrt{\det[D_0(q,\dot{q},\phi)]}$ in the path integral measure. We further show that any higher-derivative action,

$$\begin{aligned} I[q,\phi^+,\phi^-] = \int dt i\ddot{\phi}^2_+ + iV(\phi^+,\dot{\phi}^+) - i\ddot{\phi}^2_- - iV(\phi^-,\dot{\phi}^-) - \frac{D_0(q,\dot{q},\phi^+,\dot{\phi}^+)}{2}[\ddot{q} + f(q,\dot{q},\phi^+,\dot{\phi}^+)]^2 \\ - \frac{D_0(q,\dot{q},\phi^-,\dot{\phi}^-)}{2}[\ddot{q} + f(q,\dot{q},\phi^-,\dot{\phi}^-)]^2, \end{aligned} \tag{E11}$$

is also normalized. Equations (E10) and (E11) are generic type of action one gets from varying Eq. (17) with a CQ protoaction that has second-order equations of motion for the classical degree of freedom.

The steps in showing Eq. (E11) follow in the same way as the discussions of classical and quantum path integrals. Firstly, because the action is higher derivative, the CQ state is specified through $\varrho(q,\dot{q},\phi^\pm,\dot{\phi}^\pm)$.

Taking the trace at the $t_{n+1} = t_n + \delta t$ time step therefore involves identifying $\phi^+_{n+2} = \phi^-_{n+2} = \phi_{n+2}$ and $\phi^+_{n+1} = \phi^-_{n+1} = \phi_{n+1}$. We then integrate over the ϕ_{n+2} and ϕ_{n+1} variables, as well as over the q_{n+2}, q_{n+1} classical degrees of freedom.

Let us first look at the higher-derivative quantum kinetic term. This can be expanded as

$$\begin{aligned} \ddot{\phi}^2_+ - \ddot{\phi}^2_- &\sim (\phi_{n+2} - 2\phi^+_n + \phi_{n+1})^2 \\ &\quad - (\phi_{n+2} - 2\phi^-_n + \phi_{n+1})^2 \\ &= 4[(\phi^+_n)^2 - (\phi^-_n)^2 + \phi_{n+2}(\phi^-_n - \phi^+_n) \\ &\quad + \phi_{n+1}(\phi^-_n - \phi^+_n)]. \end{aligned} \tag{E12}$$

Hence, integrating over ϕ_{n+2} gives a delta function in $\delta(\phi^-_n - \phi^+_n)$. As a consequence of this, all the bra and ket fields in the path integral are identified. We are therefore left with the action

$$I'[q,\phi] = -\int dt D_0(q,\dot{q},\phi,\dot{\phi})[\ddot{q} + f(q,\dot{q},\phi,\dot{\phi})]^2. \tag{E13}$$

Since all the bra and ket quantum fields are identified, normalization of Eq. (E11) is equivalent to ensuring that Eq. (E13) is normalized.

As we saw for the classical path integrals, integrating Eq. (E13) over the $\ddot{q}$ at second time step implements a standard Gaussian integral. If D_0 is dependent on the fields, we therefore pick up a term $\sqrt{\det[D_0(q,\dot{q},\phi)]}^{-1/2}$, which we must cancel in the measure by including a $\sqrt{\det(D_0(q,\dot{q},\phi)}$ term, as in Ref. [26]. It can also be exponentiated into the action by introducing bosonic and fermionic Faddeev-Poppov fields [76]. This determinant term commonly arises in the study of Fokker-Plank-type equations when the noise is multiplicative [28,76,77]. With this in mind, once we have integrated over $\ddot{q}$, the action vanishes and we are left with the normalization of the initial CQ state. Hence the path integral preserves the normalization of CQ states.

In a similar manner, we can also show that the path integral of Eq. (E10) preserves normalization. To see this, we first take the trace of the system, setting $\phi^+_{n+1} = \phi^-_{n+1} = \phi$. Integrating over ϕ then enforces a delta function $\delta(\phi^+ - \phi^-)$. We are then left with the action

$$I'[q,\phi] = -\int dt D_0(q,\dot{q},\phi)[\ddot{q} + f(q,\dot{q},\phi)]^2, \tag{E14}$$

and we can again perform the Gaussian integral over $\ddot{q}$ to arrive at a normalized path integral if $\sqrt{\det[D_0(q,\dot{q},\phi)]}$ is included in the measure.

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