Improving student understanding of fermionic and bosonic wave function

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[This paper is part of the Focused Collection in Investigating and Improving Quantum Education through Research.] We discuss how research on student difficulties was used as a guide to develop, validate, and evaluate a Quantum Interactive Learning Tutorial (QuILT) to help students learn how to determine the completely symmetric bosonic or completely antisymmetric fermionic wave function and be able to compare and contrast them from the case when the particles can be treated as distinguishable. We discuss how explicit scaffolding is designed via guided teaching-learning sequences for two- or three-particle bosonic and fermionic systems to help students develop intuition about how to construct completely symmetric and antisymmetric wave function, both when spin part of the wave function is ignored and when both spatial and spin degrees of freedom are included.

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I. INTRODUCTION

Quantum mechanics (QM) instruction presents many challenges even at the upper level partly due to the paradigm shift from the classical to the quantum mechanical framework [1–4]. Previous investigations point to the fact that advanced students struggle while learning QM [3,4] and research-based pedagogies and learning tools can help students learn quantum concepts better [5–9]. Some prior studies have focused on conceptual surveys for gauging student understanding of quantum concepts [10–16]. Some researchers have investigated epistemological issues in learning QM or students' confidence level [17,18] while other investigations have emphasized the importance of representations commonly used in QM [19–28]. Some researchers have focused on visualizing quantum concepts [29–34] and learning QM via games [35]. Investigations have also focused on attitudes regarding teaching QM and grading student work [36,37]. Many research studies have focused on quantum measurement [38–43], relative phase in quantum states, probability distributions for measuring different outcomes, what the possible outcomes are as well as expectation values of observables and their time dependence [23,44–49]. Researchers have also investigated student understanding of the addition of angular momentum [50,51] as well as different quantum concepts in the context of quantum experiments [32,52–60]. Other research studies have focused on cognitive issues in diverse contexts [61–69]. Some researchers have investigated issues related to bound and scattering states [70–72] as well as other advanced topics [73–80]. In some preceding investigations, researchers have developed pedagogical approaches to improve student learning. Furthermore, the quantum information revolution and the potential of quantum technologies have led to efforts related to preparing students adequately for the workforce needs [81–89].

Prior research on identical particles has only focused on basics, and the number of distinct many-particle states [77,78,90]. However, except for a conference proceedings describing some of the student difficulties [91], no previous research has specifically focused on approaches to improving student understanding of many-particle wave function for systems of two or three fermions or bosons to help students learn to construct completely symmetric and antisymmetric wave functions and be able to describe how they compare and contrast with the case in which particles are treated as distinguishable. This study describes a Quantum Interactive Learning Tutorial (QuILT) designed to help upper-level undergraduate students learn these concepts both when spin is ignored and when it is included while constructing the many-particle wave functions for

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fermions and bosons. We note that the focus of the QuILT is on physics undergraduate students (not graduate students), who are almost always introduced to the many-particle systems in QM in the first quantized form. However, we were able to collect data from graduate students to compare the undergraduate students' performance with theirs. We also note that even though the first quantized notation for constructing the many-particle wave function is only convenient for few-particle cases, it is pedagogical for students to learn to construct the completely symmetric or antisymmetric wave functions for few bosons or fermions with or without spin and consider consequences of symmetrization, e.g., its connection with Pauli's exclusion principle, ground state, and first excited state of fermionic and bosonic systems, number of possible many-particle states for fermions, bosons and distinguishable particles with given constraints. Although second quantization is typically used for identical particles in physics graduate courses after a brief discussion of first quantized notation, it is not introduced in a majority of physics undergraduate courses, which this QuILT targets. Also, in addition to developing a general intuition about the consequences of symmetrization, the few-particle cases students engage with in the QuILT can be instructive both for bound-state problems such as formation of a hydrogen molecule and helium atom as well as for scattering problems.

The QuILT on identical particles focuses only on cases in which the particles are confined to one spatial dimension for simplicity and for reducing students' cognitive load. We also note that since most physics undergraduate textbooks keep the treatment of identical particles to the noninteracting case to help students focus on the consequences of the symmetrization requirements of the wave functions, we use this simplification in the QuILT even though the fermionic and bosonic wave functions would be valid even when particles interact with each other. Moreover, single-particle stationary states are one of the easiest basis sets obtained by solving the time-independent Schroedinger equation, and instructors often focus on helping students learn to calculate the ground state or first excited state wave function for few particle systems as well as the number of available many-particle states with different constraints on total energy of the system. Therefore, the QuILT asks students to only consider basis states that are energy eigenstates consistent with many undergraduate textbooks, even though the construction of the many-particle wave function students learn in the QuILT is valid for any complete set of single-particle basis states chosen. We also note that the completely antisymmetric or symmetric many-particle states for fermions and bosons are valid states for distinguishable particles since any linear superposition is a valid state for distinguishable particles although the focus of the QuILT is on helping students construct many-particle basis states. We also note that the QuILT helps students learn to explicitly consider the different permutations of the single-particle wave functions to construct the many-particle wave functions and students use Slater determinant for the fermionic case later once they have developed intuition about how to keep track of the signs of various terms by doing the permutations without using the Slater determinant. We also note that the set of QuILTs on identical particles (called the identical particles QuILT for convenience) focuses on different issues and the focus here is on helping students with many-particle wave functions by explicitly learning to construct few-particle wave functions for fermions and bosons and be able to contrast them with the case for distinguishable particles. We also note that throughout most of the QuILT, students compare the fermionic and bosonic cases with what is called the hypothetical case for identical particles that can be treated as distinguishable to help them develop intuition about how the fermionic and bosonic cases are different from this hypothetical case. However, at the very end of the QuILT, students engage in teaching-learning sequences that help them reflect, e.g., on the fact that if the wave functions of two fermions or bosons did not overlap, they can be treated as distinguishable (there is no need for symmetrization). Also, in the spirit of identical particles, when students consider the case of distinguishable particles, they only consider cases in which the particles are of the same type. Also, since all single-particle basis states are normalized, the QuILT helps students learn that the normalization constants for bosonic and fermionic wave functions depend on the number of terms in the symmetrized wave functions.

Similar to other QuILTs, the research and development of identical particles QuILT were informed by Vygotsky's zone of proximal development framework (ZPD) [92] and has been described earlier, e.g., see Ref. [90]. The zone of proximal development (ZPD) represents the gap between what a learner can accomplish independently and what they can achieve with expert guidance and peer collaboration. This framework highlights that meaningful learning occurs when activities are situated within a learner's ZPD, which changes and expands as their expertise grows. Consequently, effective instruction should align with students' existing knowledge and scaffold new learning to build on this foundation progressively. Engaging students in carefully designed instructional tasks—such as the guided inquiry-based sequences in the QuILT—can expand their ZPD and foster expertise in QM. To effectively integrate students' initial knowledge into instructional design, it is essential to identify their common difficulties with relevant concepts. By identifying prevalent student challenges, we designed the QuILT to include scaffolding that supports students and engages them in activities within their ZPD, to ensure meaningful learning experiences.

The background and methodology for the investigation of difficulties and development, validation, and in class implementation of the QuILT are similar to Refs. [77,78,91] and described in Supplementary Material (see

Supplementary Material [93] with this paper). The Supplementary Material also include the pretest and post-test questions posed to students after lecture-based instruction and after engaging with the QuILT. Also, below, we mainly discuss difficulties students had when spin degree of freedom is ignored and other difficulties are discussed in a conference proceedings [91]. All notations are standard.

II. LEARNING OBJECTIVES ASSOCIATED WITH MANY-PARTICLE WAVE FUNCTIONS

After working through the QuILT, students should be able to do the following both for cases in which spin is ignored and when spin degree of freedom is included:

- Identify the symmetrization requirements for wave function for a system of identical bosons and fermions.
- For a system of two or three fermions, bosons, or distinguishable particles, write the properly normalized many-particle wave function, such that (a) all of the particles are in different single-particle states and (b) at least two of the particles are in the same single-particle states. If a valid three-particle wave function is not possible for the given system, students should be able to explain the reason.
- Write an expression for the many-particle ground state and first-excited state wave function for a system of identical particles, e.g., in a one-dimensional infinite square well or simple harmonic oscillator well in terms of single-particle wave functions.
- Write an expression for the ground state and first-excited state energies for a system of identical particles, e.g., in the context of a one-dimensional infinite square well or simple harmonic oscillator well.

III. STUDENT DIFFICULTIES WHEN IGNORING SPIN DEGREE OF FREEDOM

All questions are included in the Supplementary Material [93]. Ignoring spin degree of freedom, the pre- and post-tests ask students to write down three-particle wave functions for fermionic and bosonic systems as well as for distinguishable particles such that in Q1, all particles are in different single-particle states and in Q2, two particles are in the same state.

One possible answer to Q1 for fermionic (Ψ_F), bosonic (Ψ_B), and distinguishable (Ψ_D) particles (with $n_1 \neq n_2 \neq n_3$) using standard notation for single-particle basis states is

$$\Psi_F(x_1,x_2,x_3) = \frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3) + \psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3) + \psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) - \psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)].$$

$$\Psi_B(x_1,x_2,x_3) = \frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) + \psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3) + \psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3) + \psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) + \psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)].$$

$$\Psi_D(x_1,x_2,x_3) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3).$$

In Q2, it is not possible for two fermions to occupy the same single-particle state ψ_{n_1} but a possible bosonic and distinguishable particle wave function (with $n_1 \neq n_2$) satisfying these conditions is

$$\Psi_B(x_1,x_2,x_3) = \frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)].$$

$$\Psi_D(x_1,x_2,x_3) = \psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)$$

Q3 and Q4 related to bosonic systems discussed later in this section were administered as clicker questions in class. Many students struggled with concepts related to many-particle wave functions for a system of indistinguishable bosons and fermions. For example, Table I shows that less than 55% of the students were able to generate the many-particle wave functions in questions Q1 and Q2 correctly on the pretest after traditional lecture-based instruction. Written responses and interviews indicate that students face several underlying challenges that hinder their ability to construct fully symmetric or antisymmetric many-particle wave functions for systems of indistinguishable particles. Below, we discuss some of these difficulties. Also, although every completely antisymmetric fermionic wave function or completely symmetric bosonic wave function is also a possible wave function for distinguishable particles (since all linear superposition of basis states are possible

TABLE I. The percentages of graduate ($N = 30$) and undergraduate ($N = 25$) students who correctly answered questions Q1 and Q2 for the given system of fermions, bosons, or distinguishable particles on pretest (after traditional lecture-based instruction in relevant concepts).

Type of particle	Graduate (%)	Undergraduate (%)
Fermions (Q1)	33	24
Bosons (Q1)	37	28
Distinguishable (Q1)	40	40
Fermions (Q2)	40	44
Bosons (Q2)	30	16
Distinguishable (Q2)	53	36

wave functions), the focus of the QuILT is on helping students construct many-particle basis states.

A. Difficulty accounting for the indistinguishability of the particles

There is no measurement that can distinguish one fermion from another fermion in a system consisting of fermions of the same type (e.g., electrons). The same is true for a system of identical bosons. Below, we discuss difficulties students had with writing a wave function or correctly identifying a wave function that accounts for the indistinguishability of identical fermions and bosons when spin is ignored.

Using the same coordinate for each particle in the many-particle wave function: Some students did not realize that each identical particle must have a unique coordinate and incorrectly generated a many-particle wave function in which the particles shared the same coordinate. For example, in Q1, for both systems of indistinguishable fermions or bosons, one student wrote $\psi_1(x)\psi_2(x)\psi_3(x)$. Some students argued that it is impossible to assign a unique coordinate to each indistinguishable particle. For example, one interviewed student stated, "we must use the coordinate x for all the indistinguishable particles since we don't know where each particle is." Besides constructing many-particle wave functions that failed to meet symmetrization requirements, students with this type of response also overlooked the need for each particle to have its own distinct coordinate. We note that these students, who expressed the wave function as the product of three separate wave functions but used the same coordinate as the argument for all three, were quite close to grasping the concept correctly. However, they became confused by the notational aspect of the problem. As a result, their responses are not entirely incorrect. In particular, they are demonstrating some valid intuition that just needs a slight adjustment to be fully accurate.

Claiming that the wave function for a system of indistinguishable particles is the same as that for a system of distinguishable particles since the particles are noninteracting: Some students provided the same answer to Q1 and Q2 for a system of indistinguishable particles and a system of the same type of particles that can be treated as distinguishable. Table II summarizes the percentages of students who wrote the same many-particle wave function for a system of identical fermions or bosons as they wrote for a system of distinguishable particles.

TABLE II. The percentages of graduate ($N = 30$) and undergraduate ($N = 25$) students who wrote the same many-particle wave function for the given system of identical particles as for a system of distinguishable particles for Q1 and Q2 on the pretest.

Question	Type of particle	Graduate (%)	Undergraduate (%)
Q1	Fermions	20	24
	Bosons	20	20
Q2	Fermions	3	16
	Bosons	20	20

One interviewed student jotted down the same many-particle wave function for a system of indistinguishable fermions, bosons, and distinguishable particle. When answering Q1 for a system of bosons, this student stated "I don't… understand how distinguishability would change the wave function if the particles are noninteracting." This student focused on the fact that the system of identical particles was noninteracting and incorrectly assumed that all systems of noninteracting particles have the same wave function. Individual discussions with some students suggest that at least some students with this type of difficulty sometimes had difficulty differentiating between "noninteracting particles" vs "nonoverlapping wave functions." In particular, the fact that when the wave functions of different particles do not overlap, the particles can be considered distinguishable was mistaken or overgeneralized to "noninteracting" particles being distinguishable.

Not realizing that in some situations, a system of indistinguishable bosons could have the same many-particle wave function as a system of particles that can be treated as distinguishable: It is possible for all bosons in a system of indistinguishable bosons to occupy the same single-particle state. For such a system of indistinguishable bosons, the many-particle wave function would be the same as that of a system of distinguishable particles. For example, if there are three indistinguishable bosons in the state ψ_{n_1}, the many-particle wave function is $\Psi(x_1, x_2, x_3) = \psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$. However, some students claimed that this could not be the many-particle wave function for a system of indistinguishable bosons since it was the same wave function as that for a system of distinguishable particles.

Overgeneralizing the case when a system of indistinguishable bosons has the same many-particle wave function as distinguishable particles: Many students correctly identified that if all the bosons are in the same

single-particle state, the many-particle wave function is the same as that of distinguishable particles. However, some students overgeneralized this case to incorrectly claim that the wave function for a system of indistinguishable bosons is always the same as a system of distinguishable particles and there is no need to symmetrize the bosonic wave function. Some of these same students claimed that one needs to only treat a system of identical particles as indistinguishable and worry about appropriately antisymmetrizing the wave function if it is a system of identical fermions. Students with this difficulty did not realize that a system of identical bosons must always have a completely symmetric many-particle wave function, although for the case when all the bosons are in the same single-particle state, the bosonic wave function is the same as that for distinguishable particles.

B. Difficulty realizing that no wave function exists for a system in which two or more fermions occupy the same single-particle state

In Q2(a), students were asked to construct the many-particle wave function for a system of three fermions in which two of the fermions occupy the same single-particle state or state the reason for why such a wave function does not exist. Below, we discuss the difficulties students had with identifying what type of particles obey Pauli's exclusion principle, applying the Pauli's exclusion principle correctly, or not making connections between Pauli's exclusion principle and the completely antisymmetric many-particle wave function for a system of identical fermions.

Not realizing Pauli's exclusion principle applies to all fermions: Some students knew that the Pauli's exclusion principle applied to certain types of particles (e.g., electrons) and forbids two particles from occupying the same single-particle state but did not know it applies to all fermions. For example, when answering Q2 for a system of indistinguishable fermions, one interviewed student initially hesitated briefly then stated, "I know Pauli's exclusion principle applies to two electrons in the same state, but does it apply to all (of the different types of) fermions? I will say it is not possible (for two fermions to be in the same single-particle state), but I don't know if that's true for all fermions." While this student correctly answered that it is not possible to write a many-particle wave function for a system in which two electrons are in the same single-particle state, he was unsure about the fact that the Pauli's exclusion principle applies to all fermions (electrons being one type of fermion).

Not applying Pauli's exclusion principle correctly: In Q2(a), students are asked to write the many-particle wave function for a system of three indistinguishable fermions in which two are in the same single-particle state. Some students struggled to apply the Pauli's exclusion principle correctly. In particular, some students had difficulty realizing that even two out of three fermions cannot be in the same single-particle state (ignoring the spin degrees of freedom), and that no wave function exists for this system. For example, one student did not provide a wave function in Q2(a) but instead stated: "…But it is not possible (to write a many-particle wave function) for all three (fermions) to be in the same (single-particle) state." Further discussion suggests that this student incorrectly reasoned that a system in which all the fermions are in the same single-particle state is the only case that is prohibited but did not realize that the Pauli's exclusion principle forbids any many-particle system in which *two* or more fermions occupy the same single-particle state as in Q2. He struggled to realize that the Pauli's exclusion principle applies to the system in Q2(a) and can be used as a justification as to why it is not possible to write the many-particle wave function for a system in which two fermions are in the same single-particle state.

Difficulty connecting the Pauli's exclusion principle to fermionic wave function: During the interview, students were asked questions concerning the possibility of two or more fermions occupying the same single-particle state. Some of the interviewed students quickly stated that two or more fermions could not occupy the same single-particle state because of the Pauli's exclusion principle. Additionally, students were asked if a possible fermionic wave function could be such that two fermions are in the same single-particle state. Again, most of the interviewed students eliminated the wave functions in which two or more fermions were in the same single-particle state as possible many-particle wave functions. For example, if one attempts to construct a system of two indistinguishable fermions in the same single-particle state ψ_i, the resulting two-particle wave function is $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_i(x_1)\psi_i(x_2) - \psi_i(x_2)\psi_i(x_1)] = 0$. Thus, there is no wave function for the system, which is consistent with Pauli's exclusion principle statement that no two fermions can be in the same quantum state. However, when explicitly asked if Pauli's exclusion principle is consistent with a completely antisymmetric many-particle wave function, some students struggled in making any connections between the two. For example, one interviewed student said "the Pauli exclusion principle says that we can't have two fermions, or more (than two fermions) in the same state. Fermions must have an antisymmetric wave function. But I don't really see how those two things are related to one another." This student viewed the Pauli's exclusion principle and completely antisymmetrizing the fermionic wave function as two disjointed facts. He did not realize that a system with a completely antisymmetric wave function in which two or more fermions are in the same single-particle state, producing a wave function equal to zero (i.e., there is no wave function for such a system), and Pauli's exclusion principle, which states that no two fermions can be in the same single-particle state, are intimately connected.

Other students were able to apply the Pauli's exclusion principle in one context but then failed to apply it in a different situation. In one question in the interview, some of the interviewed students correctly stated in response to one question that no two fermions can be in the same single-particle state due to Pauli's exclusion principle. However, these same students later attempted to generate a completely antisymmetric many-particle wave function for fermions in question Q2 (which asks about a system in which two of the fermions are in the same single-particle state) as opposed to stating that such a wave function is not possible. In Q2, some of these students went through the procedure to generate terms in the many-particle wave function by permuting the labels for either the states or the coordinates, but did not reflect upon the fact that the many-particle wave function that they generated was zero, and thus not a possible many-particle wave function. Such context dependence of student responses has also been found in introductory physics and suggests that students are not experts in these concepts and their expertise is evolving [94–96].

C. Difficulty with the symmetrization requirement

Table III shows the percentages of students who provided the same response to Q1 and Q2 in some cases. Below, we discuss two types of difficulties students had with the symmetrization requirement.

Not realizing that the wave function for fermions or bosons must obey a symmetrization requirement: Some students struggled to identify the symmetrization requirement for identical bosons and fermions or could not recognize whether a given wave function was appropriately symmetrized. For example, in Q1, 16% of the undergraduates and 10% of the graduate students after lecture-based instruction incorrectly stated the following:

$$\Psi(x_1, x_2, x_3) = \psi_{n_1}(x_1) + \psi_{n_2}(x_2) + \psi_{n_3}(x_3)$$

for at least one identical particle system and some gave this answer for all three systems.

Other students constructed a many-particle wave function that was neither symmetric nor antisymmetric, e.g., some wrote the following three-particle wave function for both indistinguishable fermions and bosons for question Q1:

$$\Psi(x_1, x_2, x_3) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3).$$

This three-particle wave function is neither symmetric nor antisymmetric. Table IV summarizes the percentages of responses in Q1 that were neither completely symmetric nor completely antisymmetric. Some of these same students also struggled to make a distinction between the wave function for a system of indistinguishable bosons and distinguishable particles.

Additionally, question Q3, asked as an in class clicker question, focuses on three identical spinless bosons and whether students agree with option (1) that says that the ground state wave function is $\psi_1(x_1)\psi_1(x_2)\psi_1(x_3)$, option (2) that says $\psi_1(x_1)\psi_1(x_2)\psi_2(x_3)$ is a first excited state of the three particle system and option (3) that says that the degeneracy of the first excited state is 3. We note that students had learned about degeneracy. Only option (1) is correct for Q3. Since the first excited state of the three-boson system must be completely symmetric, option (2) is incorrect. In particular, the three-boson first-excited state wave function consists of three terms. However, option (2) only includes one term in the wave function corresponding to the first-excited state. Option (3) is incorrect in Q3 because the degeneracy of the first-excited state is one since $\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{3}}[\psi_1(x_1)\psi_1(x_2)\psi_2(x_3) + \psi_1(x_1)\psi_2(x_2)\psi_1(x_3) + \psi_2(x_1)\psi_1(x_2)\psi_1(x_3)]$ is the only state with the first-excited energy. A correct answer (which is not correct) was selected by 71% of the undergraduate students who chose the wave function $\psi_1(x_1)\psi_1(x_2)\psi_2(x_3)$ as the first excited state of the three-particle system [option (2)] when answering individually. These students were then given the opportunity to discuss Q3 with their peers and answer the question again. This peer discussion did not help students identify option (2) as incorrect as 71% of the students still selected option (2) as correct after peer discussion. We note that since the student performance individually after traditional instruction is poor, the benefits of peer discussion are unlikely to be realized [97]. In particular, peer discussion is most beneficial when individual scores are closer to 50% correct, and it is important

TABLE III. The percentages of graduate ($N = 30$) and undergraduate ($N = 25$) students who wrote the same many-particle wave function for a system of indistinguishable fermions as for a system of indistinguishable bosons for Q1 and Q2 on the pretest.

Question	Graduate (%)	Undergraduate (%)
Q1	20	28
Q2	7	24

TABLE IV. The percentages of graduate ($N = 30$) and undergraduate ($N = 25$) students who wrote a many-particle wave function that was neither completely symmetric nor completely antisymmetric for a system of three indistinguishable fermions or bosons for Q1 on the pretest.

Question	Type of particle	Graduate (%)	Undergraduate (%)
Q1	Fermions	40	56
	Bosons	23	48

to provide additional support and hints to help students if the initial individual responses show very low performance [97]. Thus, students struggled with the concept that the first excited state wave function for a system of three identical spinless bosons must be symmetrized. Interviews suggest that students are more likely to overlook the symmetrization requirement for the first-excited state as compared to the ground state. This may partly be an issue with the cognitive load of having to consider a number of concepts for the first-excited state, such as determining the single-particle states that yield the correct energy for the many-particle first-excited state, considering the type of identical particle and identifying the appropriate symmetrization requirement. Since these students are still developing expertise in quantum mechanics, they may not have enough cognitive resources to consider each of these different aspects and coordinate them appropriately while solving problems.

Difficulty identifying the correct symmetrization requirement: Some students were unable to correctly identify which type of symmetrization requirement corresponded to each type of particle. In Q1, which explicitly asks students to ignore the spin degree of freedom using boldface letters, 25% of the undergraduates and 23% of the graduate students wrote a completely symmetric wave function for a system of identical fermions. For example, one student wrote the following as the many-particle wave function for a system of three fermions in Q1: $\psi_1(x_1)\psi_2(x_2)\psi_3(x_3)+\psi_1(x_2)\psi_2(x_3)\psi_3(x_1)+\psi_1(x_3)\psi_2(x_1)\psi_3(x_2)$. This student had difficulty writing all of the terms of the wave function and also did not antisymmetrize the wave function for the fermions. Similar difficulties were found for bosons. For instance, Q4 posed as a clicker question had three true or false options with option (1) being that bosonic spin must be an integer, option (2) being that the overall bosonic wave function is antisymmetric, and option (3) being that bosons cannot be in same single-particle state. Among the undergraduate students, 43% incorrectly chose the option indicating that the overall wave function for bosons could be antisymmetric [option (2)]. After engaging in discussions with peers, 31% still mistakenly selected option (2) as correct. This suggests that while these students may have understood that identical particles (whether bosons or fermions) require a symmetrization rule, they were unable to accurately distinguish which rule applies to which type of particle.

D. Difficulty generating a completely symmetric/antisymmetric wave function

A number of students were able to identify the many-particle wave function but some of them had difficulty generating bosonic and fermionic wave functions.

Claiming that single-particle wave functions in their product used to construct basis states for many-particle wave functions do not commute: Some students struggled to realize that the order in which the single-particle wave functions are written in the product of the single-particle states (used to construct the many-particle basis states) is irrelevant. For example, the following are all equivalent ways to express one of the basis states for a system of three noninteracting identical particles: $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$, $\psi_{n_1}(x_1)\psi_{n_3}(x_3)\psi_{n_2}(x_2)$, $\psi_{n_2}(x_2)\psi_{n_1}(x_1)\psi_{n_3}(x_3)$, $\psi_{n_2}(x_2)\psi_{n_3}(x_3)\psi_{n_1}(x_1)$, $\psi_{n_3}(x_3)\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, and $\psi_{n_3}(x_3)\psi_{n_2}(x_2)\psi_{n_1}(x_1)$. Some students focused on the order in which the labels for the single-particle states or the coordinates appeared to determine whether the products of the single-particle wave function were different. For example, when comparing the terms $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$ and $\psi_{n_1}(x_1)\psi_{n_3}(x_3)\psi_{n_2}(x_2)$, students with this type of difficulty claimed that n_2 and/or x_2 appear in the second term in the first product and in the third term in the second product, so these must be different basis states for the many-particle wave function. For each of the terms $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$, $\psi_{n_1}(x_1)\psi_{n_3}(x_3)\psi_{n_2}(x_2)$, particle 1 denoted by the coordinate x_1 is in the state ψ_{n_1}, particle 2 denoted by the coordinate x_2 is in the state ψ_{n_2}, and particle 3 denoted by the coordinate x_3 is in the state ψ_{n_3}. Thus, the terms $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$ and $\psi_{n_1}(x_1)\psi_{n_3}(x_3)\psi_{n_2}(x_2)$ are equivalent and do not represent distinctly different many-particle states. Some students, who struggled to realize that the order of the single-particle wave function in a basis state in the product space is not important, had difficulty generating a wave function with correct number of terms and determining the normalization constant. For example, some students with this type of difficulty claimed that bosonic system with all bosons in the same single-particle state is

$$\Psi(x_1,x_2,x_3)=\frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)+\psi_{n_1}(x_2)\psi_{n_1}(x_3)\psi_{n_1}(x_1)+\psi_{n_1}(x_3)\psi_{n_1}(x_1)\psi_{n_1}(x_2)] \quad (1)$$

or

$$\Psi(x_1,x_2,x_3)=\frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)+\psi_{n_1}(x_1)\psi_{n_1}(x_3)\psi_{n_1}(x_2)+\psi_{n_1}(x_2)\psi_{n_1}(x_1)\psi_{n_1}(x_3)+\psi_{n_1}(x_2)\psi_{n_1}(x_3)\psi_{n_1}(x_1)+\psi_{n_1}(x_3)\psi_{n_1}(x_1)\psi_{n_1}(x_2)+\psi_{n_1}(x_3)\psi_{n_1}(x_2)\psi_{n_1}(x_1)]. \quad (2)$$

They struggled to realize that all these terms in the sum of both these expressions are equivalent and can be simplified to a single term $\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$. Additionally, they struggled to determine the correct normalization constant. For example, the expression

$$\Psi(x_1,x_2,x_3)=\frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)+\psi_{n_1}(x_2)\psi_{n_1}(x_3)\psi_{n_1}(x_1)+\psi_{n_1}(x_3)\psi_{n_1}(x_1)\psi_{n_1}(x_2)] \quad (3)$$

reduces to $\sqrt{3}\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$ which is not the properly normalized wave function. However, we note that this type of error is a minor error.

Omitting terms: A common mistake when generating many-particle wave functions was omitting at least one term in the wave function and thus not producing a completely symmetric or antisymmetric wave function. This also created an obstacle when students determined the normalization constant for the many-particle wave function. For example, in Q1 for a system of fermions, one student wrote $|\psi\rangle=\frac{1}{\sqrt{3}}[|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle-|\psi_3\rangle|\psi_2\rangle|\psi_1\rangle-|\psi_2\rangle|\psi_3\rangle|\psi_1\rangle-|\psi_3\rangle|\psi_1\rangle|\psi_2\rangle]$. There are several mistakes in this student's response, but here we point out that the student omitted two of the terms in the many-particle wave function and the normalization constant is incorrect. In particular, this student's normalization constant is not the correct value of $\frac{1}{\sqrt{6}}$, but it is also not consistent with the number of terms he generated.

Making a sign error in at least one of the terms of the many-particle wave function for fermions: Another common mistake was writing a many-particle wave function for a system of indistinguishable fermions with the correct number of terms but making a mistake with the sign of at least one term. For example, in question Q1, one student wrote the three-particle wave function for a system of three fermions in which all the fermions are in different single-particle states as

$$\Psi(x_1,x_2,x_3)=\frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)-\psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3)\oplus\psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3)\ominus\psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3)+\psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)-\psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)]. \quad (4)$$

In this expression, the circled signs of the third and fourth terms are incorrect resulting in a wave function that is not antisymmetric. This is a common mistake that students made when generating terms in the many-particle wave function by starting with one term and then generating all the other terms by interchanging either the labels for the states (n_1, n_2, or n_3) or the labels for the coordinates (x_1, x_2, or x_3). For a system of indistinguishable fermions, each time there is an interchange of labels, the new term is multiplied by a minus sign. Thus, starting with the term $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$, all the odd permutations of the labels for the states (or coordinates) yield a term with a minus sign and all the even permutations of the labels yield a term with a plus sign. Once all the permutations are determined, they are added together to produce the many-particle wave function. Some interviewed students made mistakes when generating a new term by not multiplying every term by a -1 when exchanging two labels. They made a mistake with the sign of at least one of the terms by not carefully keeping track of the sign of the previous term or not multiplying the new term by a -1.

Incorrectly switching both the labels for the coordinates and the states: When determining the many-particle wave function for fermions or bosons starting from one product space basis state, one is free to interchange either the labels for the states (n_1, n_2, or n_3) or the labels for the coordinates (x_1, x_2, or x_3) until all the permutations are obtained. However, interchanging both the labels for the states and the coordinates will result in an incorrect many-particle wave function. Some students began to find the many-particle wave function by switching the labels for the coordinates. After finding several permutations, they interchanged the labels for the single-particle states which led to determining an incorrect many-particle wave function. They did not check their wave function to make sure it was correct with regard to exchange of any two particles.

Incorrectly applying a sign heuristic to determine a symmetric/antisymmetric wave function: Some students noted that if the signs of all terms are positive, the many-particle wave function constructed is symmetric and if there was at least one term with a negative sign, it is antisymmetric. One interviewed student who incorrectly applied this sign heuristic claimed that every term in the many-particle wave function is negative with the exception of the starting term and every term generated by permuting the labels is negative for fermions in Q1. For example, in Q1 for fermions, this student wrote $\psi_1(x_1)\psi_2(x_2)\psi_3(x_3)-\psi_1(x_2)\psi_2(x_3)\psi_3(x_1)-\psi_1(x_3)\psi_2(x_1)\psi_3(x_2)$. In addition to not generating all the terms of the many-particle wave function, he incorrectly subtracted each term found by permuting the labels of the previous term. Here the student generated all the even permutations for the coordinates, and all the terms listed should have been added while all the

terms representing the odd permutations of the coordinates are missing. This student and others with this type of difficulty struggled to realize that each new term generated by permuting the labels should have the opposite sign of the term used to find the new term by exchanging two of the labels.

When answering Q1 for a system of indistinguishable fermions, one interviewed student said he would use the alternating thing they learned. This student proceeded to jot down the following wave function:

$$\Psi(x_1,x_2,x_3)=\frac{1}{\sqrt{3!}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1) + \psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2)]. \quad (5)$$

Here the wave function does not contain all six terms, but this student also produced only terms that correspond to the even permutations of the coordinates x_1, x_2, and x_3. All the even permutations should have a positive sign, while the terms in the many-particle wave function that correspond to the odd permutations of the coordinates x_1, x_2, and x_3 should have a negative sign. This student was simply alternating the sign for each product of the single-particle wave functions rather than using a systematic reasoning to determine whether each product corresponded to an even or odd permutation of the labels.

E. Difficulty using the Slater determinant to write the many-particle wave function for a system of indistinguishable fermions

One method to help students write the completely antisymmetric wave function for a system of indistinguishable fermions is using the "Slater determinant." The method of the Slater determinant is shown below for a system of three indistinguishable fermions in the states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}.

$$\frac{1}{\sqrt{6}}\begin{vmatrix}\psi_{n_1}(x_1) & \psi_{n_2}(x_1) & \psi_{n_3}(x_1)\\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) & \psi_{n_3}(x_2)\\ \psi_{n_1}(x_3) & \psi_{n_2}(x_3) & \psi_{n_3}(x_3)\end{vmatrix} = \frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3) + \psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3) + \psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) - \psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)]. \quad (6)$$

Students had difficulty writing the Slater determinant and using it properly to determine the many-particle wave function. In order to produce the completely antisymmetric fermionic wave function, one must keep the labels for the states fixed for a given row and the labels for the coordinates fixed for a given column (or vice versa) and ensure that each label appears along one column or row. However, some students changed both the labels for the states and the labels for the coordinates in the rows/ or columns of the matrix. For example, one student wrote the following when determining the three-particle wave function for fermions in Q1 after traditional instruction in relevant concepts:

$$\frac{1}{\sqrt{6}}\begin{vmatrix}\psi_{n_1}(x_1) & \psi_{n_2}(x_2) & \psi_{n_3}(x_3)\\ \psi_{n_2}(x_2) & \psi_{n_3}(x_3) & \psi_{n_1}(x_1)\\ \psi_{n_3}(x_3) & \psi_{n_1}(x_1) & \psi_{n_2}(x_2)\end{vmatrix}.$$

The above determinant produces terms such as $\psi_{n_1}(x_1)\psi_{n_1}(x_1)\psi_{n_1}(x_1)$, $\psi_{n_2}(x_2)\psi_{n_2}(x_2)\psi_{n_2}(x_2)$, and $\psi_{n_3}(x_3)\psi_{n_3}(x_3)\psi_{n_3}(x_3)$ which have the same label x_1, x_2, or x_3 in all terms in the product of the single-particle states which cannot be a basis state for the product space of the three fermions.

Other students did not change either the label for the state or the coordinate for a given row or column. For example, one student wrote the following for a system of fermions for Q1 after traditional instruction in relevant concepts:

$$\frac{1}{\sqrt{6}}\begin{vmatrix}\psi_{n_1}(x_1) & \psi_{n_1}(x_1) & \psi_{n_1}(x_1)\\ \psi_{n_2}(x_2) & \psi_{n_2}(x_2) & \psi_{n_2}(x_2)\\ \psi_{n_3}(x_3) & \psi_{n_3}(x_3) & \psi_{n_3}(x_3)\end{vmatrix}.$$

This student left the expression with the determinant as his final answer and did not expand the determinant to produce all the terms of the three-particle wave function. However, since the determinant of a matrix in which any rows or columns are identical is equal to zero, the determinant expression written above produces no wave function. This student and others with similar difficulty failed to identify issues involved with setting up the Slater determinant and wrote a determinant in which two or more rows/columns are identical resulting in a determinant of zero and no wave function.

IV. ADDRESSING STUDENT DIFFICULTIES WITH MANY-PARTICLE WAVE FUNCTIONS IGNORING SPIN

The QuILT helps students focus on bosonic and fermionic wave functions commensurate with the learning objectives. In particular, the QuILT strives to help students (1) learn the symmetrization requirements for wave function for a system of indistinguishable fermions and bosons, (2) account for the indistinguishability of the particles by symmetrizing the many-particle wave function, and (3) construct the wave function. Below, we first summarize the case when spin degree of freedom is ignored.

Helping students realize that the many-particle wave function for a system of indistinguishable particles must obey a symmetrization requirement for both bosonic and fermionic systems: In the QuILT, students work with different systems of identical particles and receive guided support to bridge the gap between their initial ideas and the correct reasoning. For example, they construct the wave function for a system of two identical fermions and then expand their understanding to a system of three identical fermions. Students are asked to determine the many-particle wave function for the system, if possible, and explain their reasoning if it is not possible. After constructing the wave function, they are asked to reflect upon the following conversation in which three hypothetical students discuss these issues and explain their reasoning for agreeing or disagreeing with them.

Student 1: For a system of two noninteracting indistinguishable fermions, the wave function describing the system is $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, in which $\psi_{n_1}(x_1)$ and $\psi_{n_2}(x_2)$ are the single-particle wave functions for the two particles.

Student 2: I disagree. If the system consists of two fermions, there is no way to distinguish which fermion is in the state labeled by n_1 and which is in the state labeled by n_2. The wave function must reflect this symmetry.

Student 3: I agree with Student 2. The wave function describing a system of noninteracting indistinguishable fermions must be completely antisymmetric. Therefore, the normalized wave function for a system of two noninteracting fermions must be $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$.

Student 1 is incorrect, while both student 2 and student 3 are correct in the preceding conversation. In particular, statement made by student 2 is intended to help students who had difficulty identifying that it is possible to write a wave function for a system of identical fermions despite not knowing which fermion is in which single-particle state. Student 3 then provides reasoning as to how one takes into account the indistinguishability of the particles and provides a completely antisymmetric wave function for the two fermions. Further scaffolding support is provided to help students reconcile their initial ideas with the correct concepts. After working on a system of two identical fermions, students engage with several guided inquiry-based sequences for a system of three identical fermions.

Later, students consider systems of identical bosons and work through similar guided inquiry-based learning sequences that strive to help them learn to write the bosonic wave function such that it is completely symmetric.

Helping students connect the Pauli's exclusion principle to fermionic wave function: Students engage in relating the Pauli's exclusion principle to the fermionic wave function, e.g., by explaining why they agree or disagree with student 2:

Student 1: I thought the Pauli exclusion principle states that no two fermions can be in the same single-particle state. How is that consistent with the wave functions being completely antisymmetric?

Student 2: Let's suppose we have two fermions in the same single-particle state. Then $n_1 = n_2$ and the wave function would be $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)] = 0$. Thus $\Psi(x_1, x_2) = 0$ is not a possible wave function.

Student 2 is correct and their statement engages students in connecting Pauli's exclusion principle and why two fermions cannot be in the same single-particle state. They then work through a guided inquiry-based sequence focusing on a system of three identical fermions that strives to help them generalize the case for two identical fermions and learn that the completely antisymmetric three-particle wave function is also consistent with the Pauli's exclusion principle.

Helping students determine all the terms in the symmetric or antisymmetric wave function: The QuILT strives to help students develop systematic reasoning about the number of terms to expect in a symmetric or antisymmetric wave function. For example, the following hypothetical conversation is part of a guided inquiry-based learning sequence which strives to help students connect the number of permutations of the labels for the indistinguishable fermions to the number of terms in the antisymmetric many-particle wave function. In the following hypothetical conversation, the students must state whether they agree or disagree with student 2 and state their reasoning.

Student 1: When constructing the completely antisymmetric wave function for a system of three indistinguishable fermions, how do I know that I have found all the possible permutations?

Student 2: In general, for a system of N indistinguishable fermions, there are $N!$ permutations of the labels. For example, there are $N!$ permutations of the coordinates $x_1, x_2, \ldots, x_N$ or $N!$ permutations of the labels for the single-particle states $\psi_{n_1}, \psi_{n_2}, \ldots, \psi_{n_N}$. The normalization factor is $\frac{1}{\sqrt{N!}}$.

Further scaffolding is provided to help students reflect upon the fact that student 2 is correct. In particular, for a system of N indistinguishable fermions, there are $N!$ permutations of the labels for either the states or the

coordinates. Therefore, the antisymmetric many-particle wave function will have $N!$ terms.

Helping students use the Slater determinant to write the many-particle wave function for a system of indistinguishable fermions: In the QuILT, students are asked to construct the many-particle wave function for a system of two fermions in which the fermions are in different single-particle states. They are later asked to use the Slater determinant for this same system of two fermions and compare the wave function obtained using the Slater determinant method with the one obtained without it. Additionally, the students are asked to construct the Slater determinant for a system of two fermions in which the two fermions are in the same single-particle state and reflect upon the implication of what they find to whether two fermions can be in the same single-particle state. The Slater determinant for such a system is zero, consistent with Pauli's exclusion principle and there is no such wave function for this system. The following is a hypothetical student's statement from a learning sequence in the QuILT that strives to help students make this connection in which the students must state whether they agree or disagree with the student and why.

Student 1: The Slater determinant yields a many-particle wave function which is consistent with the Pauli exclusion principle. For example, for a system of two fermions, if we put both fermions in the same state, then

$$\begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_1}(x_2) \\ \psi_{n_1}(x_1) & \psi_{n_1}(x_2) \end{vmatrix} = \psi_{n_1}(x_1)\psi_{n_1}(x_2) - \psi_{n_1}(x_2)\psi_{n_1}(x_1) = 0,$$

which cannot be a possible wave function since zero represents the absence of a wave function.

This type of reflection strives to help students focus on how to set up the Slater determinant correctly and also strives to help students recognize that an antisymmetric wave function given by the Slater determinant is consistent with the Pauli's exclusion principle for a system of two identical fermions. Later the students engage with a guided inquiry-based sequence which begins by asking them to construct the many-particle wave function for a system of three identical fermions using the Slater determinant.

V. ADDRESSING DIFFICULTIES WHEN BOTH SPATIAL AND SPIN DEGREES OF FREEDOM ARE CONSIDERED

Below are some examples from the QuILT of the scaffolding support intended to help students reflect upon and learn to construct bosonic and fermionic wave functions when both spatial and spin degrees of freedom are included. All notations are standard, e.g., for spin $1/2$ case, the two-particle spin state when particles 1 and 2 are in $|\uparrow\rangle$ state (eigenstate of $\hat{S}_z$) can be abbreviated as $|\uparrow\rangle_1|\uparrow\rangle_2 = |\uparrow\uparrow\rangle$ (see Supplementary Material [93] for full details). Students had also learned about the addition of angular momentum including the coupled and uncoupled representations (e.g., for two spin-$1/2$ or two spin-1 particles) before learning about identical particles.

Helping students recognize that a given wave function is completely symmetric/antisymmetric: Students engage with examples in which they are asked to first consider the spatial or spin part of the wave function's symmetry separately. Then, they are asked to combine them appropriately to make the overall wave function symmetric or antisymmetric.

The following is a hypothetical student conversation that is aimed at helping students identify symmetric and antisymmetric spin states for a system of two spin-$1/2$ fermions. The students must provide reasoning as to why they agree or disagree with each student.

Student 1: In the uncoupled representation, the two-particle spin states $|\uparrow\rangle_1|\uparrow\rangle_2, |\downarrow\rangle_1|\downarrow\rangle_2, |\uparrow\rangle_1|\downarrow\rangle_2$, and $|\downarrow\rangle_1|\uparrow\rangle_2$ are all appropriate choices for the spin part of the wave function to satisfy the symmetrization requirement.

Student 2: I disagree with student 1. In order to satisfy the symmetrization requirement of the wave function, we must choose spin states which are either symmetric or antisymmetric. In the uncoupled representation, the two-particle spin states $|\uparrow\rangle_1|\downarrow\rangle_2$ and $|\downarrow\rangle_1|\uparrow\rangle_2$ are neither symmetric nor antisymmetric. For example, the product of the spin state $|\uparrow\rangle_1|\downarrow\rangle_2$ with the spatial part of the wave function will not produce a wave function that is completely antisymmetric. The same is true for the spin state $|\downarrow\rangle_1\uparrow\rangle_2$.

Student 3: I agree with student 2. The two-particle spin states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ in the coupled representation expressed in terms of states in the uncoupled representation, are symmetric. The two-particle spin state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is antisymmetric. Therefore, the two-particle spin states $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ are all appropriate choices for the spin part of the two-particle wave functions to be combined with suitable spatial wave functions to satisfy the symmetrization requirement.

Student 1 is incorrect while students 2 and 3 are correct in the preceding conversation. The two-particle spin states $|\uparrow\rangle_1|\downarrow\rangle_2$ and $|\downarrow\rangle_1|\uparrow\rangle_2$ are neither symmetric nor antisymmetric. One can create a spin part of the wave function with the appropriate symmetry by taking a linear combination of these two spin states. Students are also asked to reflect upon the fact that a completely symmetric spin state can be constructed from a linear combination of the symmetric triplet states (i.e., that $C_1|\uparrow\uparrow\rangle + C_2|\downarrow\downarrow\rangle + C_3\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ is a completely symmetric normalized

spin state in which C_1, C_2, and C_3 are constants such that $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$).

At the end of each guided learning sequence, students encounter checkpoints designed to help them resolve any discrepancies between their initial reasoning and the correct explanations.

Helping students identify the proper symmetry of the wave function for identical fermions or bosons: The students engage with several guided inquiry-based sequences that strive to help them identify that the many-particle wave function for a system of identical fermions must be completely antisymmetric and the many-particle wave function for a system of identical bosons must be completely symmetric. As discussed earlier, initially, they work through examples in which they only consider the spatial part of the many-particle wave function in an effort to have them focus on identifying the symmetrization requirements for a system of identical bosons or fermions. Next, students consider the symmetry of the possible spin states involved in a system of identical spin-1/2 fermions and for a system of spin-1 bosons. After engaging with examples focusing on the spatial and spin parts of the wave function separately, they work through guided inquiry-based sequences that strive to help them identify all the possible combinations of spatial and spin parts of the many-particle wave function for a system of identical fermions or bosons that satisfy the appropriate symmetrization requirement.

Helping students realize that two fermions in different spin states can occupy the same spatial state: In the QuILT, students consider various systems of identical fermions and construct the completely antisymmetric wave functions for these systems. For example, in one guided inquiry-based learning sequence, students construct the ground state and first-excited state wave functions for a system of two fermions placed in a one-dimensional harmonic oscillator potential energy well in response to the following question. For the following question, students are told that they can denote the spatial state of the ith particle in the n_ith single-particle state of the oscillator by $\psi_{n_i}(x_i)$.

QA. Two identical noninteracting spin-1/2 fermions are placed in a one-dimensional harmonic oscillator potential energy well with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. The single-particle energies are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots$$

Construct the spatial part of the two-particle *ground state* and *first-excited state* for two noninteracting particles in the one-dimensional harmonic oscillator potential energy well if the particles are

(a) Indistinguishable fermions with spin-1/2 in a total spin $s = 0$ state.

(b) Indistinguishable fermions with spin-1/2 in a total spin $s = 1$ state.

In QA(a), the total spin $s = 0$ state is the antisymmetric singlet state and therefore, the spatial part of the wave function must be symmetric. The ground state wave function is $\Psi_{00} = \psi_0(x_1)\psi_0(x_2)$, and the spatial part of the first-excited state wave function is $\Psi_{01} = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)]$. In QA(b), the total spin $s = 1$ state is one of the symmetric triplet states. and thus, the spatial part of the ground state wave function must be antisymmetric. The spatial part of ground state wave function is $\Psi_{01} = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)]$, and the spatial part of the first-excited state wave function is $\Psi_{02} = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_0(x_2)]$.

Students receive scaffolding support to learn about the spatial part of the ground state wave function. For example, the following excerpt is from a hypothetical conversation designed to engage students with the case involving the spatial part of the ground state wave function in QA(a).

Student 1: The two fermions cannot both be in the same single-particle spatial state ψ_0. For the two-particle ground state, one fermion is in the lowest single-particle spatial state ψ_0 and the other fermion is in the first-excited single-particle spatial state ψ_1, so $n_1 = 0$ and $n_2 = 1$ or $n_1 = 1$ and $n_2 = 0$. The two-particle ground state energy is $E_{10} = 2\hbar\omega$.

Student 2: I disagree with student 1. You are forgetting about the spin degrees of freedom. For a system of indistinguishable fermions, the overall two-particle state must be antisymmetric. Since the fermions are in the total spin $s = 0$ antisymmetric singlet state $|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, the spatial part of the many-particle state must be symmetric. Two fermions in the same single-particle spatial state ψ_0 correspond to the symmetric spatial state $\psi_0(x_1)\psi_0(x_2)$.

Student 1 is incorrect and student 2 is correct in the preceding conversation. The overall two-particle ground state including both spatial and spin parts is $\Psi_{00} = [\psi_0(x_1)\psi_0(x_2)][\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)]$. In the total spin $s = 0$ state, the two fermions can be in the same single-particle spatial state ψ_0 since the fermions are in different spin states with the two-particle spin state $|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ being antisymetric.

Helping students generate a completely symmetric or antisymmetric wave function: Students engage with several guided inquiry-based learning sequences in the QuILT in which they must write the overall wave function for a system of fermions or bosons. For example, students are asked to write all the possible two-particle wave functions including spin for a system of two noninteracting indistinguishable spin-1/2 fermions in different single-particle spatial states labeled by n_1 and n_2. After answering this question, students are provided scaffolding support to help them identify that the overall wave function for the two fermions is antisymmetric so that the spatial part of the

wave function is symmetric and the spin part of the wave function is antisymmetric, or vice versa. The following is an excerpt from a hypothetical conversation regarding two fermions:

Student 1: We must only ensure that the spatial part of the two-particle stationary state wave function is antisymmetric. The spatial part of the two-particle stationary state wave function must be $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$. The spin part of the two-particle stationary state wave function can be either the antisymmetric singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ or one of the three symmetric triplet states $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\}$.

Student 2: I agree with student 1 that the spatial part of the two-particle stationary state wave function must be antisymmetric. However, we must also choose the antisymmetric singlet state as the spin part of the two-particle stationary state wave function.

Student 3: I disagree with both student 1 and student 2. The overall two-particle stationary state wave function must be antisymmetric. If the spatial part of the two-particle stationary state wave function is symmetric $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$, the spin part of the two-particle stationary state wave function must be the antisymmetric singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Student 4: I agree with student 3. Additionally, the spatial part of the two-particle stationary state wave function could be antisymmetric $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ which would imply that the spin part of the two-particle stationary state wave function can be one of the symmetric triplet states $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\}$. In either case, the product of one symmetric and one antisymmetric wave function produces an overall antisymmetric two-particle stationary state wave function.

Student 3: I agree with student 4. However, remember that a linear combination of the triplet states such as $C_1|\uparrow\uparrow\rangle + C_2|\downarrow\downarrow\rangle + C_3(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ is a completely symmetric spin state where $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$.

Students 1 and 2 are incorrect while student 3 and student 4 are correct in the preceding conversation. Either the spatial part or the spin part must be antisymmetric and the other is symmetric to produce an overall antisymmetric many-particle wave function. Further scaffolding is provided to help students develop a good grasp of relevant concepts.

In the QuILT, students engage in reflecting upon and constructing the bosonic and fermionic wave functions for a general system (e.g., one particle in state n_1 and the other particle in state n_2) as well as for specific systems. For example, students construct the ground state and first-excited state wave functions for a system of two identical particles in a one-dimensional infinite square well potential energy well as well as two identical particles in a one-dimensional harmonic oscillator potential energy well. The next question, QB, focuses on two identical spin-1 bosons placed in a one-dimensional harmonic oscillator potential energy well and asks students to write the many-particle ground state and many-particle first-excited state wave functions for the two bosons.

QB. Two identical noninteracting spin-1 bosons ($s_1 = 1$, $s_2 = 1$) are placed in a one-dimensional harmonic oscillator potential energy well with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. The single-particle energies are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad n = 0, 1, 2, \ldots$$

For the following questions, you can denote the spatial state of the ith particle in the n_ith single-particle state of the oscillator by $\psi_{n_i}(x_i)$.

Construct at least two possible overall two-particle *ground state* and at least two possible *first-excited state* (including both spatial and spin parts) wave functions for two noninteracting particles in the one-dimensional harmonic oscillator potential energy well if the particles are indistinguishable bosons with spin 1.

After reflecting upon this question, students consider possible spin states for the two bosons, focus on identifying that the overall many-particle wave function must be symmetric, and learn to construct a bosonic wave function commensurate with the constraints. For example, students must agree or disagree and provide reasoning after engaging with the following hypothetical conversation regarding how to construct a two-particle ground state wave function in QB:

Student 1: The two-particle ground state for a system of two indistinguishable bosons with spin 1 ($s_1 = 1 \otimes s_2 = 1$) must be symmetric. There are two possibilities for the two-particle ground state: both the spatial part and the spin part are symmetric or both the spatial part and spin part are antisymmetric.

Student 2: While that is generally the case, the two-particle ground state must be a state with the lowest energy. The lowest energy occurs when both bosons are in the same single-particle spatial state ψ_0. Therefore, the spatial part of the two-particle ground state is $\psi_0(x_1)\psi_0(x_2)$. The two-particle ground state energy is $E_{00} = \hbar\omega$.

Student 3: I agree with student 2. Since the spatial part of the two-particle ground state is symmetric, the spin part of the two-particle ground state must also be symmetric. Six possible symmetric combinations for the spin part of the many-particle state for two indistinguishable bosons both with spin 1 ($s_1 = 1$, $s_2 = 1$) in the coupled representation are $|2, 2\rangle$, $|2, 1\rangle$, $|2, 0\rangle$, $|0, 0\rangle$, $|2, -1\rangle$, and $|2, -2\rangle$. One possible overall two-particle ground state including both spatial and spin parts is $\Psi_{00} = \psi_0(x_1)\psi_0(x_2)|2, 2\rangle$.

Student 2: I agree with student 3. We can also construct a completely symmetric spin state by taking a linear

combination of these symmetric states $C_1|2,2\rangle + C_2|2,1\rangle + C_3|2,0\rangle + C_4|0,0\rangle + C_5|2,-1\rangle + C_6|2,-2\rangle$, where $|C_1|^2 + |C_2|^2 + |C_3|^2 + |C_4|^2 + |C_5|^2 + |C_6|^2 = 1$ will yield a normalized state.

Students 2 and 3 are correct in the preceding conversation. While the statement of student 1 is true in general, it is not true for the ground state in QB that both the spatial and spin part are antisymmetric. In particular, the spatial part of the ground state must be $\psi_0(x_1)\psi_0(x_2)$, and therefore, the spin part of the ground state must also be symmetric. Next, the students consider the following hypothetical student conversation regarding how to construct the two-particle first-excited state in QB in response to which they must explain why they agree or disagree with each student:

Student 1: If the two-particle first-excited state energy is $E_{01} = 2\hbar\omega$, one boson is in the single-particle spatial state ψ_0 and the other boson is in the single-particle spatial state ψ_1. The spatial part of the two-particle first-excited state MUST be $\frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)]$ since the overall wave function must be symmetric. Therefore, the spin part of the two-particle first-excited state must be a symmetric spin state.

Student 2: The spatial part of the two-particle first-excited state can also be $\frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)]$ in which case the spin part of the two-particle first-excited state must be an antisymmetric spin state.

Student 1 is correct that both the spatial and spin part of the two-particle wave function can be symmetric to produce an overall symmetric first-excited state wave function for the two bosons. However, it is also possible that both the spatial and spin parts of the two-particle wave function can be antisymmetric resulting in an overall symmetric first-excited state wave function for the two, as stated by student 2.

Additionally, students are asked to consider a helium atom and reflect upon the fact that the wave function for the two electrons in a helium atom must be completely antisymmetric. The QuILT also asks them to consider a system of identical helium-4 (^{4}He) atoms. In this case, they reflect upon the fact that since each ^{4}He atom consists of an even number of fermions (two protons, two neutrons, and two electrons), it is a boson and a system of N identical ^{4}He atoms is an example of a system of N identical bosons.

After working through these guided inquiry-based learning sequences, students are asked to summarize what they have learned and are provided checkpoints that allow them to reconcile their initial reasoning with the correct reasoning.

VI. EVALUATION OF THE QUILT

Table V shows the performance of undergraduate and graduate students on the pretest and post-test when spin part of the wave function is ignored. Table V also includes the average normalized gain, g [98]. The normalized gain g is defined as $g = (\text{posttest\%} - \text{pretest\%})/(100 - \text{pretest\%})$. The undergraduates and the graduate students scored relatively poorly on all parts of Q1 and Q2 on the pretest. After working through the QuILT, both undergraduate and graduate students scored over 80% on all the parts of Q1 and Q2 on the post-test. Thus, the post-test scores are significantly better than the pretest scores on all of these questions for both groups.

TABLE V. Average pretest and post-test scores, and normalized gains (g) [98] for graduate students (number of students $N = 30$) and undergraduate students (number of students $N = 25$).

	Graduate students			Undergraduate students		
Question	Pre (%)	Post (%)	g	Pre (%)	Post (%)	g
Q1(a)	45	93	0.87	34	88	0.82
Q1(b)	41	95	0.92	35	93	0.89
Q1(c)	44	96	0.93	61	99	0.97
Q2(a)	44	84	0.71	44	97	0.95
Q2(b)	30	95	0.93	35	87	0.80
Q2(c)	53	97	0.94	49	99	0.98

As a measure of retention, 12 of the upper-level undergraduate students in one year of the study were asked the following question on their final exam two months after engaging with the QuILT in which they were asked to ignore the spin degrees of freedom:

QR. Suppose we have two noninteracting particles, both of mass m, in a one-dimensional infinite square well of width a (well is between $x = 0$ and $x = a$). Write down the first-excited state wave functions and energies for two-particle system (in terms of single-particle wave function and ground state energy E_0) if the particles are (a) distinguishable, (b) identical bosons, and (c) identical fermions.

Of the 12 undergraduates, 10 answered the question completely correctly. One student answered the question correctly for the fermion and boson case but incorrectly determined that the first-excited energy for the system of two distinguishable particles was $E_{21} = \frac{(1+3^2)\pi^2\hbar^2}{2ma^2} = \frac{10\pi^2\hbar^2}{2ma^2} = 10E_0$ and the corresponding first-excited state wave function was $(2/a)\sin(\frac{\pi x_1}{a})\sin(\frac{3\pi x_2}{a})$. The other student who answered QR incorrectly omitted the energies for the first-excited states for all three systems and incorrectly claimed that the first-excited state for a system of identical fermions was $\frac{1}{\sqrt{2}}[(2/a)\sin(\frac{\pi x_1}{a})\sin(\frac{2\pi x_2}{a}) - (2/a)\sin(\frac{\pi x_2}{a})\sin(\frac{2\pi x_1}{a})]$ (but provided the correct response for the wave function for a system of two bosons or distinguishable particles). Apart from these mistakes by two of the students, all of the students provided wave functions in response to QR that had the correct symmetrization and each particle had a unique coordinate in the many-particle wave function for each system of identical particles.

TABLE VI. The percentages of graduate ($N = 30$) and undergraduate ($N = 12$ in year 1 and $N = 13$ in year 2) students who provided a part of the wave function with the correct symmetry in Q5 and Q6 on the pretest and Q7, Q8, and Q9 on the post-test.

			Graduate			Undergraduate		
Question	Type of particle	Part of the wave function	N	Pre (%)	Post (%)	N	Pre (%)	Post (%)
Q5	Bosons	Spatial	30	33	…	25	48	…
Q6	Electrons	Spin	30	30	…	25	44	…
Q7	Bosons	Spin	30	…	97	12	…	92
Q8	Fermions	Spatial	30	…	87	25	…	80
Q9	Bosons	Spin	…	…	…	13	…	69

For wave functions with both spatial and spin degrees of freedom, Q5 and Q6 were posed on the pretest and Q7, Q8, and Q9 were posed on the post-test (see Supplementary Material [93] for exact wording). Questions Q7 and Q8 were posed to all the graduate students and to 12 of the undergraduate students in year 1. Question Q9 was posed to 13 undergraduate students in year 2. Questions Q5, Q7, and Q9 are similar in that students must identify the overall symmetrization requirement for a system of identical bosons and generate either the spatial or spin part of the wave function so that the product of the spatial and spin parts is completely symmetric. In particular, in Q5, students are provided the spin part that is symmetric and asked for an example of spatial part, whereas in Q7 and Q9, students are given the spatial part (spatial part of the wave function is symmetric in Q7 and antisymmetric in Q9) and asked for the spin part. Questions Q6 and Q8 are similar in that students must identify the overall symmetrization requirement for a system of identical fermions and generate either the spatial or spin part of the wave function so that the product of the spatial and spin parts is completely antisymmetric. In particular, for two fermions, Q6 gives an example of a spatial part that is symmetric and asks for the spin part whereas Q8 gives an example of the spin part that is antisymmetric and asks for the spatial part (see Supplementary Material [93] for the actual questions).

We note that in year 1, the graduate students were given the QuILT prior to the undergraduate students. The graduate students were not given the Clebsch Gordon table (which can be useful for the spin part of the wave function for two spin-1 bosons) on the pretest or the post-test, which could potentially be useful for identifying the symmetrized spin states. Many of the graduate students provided a symmetric spin state in response to Q7; however, many of them provided a spin state that was consistent with a system of two spin-1/2 fermions. After observing the difficulty that the graduate students had in generating the spin states for two spin-1 bosons on the post-test, the Clebsch Gordon tables for two spin-1/2 particles and two spin-1 particles were included with the pretest and post-test in later implementations with the undergraduate students. Despite the fact that the undergraduate students were given this table, they still struggled to correctly write the spin part of the wave function for a system of two bosons in Q7 on the post-test. Many of these undergraduate students provided similar responses to those of the graduate students in that they listed a symmetric spin part of the wave function consistent with a system of two spin-1/2 particles. Table VI gives the percentages of students who provided part of the wave functions in various questions with the correct symmetry. Although identifying the correct spin state for two bosons in Q7 proved challenging for students even when they were provided with the Clebsch Gordon table, the majority of the students were able to correctly identify the necessary symmetry. This is an area to improve upon in future refinements and implementations of the QuILT to address these difficulties more effectively.

Table VII provides the average student score on the pretest and post-test. The results are encouraging and suggest that the QuILT is effective in helping students identify the symmetrization requirements for the spatial and spin parts of the many-particle wave function for a system of identical fermions. While there is a significant improvement in the number of students who correctly identified the correct symmetrization requirement for the spatial and spin parts of the many-particle wave function (see Table VI), many students struggled to write the correct spin part of the wave function in Q7 or Q9 on the post-test (many students used the spin part of the wave function for two spin-1/2 particles as opposed to two spin-1 particles even when both Clebsch Gordon tables were provided, as in the case of undergraduates). Focusing on identifying and generating the spin part of the wave function for a system of identical bosons (and ensuring that students do not use the spin part of the wave function for two spin-1/2 particles as opposed to two spin-1 particles when both Clebsch Gordon tables were provided) is an area to improve upon in future refinements and implementations of the QuILT.

As a measure of retention for the case when both spatial and spin degrees of freedom are considered, the undergraduate students in the first year of the study were asked Q6 on their midterm examination four weeks after completing the QuILT. The percentage of students who answered Q6 correctly on the midterm examination was 68%. Here we

TABLE VII. The average student scores on questions Q5 and Q6 for the given system on the pretest and Q7, Q8, and Q9 for the given system on the post-test for graduate students ($N = 30$) and undergraduates ($N = 12$ in Year 1 and $N = 13$ in year 2).

		Graduate			Undergraduate		
Question	Type of particle	N	Pre (%)	Post (%)	N	Pre (%)	Post (%)
Q5	Bosons	30	34	…	25	51	…
Q6	Electrons	30	30	…	25	44	…
Q7	Bosons	30	…	52	12	…	54
Q8	Fermions	30	…	88	25	…	80
Q9	Bosons	…	…	…	13	…	53

find students improved on this question in the exam compared to the pretest. This suggests that the QuILT helped students gain a better understanding of the wave function for identical fermions and retain their learning.

VII. SUMMARY

We conducted an investigation into student difficulties and used this research as a guide to develop, validate, and evaluate a QuILT, along with pre- and post-tests, to provide scaffolding aligned with students' prior knowledge. The QuILT aims to help students learn concepts related to constructing many-particle wave functions for systems of identical particles, both with and without spin considerations. The guided teaching-learning sequences in the QuILT strive to help students learn to use the requirement for the wave function to be antisymmetric for fermions and symmetric for bosons, guiding them through constructing these wave functions in line with symmetrization principles. The QuILT engages students actively by using a structured, inquiry-based approach, offering checkpoints for students to revisit and reflect upon their predictions based on provided hints. The post-test results demonstrate the effectiveness of the QuILT in enhancing students' understanding of concepts related to many-particle wave functions for a system of two or three bosons and fermions with or without consideration of spin degree of freedom.

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