Comparative analysis of spins-first and wave functions-first students' understanding of expressions in quantum mechanics

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[This paper is part of the Focused Collection in Investigating and Improving Quantum Education through Research.] Instructors teaching upper-division quantum mechanics have had two primary options when it comes to textbook choice and thus curriculum sequence: starting with wave functions and the Schrödinger equation, referred to as "wave functions-first;" and starting with discrete spin-1/2 systems and Dirac notation, known as "spins-first" courses. Given the very different structures of these courses, particularly as it pertains to the notations and formalisms both emphasized and used, it begs the question as to whether and to what extent students in these different courses conceptualize symbolic expressions in Dirac and wave function notations differently. To investigate this, online surveys were administered to students in spins-first courses at six institutions and in wave functions-first courses at four institutions. As a follow-up to a prior study focused on the results from the spins-first courses, network analysis and community detection techniques were used to compare the level of conceptual similarity between expressions as viewed by the students in both curricula. Conceptual interpretations of individual expressions in both Dirac and wave function notations were also directly compared between the two populations. The primary difference observed between the two populations appears to lie in the way they interpret Dirac bras and kets: spins-first students were found to more strongly connect these expressions to vectorlike interpretations, while wave functions-first students were found to interpret them as more wave functionlike. This suggests that the choice of text and/or curricular style should be informed by the interpretation that best matches the goals of the instructor.

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I. INTRODUCTION

Student understanding of mathematical relationships is a crucial outcome of a physics education, as these expressions and relationships help students to understand and reason about the world [1]. In quantum mechanics (QM), this is particularly critical due to the inherently abstract and often nonintuitive nature of the content. Research has documented student difficulties with this abstraction and requisite mathematical sophistication, including when interpreting Dirac formalism [2], distinguishing between Euclidean and Hilbert spaces [3,4], reasoning about wave functions both graphically and symbolically [4], and studying the time evolution of quantum states [3–6]. Student interpretations of eigenequations have been studied as well, both by researchers of physics education [7–9] and mathematics education [10–12]. Multiple studies have also been conducted on the various notations that are frequently used in upper division QM and the different mathematical skills required of each [13–15]. The notations studied typically include Dirac, vector-matrix, and wave function notations—none of which have been found to be trivial for students to work with and reason about.

These varied notations are of particular interest in education research not only because of their importance in understanding and interpreting QM, but also because of the different ways that they are emphasized in different upper-division QM courses. There are two common ways that texts for these courses are structured. The first is typically referred to as "wave functions-first" due to the initial focus on wave functions and solutions to the Schrödinger equation. In contrast, the "spins-first" texts start by introducing Dirac formalism via the Stern-Gerlach experiments and spin-1/2 systems. Wave functions-first texts do eventually introduce Dirac notation, and spins-first texts do the same with wave functions, but this difference in notation introduction generally causes very different course

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structures. Though much prior work has focused on wave function notation, Dirac notation, and vector-matrix notation, we limit our investigation to the first two. This is primarily due to the inherently less-symbolic nature of vector-matrix expressions, causing direct comparisons between analogous expressions more challenging if this notation were included in the mix.

In a prior study of students enrolled in spins-first courses [16], we found that network analysis of survey data could be used to gain an understanding of when and, to some extent, how strongly students understand mathematical expressions to be conceptually related, so long as the survey is designed to elicit such connections. Students in spins-first courses viewed Dirac bras and kets as broadly similar to both generic expressions for vectors and expressions for wave functions. These students also tended to view Dirac brackets, quantum mechanical inner product integrals, and an expression for a generic vector dot product as conceptually similar. However, the students connected Dirac bras and kets more closely with the expressions for generic vectors than with the expressions for wave functions. Moreover, the concepts that they used to connect the Dirac expressions to each other were more often vectorlike concepts than those associated with quantum states. This vector-heavy conceptual understanding could be at least partially explained by the curricular focuses of a spins-first quantum mechanics course.

This raises two questions:

(1) Given the different instructional emphasis of a wave functions-first quantum mechanics course, what might one expect from a similar analysis of students enrolled in wave functions-first courses?
(2) How, if at all, do any differences align with the different emphases of the courses?

To answer these questions, network analysis and community detection techniques are used to compare the expressions viewed as most conceptually similar between the two populations. We also directly compare the conceptual interpretations of various QM expressions in both notations across populations.

In a separate, related paper included in this focused collection [17], an in-depth qualitative study of spins-first students' interpretations of symbolic QM expressions was conducted using the symbolic forms framework [18]. The symbolic forms framework posits that students develop a "vocabulary" of different symbolic templates that are ascribed particular meaning due to their particular symbolic structure or orientation. Relevant findings from that study will be referenced when they offer insight into the findings discussed below.

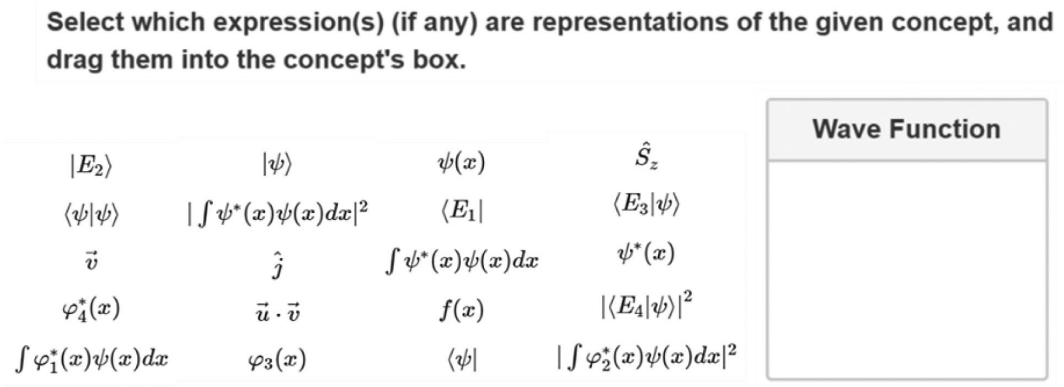
II. BACKGROUND

There has been a great deal of prior research into students' understanding of mathematical representations used in QM. This includes an analysis of the affordances and limitations of different notations for computation that found that each notation studied (Dirac, vector-matrix, and wave function) had unique properties that made them better suited for different types of computation [13]. A related study investigated spins-first students' notational preferences when calculating expectation values [14] and found that students commonly elected to use more computationally intense techniques and notations due to increased familiarity. Another study investigated the suitability of matrix-vector and Dirac notations for various applications, according to students [15]. Other work has shown that incorrectly translating between notations can cause students to struggle to develop models for calculating probabilities [19], and simulation-based instructional materials focusing on multiple representations simultaneously have been developed to help students in working among and reasoning with several representations, with some success [20,21].

Network analysis techniques have also seen extensive use in education research. This includes determining the similarity of response groupings for conceptual inventories [22–27], clustering students' knowledge, skills, and beliefs in an epistemological framing context [28], and studying social communities among communities of both educators and students [29–32]. Communities have additionally been studied in the context of different active-learning pedagogies [33] and in remote physics courses [34]. Community detection techniques have also been shown to allow for the analysis of students' conceptual interpretations of various mathematical expressions within a spins-first QM context [16].

III. RESEARCH DESIGN AND METHODOLOGY

In order to answer our research questions, an online survey (Fig. 1) was distributed to students enrolled in upper-division quantum mechanics courses at 10 different institutions in the U.S. This included students from four institutions that used wave functions-first curricula ($N = 55$) as well as students from six spins-first institutions ($N = 139$). The survey was distributed near the end of the course, after students had used and, in principle, become comfortable with both Dirac and wave function notation. Given the somewhat idiosyncratic notational choices of the various texts and instructors used in these courses, the expressions in the survey were tweaked to most closely reflect the equivalent expressions used in the text for each course. As each of these texts used consistent symbols to represent specific concepts, such as generic superposition wave functions/state vectors and specific eigenfunctions/eigenvectors, this was deemed the best course for making our survey best match the expressions that participants would have seen and used in their own courses. For example, an energy eigenfunction was expressed as either $\psi_3(x)$ or $\varphi_3(x)$, depending on the textbook used [$\psi_3(x)$ for Griffiths [35] and Townsend [36], and $\varphi_3(x)$ for McIntyre [37]. In the survey,



(a)

Concepts		
Vector	Wave function	Eigenvector
Quantum state	Unit Vector	Probability amplitude
Inner product	Basis Vector	Probability
Dot product	Eigenstate	

(b)

FIG. 1. Online survey administered to students (McIntyre notation version). (a) Example of a prompt with all expressions and one concept. (b) List of concepts for which participants were asked to select expressions.

participants were tasked with selecting expressions as representative of various concepts, as shown in Fig. 1. It should be noted that some of these conceptual identifications and distinctions are "fuzzy," even from an expert perspective; for example, both $|\psi\rangle$ and $\psi(x)$ are used to represent the same thing—namely, a quantum state—but are technically two different things: $|\psi\rangle$ is a quantum state expressed as a state vector with no basis imposed upon it, while $\psi(x) = \langle x|\psi\rangle$ is a representation of that same state within a continuous position basis. Both of these would likely be selected by most experts as representative of "quantum states," but there may be some disagreement as to whether they both represent a "wave function." Part of the reason such a spread of relatively similar concepts and expressions were chosen for inclusion in this survey was to ascertain how this expected "fuzziness" appears in student responses.

Student survey responses were used to generate networks representative of the different expressions' conceptual relatedness, using the methodology described in Ref. [16]. That is, individual student networks were formed with 20 nodes, each representing one of the expressions used in the survey. A link was created between two expressions if the student ever dragged those two expressions into the same concept box, thus demonstrating a shared conceptual understanding of those expressions for the student. This resulted in N individual unweighted networks, which were summed together to form a single weighted network [e.g., if 45 students had an edge connecting $\psi(x)$ and $|\psi\rangle$—meaning they all selected both expressions as representing at least one shared concept—the cumulative network would have a single edge connecting those two nodes, with an edge weight of 45]. Both the larger cumulative network and the individual student networks were used in the analysis.

IV. DATA ANALYSIS METHODS

Multiple analyses were conducted on the survey data, both from spins-first courses and wave functions-first courses. The first analysis investigated community structure within the networks generated from the survey results from each type of institution using edge betweenness-based methods [38]. This technique is discussed in relevant detail in Ref. [16], but in short, an edge's betweenness is representative of its likelihood to lie between two different communities. As such, the edge-betweenness community detection algorithm sequentially removes whichever edge has the highest betweenness within the network, and thus slowly pares out the "connective tissue" between communities within the network. In practice, this means that

progressively tighter-knit communities are revealed throughout the process, potentially revealing hierarchical community structure (whereby some larger communities are formed from multiple smaller, tight-knit subcommunities). This process terminates once every edge is removed and all nodes are fully disconnected into their own individual communities. As the algorithm starts with the network fully intact and ends with the network fully separated, meaningful community separations occur during the procedure and can be determined by observing the relative order of community separation (wherein those communities that separate later on are more closely connected). The question of which number of communities is "best" for a given network (i.e., in our context, when during the edge-removal procedure is it no longer meaningful to keep separating communities) can be answered several ways [16,39]. We chose a modified bootstrapping procedure for our analysis, as it has been shown to be effective at answering this exact question [40]. This bootstrapping procedure is a method of generating a large number of slightly different variations on an original network by resampling from the pool of individual student networks. The edge-betweenness community detection algorithm can then be run on each of these slightly different networks, and where there is a large agreement between their community structures, claims can be made about those community divisions being robust, while the point at which they start largely disagreeing is the point at which the communities should be divided no further.

Another analytical approach to these survey results was to compare the two populations' conceptual interpretations of the various expressions present in the survey. To do this, statistical analysis was conducted on the number of students from each curriculum that selected a given expression for each concept. By determining which expression-concept pairs showed statistically significant differences between the curricula, differences in the conceptual interpretations of these expressions were found. These comparisons, as well as simply the fraction of students that paired various concepts and expressions within a given curriculum, offer insight into the conceptual understanding of students in each curriculum as well as the cases where their interpretations differ. These conceptual interpretations can then be discussed in the context of the curricular structure and pedagogical focus of each curriculum.

V. RESULTS AND DISCUSSION

The discussion of our results is broken down by the analysis methods discussed above. First, the community structure determined by network analysis of students' responses is compared between the students enrolled in courses using the two instructional approaches. The conceptual interpretations of the expressions selected by students are then directly compared to see where the two instructional approaches differ and agree.

A. Community detection comparison

Upon running the edge betweenness algorithm and the bootstrapping procedure, the network generated from the spins-first student responses was best divided into six communities, as shown in Fig. 2(a) [16]. The requirement for this division was for at least 70% of the 1000 bootstrapped networks to agree on a specific community division. Upon performing the same bootstrapping procedure on the cumulative network created from the wave functions-first students' survey responses (and requiring the same $> 70\%$ agreement among bootstrapped networks), the community divisions shown in the dendrogram in Fig. 2(b) were found to be stable.

Upon inspection, the community structures are largely very similar. There is, however, one major difference between the two dendrograms. In the dendrogram for the spins-first students [Fig. 2(a)], the wave function terms [$\psi(x)$, $\psi^*(x)$, $\varphi_3(x)$, and $\varphi_4^*(x)$] split-off from the community containing both the Dirac bras and kets ($|\psi\rangle$, $\langle\psi|$, $|E_2\rangle$, and $\langle E_1|$) and the generic individual vector expressions ($\vec{v}$ and $\hat{j}$). The opposite occurs in the dendrogram for the wave functions-first dendrogram [Fig. 2(b)]: the generic vector expressions split-off from the community containing the Dirac bras and kets and the wave function terms. This difference in community structure suggests that the Dirac bras and kets are thought of slightly differently by the students in these different types of courses. In the spins-first courses, while the connections between the Dirac bras and kets and their associated wave functions are certainly present—as these expressions take quite a while for their communities to separate—the bras and kets are more closely connected to expressions the students think of almost exclusively in geometric vector terms. This is counter to the structure present in the wave functions-first students' network, where the conceptual connections between the Dirac bras and kets and their associated wave functions are stronger than the connections between either of these types of expressions and the generic individual vector expressions. This suggests that these different curricula have indeed affected the way that students conceptualize the meanings of both Dirac bras and kets as well as wave functions, or at the very least that they have affected the ways that these two types of expressions are viewed as conceptually connected.

B. Comparing expressions' conceptual interpretations

In analyzing the conceptual understanding of each expression individually, the survey responses for each concept can be broken down to see which expressions were chosen as representations for a given concept by students in both curricula. This section highlights the responses for concepts that are exemplary of common trends, discusses which expressions were commonly chosen and where the two populations significantly differed in the expressions they chose, and discusses potential reasons for these differences. Due to the categorical nature

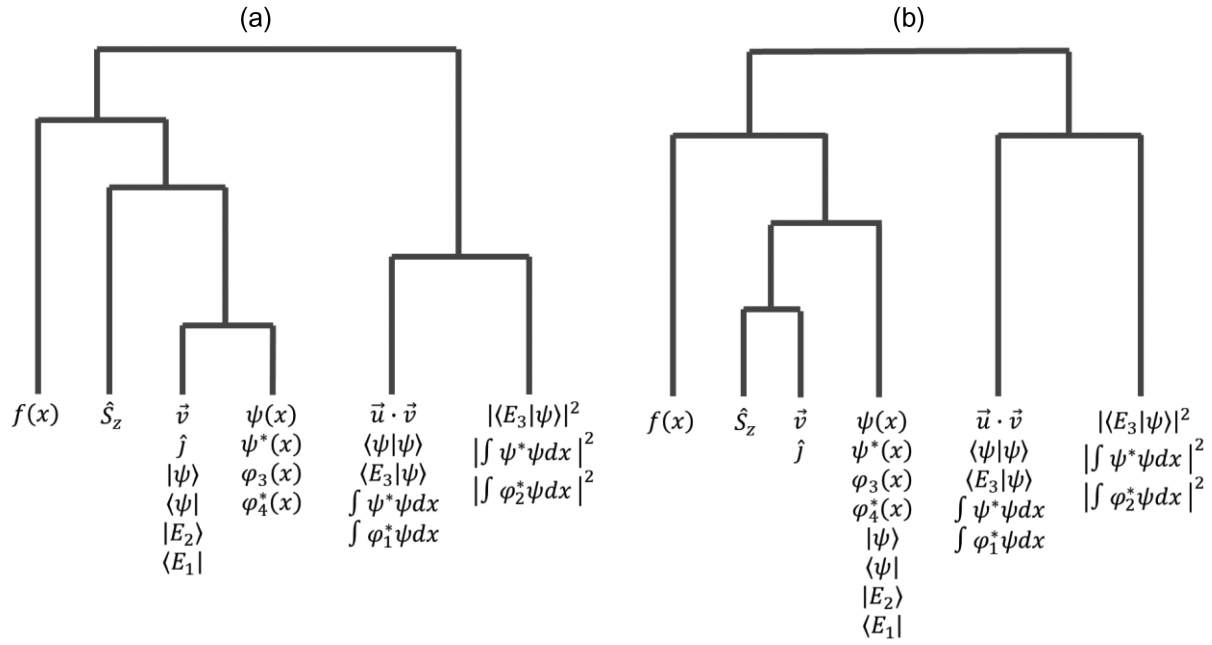


FIG. 2. Dendrograms showing the relative order of community divisions, determined by the bootstrapping process, of (a) the spins-first network and (b) the wave functions-first network. Community divisions only appear if there is $> 70\%$ agreement among bootstrapped networks.

of these data, comparisons were conducted on an expression-by-expression basis, and statistical significance was determined using the Fisher's exact test. This was necessary (rather than a simple χ^2 test) due to the low number of students who selected some expressions for a given concept. Discussion focuses on interesting expression-concept pairs where the two curricula agree, as well as where they disagree ($p < 0.05$ is shown, as are the situations where $p < 0.01\phi N$). The charts for every concept, the p-values for every comparison of expressions for each concept, as well as the -values (effect sizes) for those with statistically significant p-values can be found in Supplemental Material [41]. For the purposes of data visibility and ease of comparison between populations with different, figures will show the fraction of respondents that selected each expression for each concept rather than the raw numbers.

1. Expressions chosen for vectorlike concepts

The breakdown of student responses for expressions they viewed as representing a *vector* is shown in Fig. 3. This concept was chosen due to it being emblematic of common trends among various vectorlike concepts, particularly *basis vector* and *eigenvector*. As can be seen, both populations largely agreed that $\vec{v}$ and $\hat{\jmath}$ are representative of vectors. Given these expressions' nonquantum mechanical nature, this is to be expected and is an indication that these students have similar backgrounds. There are a few different categories of expressions with consistently statistically significant differences that suggest a theme here. The first is that $f(x)$, $\psi^*(x)$, and $\varphi_3(x)$ were all statistically more likely to be selected as representing vectors to students enrolled in wave functions-first courses, suggesting that students enrolled in wave functions-first courses were more commonly interpreting functions as potentially representative of vectors than students enrolled in spins-first courses. It is worth noting that while these functions were statistically significantly selected for this concept more by wave functions-first students, this was not the case for $\psi(x)$ or $\varphi_4^*(x)$, suggesting that while this was a consistent pattern, it was nonetheless not a terribly strong one (as can be attested by the small effect sizes). The second group of expressions of interest contains $|\psi\rangle$ and $\langle E_1|$, both of which were statistically more likely to be selected as representing vectors to students enrolled in spins-first courses; this suggests that students enrolled in spins-first courses were more likely to conceptualize Dirac bras and kets as representative of vectors than students enrolled in wave functions-first courses. As with the prior discussion of the wave function expressions, this should be tempered somewhat by the lack of a statistically different difference between the two curricula regarding $\langle\psi|$ and $|E_2\rangle$. Nonetheless, these differences could be explained by the students' respective curricula. It would appear that the fundamental expression unit of whichever notation the students learn first (Dirac bras/kets or wave functions) is more likely to be viewed as representative of vector concepts than that of the notation they learn later. It is possible that students turn to vector interpretations early in their coursework due to their familiarity with vectors from prior coursework; perhaps this conceptualization aids in interpreting inner products in their respective notation.

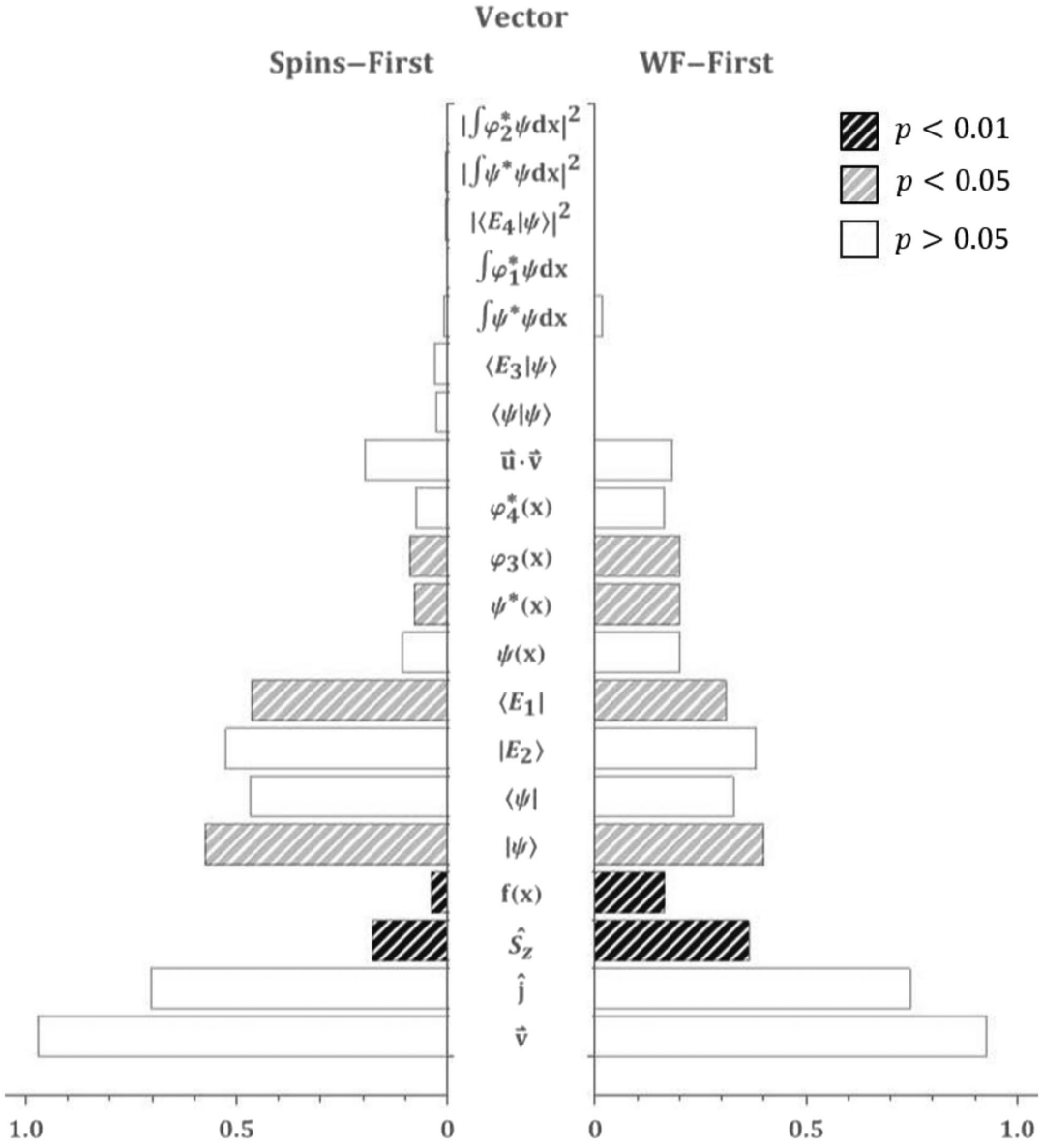


FIG. 3. Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "vector" concept. All statistically significant differences have small effect sizes ($0.1 < \phi < 0.3$).

Once a new notation system is introduced later in the course, perhaps this vector identity simply remains most closely tied to the notation they began using. It is also of interest that students in wave functions-first courses were more likely to consider the z-spin operator $\hat{S}_z$ as representing a vector, likely due to a combination of a lack of familiarity from spending significantly less time studying spin-1/2 systems within a wave functions-first course and a notational association of the "hat" with unit vectors.

2. Expressions chosen for the wave function concept

The frequencies of the two populations' choices for expressions as representative of the *wave function* concept are shown in Fig. 4. Both populations generally agreed that the various wave function expressions [$\psi(x)$, $\psi^*(x)$, $\varphi_3(x)$, and $\varphi_4^*(x)$] were representative of wave functions, which is to be expected. What is particularly interesting is that the wave functions-first students were consistently more likely to consider Dirac bras and kets as representing wave functions, particularly in the case of $\langle E_1|$ and $|E_2\rangle$ [$p < 0.01$, with a medium effect size ($0.3 < \phi < 0.5$)]. This can perhaps be explained by the different curricular structures, as wave functions-first courses establish a wave function as the fundamental object in quantum mechanics, and thus perhaps students view Dirac bras and kets in those terms, referring to them as such. Conversely, in spins-first courses, wave functions are introduced later in the course

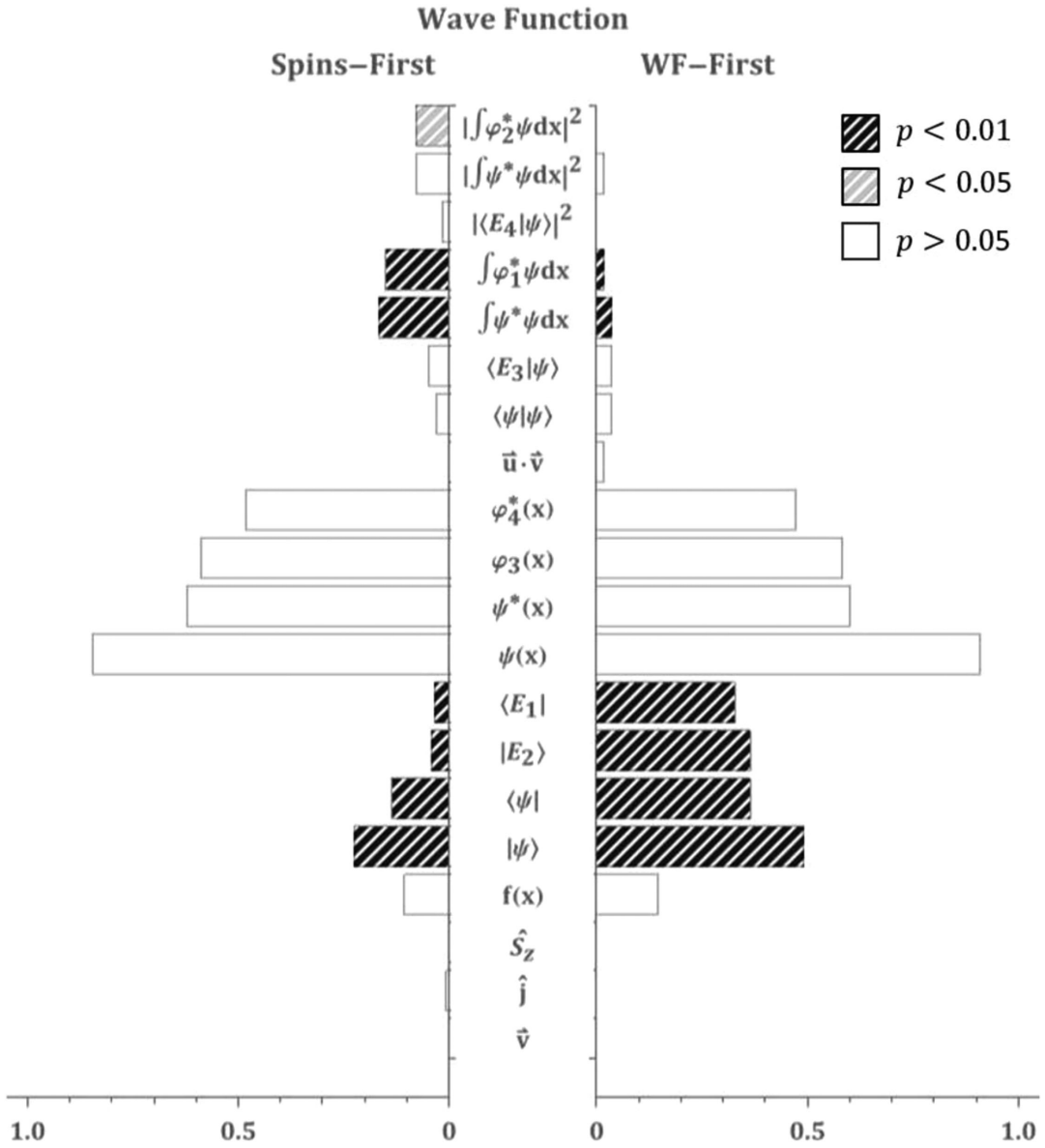


FIG. 4. Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "wave function" concept. For $\langle E_1|$ and $|E_2\rangle$, $0.3 < \phi < 0.5$ (medium effect size), and the other statistically significant differences have small effect sizes ($0.1 < \phi < 0.3$).

after developing a vocabulary for Dirac bras and kets as state vectors, eigenvectors, etc., and so perhaps this term is not viewed as strongly by students as a blanket term by the time they begin to use wave functions later in the semester. That more students in the spins-first course selected the wave function notation versions of inner products as representative of wave functions is less explicable, aside from a potential interpretation of the prompts by that population as asking for any expression involving a wave function.

3. Expressions chosen for quantum statelike concepts

As an exemplary case of the trends seen regarding student choices for representing quantum states, see Fig. 5 for the expression breakdown of *eigenstate*. Here we see a similar effect as we did for *vector*, where the different curricula are consistently more likely to select expressions in the notation they learned first in their course as representative of eigenstates. This suggests that students may better understand expressions they have spent more time with, or perhaps that they are more confident in selecting them on a survey. Regardless of curriculum, $\langle E_1|$ and $|E_2\rangle$ were selected more than $\langle\psi|$ and $|\psi\rangle$, and $\varphi_3(x)$ and $\varphi_4^*(x)$ were selected more than $\psi(x)$ and $\psi^*(x)$, suggesting that students were cuing on the difference in letter and/or the presence of a subscript when determining which expressions correspond to eigenstates. These are normatively the markers used to distinguish between an

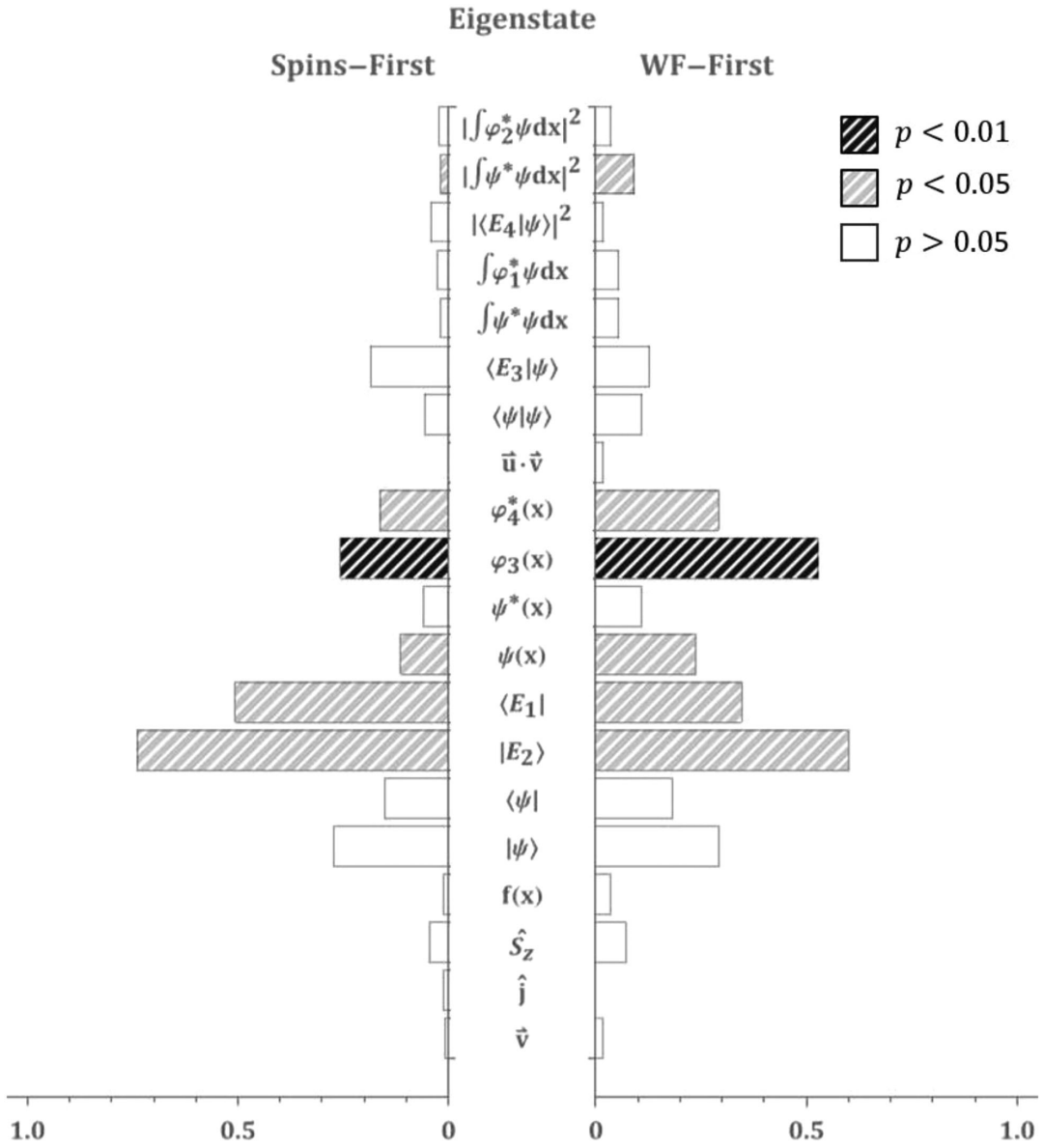


FIG. 5. Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "eigenstate" concept. All statistically significant differences have small effect sizes ($0.1 < \phi < 0.3$).

eigenstate (eigenvector/eigenfunction) of an observable (operator) and a general (unspecified) or superposition quantum state's state vector or wave function. The consistent preference for kets and noncomplex conjugate functions over bras and complex conjugate functions is also of note, as this suggests that students think of the kets and wave functions as quantum states more than they do bras and complex conjugate wave functions. This is likely due to a lack of experience with dual vectors and functions. Interestingly, the same analysis for the *quantum state* concept (see Supplemental Material [41]) instead shows broader similarity between the two curricula for most expressions. This may suggest a difference in familiarity with terminology such as *eigenstate* as opposed to a more general *quantum state*.

4. Expressions chosen for probabilitylike concepts

Finally, we discuss the differences in the expressions chosen as representative of probabilitylike concepts, using the results for the *probability amplitude* concept as an example (see Fig. 6). Unlike the distinctions between the curricula discussed in the previous sections, the divisions here are *not* along notational boundaries. Whereas in the prior comparisons, the students were seen to generally select expressions that they either learned earlier or later in

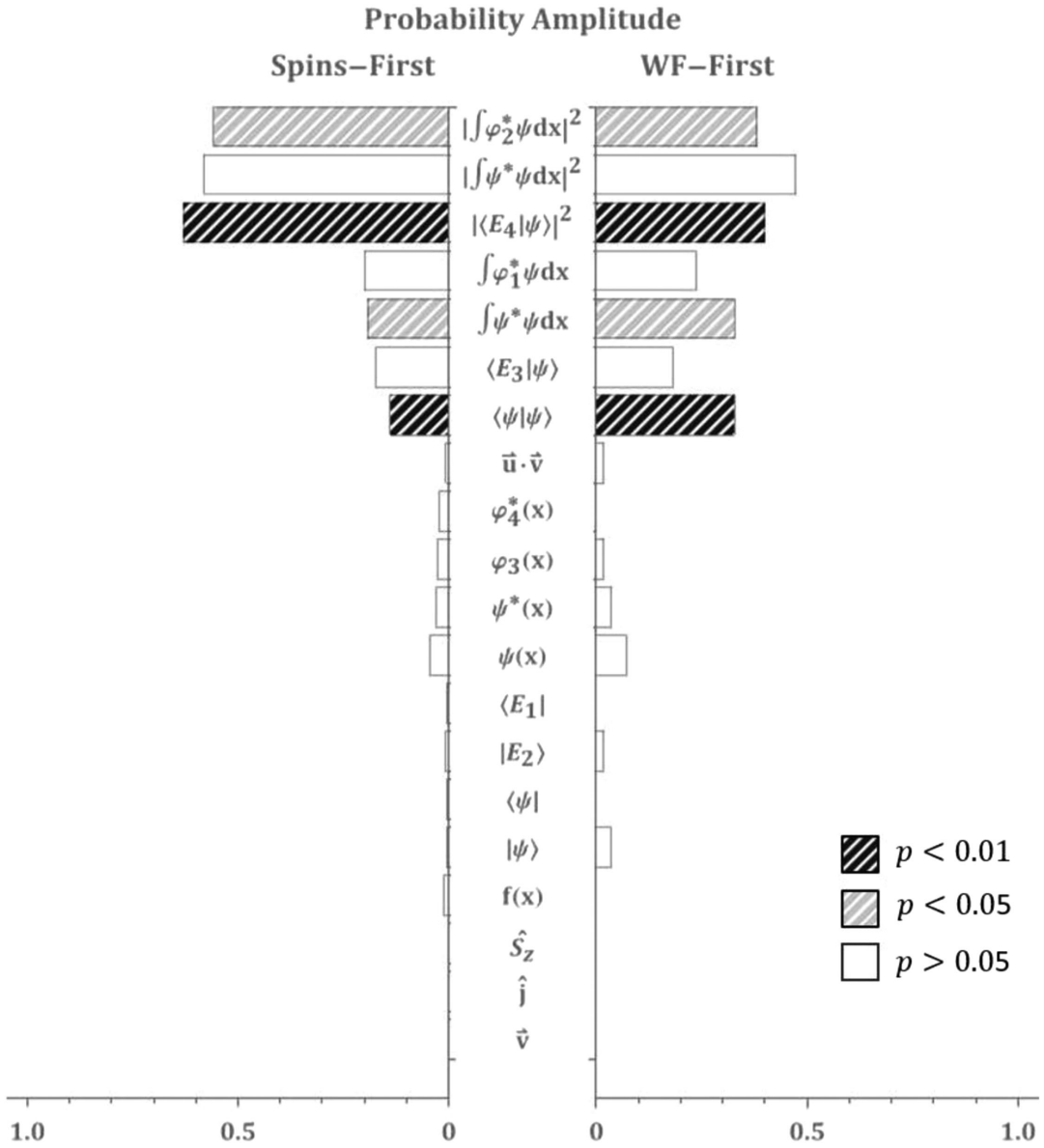


FIG. 6. Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "probability amplitude" concept. All statistically significant differences have small effect sizes ($0.1 < \phi < 0.3$).

their course as representative of various concepts, here each group of students selected some Dirac expressions more often and some wave function expressions more often, depending on the concept. The factor that instead appears to primarily distinguish the populations is whether the function has a complex square or not. Spins-first students were more likely to select $|\int \varphi_2^* \psi dx|^2$ or $|\langle E_4|\psi\rangle|^2$ than the wave functions-first students, and the wave functions-first students were more likely to select $\int \psi^* \psi dx$ and $\langle\psi|\psi\rangle$ than the spins-first students. It is worth noting that the wave functions-first students selected most expressions that included quantum mechanical inner products (including those with the complex square) with fairly similar frequency; the primary cause for the differences between the populations is the consistently larger preference for terms with a complex square among the spins-first students. This suggests that students in spins-first courses may have developed a stronger expectation that probabilitylike expressions use complex squares than students in wave functions-first courses. This could be explained by the students in spins-first courses developing a symbolic form tied to probability concepts that include a complex square in the symbol template [17], while perhaps students in wave functions-first courses do not develop a symbolic form with that requisite component or do not develop it as strongly or consistently.

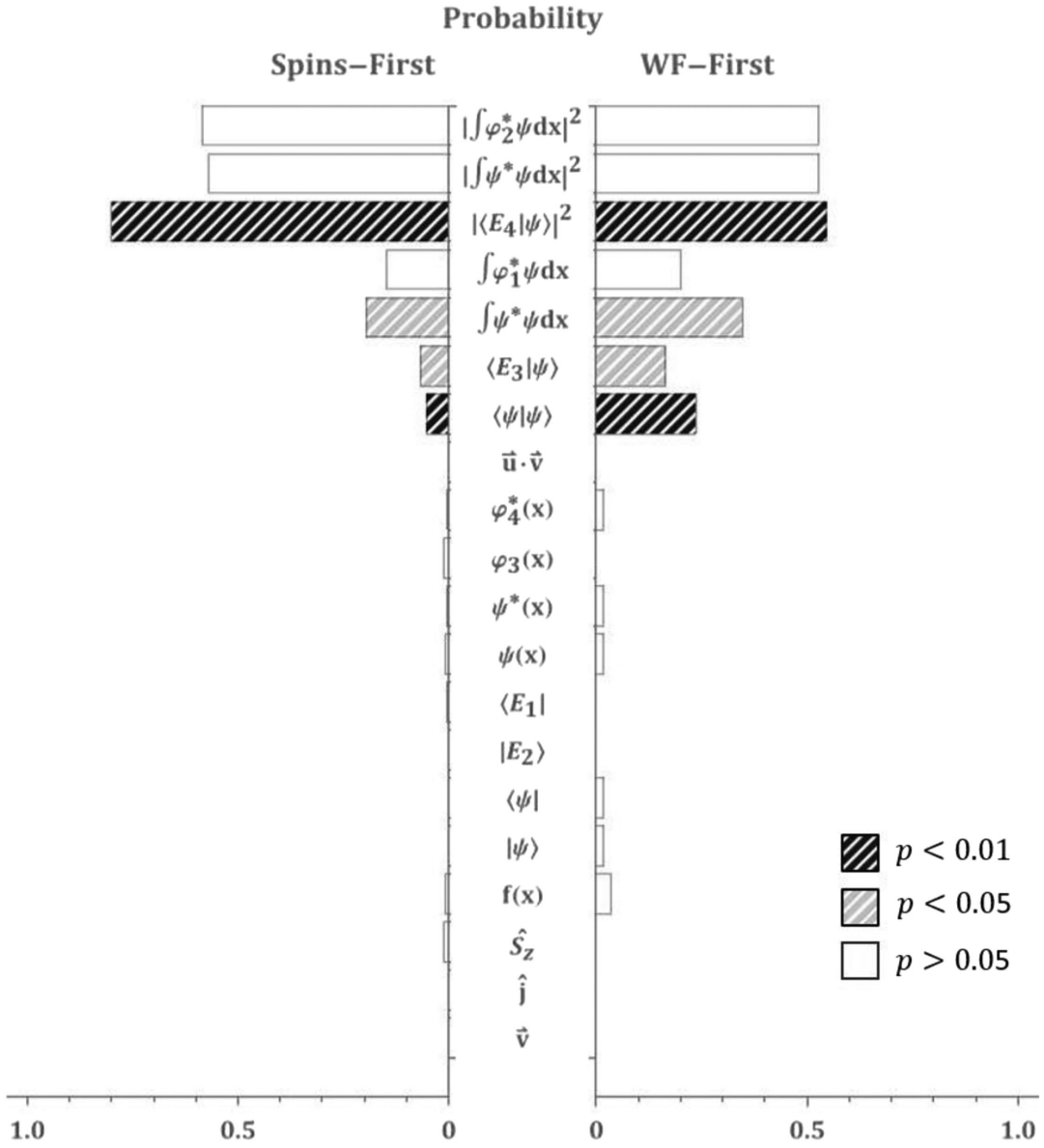


FIG. 7. Bar charts showing the fraction of respondents from spins-first (left) and wave functions-first (right) courses that selected various expressions as representative of the "probability" concept. All statistically significant differences have small effect sizes ($0.1 < \phi < 0.3$).

There are two potential alternative explanations for these findings: that perhaps the spins-first students were more likely to (mis)interpret the question as asking for *probability* than for *probability amplitude* (hence the focus on complex-square expressions), or instead that wave functions-first students selected expressions with matching symbols ($\int \psi^* \psi dx$ and $\langle\psi|\psi\rangle$) as representing probability amplitudes and that spins-first students selected expressions with different ones ($|\int \varphi_2^* \psi dx|^2$ and $|\langle E_4|\psi\rangle|^2$). These can both generally be refuted by additionally looking at the expressions chosen for the *probability* concept, as shown in Fig. 7. The general trends observed within Fig. 6 still hold here, despite this prompt asking for probability, strongly suggesting that students in both courses may not have a very clear conceptual distinction between *probability* and *probability amplitude*. This difficulty distinguishing between these related concepts has been observed by other studies of students' conceptual understanding of quantum mechanics [16,17]. Thus, even if the students were misinterpreting *probability amplitude* as *probability*, the observed trends would still hold. It can also be observed from Fig. 7 that the wave functions-first students' apparent preference for expressions without complex squares holds, even for expressions that do not have matching symbols within the inner product. Taken in conjunction with the previous point about the seeming ambiguity between *probability* and *probability amplitude*, this finding suggests that the preference is real, and thus lends credence to the

potential for strongly developed symbolic forms containing complex squares for these concepts for spins-first students, which we identified in related work [17]. The strength of these symbol templates as representative of probability concepts could potentially be explained by these students starting with Dirac notation, which has high levels of compactness and symbolic support for computation when it comes to inner products, according to Gire and Price's *structural features* framework [13]. Potentially, these features of the initial course notation allowed for a symbol template, such as $|\langle\,|\,\rangle|^2$ to form [17], aspects of which (e.g., the $|\ |^2$) they may then carry forward as an expectation for symbolic forms within other notations but with similar conceptual schemata.

VI. CONCLUSIONS

In summary, while students in spins-first and wave functions-first courses share many similarities in the ways they conceptualize common expressions in quantum mechanics, there are some interesting differences that point to potential curricular effects.

Community detection of expression networks suggests that the initial focus of spins-first courses on linear algebra and geometric interpretations of Dirac expressions causes students to conceptualize Dirac expressions more strongly in vectorlike terms than students enrolled in wave functions-first courses. This is evident from the differences in community structure between the two populations' dendrograms, where the Dirac bras and kets were conceptually connected far less strongly to the generic vector expressions by students enrolled in wave functions-first courses.

Examining the conceptual interpretations of the various expressions on the survey also highlighted some interesting differences between the curricula, with potential pedagogical explanations. One of the biggest distinguishing factors between the two populations was their conceptualization of Dirac bras and kets. While Secs. V B 1 and V B 3 showed that students in spins-first courses tended to view these expressions as more representative of vectors and quantum states than the wave functions-first students did, Sec. V B 2 showed that wave functions-first students thought of bras and kets far more as representing wave functions than the spins-first students did. These interpretations all have merit, of course, as Dirac bras and kets *are* vectors, *do* represent quantum states, and can contain the same information as (and more than) wave functions do. That students in these different curricula view them in all of these terms is encouraging, though the degree to which these interpretations are salient to the students is seemingly affected by their curriculum. The prevalence of wave functions-first students treating Dirac bras and kets as representing wave functions is reasonable, given their text's preference for discussing kets as representative of functions [35–37].

It was also seen in Sec. V B 4 that students in both curricula seemed to view "probability" and "probability amplitude" as somewhat synonymous—or at the very least conceptually ambiguous—as evidenced by their responses for the expressions representative of both concepts being largely similar. It was also noteworthy that spins-first students selected inner product expressions with complex squares as representative of these concepts more often than wave functions-first students, while the opposite was true for inner products without complex squares. This is potentially evidence of different symbolic forms having been developed in the different curricula, in light of prior findings that spins-first students develop strong symbol templates containing complex squares as representing probability concepts [17].

Overall, it can be seen that curricular focuses do in fact influence students' conceptualizations of different symbolic representations. Spins-first courses appear to make students interpret Dirac bras and kets as vectors and quantum states more than wave functions-first courses do, while wave functions-first courses (perhaps unsurprisingly) appear to make them more strongly connect bras and kets to wave functions and not as strongly to vectorlike concepts. It appears that each curriculum strengthens students' conceptual interpretations in different ways, and so a choice of a curriculum should come down to instructor preference as to which conceptions of the various expressions they wish their students to best understand and internalize. It remains to be seen whether, and the extent to which, any instructional materials in either type of course would make significant impacts on these conceptualizations, or whether a common set of conceptualizations could emerge from both courses. Anecdotally, a number of instructors have since chosen to distribute this survey in their own courses, even once we had finished collecting data. It is possible that this survey (or one like it) could serve an instructional purpose, as a sort of review task before or in conjunction with explicit time spent in lecture on these concept-expression pairings.

The geometric analogy inherent in spins-first courses provides a well-established foundation on which students can build their mathematical model of quantum mechanics, countering the abstract and often counterintuitive concepts and outcomes. The findings in this study support the notion that the connection between quantum states and vector spaces is an outcome of this paradigm. While spins-first instruction uses linear algebra as the main mathematical language, the emphasis on vector spaces and Dirac notation might be seen as a better preparation for further study, e.g., graduate-level instruction. Additionally, the familiarity with Dirac notation could reduce the cognitive load for subsequent instruction. On the other hand, students are more familiar with differential equations-based analysis for many physical systems, so starting with wave functions may yield more familiarity with solution techniques for different potentials, reducing the cognitive load in this course, at least initially. The categorization of state vectors as wave functions by wave functions-first students could be a cause for concern for subsequent instruction, although no research has yet explored this transition. There are several possible avenues for additional research, both within these courses and in the

ways in which these different curricula prepare students for subsequent study of quantum mechanics.

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[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevPhysEducRes.21.010113 for the full table of p-values and effect sizes from the comparisons across curricula for each concept-expression pair. It also contains the comparison bar charts for every concept studied in the survey.