Gauge Dependence of Matrix Elements of the Chern-Simons Current K^μ in the Schwinger Model

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The gauge dependence of matrix elements of the Chern-Simons current K^μ can be computed exactly in the $(1+1)$-dimensional Schwinger model. The matrix elements are gauge dependent even in the forward direction, i.e., when the momentum transfer vanishes. The Kogut-Susskind dipole mechanism is derived using functional-integral methods.

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The proton matrix element of K^μ was proposed as a definition of the polarized gluon distribution in the proton.[1] K^μ is the gauge-variant Chern-Simons current,

$$K^\mu = \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}A_\nu(F_{\alpha\beta} - \tfrac{2}{3}A_\alpha A_\beta)\,, \tag{1}$$

whose divergence is related to the topological charge density,

$$\partial_\mu K^\mu = \mathrm{Tr}F^{\mu\nu}\tilde{F}_{\mu\nu}\,. \tag{2}$$

The forward matrix element of K^μ was conjectured to be gauge invariant,[1] because the variation of K^μ under the gauge transformation $A^\mu \to g^{-1}A^\mu g + g^{-1}\partial^\mu g$ is

$$\begin{aligned} K^\mu \to K^\mu &- \tfrac{2}{3}\epsilon^{\mu\nu\alpha\beta}(\partial_\nu g g^{-1})(\partial_\alpha g g^{-1})(\partial_\beta g g^{-1}) \\ &- 2\epsilon^{\mu\nu\alpha\beta}\partial_\nu(\partial_\alpha g g^{-1}A_\beta)\,. \end{aligned} \tag{3}$$

The last term is a total derivative, and vanishes in the limit of zero momentum transfer. The preceding term is locally a total derivative, and hence also vanishes. In Ref. 2, it was argued that the forward matrix element of K^μ must be gauge dependent, even in the forward direction, because of nonperturbative effects connected with the η' mass. In this Letter, I will show that forward matrix elements of K^μ can be computed exactly in the Schwinger model and are gauge dependent because of intermediate Kogut-Susskind unphysical states, which are produced when the η' gets a mass.

The Schwinger model, massless quantum electrodynamics in $1+1$ dimensions, has many of the qualitative features of the $\mathrm{U}(1)_A$ sector of QCD in $3+1$ dimensions. The classical $\mathrm{U}(1)_A$ symmetry is not present in the quantum theory because of the anomaly

$$\partial_\mu j_A^\mu = (e/2\pi)\epsilon_{\mu\nu}F^{\mu\nu}, \quad j_A^\mu = \bar\psi\gamma^\mu\gamma_5\psi\,. \tag{4}$$

The axial U(1) current is spontaneously broken, but there is no U(1) Goldstone boson, because the current is not conserved (4). There is a gauge-variant conserved current, $j_A^\mu - (e/2\pi)K^\mu$, which is spontaneously broken, so there is a Goldstone boson which does not contribute to Green's functions involving only gauge-invariant operators. This is the well-known result of Kogut and Susskind.[3] The same qualitative features are also believed to hold in $(3+1)$-dimensional QCD.

The definition of K^μ for $1+1$ dimensions is

$$K^\mu = 2\epsilon^{\mu\nu}A_\nu, \quad \partial_\mu K^\mu = \epsilon_{\mu\nu}F^{\mu\nu}\,. \tag{5}$$

The gauge variation of K^μ under the gauge transformation $A^\mu \to A^\mu + \partial^\mu\lambda$ is

$$K^\mu \to K^\mu + 2\partial_\nu(\epsilon^{\mu\nu}\lambda)\,, \tag{6}$$

which is a total derivative, and naively vanishes in the forward direction, as in the $(3+1)$-dimensional case, Eq. (3); this naive expectation will be shown to be incorrect.

The Schwinger-model Lagrangian is

$$\mathcal{L}_0 = i\bar\psi\partial\!\!\!/\psi - e\bar\psi\gamma^\mu\psi A_\mu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{7}$$

The theory can be solved exactly by bosonization. The fermion field ψ can be replaced by a boson field ϕ, with the identifications

$$\begin{aligned} &i\bar\psi\partial\!\!\!/\psi = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi, \quad \bar\psi\gamma^\mu\psi = \frac{\epsilon^{\mu\nu}\partial_\nu\phi}{\sqrt{\pi}}\,, \\ &\bar\psi\gamma^\mu\gamma_5\psi = \frac{\partial^\mu\phi}{\sqrt{\pi}}\,. \end{aligned} \tag{8}$$

The Schwinger model will be solved in $n\cdot A = 0$ gauge. Gauge fixing does not introduce a Jacobian which depends on the quantum fields, so we do not need to include it in the functional integral. The gauge field in $n\cdot A = 0$ gauge can be written in terms of the field σ,

$$A^\mu = (\epsilon^{\mu\nu}n_\nu)\sigma\,. \tag{9}$$

The bosonized Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + m\sigma(n\cdot\partial)\phi + \tfrac{1}{2}[(n\cdot\partial)\sigma]^2\,, \tag{10}$$

where $m = e/\sqrt{\pi}$ has the dimensions of a mass, and we have used the identity

$$\epsilon^{\alpha\mu}\epsilon_{\alpha\nu} = -\delta^\mu_\nu\,. \tag{11}$$

This is a quadratic Lagrangian, so the Schwinger model is a free field theory. The two-point correlation functions can be written as a matrix

$$\frac{1}{(q\cdot n)^2(q^2-m^2)}\begin{bmatrix}(n\cdot q)^2 & im(n\cdot q)\\ -im(n\cdot q) & q^2\end{bmatrix}, \qquad (12)$$

where index 1 refers to ϕ and index 2 to σ. The $\langle\phi\phi\rangle$ correlation function has a pole at $q^2=m^2$, so the field ϕ creates a gauge-invariant physical state with mass m. This is the massive photon. There is a gauge-variant state which produces the $q\cdot n$ pole. This state must exist, because there is a (gauge-variant) conserved current which is spontaneously broken. (Goldstone's theorem implies that if a conserved current is spontaneously broken, then there must exist an excitation above the vacuum which has zero energy and zero momentum. Lorentz invariance is not required for the proof.) The state has a dispersion relation which is not Lorentz invariant, because Lorentz invariance has been broken by the choice of gauge. Gauge-invariant correlation functions such as $\langle j_A^\mu j_A^\nu\rangle$ can be expressed in terms of the $\langle\phi\phi\rangle$ correlation function, which is gauge invariant (i.e., independent of n), as can be seen from (12). Gauge-variant correlation functions such as $\langle K^\mu j^\nu\rangle$ involve the correlations $\langle\phi\sigma\rangle$ and/or $\langle\sigma\sigma\rangle$ and depend explicitly on n.

The matrix element

$$\begin{aligned}\langle 0|K^\mu|\phi(q)\rangle &= \lim_{q^2\to m^2}(q^2-m^2)\langle 0|T(K^\mu\phi)|0\rangle\\ &=2imn^\mu/(q\cdot n) \qquad (13)\end{aligned}$$

is explicitly gauge dependent. Continuing the matrix element off shell to $q=0$, one finds that the matrix element is not gauge invariant even in the forward direction. The reason for this is clear from (12). The matrix element of K^μ has a gauge dependence proportional to $q\cdot n$, which would vanish in the forward direction, except that the unphysical state produces a $1/(q\cdot n)^2$ pole, to give the $1/(q\cdot n)$ dependence in (13). Note that the matrix element of $\epsilon_{\mu\nu}F^{\mu\nu}$,

$$\langle 0|\epsilon_{\mu\nu}F^{\mu\nu}|\phi(q)\rangle = -iq_\mu\langle 0|K^\mu|\phi(q)\rangle = 2m\,, \qquad (14)$$

is n independent, even at nonzero q.

To avoid continuing a state off shell, and to make the connection with the proton matrix element of K^μ even more explicit, couple the Schwinger-model Lagrangian (7) to a neutral massive fermion χ,

$$\mathcal{L}=\mathcal{L}_0+i\bar\chi\not\partial\chi-M\bar\chi\chi+\sqrt{\pi}\zeta\bar\psi\gamma^\mu\psi\bar\chi\gamma_\mu\chi\,, \qquad (15)$$

where ζ is an infinitesimal coupling constant. The bosonized Lagrangian is

$$\mathcal{L}=\mathcal{L}_0+i\bar\chi\not\partial\chi-M\bar\chi\chi+\zeta\epsilon^{\mu\nu}\partial_\nu\phi\bar\chi\gamma_\mu\chi\,. \qquad (16)$$

Let $|\chi(p)\rangle$ denote the state of a χ particle with momentum p. Then from the graph in Fig. 1, one finds

$$\langle\chi(p')|K^\mu|\chi(p)\rangle = \frac{2\zeta mn^\mu\epsilon^{\alpha\beta}q_\alpha}{(q\cdot n)(q^2-m^2)}\bar u(p')\gamma^\beta u(p)\,, \qquad (17)$$

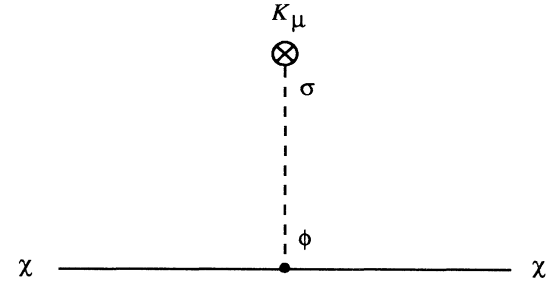

FIG. 1. The graph contributing to the matrix element of K^μ between two χ states.

where $u(p)$ is a spinor, and $q=p'-p$. The forward matrix element is

$$\langle\chi(p)|K^\mu|\chi(p)\rangle = -\frac{4\zeta}{m}n^\mu\lim_{q\to 0}\frac{\epsilon^{\alpha\beta}q_\alpha p_\beta}{q\cdot n}\,, \qquad (18)$$

using $\bar u(p)\gamma^\mu u(p)=2p^\mu$. This is clearly n dependent and singular as $q\to 0$.

The class of gauges considered so far, $n\cdot A=0$, includes light-cone gauge and axial gauge. It is also instructive to compute the matrix element of K_μ in a Lorentz-invariant gauge. Consider the gauge choice $\partial_\mu A^\mu=0$. The vector potential can be written as

$$eA_\mu=\sqrt{\pi}\epsilon_{\mu\nu}\partial^\nu\psi\,, \qquad (19)$$

which automatically enforces the gauge constraint. The Jacobian for the change of variables is a constant, and can be ignored. The bosonized Schwinger-model Lagrangian (7) in Lorentz gauge is obtained using (8) and (19),

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\partial_\mu\phi\partial^\mu\psi+\frac{1}{2m^2}(\partial^2\psi)^2\,, \qquad (20)$$

where $m=e/\sqrt{\pi}$. To eliminate the four-derivative term in (20), it is convenient to introduce an auxiliary field χ, and modify (20) to

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\partial_\mu\phi\partial^\mu\psi+\frac{1}{2m^2}(\partial^2\psi)^2\\ &-\tfrac{1}{2}m^2\left(\chi+\frac{\partial^2\psi}{m^2}\right)^2. \qquad (21)\end{aligned}$$

The functional integral over χ produces an irrelevant constant. The modified Lagrangian (21) can be written as

$$\begin{aligned}\mathcal{L}&=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\partial_\mu\phi\partial^\mu\psi+\partial_\mu\chi\partial^\mu\psi-\tfrac{1}{2}m^2\chi^2\\ &=\tfrac{1}{2}\partial_\mu\xi_1\partial^\mu\xi_1-\tfrac{1}{2}\partial_\mu\xi_2\partial^\mu\xi_2+\tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi-\tfrac{1}{2}m^2\chi^2\,, \qquad (22)\end{aligned}$$

where $\xi_1=\phi+\psi$ and $\xi_2=\chi-\psi$. The theory therefore has one physical particle with mass m, and two massless particles $\xi_{1,2}$ with opposite kinetic-energy terms. The ξ's are the Kogut-Susskind dipole. It is easy to verify that there are no unphysical massless poles in Green's functions involving only gauge-invariant operators, such as $\bar{\psi}\gamma^\mu\psi = \epsilon^{\mu\nu}\partial_\nu(\xi_1+\xi_2-\chi)$, or $mF_{\mu\nu}=\epsilon_{\mu\nu}\partial^2(\chi-\xi_2)$. However, there are unphysical singularities in matrix elements of the gauge-variant operator $K_\mu = 2\partial_\mu\psi/m$. Thus the forward matrix element of K_μ for the theory (15) considered before is

$$\langle\chi(p')|K^\mu|\chi(p)\rangle = -\frac{2\zeta m q^\mu \epsilon^{\alpha\beta} q_\alpha}{q^2(q^2-m^2)}\bar{u}(p')\gamma^\beta u(p) . \quad (23)$$

The forward matrix element is

$$\langle\chi(p)|K^\mu|\chi(p)\rangle = \frac{4\zeta}{m}\lim_{q\to 0}\frac{q^\mu\epsilon^{\alpha\beta}q_\alpha p_\beta}{q^2} , \quad (24)$$

which is singular as $q\to 0$, and different from the result for the gauge choice $n\cdot A=0$. The matrix element of a gauge-invariant operator such as $F_{\mu\nu}$ in Lorentz gauge is the same as for $n\cdot A=0$ gauge.

The calculations in this paper have been performed for an exactly soluble model in $1+1$ dimensions. The matrix elements of K^μ are gauge dependent because K^μ has a nonvanishing coupling to unphysical massless states. The existence of these states is guaranteed in $(3+1)$-dimensional QCD, because there is a conserved (but gauge-variant) current which is spontaneously broken, but there is no U(1)-singlet massless particle in the physical spectrum. These massless unphysical states exist even in "physical" gauges, such as axial gauge, as has been shown explicitly in the Schwinger model. Thus the matrix elements of K_μ will be gauge dependent in QCD, even at zero momentum transfer.

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