Cascade of mesostrophy in turbulence with reduced vortex stretching

Wouter J. T. Bos
CNRS, Ecole Centrale de Lyon, INSA Lyon, Université Claude Bernard Lyon 1, LMFA, UMR5509, 69134 Écully, France

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In three-dimensional turbulence, vortex stretching is the central mechanism enabling the transfer of kinetic energy toward smaller scales. If vortex stretching is absent in three-dimensional turbulence, energy is no longer conserved and enstrophy cascades to smaller scales. In this paper we propose a system that interpolates between these two cases by reducing the strength of vortex stretching in the Navier-Stokes equations. The resulting dynamics yield a cascade that is neither a pure energy cascade nor a pure enstrophy cascade. We refer to this process as a mesostrophy cascade. We formulate a consistent picture describing this cascade and the characteristic scales involved in the mesostrophy balance. We illustrate this picture via numerical integrations of the eddy-damped quasinormal Markovian model with reduced vortex stretching.

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I. INTRODUCTION

A key feature of three-dimensional turbulence is its dissipative nature. Irrespective of the value of the viscosity, energy is dissipated at a rate determined by the large-scale flow structure [1,2]. The underlying process is the cascade of energy toward small scales [3–5], associated with the nonlinearity of the Navier-Stokes equations. Simultaneously, this nonlinear interaction produces and transfers enstrophy [6,7]. The energy cascade adapts its length in order to balance energy input into the system on one side, and dissipation, governed by the enstrophy, produced by vortex stretching on the other. Indeed, the product of enstrophy and viscosity determines the viscous dissipation. Decreasing the viscosity will lead to an extension of the energy cascade down to smaller scales, allowing for more enstrophy to be produced by vortex stretching, in order to keep this product constant.

The investigation of three-dimensional turbulence *without* vortex stretching was motivated by the academic curiosity about how the absence of vortex stretching would alter the dynamics of isotropic turbulence [8]. It was shown that, in the absence of vortex stretching, enstrophy W is conserved and transferred toward small scales [8,9] very much like in the case of two-dimensional (2D) turbulence [10–12]. It was, however, realized that energy is not conserved, but helicity is [13]. These ingredients lead to an inverse cascade of helicity [9] and the formation of large-scale structures, predictable by point-vortex statistical mechanics [14,15]. Interestingly, this scenario was proposed in Newtonian turbulence [16] as a possibility, but not observed [17–19]. In the limit of conserved enstrophy, this scenario becomes thus relevant. At this point it seemed that we had sufficiently explored this, somewhat unphysical, system.

However, recently we investigated the behavior of the system, without suppressing all of the vortex stretching, in channel flow. It was realized that this system shows remarkable similarities with viscoelastic flows [20]. In particular, starting from the equation with partially suppressed vortex stretching (1) below, one can analytically predict how the near-wall scaling is modified by following von Kármán's arguments [21]. The resulting velocity profile and the friction factor were shown to be close to those observed in experiments on pipe flow in the presence of polymer additives [22,23], and we anticipate that the ideas might be relevant for microbubble drag reduction [24–26].

The considered equation reads

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = (1-\lambda)\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} - \lambda \nabla P_{\omega} + \nu \Delta \boldsymbol{\omega} + \nabla \times \boldsymbol{f}, \tag{1}$$

where the vorticity $\boldsymbol{\omega}$ is the curl of the velocity $\boldsymbol{u}$, ν the kinematic viscosity, and P_{ω} a Lagrange multiplier, ensuring $\nabla \cdot \boldsymbol{\omega} = 0$. We also add an external force term $\boldsymbol{f}$ to allow for the investigation of statistically steady flow in the absence of mean gradients (this will be useful in the current study, but was absent in [20]). The parameter λ will be varied between 0 and 1 in the present investigation.

The unexpected application of turbulence with partial suppression of vortex stretching to drag reduction by polymers motivates one to have a closer look at the dynamics of Eq. (1). For instance, a key quantity in drag reduction turns out to be the modification of the enstrophy production. Since the term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ is responsible for the production of enstrophy, it seems clear from Eq. (1) that enstrophy production will be reduced when λ is increased, but to what extent? How does the enstrophy depend on λ in the simplest case of mirror-symmetric (helicity-free) isotropic turbulence governed by Eq. (1)? Since we were not able to answer this question from dimensional considerations alone, we decided to undertake the present study of isotropic turbulence with reduced vortex stretching.

We will show in the following that the system of reduced vortex stretching can be understood in the framework of the Kolmogorov-Richardson cascade once we realize that Eq. (1) conserves a quadratic quantity for $0 < \lambda < 1$, intermediate between energy and enstrophy. As a variation on the words enstrophy and palinstrophy, we will call this quantity mesostrophy, from the greek words $\mu\varepsilon\sigma o$ (middle) and $\sigma\tau\rho o\varphi\acute{\eta}$ (turn), since its behavior is somewhere in the middle between energy and enstrophy.

In the next section we will develop the theory allowing us to describe the mesostrophy cascade. In Sec. III we present the eddy-damped quasinormal Markovian (EDQNM) equations for the present system. In Sec. IV these equations are numerically integrated. Section V presents the conclusions.

II. ANALYTICAL CONSIDERATIONS

We will describe in this section analytically how the dynamics of Eq. (1) can be described on the level of the kinetic energy spectrum. For that purpose, we first write down the modified Lin equation. Subsequently, in Sec. II B, we prescribe the simplest possible closure for the nonlinear transfer. This closure is associated with the mesostrophy cascade, which we discuss in Sec. II C. In Sec. II D we show what the mesostrophy cascade implies for second- and third-order statistics.

A. Spectral dynamics

The starting point is the system (1). In particular, in the present investigation we will be interested in the characterization of the energy spectrum $E(k)$, defined as

$$E(k) = \frac{1}{2} \int_{|\boldsymbol{k}|=k} \langle |\boldsymbol{u}(\boldsymbol{k})|^2 \rangle dS(k), \tag{2}$$

with $\boldsymbol{u}(\boldsymbol{k})$ the Fourier transform of $\boldsymbol{u}(\boldsymbol{x})$ where the integral is carried out over spherical shells of radius k, such that

$$\int E(k)dk = \frac{\langle |\boldsymbol{u}|^2 \rangle}{2}, \tag{3}$$

with $\boldsymbol{k}$ the wave vector and k its norm, the wave number.

For isotropic turbulence, one can write the Lin equation [27,28]

$$\frac{\partial E(k)}{\partial t} = P(k) + T(k) - 2\nu k^2 E(k). \tag{4}$$

In this equation $P(k)$ is the energy injection spectrum and $T(k)$ the nonlinear transfer term. The enstrophy is obtained from the energy spectrum by

$$W = \int k^2 E(k) dk = \frac{\langle |\boldsymbol{\omega}|^2 \rangle}{2}. \tag{5}$$

For the enstrophy spectrum we have, consequently, the equation

$$\frac{\partial k^2 E(k)}{\partial t} = k^2 P(k) + k^2 T(k) - 2\nu k^4 E(k). \tag{6}$$

The nonlinear term of the Navier-Stokes equations conserves energy so that we can write the energy transfer as the divergence of a flux.

$$T(k) = -\frac{\partial}{\partial k} \Pi_E(k), \tag{7}$$

with $\Pi_E(k)$ the flux of energy. If this flux is zero at $k = 0$ and $k = \infty$, the integral of the transfer is zero:

$$\int T(k) dk = 0. \tag{8}$$

Similarly, if enstrophy is conserved, we can write for a flux of enstrophy $\Pi_W(k)$,

$$k^2 T(k) = -\frac{\partial}{\partial k} \Pi_W(k), \tag{9}$$

so that

$$\int k^2 T(k) dk = 0 \tag{10}$$

holds.

For three-dimensional turbulence Eq. (8) holds, whereas for two-dimensional turbulence both relations (8) and (10) are simultaneously satisfied. It was shown that for three-dimensional turbulence without vortex stretching, energy conservation breaks down, but enstrophy is strictly conserved [13]. We then come back to the question: when a part of the vortex stretching is removed [$0 < \lambda < 1$ in Eq. (1)], can we still describe the system by a cascade picture, and what is the quantity that is transferred?

To show how we can combine the expressions for an enstrophy flux and an energy flux, we start by rewriting Eq. (1):

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (1 - \lambda) \underbrace{(-\boldsymbol{u} \cdot \nabla \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \boldsymbol{u})}_{\mathrm{I}} + \lambda \underbrace{(-\boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \nabla P_\omega)}_{\mathrm{II}} + \nu \Delta \boldsymbol{\omega} + \nabla \times \boldsymbol{f}. \tag{11}$$

At this point the equation is identical to (1), but we have arranged the terms in such a form that the terms denoted by I correspond to the Navier-Stokes equations and the terms denoted by II to turbulence without vortex stretching.

In statistically homogeneous turbulence, the kinetic energy spectrum is readily obtained from the vorticity, by the relation

$$E(k) = \frac{1}{2k^2} \int_{|\boldsymbol{k}|=k} \langle |\boldsymbol{\omega}(\boldsymbol{k})|^2 \rangle dS(k), \tag{12}$$

where the integral is again carried out over spherical shells of radius k. Therefore, we can derive the Lin equation starting from Eq. (11). Combining the ideas of a conserved energy flux for I and a conserved enstrophy flux for II, this yields for the nonlinear interaction term,

$$T(k) = (1 - \lambda) \frac{\partial}{\partial k} \Pi_E(k) + \lambda k^{-2} \frac{\partial}{\partial k} \Pi_W(k). \tag{13}$$

Up to here we have not made any assumptions and Eq. (13) describes exactly the nonlinear interactions associated with Eq. (1), once the correct expressions for Π_E and Π_W are prescribed. No exact, closed expressions as a function of statistical quantities exist so that to advance analytically, we need to introduce a model.

B. Generalization of simple flux models

If we assume a local cascade, and use dimensional analysis, the fluxes can be modeled by

$$\Pi_E(k) \sim (k^{5/3}E(k))^{3/2} \quad \Pi_W(k) \sim (k^3 E(k))^{3/2}. \tag{14}$$

These are the simplest models [29,30], describing the limiting cases of isotropic turbulence with and without vortex stretching. While these expressions are not proposed as a precise model, they should yield correct qualitative behavior in a large inertial range, if energy transfer is dominantly local. It is then insightful to model, qualitatively, the intermediate case (13) using these expressions. This yields

$$T(k) = (1-\lambda)\frac{\partial}{\partial k}\left[C_K^{-1}k^{5/3}E(k)\right]^{3/2} + \lambda k^{-2}\frac{\partial}{\partial k}\left[C_W^{-1}k^3 E(k)\right]^{3/2}. \tag{15}$$

In Sec. IV we will verify whether the insights obtained analytically using this model are in agreement with more sophisticated turbulence closure. For the moment we consider some properties of this elementary representation of $T(k)$. The first question that we address is whether we can determine constant flux solutions of this model. To this end, we multiply $T(k)$ by $k^{2\beta}$, and attempt to find a solution that can be written in the form of the divergence of a flux.

$$k^{2\beta}T(k) = \frac{\partial}{\partial k}\Pi_\mu(k). \tag{16}$$

Dimensionally, the flux Π_μ is

$$\Pi_\mu(k) = \left[C_\mu^{-1}k^{(5/3)+(4/3)\beta}E(k)\right]^{3/2}. \tag{17}$$

Manipulation of these expressions shows that such a constant flux solution exists where $C_\mu(\lambda)$ is given by

$$C_\mu(\lambda)^{-3/2} = C_K^{-3/2}(1-\lambda) + \lambda C_W^{-3/2}, \tag{18}$$

which yield $C_\mu = C_K$ for $\lambda = 0$ and $C_\mu = C_W$ for $\lambda = 1$. For the mesostrophy exponent we find

$$\beta = \frac{\lambda}{C_W^{3/2}/C_K^{3/2}(1-\lambda)+\lambda}. \tag{19}$$

For small amounts of suppression, $\lambda \ll 1$, we have

$$\beta \approx \left(\frac{C_K}{C_W}\right)^{3/2}\lambda. \tag{20}$$

Note that in the case $C_K = C_W$, expressions (18) and (19) simplify considerably, to $C_\mu = C_K$ and $\beta = \lambda$. Yielding the model

$$\Pi_\mu(k) = \left[C_K^{-1}k^{(5/3)+(4/3)\lambda}E(k)\right]^{3/2}. \tag{21}$$

However, there is no reason to suppose that the constants measuring the enstrophy and energy transfer are equal. Nevertheless, for an order of magnitude estimate, this simplification can be convenient.

In the case of zero production and dissipation of mesostrophy, we write

$$\frac{\partial k^{2\beta}E(k)}{\partial t} = -\frac{\partial}{\partial k}\Pi_\mu(k). \tag{22}$$

Since Eq. (16) is written in the form of the divergence of a flux, this implies that the integral over k of $k^{2\beta}E(k)$ is conserved,

$$\frac{d\mu}{dt} = 0, \tag{23}$$

where

$$\mu = \int_0^\infty k^{2\beta} E(k) dk. \tag{24}$$

Equation (23) shows that the mesostrophy μ is an invariant of the system. This is the most important observation of the present investigation. Indeed, this observation allows us to predict the scaling of the kinetic energy spectrum for arbitrary values of λ in the limit of large Reynolds numbers. It allows us also to predict that, assuming ergodicity, equipartition of mesostrophy is a statistical equilibrium solution of the truncated nondissipative system (see Sec. IV D).

C. Cascade of mesostrophy

We start by characterizing the cascade of mesostrophy in a forced-dissipative system. Let us assume that the cascade is toward large k and that it is forced, at small wave numbers around k_0 (large scales) only, by $P(k) = \epsilon_0 \delta(k - k_0)$. Adding production and dissipation to Eq. (16), and considering a steady state,

$$0 = k^{2\beta} P(k) + \frac{\partial}{\partial k} \Pi_\mu(k) - 2\nu k^{2\beta+2} E(k). \tag{25}$$

Integrating Eq. (25) from $k > k_0$ to infinity, we have

$$\left(C_\mu^{-1}(\lambda) k^{(5+4\beta)/3} E(k)\right)^{3/2} = \epsilon_\mu, \tag{26}$$

where

$$\epsilon_\mu = \int 2\nu k^{2\beta+2} E(k) dk. \tag{27}$$

Therefore, the energy spectrum associated with the mesostrophy cascade in the inertial range should be

$$E(k) = C_\mu \epsilon_\mu^{2/3} k^{-(5+4\beta)/3}, \tag{28}$$

which is consistent with our closure assumption (15). Expression (28) is the generalization of Kolmogorov's energy spectrum for the mesostrophy cascade. This expression will be instrumental in obtaining expressions for the integral quantities as a function of β and the Reynolds number.

A further consequence of the existence of a mesostrophy cascade is that we can generalize the viscous (Kolmogorov) scale to

$$k_\mu = \left(\frac{2+\beta}{3C} \frac{\epsilon_\mu^{1/3}}{\nu}\right)^{3/(4+2\beta)}, \tag{29}$$

which is a quantity that we will need in order to estimate the influence of λ on the enstrophy and skewness.

D. Energy, enstrophy, and velocity derivative skewness

Now that we have determined the self-similar scaling of the mesostrophy cascade, we can also quantify how integral quantities vary as a function of λ or β and the Reynolds number. We will compute three key quantities, associated with second- and third-order moments of the velocity and its gradients. These quantities are the kinetic energy, enstrophy, and velocity derivative skewness. These are defined as

$$E \equiv \frac{1}{2}\langle |\boldsymbol{u}|^2 \rangle = \int_0^\infty E(k)dk, \tag{30}$$

$$W \equiv \frac{1}{2}\langle |\boldsymbol{\omega}|^2 \rangle = \int_0^\infty k^2 E(k)dk, \tag{31}$$

$$-S \equiv \frac{\left\langle \left(\frac{\partial u}{\partial x}\right)^3 \right\rangle}{\left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle^{3/2}} = a\frac{\int_0^\infty k^2 T(k)dk}{\left(\int_0^\infty k^2 E(k)dk\right)^{3/2}}, \tag{32}$$

where $a = 15^{3/2}/(2^{1/2}35) \approx 0.918$. In the present investigation, the control parameter is λ. However, when considering a physical system, such as polymer turbulence, the parameter λ is unknown and the parameter that can be measured from the data is β (which should be of the same order of magnitude as λ). For the sake of later convenience, we will express in this section all quantities as a function of β.

We evaluate, assuming a steady state, the energy input to determine the dissipation rate. Indeed, we should have

$$\int (k^{2\beta} P(k) - 2\nu k^{2\beta+2} E(k))dk = 0, \tag{33}$$

and since in our case $P(k) = \epsilon_0 \delta(k - k_0)$, we have

$$\epsilon_\mu = k_0^{2\beta}\epsilon_0. \tag{34}$$

Since ϵ_μ and ϵ are fixed and proportional, for a fixed forcing, we will use the input rate and scale to define a Reynolds number

$$\mathrm{Re} = \frac{\epsilon_0^{1/3} L^{4/3}}{\nu}, \tag{35}$$

where we define $L = 1/k_0$.

To determine scaling as a function of integral quantities, we replace the integration bounds in Eq. (30) by k_0 and k_μ, respectively. Integrating the theoretical spectra, we find

$$\frac{\langle |\boldsymbol{u}|^2 \rangle_\lambda}{\langle |\boldsymbol{u}|^2 \rangle_{\lambda=0}} = \frac{C_\mu(\beta)/C_K}{1+2\beta} \sim \mathrm{Re}^0. \tag{36}$$

This shows that changing β from 0 to 1 might change the kinetic energy, but not dramatically. Indeed, this expression shows that the kinetic energy is independent of the Reynolds number, if the Reynolds number is large enough.

This is different for the enstrophy. We find, by integrating $k^2 E(k)$,

$$\frac{\langle |\boldsymbol{\omega}|^2 \rangle_\lambda}{\langle |\boldsymbol{\omega}|^2 \rangle_{\lambda=0}} = \frac{3}{2C_\mu}\frac{\{[(B\,\mathrm{Re})^\gamma]^{1-\beta} - 1\}}{(1-\beta)\mathrm{Re}}, \tag{37}$$

where $\gamma = 2/(2+\beta)$ and $B = (2+\beta)/3C_\mu$. The enstrophy is thus a function of both the Reynolds number and β. This somewhat complicated expression will be shown to correctly describe the data obtained in Sec. IV. At large Reynolds numbers, it simplifies, for $0 < \beta < 1$, to

$$\frac{\langle |\boldsymbol{\omega}|^2 \rangle_\lambda}{\langle |\boldsymbol{\omega}|^2 \rangle_{\lambda=0}} \sim \mathrm{Re}^{-3\beta/(2+\beta)}. \tag{38}$$

The skewness of the longitudinal velocity gradients is a key quantity in the analysis of turbulence, since it links the shape of the probability density function to the kinetic energy spectrum and the nonlinear transfer [31,32]. In an isotropic turbulent flow, this quantity is related to the energy transfer by relation (32). Using expressions (15) and (29) one can then estimate for the skewness,

$$\frac{S(\lambda)}{S(0)} = (1-\lambda)[1-\beta(\lambda)]^{1/2}. \tag{39}$$

As an order of magnitude estimate we can use $\lambda \approx \beta$, allowing for the approximation,

$$\frac{S(\beta)}{S(0)} \approx (1-\beta)^{3/2}. \tag{40}$$

To illustrate these expressions, we will compute the energy, enstrophy, and skewness using the EDQNM closure. This closure is discussed in the following section.

III. SPECTRAL MODELS FOR THE MESOSTROPHY CASCADE

A. Diffusion approximation for the mesostrophy cascade

The model (15) is the simplest possible model compatible with the requirements that fluxes of energy and enstrophy are conserved in the limits $\lambda = 0, 1$. It is not compatible with the equipartition property [Sec. IV D, Eq. (56)]. A more complicated expression can be formulated, by generalizing Leith's diffusion model [33], compatible with both equipartition and flux solutions,

$$\Pi_\mu \sim E(k)^{1/2} k^{(11-4\lambda)/2} \frac{\partial}{\partial k}\left(\frac{E(k)}{k^{2(1-\lambda)}}\right). \tag{41}$$

This model represents the idea of local transfer, compatible with the foregoing assumptions. Additional constraints can be added, leading to even more sophisticated expressions (e.g., [34]). Nonlocal interactions can be added by using closures first proposed by Heisenberg [30,35,36]. These types of models are useful to analyze certain key features of turbulent flows, once they are fairly well understood. Indeed, the link with the governing equations is tenuous and since we have modified the governing equations, we will use a more sophisticated approach, which can be systematically derived from the governing equations.

B. EDQNM approximation for the mesostrophy cascade

In the present investigation we will go beyond the classical flux closures and use analytical closure theory to further investigate the spectral behavior of Eq. (1). Analytical closures such as the Direct Interaction Approximation [37,38] or EDQNM [28,39,40] are useful tools to efficiently explore the spectral dynamics of turbulent systems. Furthermore, they allow one to also investigate modified Navier-Stokes turbulence, such as turbulence in noninteger dimensional space [41], turbulence with a modified Lagrangian time correlation [42], helically decimated turbulence [43], time-reversible turbulence, or turbulence in higher dimensions [44,45]. Even though most of these systems can be addressed by direct numerical simulations (DNS) [46–48], closure often allows for additional insights due to its ability to address high Reynolds numbers, since it directly models statistical moments. This allows one to carry out parameter sweeps unrealizable, or very expensive, if carried out using DNS.

Nevertheless, the EDQNM approach does have a certain number of limitations. First of all, to keep the closure approach at an acceptable level of complexity, only statistically homogeneous flows can be considered. Secondly, closures can at present not take into account the imprint on the statistics of long-living coherent structures. Furthermore, by directly modeling the ensemble average of a turbulent flow, all information on instantaneous realizations is lost. The use of closure is therefore ideally accompanied by direct numerical simulations at moderate resolution. For the present case, we stress the fact that information on the two limiting cases, $\lambda = 0, 1$ is available in literature. For $\lambda = 0$, this is conventional Navier-Stokes turbulence and for $\lambda = 1$, it is turbulence without vortex stretching, investigated in a recent study [9]. The role of closure in the present context is therefore the interpolation between the two known limiting cases. We leave a numerical assessment by direct numerical simulation for further research.

In Ref. [8] we derived the EDQNM closure for turbulence without vortex stretching. The more general expression of reduced vortex stretching, considered in the present investigation, was also derived but not analyzed. There the only cases investigated were $\lambda = 0$ and $\lambda = 1$. We recall here the expression for variable λ, and for the derivation we refer to [8]. A word of caution is that the definition of λ is opposite in [8] so that the cases $\lambda = 0$ and $\lambda = 1$ are inverted. Using our current definition of λ the expression for the transfer is

$$T(k) = \frac{1}{2}\int_\Delta \frac{dp}{p}\frac{dq}{q}\Theta_{k,p,q}\{[f^\lambda_{(1)}X_{(1)} + f^\lambda_{(2)}X_{(2)}]k^3E(p)E(q) - [f^\lambda_{(3)}X_{(3)} + f^\lambda_{(4)}X_{(4)}]p^3E(k)E(q) - [f^\lambda_{(5)}X_{(5)} + f^\lambda_{(6)}X_{(6)}]q^3E(p)E(k)\}, \tag{42}$$

where the Δ denotes the integration domain in the p-q plane, where k, p, q can form the sides of a triangle. The X terms are

$$X_{(1)} = (1 - z^2)(1 + y^2), \quad X_{(2)} = -xyz - y^2z^2, \tag{43}$$

$$X_{(3)} = xy(1 - z^2), \quad X_{(4)} = z(-y^2 - xyz), \tag{44}$$

$$X_{(5)} = y(-x^2 - xyz), \quad X_{(6)} = (y + zx)(1 + y^2) \tag{45}$$

and the f^λ terms read

$$f^\lambda_{(1)} = \left(\frac{q}{k}y\lambda + (1 - \lambda)\right)^2, \tag{46}$$

$$f^\lambda_{(2)} = \left(\frac{p}{k}z\lambda + (1 - \lambda)\right)\left(\frac{q}{k}y\lambda + (1 - \lambda)\right), \tag{47}$$

$$f^\lambda_{(3)} = \left(\frac{q}{p}x\lambda + (1 - \lambda)\right)\left(\frac{q}{k}y\lambda + (1 - \lambda)\right), \tag{48}$$

$$f^\lambda_{(4)} = \left(\frac{k}{p}z\lambda + (1 - \lambda)\right)\left(\frac{q}{k}y\lambda + (1 - \lambda)\right), \tag{49}$$

$$f^\lambda_{(5)} = \left(\frac{p}{q}x\lambda + (1 - \lambda)\right)\left(\frac{q}{k}y\lambda + (1 - \lambda)\right), \tag{50}$$

$$f^\lambda_{(6)} = \left(\frac{k}{q}y\lambda + (1 - \lambda)\right)\left(\frac{q}{k}y\lambda + (1 - \lambda)\right). \tag{51}$$

The triad interaction time is defined as

$$\Theta(k, p, q) = \frac{1 - \exp[-(\theta_k + \theta_p + \theta_q)t]}{\theta_k + \theta_p + \theta_q} \tag{52}$$

with

$$\theta_k = \alpha\sqrt{\int_0^k s^2E(s)ds} + \nu k^2, \tag{53}$$

and $\alpha = 0.5$. Changing the value of α changes the Kolmogorov constant and its enstrophy equivalent, but is not expected to change the dynamics qualitatively.

IV. NUMERICAL INVESTIGATION OF THE MESOSTROPHY CASCADE

In this section we numerically integrate the eddy-damped quasinormal Markovian model to assess the following ideas:

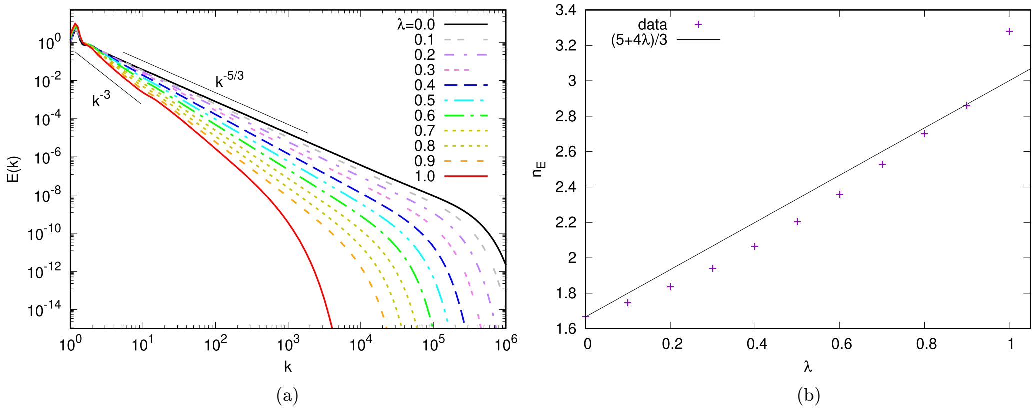

FIG. 1. (a) Energy spectra for k larger than the forcing range, for values of $0 \leqslant \lambda \leqslant 1$. Also shown are the expected scaling behaviors for $\lambda = 0, 1$. (b) Spectral slopes as a function of λ.

(1) We can define a quadratic quantity (mesostrophy), which is conserved by the nonlinearity of turbulence with partially suppressed vortex stretching.

(2) In the presence of forcing and dissipation, an inertial range will establish itself with a scaling $E(k) = C_\mu \epsilon_\mu^{2/3} k^{-(5+4\beta)/3}$.

(3) Integrating moments of this spectrum allows one to estimate the value of the energy and the enstrophy and derivative skewness characterizing the modified system.

(4) The conservation of mesostrophy admits an equilibrium solution for the undamped, unforced system, corresponding to an equipartition of μ, associated with an equilibrium spectrum $E(k) \sim k^{2(1-\beta)}$.

We numerically integrate the closure equations presented in the previous section, typically using a grid of 200 logarithmically equidistant wave numbers. The viscosity is varied in between 10^{-2} and 10^{-8}. The injected energy ϵ_0 is set to unity and the forcing wave number is $k_0 = 1$. The Reynolds number is then directly ν^{-1}. All results are reported in a steady state.

A. Energy spectra

In Fig. 1(a) we show energy spectra. We focus on the range $k \geqslant k_0$ and viscosity $\nu = 10^{-8}$. Results illustrate a smooth transition from a $-5/3$ spectrum toward a -3 spectrum (with logarithmic correction [9,49]). Figure 1(b) compares the slopes of the spectra, obtained as the logarithmic derivative at $k = 100$. Compensating the spectra at this same wave number yields for the dimensionless constants $C_K = 1.70$ and $C_W = 3.8$.

It is observed that the relation $n_E = (5 + 4\lambda)/3$ describes the data qualitatively, but deviations from the prediction are significant using the assumption $\beta = \lambda$. There is thus no linear relation between β and λ. In particular, the slope deviates significantly from $n_E = (5 + 4\lambda)/3$ for $\lambda = 1$. There are several possible reasons for this behavior.

First, as also observed in 2D turbulence, around this limit the cascade becomes dominated by nonlocal interactions, yielding logarithmic corrections to the slope of the spectrum [49]. Nonlocal interactions converge more slowly than local ones in the numerical scheme used [40]. A second reason is that, as observed in simulations of 2D turbulence [50], the -3 power-law scaling is approached only very slowly when the Reynolds number is increased. Finally, as observed in turbulence with noninteger dimension, a reversal of the energy transfer takes place near the 2D limit [41].

This limit, $\lambda \to 1$, is therefore an interesting case to investigate using direct numerical simulations. However, we anticipate that for the application of the current system to understand drag

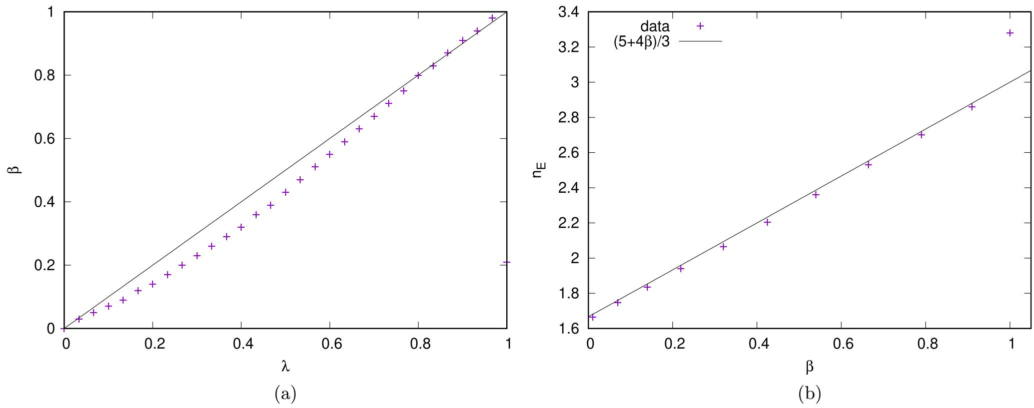

FIG. 2. (a) The relation between β and λ for (b) The slope of the kinetic energy spectrum as a function of the mesostrophy exponent β.

reduction, it is rather the other limit, associated with small λ, that is most relevant. That limit seems better behaved in the present closure results.

B. Determination of the mesostrophy exponent

Determining scaling exponents is not straightforward. Very high values of the Reynolds number are needed to determine the exponent of $E(k)$ and thereby β [51–53], in particular in the case of an enstrophy cascade [50]. In future studies if experiments or DNS are considered, this can lead to a substantial problem, since very high Reynolds numbers are rarely available.

There is, however, an alternative way to determine β even at low values of the Reynolds number. In the present investigation, β is defined as the exponent for which the transfer of mesostrophy is conservative. This constraint is in principle satisfied for arbitrary Reynolds numbers. Since mesostrophy is conserved by the nonlinearity, we should have

$$\int k^{2\beta} T(k) dk = 0, \tag{54}$$

which allows one to determine β from the transfer spectrum. An alternative way to determine β is by assessing the balance

$$\int 2\nu k^{2+2\beta} E(k) dk = \int k^{2\beta} P(k) dk, \tag{55}$$

which should yield the same value for β in a steady state as (54).

We therefore assess in our numerical integration Eq. (4) with $T(k)$ closed by Eq. (42) and determine for each value of λ the corresponding value of β leading to conservation of the mesostrophy. In Fig. 2(a) we show for each value of λ the corresponding value of β.

The deviation from the linear relation is significant. Indeed, if for the sake of simplicity, dimensional arguments can start from the assumption $\lambda = \beta$, corresponding to $C_E = C_W = C_\mu$ and to Eq. (21), this can yield significant errors if more than an order of magnitude estimate is required. Now we will verify the relation between β and the scaling exponent. In Fig. 2(b) we show that the relation $n_E = (5 + 4\beta)/3$ describes the data more precisely than for $\lambda = \beta$. For $\beta = \lambda = 1$, again the logarithmic corrections to the spectrum start influencing the slope significantly and the spectrum is steeper than k^{-3}.

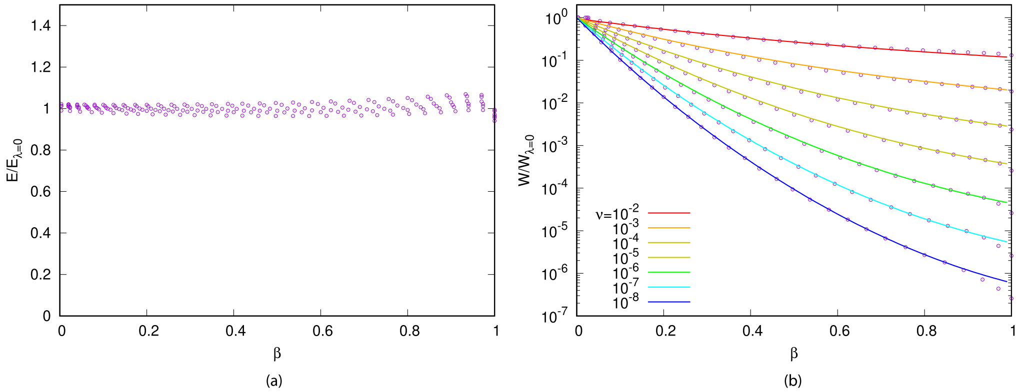

FIG. 3. (a) Energy as a function of β for different values of ν. (b) Enstrophy as a function $0 \leqslant \lambda \leqslant 1$, compared to its Newtonian value. Also shown is the expected scaling behavior assuming a conserved mesostrophy flux.

C. Integral quantities

The Results for more than 200 simulations of our system are compared to the predictions in Figs. 3(a) and 3(b). The kinetic energy is shown to be roughly constant, independent of both the Reynolds number and the value of β. Indeed, as long as a forward cascade is present with steep enough inertial range, the energy will be determined by the forcing scale and injection rate.

The enstrophy as a function of β and Re is shown in Fig. 3(b). The predicted dependence on β, Re [See Eq. (37)] is well reproduced for a fixed value $C_\mu = 3.5$. This was the main question that triggered this investigation. It is clear from the nontrivial form of (37) that it is not surprising that we could not obtain this dependence from dimensional reasoning. An interesting observation is that the dependence on β is amplified at higher Re and this should be kept in mind when one wants to compare the present results to viscoelastic turbulence.

We compute the skewness for values of λ between zero and unity and varying the viscosity from 0.01 to 10^{-5} and we show the results in Fig. 4. We compare to the function $S(0)(1-\beta)^m$ with $S(0) = 0.45 \pm 0.003$ and $m = 1.41 \pm 0.02$, which reasonably describes the data. The value 1.41 can be compared to the rough estimate $m = 1.5$ obtained assuming $\lambda = \beta$ [See expression (40)].

D. Infrared energy distribution

Since we have found the invariant of the system (the mesostrophy), we can predict the equilibrium of a truncated Fourier system following Kraichnan's ideas. We consider a finite number of Fourier modes in the range $k \in [k_{\min}, k_{\max}]$ and assume that the system contains an amount μ of mesostrophy. If the only constraint is conservation of μ, the statistically most probable state, assuming ergodicity, is that we have equipartition of μ over all modes [54]. The energy spectrum, associated with this equipartition is

$$E(k, t = \infty) \sim k^{2(1-\beta)}. \tag{56}$$

This shows that if the system is forced at large scales by an input of μ, this quantity is dominantly dissipated at large k and, to restore equilibrium, a flux of mesostrophy toward large k is expected.

In the scales larger than the forcing scale ($k < k_f$) we expect that the spectrum will be proportional to $k^{2(1-\beta)}$, associated with this mesostrophy equipartition. We illustrate this numerically for a case with a restricted forward mesostrophy range (Fig. 5). Again, the scaling smoothly evolves from energy equipartition [$E(k) \sim k^2$] to enstrophy equipartition [$k^2 E(k) \sim k^2$, or $E(k) \sim k^0$].

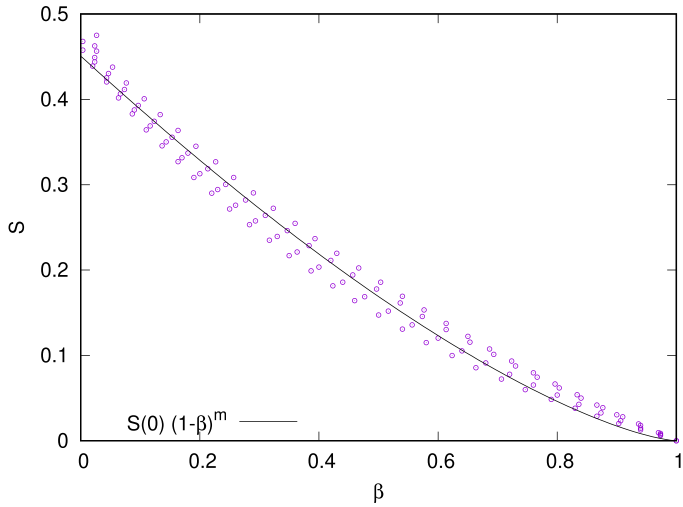

FIG. 4. Velocity gradient skewness for values of $0 \leqslant \lambda \leqslant 1$ and $10^{-2} \geqslant \nu \geqslant 10^{-5}$. Also plotted is the expression fit to the data $S(0)(1-\beta)^m$ with $S(0) = 0.45 \pm 0.003$ and $m = 1.41 \pm 0.02$.

We do stress here that we have considered the mirror-symmetric case. In the case of helicity injection, it is known that this quantity is conserved by the nonlinear interaction [55,56]. Whereas helicity is transferred to small scales in three-dimensional Navier-Stokes turbulence [40], it cascades to large scales in turbulence without vortex stretching [9]. For the present system it is not clear how the cascade changes its direction in between the two extremes. The EDQNM closure becomes substantially more involved for this case but is still analytically treatable [17,57]. This is left for further research.

Furthermore, statistical equilibria can show strong deviations from the equipartition solutions when coherent structures or condensates are present, leading to violation of ergodicity and of

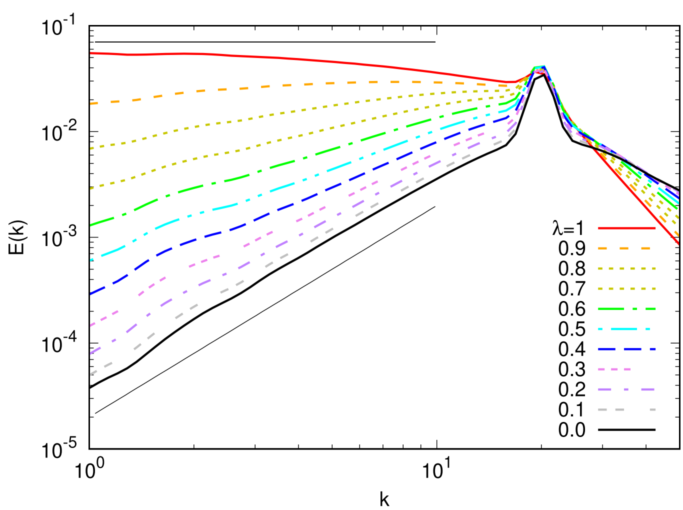

FIG. 5. Energy spectra for k smaller than the forcing range (infrared range), for values of $0 \leqslant \lambda \leqslant 1$. Also shown are the expected scaling behaviors for $\lambda = 0, 1$.

equipartition [58,59]. For these cases, a separate treatment of the coherent structures is needed to describe the statistical equilibrium spectra.

V. CONCLUSION

The main motivation to undertake the current investigation was to prepare, theoretically, further comparison between turbulence with reduced vortex stretching and turbulent flow in the presence of polymer molecules, or microbubbles. Indeed, at least in the case of polymer drag reduction, the principle modification of the turbulent dynamics turns out to be the modification of the enstrophy production. This effect is related to the increase of the extensional viscosity [60]. For microbubbles, we expect that it is the tendency of the bubbles to concentrate in vortex lines [61], which modifies the enstrophy.

Possibly the most interesting implication from the comparison of these systems with the present system, Eq. (1), is that a small modification of the vortex stretching can have large implications. An insightful way to illustrate this is to simplify expression (37) for small β, yielding

$$\frac{\langle|\boldsymbol{\omega}|^2\rangle_\lambda}{\langle|\boldsymbol{\omega}|^2\rangle_{\lambda=0}} \approx \mathrm{Re}^{-3\beta/2} \quad \text{for } \beta \ll 1. \tag{57}$$

Indeed, since the enstrophy ratio is a decreasing function of the Reynolds number, with an exponent dependent on the amount of vortex stretching through β, the suppression of enstrophy production becomes more important for large values of the Reynolds number. For larger values of β we saw a reduction of the enstrophy down to 6 orders of magnitude at the highest values of the Reynolds number we considered.

In the near future the present theory will be compared to the behavior of viscoelastic turbulence. For that we plan to assess the validity of the present system to describe the scaling of turbulent statistics in the finite-extensible nonlinear elastic system with Peterlin approximation in isotropic turbulence [62,63].

DATA AVAILABILITY

The data that support the findings of this article are available upon request from the author.

[1] G. I. Taylor, Statistical theory of turbulence, Proc. R. Soc. London, Ser. A **151**, 421 (1935).

[2] L. Onsager, Statistical hydrodynamics, Il Nuovo Cimento **6**, 279 (1949).

[3] A. N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk. SSSR **30**, 301 (1941).

[4] L. F. Richardson, Atmospheric diffusion shown on a distance-neighbour graph, Proc. R. Soc. London, Ser. A **110**, 709 (1926).

[5] U. Frisch, *Turbulence, the Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, UK, 1995).

[6] P. A. Davidson, K. Morishita, and Y. Kaneda, On the generation and flux of enstrophy in isotropic turbulence, J. Turbul. **9**, N42 (2008).

[7] P. Baj, F. A. Portela, and D. W. Carter, On the simultaneous cascades of energy, helicity, and enstrophy in incompressible homogeneous turbulence, J. Fluid Mech. **952**, A20 (2022).

[8] W. J. T. Bos, Three-dimensional turbulence without vortex stretching, J. Fluid Mech. **915**, A121 (2021).

[9] T. Wu and W. J. T. Bos, Cascades of enstrophy and helicity in turbulence without vortex stretching, Phys. Rev. Fluids **7**, 094601 (2022).

[10] G. K. Batchelor, Computation of the energy spectrum in homogeneous two-dimensional turbulence, Phys. Fluids **12**, II-233 (1969).

[11] C. E. Leith, Diffusion approximation for two-dimensional turbulence, Phys. Fluids **11**, 671 (1968).
[12] R. H. Kraichnan, Inertial ranges in two-dimensional turbulence, Phys. Fluids **10**, 1417 (1967).
[13] T. Wu and W. J. T. Bos, Statistical mechanics of the Euler equations without vortex stretching, J. Fluid Mech. **929**, A11, (2021).
[14] G. Joyce and D. Montgomery, Negative temperature states for the two-dimensional guiding center plasma, J. Plasma Phys. **10**, 107 (1973).
[15] T. Wu, T. David, and W. J. T. Bos, Point-vortex statistical mechanics applied to turbulence without vortex stretching, J. Stat. Mech. (2023) 113203.
[16] A. Brissaud, U. Frisch, J. Léorat, M. Lesieur, and A. Mazure, Helicity cascades in fully developed isotropic turbulence, Phys. Fluids **16**, 1366 (1973).
[17] J. C. André and M. Lesieur, Influence of helicity on the evolution of isotropic turbulence at high Reynolds number, J. Fluid Mech. **81**, 187 (1977).
[18] Q. Chen, S. Chen, and G. L. Eyink, The joint cascade of energy and helicity in three-dimensional turbulence, Phys. Fluids **15**, 361 (2003).
[19] F. Plunian, A. Teimurazov, R. Stepanov, and M. K. Verma, Inverse cascade of energy in helical turbulence, J. Fluid Mech. **895**, A13 (2020).
[20] W. J. T. Bos, X. Shao, T. Wu, and L. Fang, The role of vortex stretching in drag reduction of polymer-laden turbulent flow, J. Fluid Mech. **1019**, A32 (2025).
[21] T. von Kármán, Mechanische aenlichkeit und turbulenz, Nachr. Ges. Wiss. Goettingen, Math.-Phys. Kl. **1930**, 58 (1930).
[22] B. E Owolabi, D. J. C. Dennis, and R. J. Poole, Turbulent drag reduction by polymer additives in parallel-shear flows, J. Fluid Mech. **827**, R4 (2017).
[23] P. S. Virk, Drag reduction fundamentals, AIChE J. **21**, 625 (1975).
[24] M. E. McCormick and R. Bhattacharyya, Drag reduction of a submersible hull by electrolysis, Nav. Eng. J. **85**, 11 (1973).
[25] J.-L. Marié, A simple analytical formulation for microbubble drag reduction, Physicochem. Hydrodyn. **8**, 213 (1987).
[26] X. Shen, S. L Ceccio, and M. Perlin, Influence of bubble size on micro-bubble drag reduction, Exp. Fluids **41**, 415 (2006).
[27] J. Mathieu and J. F. Scott, *An Introduction to Turbulent flow* (Cambridge University Press, Cambridge, UK, 2000).
[28] P. Sagaut and C. Cambon, *Homogeneous Turbulence Dynamics* (Springer, New York, 2008), Vol. 10.
[29] L. S. G. Kovaznay, Spectrum of locally isotropic turbulence, J. Aeronaut. Sci. **15**, 745 (1948).
[30] R. Rubinstein and T. Clark, *Reassessment of the Classical Turbulence Closures* (Cambridge Scholars Publishing, Newcastle upon Tyne, England, 2022).
[31] G. K. Batchelor, *The theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, UK, 1953).
[32] F. Liu, L. Fang, and L. Shao, The role of velocity derivative skewness in understanding non-equilibrium turbulence, Chin. Phys. B **29**, 114702 (2020).
[33] C. E. Leith, Diffusion approximation to inertial energy transfer in isotropic turbulence, Phys. Fluids **10**, 1409 (1967).
[34] V. S. L'vov and S. Nazarenko, Differential model for 2D turbulence, JETP Lett. **83**, 541 (2006).
[35] W. Heisenberg, Zur statistischen Theorie der Turbulenz, Z. Phys. **124**, 628 (1948).
[36] R. Rubinstein and T. T. Clark, A generalized Heisenberg model for turbulent spectral dynamics, Theor. Comput. Fluid Dyn. **17**, 249 (2004).
[37] R. H. Kraichnan, The structure of isotropic turbulence at very high Reynolds numbers, J. Fluid Mech. **5**, 497 (1959).
[38] R. H. Kraichnan, Lagrangian-history closure approximation for turbulence, Phys. Fluids **8**, 575 (1965).
[39] S. A. Orszag, Analytical theories of turbulence, J. Fluid Mech. **41**, 363 (1970).
[40] M. Lesieur, *Turbulence in Fluids* (Kluwer, Dordrecht, 1990).

[41] U. Frisch, M. Lesieur, and P. L. Sulem, Crossover dimensions for fully developed turbulence, Phys. Rev. Lett. **37**, 1312 (1976).
[42] W. J. T. Bos and L. Fang, Dependence of turbulent advection on the Lagrangian correlation time, Phys. Rev. E **91**, 043020 (2015).
[43] A. Briard, L. Biferale, and T. Gomez, Closure theory for the split energy-helicity cascades in homogeneous isotropic homochiral turbulence, Phys. Rev. Fluids **2**, 102602(R) (2017).
[44] T. Gotoh, Y. Watanabe, Y. Shiga, T. Nakano, and E. Suzuki, Statistical properties of four-dimensional turbulence, Phys. Rev. E **75**, 016310 (2007).
[45] D. Clark, R. D. J. G. Ho, and A. Berera, Effect of spatial dimension on a model of fluid turbulence, J. Fluid Mech. **912**, A40 (2021).
[46] G. Sahoo, A. Alexakis, and L. Biferale, Discontinuous transition from direct to inverse cascade in three-dimensional turbulence, Phys. Rev. Lett. **118**, 164501 (2017).
[47] U. Frisch, A. Pomyalov, I. Procaccia, and S. S. Ray, Turbulence in noninteger dimensions by fractal Fourier decimation, Phys. Rev. Lett. **108**, 074501 (2012).
[48] A. Alexakis and L. Biferale, Cascades and transitions in turbulent flows, Phys. Rep. **767**, 1 (2018).
[49] R. H. Kraichnan, Inertial-range transfer in two-and three-dimensional turbulence, J. Fluid Mech. **47**, 525 (1971).
[50] G. Boffetta and S. Musacchio, Evidence for the double cascade scenario in two-dimensional turbulence, Phys. Rev. E **82**, 016307 (2010).
[51] L. Mydlarski and Z. Warhaft, On the onset of high-Reynolds-number grid-generated wind tunnel turbulence, J. Fluid Mech. **320**, 331 (1996).
[52] W. J. T. Bos, L. Chevillard, J. F. Scott, and R. Rubinstein, Reynolds number effect on the velocity increment skewness in isotropic turbulence, Phys. Fluids **24**, 015108 (2012).
[53] S. Tang, L. Danaila, and R. A. Antonia, Finite Reynolds number effect on small-scale statistics in decaying grid turbulence, Atmosphere **15**, 540 (2024).
[54] T. D. Lee, On some statistical properties of hydrodynamical and magnetohydrodynamical fields, Quart. Appl. Math. **10**, 69 (1952).
[55] H. K. Moffatt, The degree of knottedness of tangled vortex lines, J. Fluid Mech. **35**, 117 (1969).
[56] R. H. Kraichnan, Helical turbulence and absolute equilibrium, J. Fluid Mech. **59**, 745 (1973).
[57] A. Briard and T. Gomez, Dynamics of helicity in homogeneous skew-isotropic turbulence, J. Fluid Mech. **821**, 539 (2017).
[58] A. Venaille, T. Dauxois, and S. Ruffo, Violent relaxation in two-dimensional flows with varying interaction range, Phys. Rev. E **92**, 011001(R) (2015).
[59] W. Agoua, X.-Y. Yin, T. Wu, and Wouter J. T. Bos, Coexistence of two equilibrium configurations in two-dimensional turbulence, Phys. Rev. Fluids **10**, 034604 (2025).
[60] E. J. Hinch, Mechanical models of dilute polymer solutions in strong flows, Phys. Fluids **20**, S22 (1977).
[61] S. Douady, Y. Couder, and M. E. Brachet, Direct observation of the intermittency of intense vorticity filaments in turbulence, Phys. Rev. Lett. **67**, 983 (1991).
[62] P. C. Valente, C. D. Silva, and F. T. Pinho, The effect of viscoelasticity on the turbulent kinetic energy cascade, J. Fluid Mech. **760**, 39 (2014).
[63] P. C. Valente, C. B. da Silva, and F. T. Pinho, Energy spectra in elasto-inertial turbulence, Phys. Fluids **28**, 075108 (2016).