Null-divergence nature of the odd viscous stress for an incompressible liquid

E. Kirkinis *

Department of Materials Science & Engineering, Robert R. McCormick School of Engineering and Applied Science, Northwestern University, Evanston, Illinois 60208, USA
and Center for Computation and Theory of Soft Materials, Northwestern University, Evanston, Illinois 60208, USA

(Received 30 June 2022; accepted 4 January 2023; published 26 January 2023)

What constitutes the "extra stress" exerted by a liquid endowed with odd viscosity [for instance, the tensile or compressive stress imparted on a cylinder rotating in a viscous liquid as was calculated by Avron, J. Stat. Phys. **92**, 543 (1998)], requires clarification. The nature of this extra stress depends on the character of the boundary conditions. We show that when only velocities are prescribed on the boundaries, it is *not* the full anomalous (odd) stress tensor that generates this "extra" stress but only its null divergence part, that is, the part of the odd stress whose divergence vanishes identically. We demonstrate this fine point by calculating the extra stress for a viscous liquid between concentric rotating cylinders and the corrections to the viscosity coefficient in liquid suspensions. Similar conclusions can be reached for a liquid with periodic boundary conditions. On the other hand, when stresses are prescribed on some part of the boundary, the state of stress due to odd viscosity becomes ambiguous. In general, it is the whole Cauchy stress tensor that now depends on odd viscosity. There are exceptions, however, to this rule which we briefly discuss. Finally, the decomposition of the Cauchy stress tensor into a part that is operative in the Navier-Stokes equations, and into a null part (whose divergence vanishes identically) gives rise to new fluid flow behavior and recovers previous results from the literature as special cases.

DOI: 10.1103/PhysRevFluids.8.014104

I. INTRODUCTION

Avron and colleagues [1,2] pointed out that the hydrodynamic equations are endowed with an additional viscosity coefficient, termed odd or Hall viscosity, when time-reversal symmetry is broken either spontaneously or due to an external magnetic field or rotation. The Cauchy stress tensor, apart from its usual rate-of-strain part, acquires additional terms (the anomalous or odd stress), of nondissipative nature, that depend on field direction and which break the time-reversal symmetry [3], (§13). There is a number of striking physical manifestations associated with the presence of odd viscosity: a disk rotating in a viscous liquid experiences a normal compressive or tensile stress [2] in addition to the shear stresses caused by the shear (even) viscosity; a swimmer experiences a torque proportional to the rate-of-change of its area [4]; an expanding bubble will promote an azimuthal flow on its surrounding liquid [5], in addition to the radial flow existing in the absence of odd viscosity; chiral matter interfacial deformations level off in a manner akin to surface tension [6]; and viscous unstable thin liquid films can become stabilized by the generation of nonlinear capillary waves [7].

Odd viscosity is not merely introduced because of an abstract academic interest but has been observed in experiment [6]: in a colloidal suspension of spinning magnets, a leveling off of the

*kirkinis@northwestern.edu

TABLE I. Explicit formulas for even and odd stress defined in Eqs. (2) and (3) respectively, and in Cartesian and plane polar coordinates, where $\mathbf{v} = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y$ and $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta$, with respect to the corresponding basis vectors. For completeness we include the vorticity and continuity in the latter system.

Stress component	even stress σ^e	odd stress σ^o
σ_{xx}	$2\eta_e\partial_x u$	$-\eta_o(\partial_x v + \partial_y u)$
σ_{xy}	$\eta_e(\partial_x v + \partial_y u)$	$\eta_o(\partial_x u - \partial_y v)$
σ_{yy}	$2\eta_e\partial_y v$	$\eta_o(\partial_x v + \partial_y u)$
σ_{rr}	$2\eta_e\partial_r v_r$	$-\eta_o(\partial_r v_\theta - \frac{1}{r}v_\theta + \frac{1}{r}\partial_\theta v_r)$
$\sigma_{r\theta}$	$\eta_e(\partial_r v_\theta - \frac{1}{r}v_\theta + \frac{1}{r}\partial_\theta v_r)$	$2\eta_o\partial_r v_r$
$\sigma_{\theta\theta}$	$2\eta_e(\frac{1}{r}\partial_\theta v_\theta + \frac{1}{r}v_r)$	$\eta_o(\partial_r v_\theta - \frac{1}{r}v_\theta + \frac{1}{r}\partial_\theta v_r)$
Cauchy stress [Eq. (1)]	$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^o,$	
vorticity	$\omega = \partial_r v_\theta + \frac{1}{r}v_\theta - \frac{1}{r}\partial_\theta v_r$	
continuity	$\partial_r v_r + \frac{1}{r}v_r + \frac{1}{r}\partial_\theta v_\theta = 0$	

dissipation rate was observed at short wavelengths that could not be accounted for by the existing hydrodynamic theory. Incorporation of the anomalous (or odd) stress into the theory, led to the theoretical explanation of the aforementioned observation in the form of a boundary stress that flattens surface deformations in a manner akin to surface tension.

Odd viscosity is compatible with isotropy in two but not three spatial dimensions [2]. In such an incompressible liquid with broken time-reversal symmetry the Cauchy stress tensor $\boldsymbol{\sigma}$ becomes [2]

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^o, \tag{1}$$

where $\boldsymbol{I}$ is the unit tensor and p is the liquid pressure. In two dimensions

$$\sigma^e_{ij} = \eta_e\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i = 1, 2, \ j = 1, 2, \tag{2}$$

is the standard (even) deviatoric part of the Cauchy stress tensor with standard (even) shear viscosity η_e.

$\boldsymbol{\sigma}^o$ in Eq. (1) is the odd part of the Cauchy stress tensor (also known as the anomalous stress) which arises in the presence of time-reversal symmetry breaking due, for instance, to a magnetic field. A constitutive law that contains the odd viscous stress was introduced Ref. [3], (§13). Here, we will assume that a constant magnetic field is normal to the plane of a two-dimensional flow. In this case the odd part of the Cauchy stress tensor is characterized by a single odd viscosity coefficient η_o, and its components read [2,4]

$$\sigma^o_{ik} = -\eta_o(\delta_{i1}\delta_{k1} - \delta_{i2}\delta_{k2})\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + \eta_o(\delta_{i1}\delta_{k2} + \delta_{i2}\delta_{k1})\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right), \tag{3}$$

where $i = 1, 2$ and $j = 1, 2$. The components of the even and odd parts of the stress are collected in Table I in two-coordinate systems.

Avron discovered that a solid disk (or cylinder) of finite radius R rotating with angular velocity Ω in an infinite viscous liquid, will experience the following *extra* tensile or compressive stress (depending on the cylinder's sense of rotation):

$$\sigma^{\text{extra}}_{rr}(R) = 2\eta_o\Omega, \tag{4}$$

([2], Eq. (61)) in the presence of odd viscosity. Is this the σ^o_{rr} component of the odd stress in Table I? The answer is negative, in general.

In this article, *inter alia*, we systematize the search for this extra stress in viscous incompressible flow in two dimensions. When it is the liquid velocity that is prescribed on the boundaries (and not the stress), for instance through the no-slip boundary condition, we identify the extra stress (4) discovered by Avron [2], not with the full odd stress tensor $\boldsymbol{\sigma}^o$ in (3) but with the null stress tensor $\boldsymbol{\sigma}^{o,\text{null}}$, that is, the part of the odd stress tensor whose divergence vanishes identically. Thus, we employ the decomposition

$$\boldsymbol{\sigma}^o = -\omega \boldsymbol{I} + \boldsymbol{\sigma}^{o,\text{null}}, \tag{5}$$

of Eq. (3), to be made precise below, in Eq. (14), where ω is the scalar liquid vorticity in two dimensions. Realizing this fine point is important in order to avoid reaching erroneous conclusions regarding the state of stress in the presence of odd viscosity. We demonstrate this point in the calculation of the Couette flow between two rotating cylinders and the more complex problem of calculating corrections to the viscosity in a suspension. A problem with periodic boundary conditions falls into the same category: it is the null part of the odd stress tensor that describes the state of stress in such a liquid.

The situation is very different when it is the stresses that are prescribed on the boundary or on some part of it. Now, *every term* of the total stress in Eq. (1) depends on the coefficient of odd viscosity η_o and the interpretation of what constitutes the "extra" stress associated with odd viscosity becomes ambiguous. Yet, this behavior is not universal and exceptions do exist. Such an exception is the motion of liquid layer on an inclined solid substrate with a flat liquid-gas interface. Finally, we show that the null divergence of the total stress [Eq. (1)] can give rise to new fluid-flow behavior. For example, an expanding bubble in a viscous liquid will acquire an azimuthal velocity component [5].

This article is organized as follows. In Sec. II we decompose the total stress tensor [Eq. (1)] in a vortex part (that appears in the Navier-Stokes equations) and a null part (whose divergence vanishes identically). In Sec. III we investigate the nature of the extra stress associated with odd viscosity in problems when only the velocity is prescribed on the boundary. Thus, in Sec. III A we consider the liquid flow between two rotating concentric cylinders. Here, and in any problem where only velocities are prescribed on the boundaries, the "extra" stress due to odd viscosity is unambiguously described by the null part of the odd stress tensor $\boldsymbol{\sigma}^{o,\text{null}}$. In Sec. III B we show that it is only the null divergence part of the odd stress tensor that needs to be employed in order to calculate corrections to the odd viscosity coefficient in a suspension of cylinders or disks. Likewise, in Sec. III C we show that it is only the null part of the odd stress tensor that determines the state of stress associated with odd viscosity when periodic boundary conditions are imposed. In Sec. III D we revisit Avron's problem [2] about the flow around a rotating cylinder in a liquid with odd viscosity leading to Eq. (4) because it contains an ambiguity: the values of the null odd stress $\boldsymbol{\sigma}^{o,\text{null}}$ and the full odd stress $\boldsymbol{\sigma}^o$ incidentally coincide. We show why this is the case.

In Sec. IV we show that when it is the stress that is prescribed on the boundary or some part of it, the interpretation of what constitutes "extra" stress associated with odd viscosity is ambiguous. Two free surface problems are employed: first, a liquid in a corner between a solid substrate and a gas phase under a surface shear and second, a liquid layer flowing down an inclined plane. In the first case it is the total stress tensor [Eq.(1)] that depends on η_o. In the second case only the pressure p depends on odd viscosity and thus one could resort to the formulation in Sec. III A to investigate the state of stress associated with odd viscosity.

In Sec. V we provide the theoretical framework of null divergences as it was developed by Olver [8] for the general case of n-dimensional fields. With the aid of the decomposition developed in Sec. II, new fluid flow behavior can be discovered. We show, for instance, that the azimuthal component of the liquid velocity field acquired by a radially expanding bubble [5], is recovered as a special case of this formulation.

II. THE NULL DIVERGENCE OF THE NAVIER-STOKES EQUATIONS

In this section the full stress tensor σ_{ij} in Eq. (1) is decomposed into two parts. We separate the part of the stress that appears in the Navier-Stokes equations (and which modifies the observables

TABLE II. Components of the stresses $\boldsymbol{\sigma}^{\mathrm{NS}}$ appearing in the Navier-Stokes equations and $\boldsymbol{\sigma}^{\mathrm{null}}$ (the null stress components) that do not appear explicitly in the Navier-Stokes equations cf. Eqs. (9) and (10). Grouped together, they form the Cauchy stress tensor (8) $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{NS}} + \boldsymbol{\sigma}^{\mathrm{null}}$ of an incompressible liquid.

Null Cartesian	Null plane polar	Navier-Stokes
$\sigma_{xx} = 2\eta_e\partial_x u - 2\eta_o\partial_y u$	$\sigma_{rr} = 2\eta_e\partial_r v_r - 2\eta_o\frac{1}{r}(\partial_\theta v_r - v_\theta)$	$-p - \eta_o\omega$
$\sigma_{xy} = 2\eta_e\partial_x v + 2\eta_o\partial_x u$	$\sigma_{r\theta} = 2\eta_e\partial_r v_\theta + 2\eta_o\partial_r v_r$	$-\eta_e\omega$
$\sigma_{yx} = 2\eta_e\partial_y u + 2\eta_o\partial_x u$	$\sigma_{\theta r} = 2\eta_e\frac{1}{r}(\partial_\theta v_r - v_\theta) + 2\eta_o\partial_r v_r$	$\eta_e\omega$
$\sigma_{yy} = 2\eta_e\partial_y v + 2\eta_o\partial_x v$	$\sigma_{\theta\theta} = 2\eta_e\frac{1}{r}(\partial_\theta v_\theta - v_r) + 2\eta_o\partial_r v_\theta$	$-p - \eta_o\omega$

u, v, and p), and the part of the stress that does not. The former depends on the (scalar) vorticity ω of the liquid. Motivation for this decomposition comes from the fact that the terms multiplied by the coefficient of odd viscosity in the Navier-Stokes equations for an incompressible liquid appear as the gradient of the (scalar) vorticity and thus, they can be "absorbed" into the pressure.

It is not difficult to show that the Navier-Stokes equations (subject to the incompressibility condition) can be written in the form

$$\rho[\partial_t u + u\partial_x u + v\partial_y u] = -\partial_x p - \eta_e\partial_y\omega - \eta_o\partial_x\omega, \tag{6}$$

$$\rho[\partial_t v + u\partial_x v + v\partial_y v] = -\partial_y p + \eta_e\partial_x\omega - \eta_o\partial_y\omega, \tag{7}$$

where $\omega = \partial_x v - \partial_y u$ is the vorticity of the liquid in the z direction.

Because it was observed in the literature [5] that the odd viscosity terms in the Navier-Stokes equations of an incompressible liquid can be expressed as the gradient of the scalar vorticity [see Eqs. (6) and (7)], we decompose the Cauchy stress (1) in two parts

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{NS}} + \boldsymbol{\sigma}^{\mathrm{null}}, \tag{8}$$

the first of which displays this vorticity dependence explicitly

$$\sigma_{ij}^{\mathrm{NS}} = -p\delta_{ij} - \omega(\eta_e\epsilon_{ij} + \eta_o\delta_{ij}), \quad i = 1, 2, \; j = 1, 2. \tag{9}$$

The second part reads

$$\sigma_{ij}^{\mathrm{null}} = 2(\eta_e\delta_{ik} - \eta_o\epsilon_{ik})L_{kj}^T, \quad i = 1, 2, \; j = 1, 2, \tag{10}$$

where $L_{ij} = \partial u_i/\partial x_j$ is the velocity gradient, $\epsilon_{12} = 1 = -\epsilon_{21}$, $\epsilon_{11} = \epsilon_{22} = 0$ is the two-dimensional alternating tensor and T stands for the transpose of a matrix. Explicit formulas for Eqs. (9) and (10) appear in Table II. The divergence of $\boldsymbol{\sigma}^{\mathrm{null}}$ is identically zero, and this part of the stress tensor does not explicitly appear in the Navier-Stokes equations. Only $\boldsymbol{\sigma}^{\mathrm{NS}}$ will appear in the Navier-Stokes equations.

The above can be formulated in terms of the stretching (rate-of-strain) tensor D and spin tensor W of continuum mechanics [9] obtained by decomposing the velocity gradient L into symmetric and antisymmetric parts

$$L = D + W. \tag{11}$$

Expressing W as $W_{kj} = \frac{1}{2}\omega\epsilon_{jk}$, we have

$$\sigma_{ij}^{\mathrm{NS}} = -p\delta_{ij} + 2[\eta_e\delta_{ik} + \eta_o\epsilon_{ik}]W_{kj}. \tag{12}$$

Another route to obtain results in Eqs. (10) and (12) is as follows. Collecting Eqs. (2) and (3) in a single compact formula shows that the total stress [Eq. (1)] can be written in the form

$$\sigma_{ij} = -p\delta_{ij} + 2\eta_{ik}D_{kj}, \quad i = 1, 2, \; j = 1, 2, \quad \text{where} \quad \eta_{ik} = \begin{pmatrix} \eta_e & -\eta_o \\ \eta_o & \eta_e \end{pmatrix}, \tag{13}$$

and $D_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the rate-of-strain tensor. Thus, replacing D by $L^T - W^T$ leads to the results in Eqs. (10) and (12).

III. THE EXTRA STRESS ASSOCIATED WITH ODD VISCOSITY

Consider the flow of a liquid with boundary conditions specified only by the velocities and not the stresses. Collecting together the second terms of each of Eqs. (9) and (10) we express the odd stress tensor [Eq. (3)] in the form

$$\sigma_{ij}^{o} = -\eta_o \omega \delta_{ij} + \sigma_{ij}^{o,\mathrm{null}}, \quad i = 1, 2, \;\; j = 1, 2, \tag{14}$$

where ω is the scalar vorticity in two dimensions and

$$\sigma_{ij}^{o,\mathrm{null}} = -2\eta_o \epsilon_{ik} L_{kj}^T, \quad i = 1, 2, \;\; j = 1, 2. \tag{15}$$

In Cartesian and plane polar coordinates, respectively, where the velocity fields read $\mathbf{v} = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y$ and $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta$ with respect to their corresponding basis vectors, $\sigma_{ij}^{o,\mathrm{null}}$ obtains the form

$$\sigma_{ij}^{o,\mathrm{null}} = 2\eta_o \begin{pmatrix} -\partial_y u & \partial_x u \\ -\partial_y v & \partial_x v \end{pmatrix} \quad \text{and} \quad \sigma_{ij}^{o,\mathrm{null}} = 2\eta_o \begin{pmatrix} -\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{\partial v_r}{\partial r} \\ \frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} \end{pmatrix}. \tag{16}$$

It is only the vorticity part $-\eta_o \omega \delta_{ij}$ of the stress σ_{ij}^{o} in Eq. (14) that appears in the Navier-Stokes equations because the divergence of the null part $\sigma_{ij}^{o,\mathrm{null}}$ vanishes identically

$$\partial_j \sigma_{ij}^{o,\mathrm{null}} \equiv 0, \quad i = 1, 2. \tag{17}$$

We identify the extra stress in Eq. (4) found by Avron, with the tensor $\sigma_{ij}^{o,\mathrm{null}}$ and not with the full odd stress tensor σ_{ij}^{o}. This is a general result that is valid whenever the boundary conditions of the liquid are specified by velocities and not the stresses. This is justified with the following elementary argument: In the presence of odd viscosity and subject to the incompressibility condition, the Navier-Stokes equations can be written as

$$\rho[\partial_t u + u\partial_x u + v\partial_y u] = -\partial_x \tilde{p} + \eta_e \nabla^2 u, \tag{18}$$

$$\rho[\partial_t v + u\partial_x v + v\partial_y v] = -\partial_y \tilde{p} + \eta_e \nabla^2 v, \tag{19}$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$ and the modified pressure $\tilde{p}$ is defined through

$$\tilde{p} = p + \eta_o \omega. \tag{20}$$

Prescribing only velocities at the boundaries and solving the Navier-Stokes equations gives rise to fields u, v, and $\tilde{p}$ that are independent of η_o. The pressure p on the other hand depends on odd viscosity through relation [Eq. (20)].

With the aid of Eqs. (14) and (20), the *total* stress $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^o$ in Eq. (1) can be now written in the form

$$\sigma_{ij} = -\tilde{p}\delta_{ij} + \sigma_{ij}^{e} + \sigma_{ij}^{o,\mathrm{null}}. \tag{21}$$

Since the first two terms on the right hand side of Eq. (21) do not depend on η_o, it is only the third term, the null part $\sigma_{ij}^{o,\mathrm{null}}$ of the odd stress in Eq. (21), that will give rise to the extra stress observed by Avron. We note that decomposition in Eq. (21) is identical to (Ref. [5], Eq. (6)), not developed however in the context of null divergencies.

A. Flow between two rotating cylinders

In this section we show that the stresses $\sigma_{rr}^{o,\mathrm{null}}$ and σ_{rr}^{o} are different and thus care must be exercised when one needs to identify the "extra" stress associated with odd viscosity.

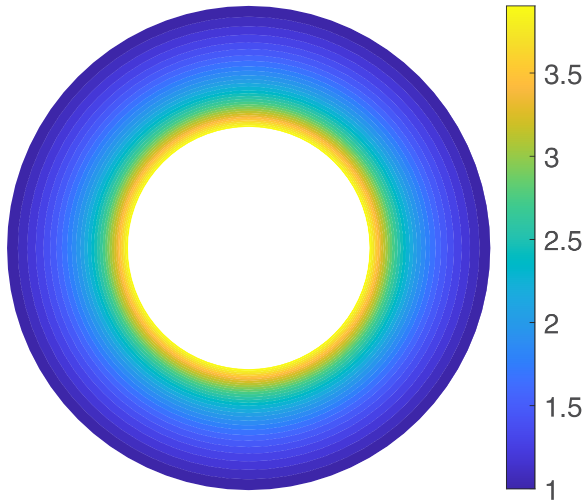


FIG. 1. Strength of the component $\sigma_{rr}^{o,\text{null}} \sim \frac{1}{r^2}$ in Eq. (24) of the null part of the odd stress tensor for fluid flow between two rotating cylinders, as an example of the case where only velocities are prescribed on the boundaries (here this is the no-slip condition). The "extra" stress associated with odd viscosity is $\sigma_{rr}^{o,\text{null}}$ and not the full anomalous stress σ_{rr}^{o}. This is discussed in Sec. III A.

Consider two concentric cylinders of radii $R_1 < R_2$ rotating with angular velocities Ω_1 and Ω_2, respectively, and containing a viscous liquid with odd viscosity η_o. The liquid velocity is azimuthal ([10], §18)

$$\mathbf{v} = v_\theta \hat{\mathbf{e}}_\theta, \quad \text{where} \quad v_\theta = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2} \frac{1}{r}, \tag{22}$$

where r is the distance from the mutual center of the cylinders. Then, from Eq. (16) we find that $\sigma_{rr}^{o,\text{null}} = 2\eta_o \frac{v_\theta}{r}$, thus,

$$\sigma_{rr}^{o,\text{null}}(R_1) = 2\eta_o \Omega_1, \quad \text{and} \quad \sigma_{rr}^{o,\text{null}}(R_2) = 2\eta_o \Omega_2. \tag{23}$$

Equation (23) describes the extra normal compressive or tensile stress (depending on the cylinder's sense of rotation and the orientation of the normal to their lateral surfaces) experienced by the two cylinders.

The stresses σ_{rr}^{o} on the two cylinders are markedly different from Eq. (23). Since $\sigma_{rr}^{o} = -\eta_o(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r})$ these stresses read $\sigma_{rr}^{o}(R_1) = 2\eta_o \frac{(\Omega_1 - \Omega_2) R_2^2}{R_2^2 - R_1^2}$ and $\sigma_{rr}^{o}(R_2) = 2\eta_o \frac{(\Omega_1 - \Omega_2) R_1^2}{R_2^2 - R_1^2}$ at the cylinder surfaces. Thus, according to Eq. (21), it would be incorrect to identify the latter stresses as being the "extra" stress associated with odd viscosity.

In Fig. 1 we display the strength of the rr component of the null odd stress

$$\sigma_{rr}^{o,\text{null}} = 2\eta_o \frac{v_\theta}{r} \sim \frac{1}{r^2}. \tag{24}$$

We reiterate here that, when the boundary conditions are specified by velocities and not stresses, the extra stress associated with odd viscosity is not given by the full expression of the odd stress tensor $\boldsymbol{\sigma}^{o}$ as displayed in Table I and Eq. (3) but by the null stress $\boldsymbol{\sigma}^{o,\text{null}}$ as this is described in Eq. (16) and tabulated in Table II.

B. The viscosity of suspensions

Another example of the appearance of null stresses is in the calculation of corrections to the even and odd viscosity coefficients in a suspension of cylinders (or disks) of radius a in a two-dimensional incompressible flow due to a pure straining motion with rate-of-strain tensor E_{ij}. The isochoric constraint requires that $\mathrm{tr}(E) = 0$. In a liquid endowed with two viscosity coefficients η_e and η_o the presence of the cylinders modifies the flow. Following Ref. [11], the liquid velocity and pressure read

$$u_i(\mathbf{x}) = E_{ij}x_j + V_{ijk}E_{jk} \quad \text{and} \quad p = p^\infty + P_{jk}E\,jk - \eta_o\omega, \tag{25}$$

respectively, where

$$V_{ijk} = -\frac{1}{2}(\delta_{ij}x_k + \delta_{ik}x_j)\left(\frac{a}{r}\right)^4 - 2a^2\frac{x_ix_jx_k}{r^4}\left[1 - \left(\frac{a}{r}\right)^2\right], \tag{26}$$

$$P_{jk} = -4\eta_e a^2\frac{x_jx_k}{r^4}, \tag{27}$$

$$\omega = \partial_{x_1}u_2 - \partial_{x_2}u_1, \tag{28}$$

$r^2 = x_i^2$ and $\tilde{p} = p^\infty + P_{jk}E\,jk$. In the above expressions we employed the Einstein summation convention on repeated indices and p^∞ is the constant pressure of the straining flow. We note that u_i and p, as described in Eq. (25) are the solutions of the Stokes equations with odd viscosity.

To compute corrections to the even and odd viscosity coefficients we need to form the average of the stress tensor σ in Eq. (1). This constitutes a generalization of the corresponding formula in (Ref. [11], Eq. (5))

$$\langle\sigma_{ij}\rangle = \mathrm{I.T.} + 2\eta_{ik}\langle E_{kj}\rangle + \bar{N}\langle S_{ij}\rangle, \tag{29}$$

where

$$\eta_{ik} = \begin{pmatrix} \eta_e & -\eta_o \\ \eta_o & \eta_e \end{pmatrix},$$

$\langle\cdot\rangle$ is average over the entire dispersion volume (particles plus fluid), $\bar{N}$ is the number density of the cylinders, I.T. are isotropic terms that do not play a role in the calculation, and $\langle S_{kj}\rangle$ is the average extra particle stress expressed as a sum over all particles in a volume. Following (Refs. [11], Eq.(6), [12], Eq. (4.5), (4.11a), and (5.11)), ([13], (§4.11)), and ([10], (§22), S_{ij} obtains the form

$$S_{ij} = \int \{[-p\delta_{ik} + \sigma_{ik}^e + \sigma_{ik}^o]x_jn_k\}dS, \tag{30}$$

where $n_i = x_i/r$ is the normal vector at the cylinder surface (at $r = a$). The fields have been defined in Eqs. (25)–(28). Carrying out the integration over the disk surface we obtain the following three contributions related to the three respective terms in Eq. (30):

$$S_{ij} : -\eta_o 2\pi\alpha^2\epsilon_{ik}E_{kj}, \quad 4\pi\alpha^2\eta_e\delta_{ik}, \quad \eta_o 2\pi\alpha^2\epsilon_{ik}E_{kj}. \tag{31}$$

The second term leads to the average stress in the form

$$\langle\sigma_{ij}\rangle = 2\eta_e(1 + 2\pi a^2\bar{N})E_{ij}, \tag{32}$$

which gives rise to the usual correction to the shear viscosity coefficient η_e in two dimensions:

$$\eta_e^{\mathrm{eff}} = \eta_e(1 + 2\phi), \tag{33}$$

expressed in terms of the small ratio of the total volume of the cylinders to the total volume of the suspension

$$\phi = \pi a^2\bar{N}. \tag{34}$$

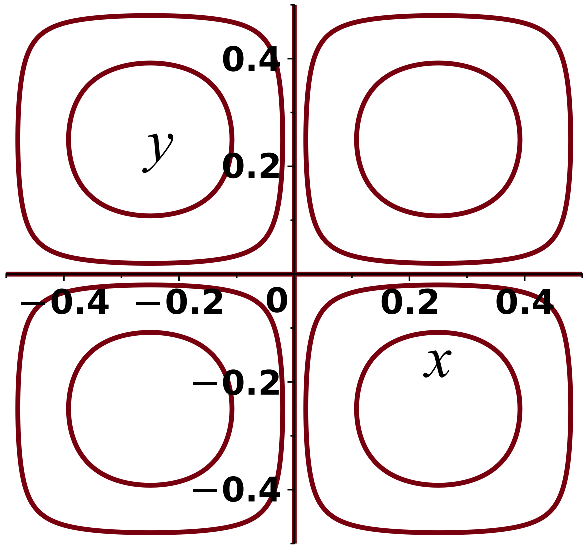


FIG. 2. The Taylor-Green vortex described in Sec. III C as an example of a flow with periodic boundary conditions. Here only the pressure p depends on odd viscosity, see Eq. (37). The velocity fields are independent of odd viscosity. The "extra stress" associated with odd viscosity is expressed by $\sigma^{o,\mathrm{null}}$ and not σ^{o} [cf. Eqs. (14), (15), and (38)].

On the other hand, the first term in Eq. (31) cancels the third and we obtain no corrections to the odd viscosity (at this order).

The above result could have been anticipated on the basis of the form of the null stress tensor. If we replace the stress expression in the integrand of Eq. (30) by Eq. (21), it is clear that only $\sigma^{o,\mathrm{null}}$ depends on odd viscosity. A simple calculation then gives

$$\int \left\{\sigma_{ik}^{o,\mathrm{null}} x_j n_k\right\} dS = 0. \tag{35}$$

It would then be incorrect to calculate corrections to the odd viscosity of suspensions by employing the odd stress tensor σ^{o} alone in Eq. (35). Had we done so, we would have obtained the erroneous result $\eta_o^{\mathrm{eff}} = \eta_o(1+\phi)$.

C. The Taylor-Green vortex: periodic boundary conditions

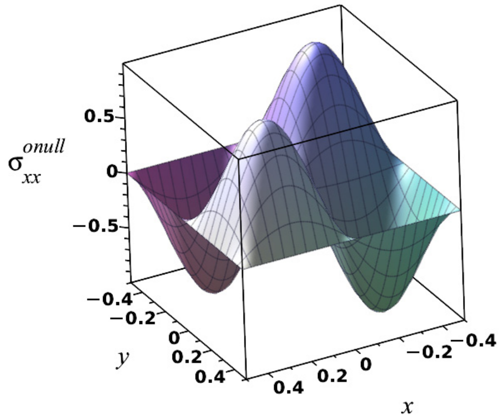
When periodic boundary conditions are prescribed in a viscous incompressible liquid endowed with odd viscosity, then the fields u, v, and $\tilde{p}$ are independent of η_o and these problems must be described with the formalism developed in Sec. III A.

Consider the Taylor-Green vortex (cf. Fig. 2), now endowed with odd viscosity, which is an exact solution of the two-dimensional Navier-Stokes equations, employed as a benchmark test for the validity of numerical schemes, valid for any Reynolds number (all variables and parameters should be considered here dimensionless). The velocity fields and pressure read

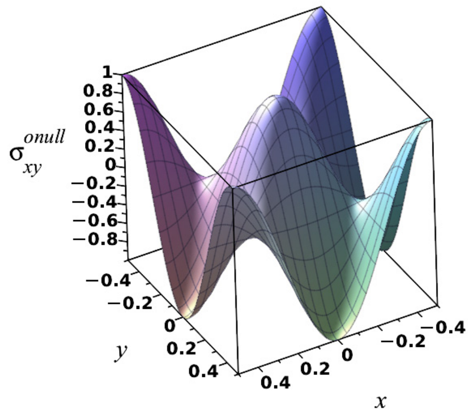
$$(u, v) = e^{-8\pi^2\eta_e t}(\sin 2\pi x \cos 2\pi y, -\cos 2\pi x \sin 2\pi y), \tag{36}$$

and

$$p = e^{-16\pi^2\eta_e t}\frac{\cos 4\pi x + \cos 4\pi y}{4} - 4\pi\eta_o e^{-8\pi^2\eta_e t}\sin 2\pi x \sin 2\pi y, \tag{37}$$



(a) $\sigma_{xx}^{o,\text{null}}$ or $\sigma_{yy}^{o,\text{null}}$



(b) $\sigma_{xy}^{o,\text{null}}$

FIG. 3. Plot of the "extra" stress generated by odd viscosity in a Taylor-Green vortex. It is only the null component of the stress responsible for this contribution as was discussed in Sec. III A. (a) Plot of the $\sigma_{xx}^{o,\text{null}}$ stress (which is identical to the $\sigma_{yy}^{o,\text{null}}$ stress). (b) Plot of the $\sigma_{xy}^{o,\text{null}}$ stress. Both stresses are given analytically in Eq. (38).

and $\tilde{p}$ corresponds to the odd viscosity-independent term of Eq. (37). The state of stress associated with odd viscosity can now be read off from Eq. (21). It is the null part of the odd stress tensor that describes the effects odd viscosity has on the liquid.

In Figs. 3(a) and 3(b) we plot the dimensionless null part of the odd stress tensor [Eq. (16)]

$$\sigma_{xx}^{o,\text{null}} = \sigma_{yy}^{o,\text{null}} = 4\pi\eta_o e^{-8\pi^2\eta_e t}\sin 2\pi x \sin 2\pi y, \quad \sigma_{xy}^{o,\text{null}} = 4\pi\eta_o e^{-8\pi^2\eta_e t}\cos 2\pi x \cos 2\pi y, \tag{38}$$

which constitutes the "extra stress" associated with odd viscosity.

It is clear that periodic problems should be described by the framework developed in Sec. III A since the pressure $\tilde{p}$ and the fields u and v are independent of odd viscosity η_o.

D. The Avron extra stress on a rotating cylinder

With the foregoing discussion in mind, we revisit Avron's problem and calculate the stress Eq. (4), cf. Ref. ([2], Eq. (61)). From Landau & Lifshitz ([10], §18) we know that the velocity field outside a solid disk of radius R rotating with angular velocity Ω in a viscous incompressible liquid is azimuthal

$$\mathbf{v} = v_\theta \hat{\mathbf{e}}_\theta, \quad \text{where} \quad v_\theta = \frac{\Omega R^2}{r}, \tag{39}$$

and r is the distance from the center of the disk. Then, the rr component of the $\sigma^{o,\text{null}}$ tensor in Eq. (16) evaluated at the surface $r = R$ of the cylinder reads $\sigma_{rr}^{o,\text{null}}(R) = 2\eta_o\Omega$, which recovers the Avron result [Eq. (4)].

It is only incidental however, that σ_{rr}^o gives exactly the same result: $\sigma_{rr}^o = 2\eta_o\Omega$; this happens because for this special problem we have $\omega \equiv 0$ and thus from Eq. (14) $\sigma_{rr}^{o,\text{null}} \equiv \sigma_{rr}^o$ everywhere.

IV. THE EXTRA STRESS WITH FREE BOUNDARIES

When the stress is specified on the boundaries, then what constitutes "extra" stress due to odd viscosity becomes ambiguous. This is because all the fields u, v, and $\tilde{p}$ and their derivatives that appear in Eq. (14) may depend on η_o and likewise for every term in the total stress expression

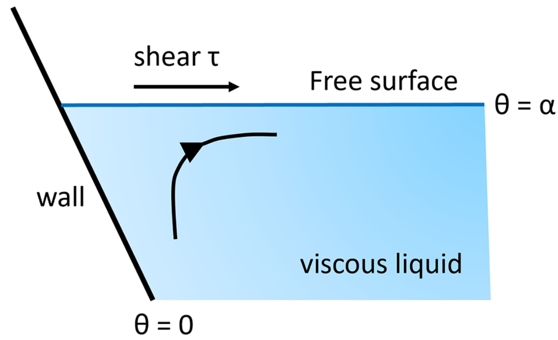


FIG. 4. Viscous liquid in a corner of angle α under a surface shear τ on its free surface described in Sec. IV (cf. Ref. ([14], Fig. 2(b))). This is an example where it is the stresses that are prescribed on a part of the boundary. All fields and stress components of the Cauchy stress tensor [Eq. (1)] become odd viscositydependent, making the definition of the "extra" stress ambiguous.

[Eq. (1)]. This has also been observed in Ref. [5] and was systematically developed analytically in Ref. [15] for a number of free-surface problems of an odd viscous liquid.

As an illustration, consider a viscous liquid endowed with odd viscosity in a corner of angle α between a solid substrate (at $\theta = 0$) and a free surface (at $\theta = \alpha$) in Fig. 4 under surface shear τ. The boundary conditions for this flow are

$$v_r = v_\theta = 0, \quad \text{at} \quad \theta = 0, \quad \text{and} \quad v_\theta = 0, \quad \sigma_{r\theta} = \tau \quad \text{on} \quad \theta = \alpha, \tag{40}$$

applied on the streamfunction $\psi = \tau r^2[A\cos 2\theta + B\sin 2\theta + C\theta + D]$, where the constants A to D will be determined from the boundary conditions and r is distance from the corner tip. This problem is identical to the one considered by Moffatt [14] but here the shear stress condition also accounts for odd viscosity:

$$\sigma_{r\theta} \equiv \eta_e \frac{1}{r}\partial_\theta v_r + 2\eta_o \partial_r v_r = \tau \quad \text{on} \quad \theta = \alpha. \tag{41}$$

We can thus determine the streamfunction ψ and modified pressure $\tilde{p}$ which now all depend on η_o:

$$\psi = -\frac{r^2\tau(-2\alpha + 2\theta - \sin(2\alpha - 2\theta) - 2\theta\cos(2\alpha) + \sin(2\alpha) + 2\cos(2\theta)\alpha - \sin(2\theta))}{4((2\sin(2\alpha)\alpha + 2\cos(2\alpha) - 2)\eta_o + (2\cos(2\alpha)\alpha - \sin(2\alpha))\eta_e)}, \tag{42}$$

and

$$\tilde{p} = \frac{2\eta_e\tau\ln(r)(\cos(2\alpha) - 1)}{(2\sin(2\alpha)\alpha + 2\cos(2\alpha) - 2)\eta_o + (2\cos(2\alpha)\alpha - \sin(2\alpha))\eta_e}. \tag{43}$$

Likewise, the velocity fields $v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ and $v_\theta = -\frac{\partial\psi}{\partial r}$, the pressure p, and the stress tensors $\boldsymbol{\sigma}^e$ and $\boldsymbol{\sigma}^o$ all depend on odd viscosity η_o.

There are exceptions however, in which the fields and the stresses may remain independent of η_o, although the boundary conditions are (partly) specified by the stresses. Consider for instance the case of a layer of fluid above a solid substrate inclined at angle α to the horizontal and surrounded by an ambient gas phase, cf. Fig. 5. Its free surface lies at constant elevation $y = h$ (cf. Ref. [10], (p. 54)). Now the shear stress at the free surface also accounts for odd viscosity

$$\sigma_{xy} \equiv -p - \eta_o \frac{\partial u}{\partial y} = -p_0, \tag{44}$$

where p and p_0 are the liquid and gas pressures, respectively. Solving the Navier-Stokes equations we find the same velocity field $u = \frac{\rho g \sin\alpha}{2\eta_e} y(2h - y)$, to the one derived in the literature [10], but a

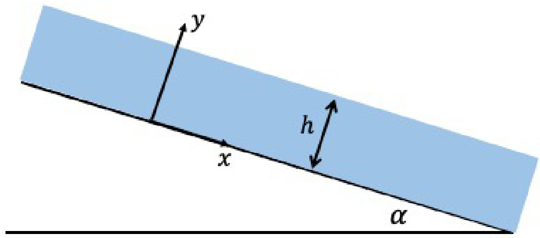


FIG. 5. Liquid layer of thickness h with a flat free surface on an inclined substrate. Although stresses are prescribed on the boundaries, the rate-of-strain tensor remains independent of η_o. Thus, the state of the "extra" stress associated with odd viscosity is given by the null part of the odd stress tensor $\sigma_{ij}^{o,\mathrm{null}}$, (as described in Sec. III) and not the full odd stress tensor σ_{ij}^{o}.

different pressure

$$p = p_0 + \rho g(h - y)\frac{\sqrt{\eta_e^2 + \eta_o^2}}{\eta_e}\cos(\alpha + \beta), \tag{45}$$

where $\beta = \arctan\frac{\eta_o}{\eta_e}$. Here $\tilde{p} = p_0 + \rho g(h - y)\cos\alpha$. Thus, it is only the pressure p that depends on odd viscosity. Thus, this problem falls into the category treated in Sec. III A. In these cases, the state of the "extra" stress associated with odd viscosity is given by the null part of the odd stress tensor $\sigma_{ij}^{o,\mathrm{null}}$ and not the full odd stress tensor σ_{ij}^{o}.

V. THE STRESS POTENTIAL AND NEW FLUID-FLOW BEHAVIOR

According to Olver [8], a vector $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ in an n-dimensional Euclidean space is a null divergence ($\mathrm{div}\sigma \equiv 0$) if and only if there is an antisymmetric matrix Q ($Q_{ij} = -Q_{ji}$) such that $\sigma_i = \partial_j Q_{ij}$.

We can employ this theorem to reach certain conclusions about the flow structure of liquids endowed with odd viscosity. The null stress σ^{null} can be written as a null divergence. For the Cauchy stress tensor components $\sigma_{11}^{\mathrm{null}}$ and $\sigma_{12}^{\mathrm{null}}$ we calculate, resorting to Table II,

$$\left(\sigma_{11}^{\mathrm{null}}, \sigma_{12}^{\mathrm{null}}\right) = \left(\frac{\partial Q_{11}}{\partial x} + \frac{\partial Q_{12}}{\partial y}, \frac{\partial Q_{21}}{\partial x} + \frac{\partial Q_{22}}{\partial y}\right) = (-2\partial_y(\eta_e v + \eta_o u), 2\partial_x(\eta_e v + \eta_o u)). \tag{46}$$

In other words we find

$$Q_{12} = -2(\eta_e v + \eta_o u) = -Q_{21}. \tag{47}$$

Repeating for the stress components $\sigma_{21}^{\mathrm{null}}$ and $\sigma_{22}^{\mathrm{null}}$ we also obtain

$$Q_{12} = 2(\eta_e u - \eta_o v) = -Q_{21}. \tag{48}$$

Of course, in two dimensions the off-diagonal components of the matrix Q correspond to the vector potential of a field, so that $(\sigma_{11}^{\mathrm{null}}, \sigma_{12}^{\mathrm{null}}, 0) = -2\nabla \times (\eta_e v + \eta_o u)\hat{\mathbf{e}}_z$ and $(\sigma_{21}^{\mathrm{null}}, \sigma_{22}^{\mathrm{null}}, 0) = 2\nabla \times (\eta_e u - \eta_o v)\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ is the unit vector in the z direction in a Cartesian frame of reference.

With this theoretical framework in mind, one can invent new fluid-flow behavior. For instance, consider the expanding bubble problem [5]. The null stress tensor can be read off from Eq. (10) or directly from Table II. The boundary condition of zero shear stress $\sigma_{r\theta} = 0$ at the bubble-liquid interface at $r = R$ can thus be written as

$$\sigma_{r\theta} \equiv 2\partial_r[\eta_o v_r + \eta_e v_\theta] + \eta_e \omega = 0 \quad \text{at} \quad r = R. \tag{49}$$

Since there is no mechanism that generates vorticity at the boundary, we can set $\omega = 0$. The obvious solution then, that satisfies Eq. (49) is

$$v_\theta = -\frac{\eta_o}{\eta_e} v_r, \tag{50}$$

i.e., exactly the solution that was obtained in the literature [5] by other means. This constitutes new fluid flow behavior induced by odd viscosity, superposed on the radial expansion of a bubble [13]. It is possible to also calculate the matrix Q, which would lead to the same result.

VI. DISCUSSION AND CONCLUDING REMARKS

In this article we showed that when only velocities are prescribed on the boundaries, the stress associated with odd viscosity is identified with the null stress tensor [Eqs. (15) and (16)], that is, the part of the stress tensor whose divergence vanishes identically. We demonstrate this point in the calculation of the Couette flow between two rotating cylinders and the more complex problem of viscosity of suspensions. In general, problems like the Stokes oscillating plate, Couette and Poiseuille flow in a channel, the Rayleigh impulsive motion of a boundary, and Hiemenz flow will all have an odd viscosity dependence only through the null odd stress tensor whose theory was developed in Sec. III.

The situation is very different when it is the stresses that are prescribed on the boundary or some part of it. Now, *every term* of the total stress [Eq. (1)] depends on the coefficient of odd viscosity η_o and the interpretation of what constitutes the "extra" stress associated with odd viscosity is ambiguous. Nevertheless we show that the null divergence of the total stress [Eq. (1)] can give rise to new fluid-flow behavior. The expanding bubble in a viscous liquid is such an example.

We note that the expression of the stress tensor including the null part [Eq. (21)] has also appeared in the recent literature ([5], Eq. (6)) but in the context of calculating the odd viscosity-related force and torque acting on a closed contour when only velocities are prescribed on the boundaries. Reference [5] also pointed out the importance of clearly recognizing the contributions of odd viscosity in order to avoid reaching erroneous conclusions (see the discussion in Ref. [5], below Eq. (8)).

The question may arise as to whether the theory developed in this article can prove useful from a practical point of view. To this end we conclude with the following comments. When odd viscosity gives rise to new fluid-flow behavior, and the theoretical development of Sec. V can be employed, determining the value of odd viscosity η_o could proceed simply by measuring the magnitude and direction of the flow by a probe and decomposing the measurements with respect to a suitable coordinate system [for instance, $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta$ for the problem leading to the flow, Eq. (50)]. This would require the determination of the antisymmetric matrix Q of Sec. V or equivalently, an interfacial condition of the form [Eq. (49)].

Although the development of the odd stress theory for free-surface problems in Sec. IV combines logical consistency with completeness, practically, this detailed knowledge might not be necessary to determine the value of the odd viscosity coefficient η_o. This was demonstrated amply in the experimental work of Soni *et al.* [6] where the effects of odd viscosity were associated only with the null part of the odd stress tensor [that is, with $\sigma_{ij}^{o,\text{null}}$ in Eq. (21) where $\sigma_{ij} = -\tilde{p}\delta_{ij} + \sigma_{ij}^e + \sigma_{ij}^{o,\text{null}}$], although *all* terms in the Cauchy stress tensor depend on odd viscosity as manifested by the analysis of Sec. IV. There might be other occasions however, where the odd viscosity dependence of the full stress tensor might need to be accounted for, but at this moment we possess no such insight.

Finally, in problems expressed exclusively through the no-slip boundary condition, a determination of the pressure $\tilde{p}$ with and without odd viscosity, while taking into account its vorticity dependence in the latter case ($\tilde{p} = p + \eta_o\omega$) could, in principle, be another approach to experimentally determine the value of the odd viscosity coefficient η_o.

We reiterate here that odd viscosity is not introduced as an academic subject in the dynamics of classical fluids, but it has observable effects and its magnitude was recently measured in a suspension of spinning magnets [6]. We thus expect it to find its way in systems with broken time-reversal symmetry from the application of a magnetic field. In biomedical applications where such a field has been employed to destroy malignant cells, either mechanically or through local heating, one would like to know the effect of odd viscosity on suspensions of magnetic nanoparticles [16]. Odd viscosity might be intimately connected to the theoretical explanation of a number of experimental observations, for instance, interface patterns propagating in a direction opposite to the bulk liquid [17]. Since these suspensions are currently employed in a number of biomedical applications and theory [16,18], the effect of odd viscosity on their behavior may prove to be critical in the design of experiments and their commercial adaptation. Finally, it is of fundamental interest to study the role of odd viscosity in turbulent flows where a collection of vortices is also present. This type of research was initiated at the University of Chicago [19].

For industrial applications one would like to know how odd viscosity modifies the flow when the contact line of a liquid droplet is moving, and in particular, to determine whether the slip law governing contact line motion [20] is modified by the presence of odd viscosity.

ACKNOWLEDGMENTS

This work was supported by the Center for Computation & Theory of Soft Materials at Robert R. McCormick School of Engineering and Applied Science, Northwestern University.

---

[1] J. E. Avron, R. Seiler, and P. G. Zograf, Viscosity of Quantum Hall Fluids, Phys. Rev. Lett. **75**, 697 (1995).
[2] J. E. Avron, Odd viscosity, J. Stat. Phys. **92**, 543 (1998).
[3] E. M. Lifshitz and L. P. Pitaevskii, *Course of Theoretical Physics. Vol. 10: Physical Kinetics* (Pergamon Press, 1981).
[4] M. F. Lapa and T. L. Hughes, Swimming at low Reynolds number in fluids with odd, or Hall, viscosity, Phys. Rev. E **89**, 043019 (2014).
[5] S. Ganeshan and A. G. Abanov, Odd viscosity in two-dimensional incompressible fluids, Phys. Rev. Fluids **2**, 094101 (2017).
[6] V. Soni, E. S. Bililign, S. Magkiriadou, S. Sacanna, D. Bartolo, M. J. Shelley, and W. T. M. Irvine, The odd free surface flows of a colloidal chiral fluid, Nat. Phys. **15**, 1188 (2019).
[7] E. Kirkinis and A. V. Andreev, Odd viscosity-induced stabilization of viscous thin liquid films, J. Fluid Mech. **878**, 169 (2019).
[8] P. J. Olver, Conservation laws and null divergences, Math. Proc. Cambridge Philos. Soc. **94**, 529 (1983).
[9] C. Truesdell and W. Noll, *The Nonlinear Field Theories of Mechanics* (2nd ed.) (Springer-Verlag, Berlin, 1992); P. Chadwick, *Continuum Mechanics* (Halsted Press [John Wiley & Sons], New York (Now in Dover), 1976), Concise theory and problems.
[10] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics. Course of Theoretical Physics* (Pergamon Press Ltd., London-Paris, 1987), Vol. 6.
[11] J. F. Brady, The Einstein viscosity correction in n dimensions, Int. J. Multiphase Flow **10**, 113 (1983).
[12] G. K. Batchelor, The stress system in a suspension of force-free particles, J. Fluid Mech. **41**, 545 (1970).
[13] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).
[14] H. K. Moffatt, Viscous and resistive eddies near a sharp corner, J. Fluid Mech. **18**, 1 (1964).
[15] E. Kirkinis, J. Mason, and M. Olvera de la Cruz, Odd viscosity-induced passivation of Moffatt vortices, J. Fluid Mech. **950**, A19 (2022).
[16] I. Torres-Díaz and C. Rinaldi, Recent progress in ferrofluids research: Novel applications of magnetically controllable and tunable fluids, Soft Matter **10**, 8584 (2014).
[17] C. Rinaldi and M. Zahn, Effects of spin viscosity on ferrofluid flow profiles in alternating and rotating magnetic fields, Phys. Fluids **14**, 2847 (2002); A. Chaves, M. Zahn, and C. Rinaldi, Spin-up flow of ferrofluids: Asymptotic theory and experimental measurements, *ibid.* **20**, 053102 (2008).

[18] C. Rinaldi, Continuum modeling of polarizable systems, Ph.D. thesis, Massachusetts Institute of Technology, 2002; A. Aggarwal, E. Kirkinis, and M. Olvera de la Cruz, Activity-induced migration of viscous droplets on a solid substrate, J. Fluid Mech. **955**, A10 (2023).

[19] P. Wiegmann and A. G. Abanov, Anomalous Hydrodynamics of Two-Dimensional Vortex Fluids, Phys. Rev. Lett. **113**, 034501 (2014).

[20] E. Kirkinis and S. H. Davis, Hydrodynamic Theory of Liquid Slippage on a Solid Substrate Near a Moving Contact Line, Phys. Rev. Lett. **110**, 234503 (2013); Moffatt vortices induced by the motion of a contact line, J. Fluid Mech. **746**, R3 (2014).