

## LES wall modeling for heat transfer at high speeds

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A practical application of universal wall scalings is near-wall turbulence modeling. In this paper, we exploit the semilocal scaling [Patel, Boersma, and Pecnik, *Phys. Rev. Fluids*, **2**, 084604 (2017)] and derive an eddy conductivity closure for wall-modeled large-eddy simulation of high-speed flows. We show that while the semilocal scaling does not collapse high-speed direct numerical simulation (DNS) data, the resulting eddy conductivity and the wall model work fairly well. The paper attempts to answer the following outstanding question: why the semilocal scaling fails but the resulting eddy conductivity works well. We conduct DNSs of Couette flows at Mach numbers from  $M = 1.4$  to 6. We add a source term in the energy equation to get a cold wall, a close-to-adiabatic wall, and a hot wall. Detailed analysis of the flows' energy budgets shows that aerodynamic heating is the answer to our question: Aerodynamic heating is not accounted for in Patel *et al.*'s semilocal scaling but is modeled in the equilibrium wall model. We incorporate aerodynamic heating in the semilocal scaling and show that the new scaling successfully collapses the high-speed DNS data. We also show that incorporating aerodynamic heating or not, the semilocal scaling gives rise to the exact same eddy conductivity, thereby answering the outstanding question.

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## I. INTRODUCTION

Engineering designs of aero-vehicles at high speeds are often aided by computational tools [1–4]. Compared to wall-resolving methods like direct numerical simulation (DNS) and wall-resolved large-eddy simulation, wall-modeled methods like hybrid RANS/LES method and wall-modeled large-eddy simulation (WMLES) are much more cost-effective at high Reynolds numbers [5,6]. Thanks to the recent developments in high-performance computing [7], the scale-resolving tool of WMLES has seen increased use for real-world engineering problems at low speeds [8,9]. For example, WMLES of atmospheric boundary-layer flows are reported in Refs. [10–15]; Xu *et al.* reported WMLES of a compressor's return channel [16]; Park *et al.* presented WMLES results for the NASA common research model [17]; Ghate *et al.* compared WMLES, DDES, and RANS for the NASA juncture flow [18]; Huang, Xu, and coauthors reported WMLES of flow around complex and moving boundaries [19–21], to name a few. Because of a lack of model validation, there are fewer uses of WMLES for high-speed flows, see, e.g., Refs. [22,23], and even fewer for heat transfer problems, see, e.g., Refs. [24,25]. Two open questions in the field are as follows: How to close the eddy conductivity for high-speed flows? How does the wall model do and why does it work/not work? The objective of this work is to answer the above two questions. In the following, we briefly

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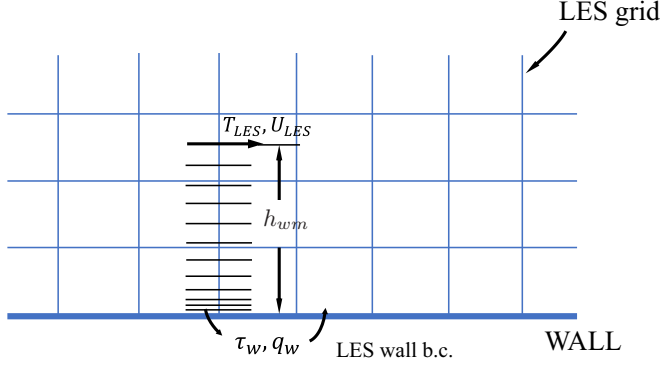


FIG. 1. A sketch of WMLES. The wall model is a stress-based one.

review the existing literature on this topic. We would also show some data to better motivate the present work.

Figure 1 is a sketch of a WMLES. The wall model models the wall-shear stress and the wall heat transfer rate. The most extensively used wall model is the equilibrium wall model (EWM) [26], which reads

$$\frac{d}{dy} \left[ (\mu + \mu_{t,wm}) \frac{du_{||}}{dy} \right] = 0, \quad (1a)$$

$$\frac{d}{dy} \left[ (\mu + \mu_{t,wm}) u_{||} \frac{du_{||}}{dy} + c_p \left( \frac{\mu}{Pr} + \frac{\mu_{t,wm}}{Pr_t} \right) \frac{dT}{dy} \right] = 0, \quad (1b)$$

where  $\mu$  is the molecular dynamic viscosity,  $\mu_{t,wm}$  is wall model's eddy viscosity,  $u_{||}$  is the wall-parallel velocity,  $c_p$  is the specific heat,  $Pr$  is the molecular Prandtl number,  $Pr_t$  is the turbulent Prandtl number ( $\mu_{t,wm}/Pr_t$  is the eddy conductivity), and  $T$  is the temperature. The eddy viscosity and the eddy conductivity in the wall model equations must be closed, and a commonly used closure is

$$\mu_{t,wm} = \sqrt{\rho \tau_w} \kappa y D, \quad (2)$$

and  $Pr_t = 0.9$ . Here  $\rho$  is the fluid density,  $\tau_w$  is the wall-shear stress,  $\kappa = 0.41$  is the von Karman constant, and

$$D = [1 - \exp(-y^+/A^+)]^2 \quad (3)$$

is the van Driest damping function with  $A^+ = 17$  being the van-Driest damping function constant. Equations (1a) and (1b) are solved on a one-dimensional (1D) grid between the wall  $y = 0$  and a matching location  $y = h_{wm}$  with the following matching condition at  $y = h_{wm}$ :

$$u_{||}(y = h_{wm}) = U_{LES}, \quad T(y = h_{wm}) = T_{LES}, \quad (4)$$

where the subscript LES denotes large-eddy simulation (LES) quantities. Although there are complications like log-layer mismatch [26,27] and near-wall resolution [22,28,29] when applying the EWM in a WMLES, the model gives accurate estimates of the mean flow if a constant-stress-layer/surface-layer exists. In fact, at low speeds, the energy equation Eq. (1b) degenerates to  $T = \text{const}$ , and integrating Eqs. (1a) leads to  $U^+ = \log(y^+)/\kappa + B$ , i.e., the well-established logarithmic law of the wall [30,31], which is a direct *a priori* validation of the EWM at low speeds. Here the superscript  $+$  denotes normalization by the wall units and  $B \approx 5.2$  is a constant. The largest uncertainty comes from the von Kármán constant, which is responsible for a 5% error. While *a priori* validation of the EWM at low speed is quite straightforward, *a priori* validation at high speeds is

much more difficult because there is no well-established universal law of the wall. Considering that one relies on the law of the wall to close the wall model equations, the lack of a well-established universal law of the wall partly explains why WMLES is considered to be less reliable for flows at high speeds than at low speeds.

Finding a universal law of the wall for the fluid velocity and the fluid temperature in high-speed boundary layer flows is a continuous effort in fundamental flow physics research. The research on this topic dates back to van Driest [32], Morkovin [33], and Huang *et al.* [34,35]. The more recent developments are by Trettel and Larsson on the velocity scaling [36] and Patel *et al.* on the scalar scaling [37]. Patel *et al.* hypothesized that turbulent transport and viscous diffusion are the dominant terms in the Reynolds averaged NS equations. Their hypothesis gives rise to the following semilocal scaling:

$$\begin{aligned} \frac{dU^*}{dU} &= \left(1 + \frac{y}{\text{Re}_\tau^*} \frac{d\text{Re}_\tau^*}{dy}\right) \sqrt{\frac{\rho}{\rho_w}} / u_\tau, \\ U^* &\equiv y^*, \quad \text{in the viscous sublayer,} \\ U^* &\equiv \frac{1}{\kappa} \log(y^*) + B, \quad \text{in the log layer;} \end{aligned} \quad (5)$$

and

$$\begin{aligned} \frac{d\theta^*}{d\theta} &= \left(1 + \frac{y}{\text{Re}_\tau^*} \frac{d\text{Re}_\tau^*}{dy}\right) \sqrt{\frac{\rho}{\rho_w}} / \theta_\tau, \\ \theta^* &\equiv \text{Pr}^* y^*, \quad \text{in the viscous sublayer,} \\ \theta^* &\equiv \frac{1}{\kappa_T} \log(y^*) + f(\text{Pr}^*), \quad \text{in the log layer;} \end{aligned} \quad (6a)$$

$$\begin{aligned} \frac{d\theta_T}{d\theta^*} &= \frac{1/\text{Pr}^* + \alpha_t/\mu}{1 + \alpha_t/\mu}, \\ \theta_T &\equiv y^*, \quad \text{in the viscous sublayer,} \\ \theta_T &\equiv \frac{1}{\kappa_T} \log(y^*) + B', \quad \text{in the log layer.} \end{aligned} \quad (6b)$$

The above semilocal scaling maps the velocity  $U$  and the temperature  $T$  (or  $T - T_w$ ) to  $U^*$  and  $\theta_T$ , both of which are universal functions of  $y^* = \sqrt{\rho \tau_w} y / \mu$ . Here  $1/\kappa_T \approx 2.12$  [38],  $f(\text{Pr}^*)$  is a function of  $\text{Pr}^*$ ,  $\text{Pr}^* = \mu/\lambda$ ,  $B' \approx 6.5$  is a constant,  $\theta = T - T_w$ ,  $\theta_\tau = q_w / (\rho_w c_{pw} u_\tau)$ ,  $q_w$  is the wall heat flux,  $u_\tau = \sqrt{\tau_w / \rho_w}$  is the friction velocity,  $\text{Re}_\tau^* = \sqrt{\rho \rho_w} u_\tau h / \mu$ ,  $\lambda$  is the molecular conductivity, and  $\alpha_t = \mu_t / \text{Pr}_t$  is the eddy conductivity. The subscript  $w$  denotes quantities evaluated at the wall, and the superscript  $*$  denotes normalization by local quantities (as opposed to wall quantities). All quantities in Eqs. (5) and (6) are Reynolds averaged. Compared to the velocity scaling, which has received much attention [39–43], the temperature (scalar) scaling has received much less attention. The temperature transformation in Ref. [37] is found to work in low-speed variable property flows, i.e., at supercritical flow conditions. Figures 2(a) and 3 compare  $\theta_T$  to the expected universal scaling in incompressible [45] and compressible flows. Evaluating  $\theta_T$  proves to be not possible for many of the online DNS databases [39,46]: First, the evaluation of  $\theta_T$  involves the evaluation of the eddy conductivity,  $\alpha_t = -\rho \overline{\theta'' v''} / d\theta/dy$ , but  $\overline{\theta'' v''}$  is not available. Second, we could not check the statistical convergence of the thermal fields. We would revisit this point in Secs. IV and V. There we shall see that the thermal field is more challenging to compute than the momentum field. In all, we conclude from Figs 2(a) and 3 that while the semilocal scaling collapses the low-speed data, it fails to do so for high-speed flows.

In addition to collapsing data, a transformation that maps  $U$  and  $T$  to a universal scaling gives closure models for the eddy viscosity/conductivity. Yang and Lv [42] exploited the semilocal

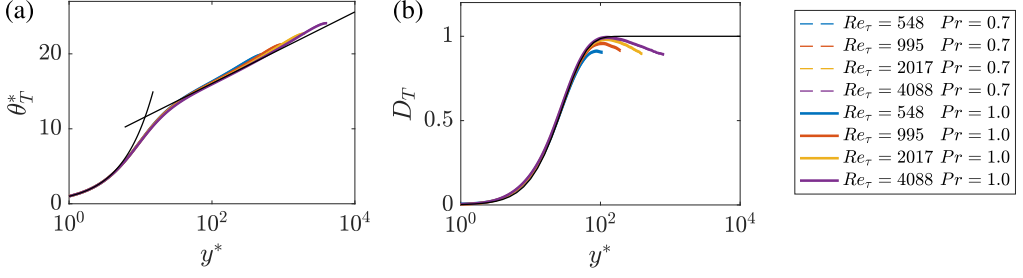


FIG. 2. (a)  $\theta_T$  as a function of  $y^*$  for incompressible channel flow at  $Re_\tau = 548, 995, 2017$ , and  $4088$  and for  $Pr = 0.7$  and  $1.0$  [45]. The two thin black lines are  $\theta_T = y^*$  and  $\theta_T = 1/\kappa_T \log(y^*) + B'$ . For incompressible flows,  $\theta^* = \theta/\theta_\tau$ ,  $y^* = y^+$ . (b)  $D_T = [(d\theta_T/dy^*)^{-1} - 1]/(\kappa y^*)$  for  $y/\delta < 0.2$ , i.e., below the logarithmic layer. The thin black line corresponds to  $D_T(y) = [1 - \exp(-y/A_T)]^2$ , where  $A_T = 20$ .

scaling and concluded that the damping function must be

$$D_{sl} = [1 - \exp(-y^*/A^+)]^2. \quad (7)$$

The use of semilocal scaling in the damping function was also reported in Ref. [47]—although there its use is entirely empirical. In a recent work, Iyer *et al.* [48] proposed a mixed scaling, which gives more accurate results than Eq. (7) in a WMLES context. To date, research on this topic is limited to the momentum field and is yet to be extended to the thermal field.

In Sec. II, we exploit Eq. (6) and close the energy equation. Given the poor performance of the semilocal scaling in Figs. 2(a) and 3, it would seem that relying on the scaling in Eq. (6) for closure models is an ill-conceived idea. The results, however, are surprising. In fact, Iyer and Malik [48] assessed a similar model and concluded that the EWM works well for high-speed flows: The error is less than about 5% if the LES/wall-model matching location is not above about 10% of the half channel height and if one employs the semilocal scaling. The outstanding question is why the semilocal scaling fails but the resulting wall model works.

The rest of the paper is organized as follows. In Sec. II, we exploit the semilocal scaling to close the energy equation. We conduct DNSs and WMLESs of turbulent Couette flows for *a priori* and *a posteriori* analyses. Details of the DNSs and WMLESs are shown in Sec. III. In Sec. IV,

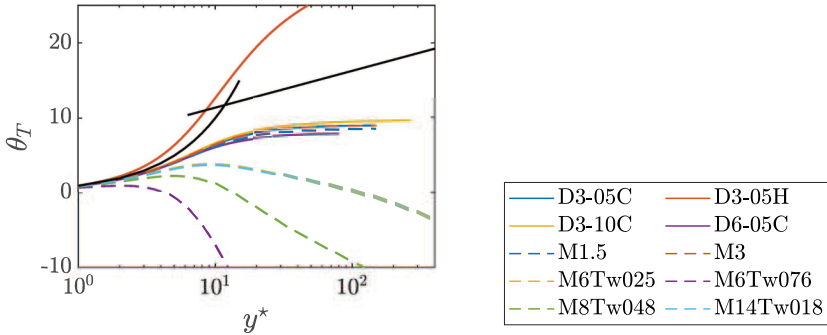


FIG. 3.  $\theta_T$  as a function of  $y^*$  for our DNSs: D3-05C, D3-05H, D3-10C, and D6-05C, Zhang *et al.*'s DNSs: M6Tw025, M6Tw076, M8Tw048, and M14Tw018 [41], and Coleman *et al.*'s DNSs: M1.5 and M3. Table I tabulates our DNSs' details. The nomenclature of Zhang *et al.*'s DNSs is as follows: M[A]Tw[B], where A is the freestream Mach number,  $B = T_w/T_r \times 100$ , and  $T_r$  is the recovery temperature. The nomenclature of Coleman *et al.*'s DNSs is as follows: M[A], where A is the bulk Mach number [44]. The two thin black lines are  $\theta_T = y^*$  and  $\theta_T = 1/\kappa_T \log(y^*) + B'$ .

we show that the aerodynamic heating term  $(\mu + \mu_{t,wm})u_{||}du_{||}/dy$  is the answer to our question. Incorporating the aerodynamic heating term in the semilocal scaling, we show in Sec. V that the semilocal scaling collapses the high-speed data. Finally, conclusions are given in Sec. VI.

## II. SEMI-LOCAL SCALING AND EDDY CONDUCTIVITY

In this section, we exploit the semilocal scaling in Eq. (6) and model the eddy conductivity in the constant-stress-layer/surface-layer. In reality, the constant stress layer may be much thinner than the layer where the velocity follows the logarithmic law. However, the constant stress layer is the starting point of the derivations in Refs. [36,37], and we will follow the previous authors. This section's basic logic is to accept all the assumptions in Ref. [37] and derive the eddy conductivity. We will revisit the validity of some assumptions in Sec. V. Following Patel *et al.* [37], we neglect aerodynamic heating and mean flow convection. By definition, the eddy conductivity is

$$\alpha_t = -\rho \overline{v''\theta''} / \frac{d\theta}{dy}, \quad (8)$$

where  $-\rho \overline{v''\theta''}$  is the turbulent heat flux, and  $\overline{\cdot}$  denotes Reynolds averaged. The energy equation in the surface layer reads [37]:

$$-\rho \overline{v''\theta''} + \frac{\mu}{Pr^*} \frac{d\theta}{dy} = q_w / c_{pw}, \quad (9)$$

and therefore

$$-\rho \overline{v''\theta''} = q_w / c_{pw} - \frac{\mu}{Pr^*} \frac{d\theta}{dy} = \rho_w u_\tau \theta_\tau - \frac{\mu}{Pr^*} \frac{d\theta}{dy}. \quad (10)$$

Per Eq. (6), we have

$$\begin{aligned} \frac{d\theta}{dy} &= \frac{d\theta}{d\theta^*} \frac{d\theta^*}{d\theta_T} \frac{d\theta_T}{dy^*} \frac{dy^*}{dy} \\ &= \frac{\theta_\tau}{\left(1 + \frac{y}{Re_\tau^*} \frac{dRe_\tau^*}{dy}\right) \sqrt{\frac{\rho}{\rho_w}}} \times \frac{1 + \alpha_t / \mu}{1 / Pr^* + \alpha_t / \mu} \times \frac{d\theta_T}{dy^*} \times \left(1 + \frac{y}{Re_\tau^*} \frac{dRe_\tau^*}{dy}\right) \frac{\sqrt{\rho \tau_w}}{\mu} \\ &= \rho_w u_\tau \theta_\tau \frac{d\theta_T}{dy^*} \frac{1 + \alpha_t / \mu}{1 / Pr^* + \alpha_t / \mu} \frac{1}{\mu}. \end{aligned} \quad (11)$$

In Eq. (11),  $d\theta_T / dy^*$  is (supposedly) a universal function of  $y^*$ , and according to Eq. (6b)

$$\frac{d\theta_T}{dy^*} = 1 \quad \text{in the viscous sublayer,} \quad \frac{d\theta_T}{dy^*} = \frac{1}{\kappa_T y^*} \quad \text{in the logarithmic layer.} \quad (12)$$

We make an analogy to the velocity scaling and define a damping function  $D_T(y^*)$  such that

$$\frac{d\theta_T}{dy^*} = \frac{1}{1 + \kappa_T y^* D_T(y^*)}. \quad (13)$$

This damping function  $D_T$  is 0 when  $y^* \rightarrow 0$  and 1 when  $y^* \gg 1$ . Thanks to the universality of  $\theta_T$ , we can measure  $D_T$  in an incompressible flow and expect the measured  $D_T$  to work at high speeds. Figure 2(b) shows  $D_T$ , i.e.,  $D_T = [(d\theta_T / dy^*)^{-1} - 1] / (\kappa_T y^*)$ , as a function of  $y^*$  in flows at Reynolds numbers from  $Re_\tau = 548$  to 4088. There is an apparent Reynolds number effect, but at a sufficiently high Reynolds number and for  $y \ll h$ ,

$$D_T(y^*) = [1 - \exp(-y^* / A_T)]^2, \quad A_T = 20, \quad (14)$$

seems to be a good working approximation. Here we emphasize that  $y$ 's scaling in the damping function  $D_T$  is solely controlled by  $\theta_T$ 's scaling. While one may employ other scalings for  $y$  in

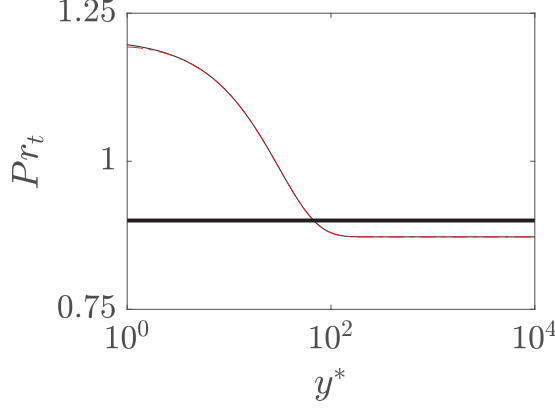


FIG. 4. Turbulent Prandtl number model. Here the solid blank line and the red dashed line correspond to Eqs. (17) and (19).  $\epsilon = 10^{-6}$  in Eq. (19).

the damping function, and it may give more accurate results, such efforts are necessarily empirical [48,49].

Equations (10), (11), and (13) give

$$\alpha_t = \frac{(1 + \kappa_T y^* D_T)(\mu/\text{Pr}^* + \alpha_t)}{1 + \alpha_t/\mu} - \frac{\mu}{\text{Pr}^*}. \quad (15)$$

Equation (15) is a quadratic equation of  $\alpha_t$ , whose only physical solution is

$$\alpha_t/\mu = \kappa_T y^* D_T(y^*). \quad (16)$$

Alternatively,

$$\text{Pr}_t = \frac{\mu_t}{\alpha_t} = \frac{\kappa}{\kappa_T} \frac{D_{sl}}{D_T} = \frac{\kappa}{\kappa_T} \left[ \frac{1 - \exp(-y^*/A)}{1 - \exp(-y^*/A_T)} \right]^2. \quad (17)$$

Although the denominator  $[1 - \exp(-y^*/A_T)]^2 = 0$  when  $y^* = 0$ , Eq. (17) is nonsingular at  $y^* = 0$ :

$$\lim_{y^* \rightarrow 0} \text{Pr}_t = \frac{\kappa}{\kappa_T} \left[ \frac{y^*/A}{y^*/A_T} \right]^2 = \frac{\kappa}{\kappa_T} \left( \frac{A_T}{A} \right)^2. \quad (18)$$

Nonetheless, one may still encounter numerical difficulty if (s)he directly implements Eq. (17), in which case, the following formulation should be used:

$$\text{Pr}_{t,\epsilon} = \frac{\mu_t}{\alpha_t} = \frac{\kappa}{\kappa_T} \frac{[1 - \exp(-y^*/A)]^2 + A_T^2 \epsilon}{[1 - \exp(-y^*/A_T)]^2 + A^2 \epsilon}, \quad (19)$$

where  $\epsilon$  is a small number. Figure 4 shows the turbulent Prandtl number in Eqs. (17) and (19) as well as that in the conventional viscous scaled WM, i.e.,  $\text{Pr}_t = 0.9$ . The turbulent Prandtl in Eq. (17) is a decreasing function of  $y^*$  and asymptotes to  $\kappa/\kappa_T = 0.85$  for sufficiently large  $y^*$ . We are certainly not the first to try to model the turbulent Prandtl number. Such work dates back to Reynolds 1901 [50] if not earlier. The more recent work is reviewed in Ref. [51] and the earlier work is reviewed in Ref. [52]. Considering Eq. (17) is intended for low-speed flows as well, it is instructive to review the previous work. Cebeci [53] argued that the scaling of the turbulent Prandtl number in the wall layer is a result of the disparate wall damping to the eddy conductivity and the eddy viscosity, arriving at a turbulent Prandtl number similar to the one in Eq. (17). Weigand *et al.* [54] argued that it is not always possible to find an explicit formulation for the turbulent Prandtl number  $\text{Pr}_t$  as a function of the wall-normal distance  $y$  and proposed a Prandtl number that is a function of the

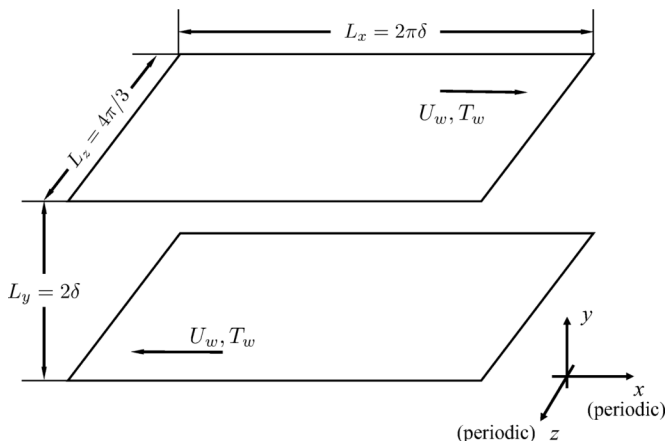


FIG. 5. A sketch of the flow configuration.  $x$ ,  $y$ , and  $z$  are the streamwise, wall-normal, and spanwise coordinates. The flow is periodic in the streamwise and the spanwise directions. The two walls are at the same temperature  $T_w$ . The wall velocity is  $U_w$ .  $\delta$  is the half channel height.

eddy viscosity. Furthermore, Bae [55] made  $Pr_t$  a function of the fluid properties. Generally,  $Pr_t$  is a human-made concept, and each model has its own applicable range. Here, we will refrain from making any modeling recommendations.

Equations (1), (2), (7), and (17) constitute our wall model. Equation (17) is the contribution of this work. The wall model closures, i.e., the eddy viscosity and the eddy conductivity (the turbulent Prandtl number), are direct consequences of the semilocal scaling. Like other wall models, our model is valid only in the viscous layer and the logarithmic layer. Admittedly, a wall model is sometimes applied with the LES/matching location at  $y/\delta \approx 0.3$ , i.e., above the logarithmic layer. However, that should not be concern for LES wall modeling. Wall-modeled LES, by definition, is LES. An LES must resolve the large scales, and for boundary-layer flows, an LES must resolve the logarithmic layer. LES wall modeling ought to focus on wall modeling in a proper LES rather than an under-resolved LES. This is a practice that most, if not all, wall model developers follow. Moreover, the flow above the logarithmic layer is not expected to be universal, and one cannot model something that is different from one flow to another.

### III. COMPUTATIONAL SETUP

Rather than entirely relying on the online databases, we will conduct our own DNSs. The online DNS databases is not sufficient because they do not report all the statistics we need. Figure 5 shows a sketch of the flow configuration. The mean pressure gradient is 0 in Couette flow, and the logarithmic layer extends closer to the channel centerline than, e.g., in a channel flow [56]. The absence of an apparent wake layer allows a direct comparison between the flow and the EWM [24,42]. In the next two subsections, we present the detailed setups of our DNSs and LESs.

#### A. DNS

We use the in-house high-order finite-difference code Hoam-OpenCFD for our DNSs. The code solves the full compressible Navier-Stokes equations. The working fluid is an ideal gas. The molecular viscosity varies with the temperature according to Sutherland's law,

$$\frac{\mu}{\mu_{\text{ref}}} = \left( \frac{T}{T_{\text{ref}}} \right)^{3/2} \frac{T_{\text{ref}} + T_s}{T + T_s}, \quad (20)$$

TABLE I. DNS details.  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$  are the grid spacings in streamwise, wall-normal, and spanwise directions. We show both the grid resolution at the wall (the first number), i.e.,  $\Delta x_w^+$ ,  $\Delta y_w^+$ , and  $\Delta z_w^+$ , and the grid resolution at the channel centerline, i.e.,  $\Delta x_c^*$ ,  $\Delta y_c^*$ , and  $\Delta z_c^*$  (the second number).  $a_{\text{ref}} = \sqrt{\gamma R T_{\text{ref}}}$  is the speed of sound at the reference temperature, where  $\gamma = 1.4$  is the specific heat ratio.  $M$  is the wall Mach number. The molecular Prandtl number is  $\text{Pr} \equiv \text{Pr}^* = 0.7$ .  $\text{Re}_b = \rho_b U_w \delta / \mu_{\text{ref}}$  is the bulk Reynolds number, where  $\rho_b$  is the bulk density. The computational domain size is  $L_x \times L_y \times L_z = 2\pi\delta \times 2\delta \times 4/3\pi\delta$ .  $\varphi$  is a heat source.  $b_g$  controls the grid stretching and is defined in Eq. (21).

| Case   | $U_w/a_{\text{ref}}$ | $M$ | $\text{Re}_b$ | $T_w/T_{\text{ref}}$ | $\varphi\delta/\rho_b U_w^3$ | $N_x \times (N_y, b_g) \times N_z$ | $\Delta x$ | $\Delta y$ | $\Delta z$ |
|--------|----------------------|-----|---------------|----------------------|------------------------------|------------------------------------|------------|------------|------------|
| D3-05C | 3                    | 3   | 5000          | 1.0                  | 0                            | $256 \times (257, 2.0) \times 256$ | 9.8, 3.5   | 0.47, 2.3  | 6.56, 2.3  |
| D3-05A | 3                    | 1.9 | 5000          | 2.5                  | -0.0029                      | $256 \times (161, 2.0) \times 256$ | 2.3, 6.7   | 0.18, 7.1  | 1.54, 4.5  |
| D3-05H | 3                    | 1.4 | 5000          | 4.5                  | -0.0050                      | $256 \times (161, 2.0) \times 256$ | 1.7, 5.0   | 0.13, 5.2  | 1.13, 3.3  |
| D3-10C | 3                    | 3   | 10000         | 1.0                  | 0                            | $512 \times (241, 2.3) \times 512$ | 9.4, 3.3   | 0.59, 5.26 | 6.30, 2.2  |
| D3-10A | 3                    | 1.8 | 10000         | 2.8                  | -0.0025                      | $416 \times (209, 1.0) \times 400$ | 2.5, 6.8   | 0.85, 5.7  | 1.68, 4.7  |
| D6-05C | 6                    | 6   | 5000          | 1.0                  | 0                            | $384 \times (385, 2.0) \times 512$ | 11, 1.3    | 0.53, 0.86 | 8.4, 0.98  |

where  $T_s = 110.4$  K and  $T_{\text{ref}} = 288.15$  K are the Sutherland temperature and the reference temperature, respectively. Further details of the code can be found in Ref. [57] and the references cited therein. The size of the computational domain is  $L_x \times L_y \times L_z = 2\pi \times 2 \times 4\pi/3(\delta)$  [58]. We use a uniform grid in both the streamwise and the spanwise directions. The grid in the wall-normal direction follows

$$y_j/\delta = \tanh \left[ b_g \left( 2 \frac{j-1}{N_y-1} - 1 \right) \right] / \tanh(b_g), \quad (21)$$

where  $j = 1, 2, \dots, N_y$ ;  $N_y$  is the number of wall-normal grid points;  $b_g$  controls the grid stretching. The grid is such that the resolution is comparable/finer than the previous DNSs [46, 59–62]. In the absence of any heat source, the viscous heating is balanced by the heat transfer at the wall, and the two walls are cold. We include a negative heat source such that the wall is hot, nearly adiabatic, and cold. Fully adiabatic walls need no wall modeling because the heat flux is 0. Also, there is no temperature scaling that can handle adiabatic walls ( $\theta_\tau = 0$  for adiabatic walls, and  $\theta/\theta_\tau$  is infinity). Table I shows the other details of our DNSs. The nomenclature is D[ $U_w/a_{\text{ref}}$ ]-[ $\text{Re}_b/1000$ ]C/A/H. The first letter is for “DNS.” The last letter “C” is for “cold wall,” “A” is for “nearly adiabatic wall,” and “H” is for “hot wall.”

Following Ref. [63], we check the convergence of our DNSs by examining the momentum budget. In a Couette flow,  $-\rho \overline{u''v''} + \mu dU/dy = \tau_w$ . Figures 6(a) and 6(b) shows the turbulent flux and the viscous flux as a function of the wall-normal coordinate in cases D3-10C and D3-05H. There is no heat source in D3-10C, and D3-05H has a nonzero heat source. Results of other DNSs are

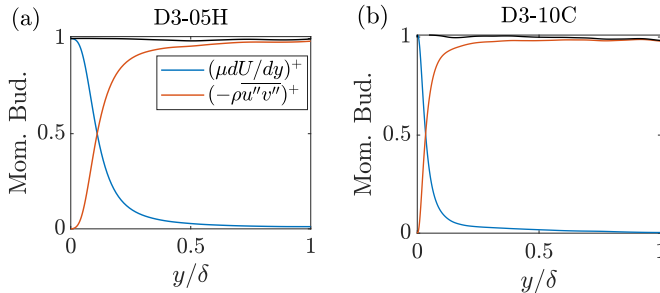


FIG. 6. Viscous and turbulent momentum fluxes as a function of  $y/\delta$  in (a) Case D3-05H and (b) Case D3-10C. “Mom. Bud.” denotes “momentum budget.” The thin black line is the sum of the two flux.

TABLE II. Further DNS details.  $\text{Re}_\tau = \rho_w u_\tau \delta / \mu_w$  is the friction Reynolds number.  $\text{Re}_\tau^* = \sqrt{\rho_w \rho_c} u_\tau \delta / \mu_c$  is the semilocal Reynolds number.  $B_q = q_w / \rho_w c_{pw} u_\tau T_w$  is the heat transfer rate.  $\text{Br} = q_w / \tau_w U_w$  is the Brinkman number, which measures the ratio of heat transfer and aerodynamic heating. Here a positive Brinkman number corresponds to heat transfer from the fluid to the wall, and a negative Brinkman number corresponds to heat transfer from the wall to the fluid.

|                    | D3-05C | D3-05A | D3-05H | D3-10C | D3-10A | D6-05C |
|--------------------|--------|--------|--------|--------|--------|--------|
| $\text{Re}_\tau$   | 401.09 | 93.99  | 69.16  | 774.80 | 159.85 | 685.05 |
| $\text{Re}_\tau^*$ | 141.02 | 272.63 | 202.16 | 270.51 | 452.81 | 79.97  |
| $B_q$              | 0.12   | -0.013 | -0.042 | 0.12   | -0.012 | 0.34   |
| $\text{Br}$        | 1.0    | -0.1   | -0.6   | 1.0    | -0.13  | 1.0    |

similar and are not shown here for brevity. The error in the total flux is about 1% and is comparable to the previous DNSs [45,64]. We may also check the convergence by examining the energy (the total energy, not the turbulent kinetic energy) budget. The results are shown in Fig. 13. There, the error is also less than 1%. Table II shows the further details of our DNSs. The Reynolds numbers of the DNSs are comparable to, if not higher than, the ones in Ref. [37]. Because the semilocal scaling works very well for the cases in Ref. [37], we expect the low Reynolds number not to be a big issue. The low Reynolds number is also partly justified by the fact that aerodynamic heating is most pronounced in the viscous layer. We have reported the Brinkman number,  $\text{Br} = q_w / (\tau_w U_w) = \text{heat transfer/aerodynamic heating}$ . For the *C* cases, i.e., D3-05C, D3-10C, and D6-05C, the heat source is 0 and energy conservation dictates that  $q_w = U_w \tau_w$  and  $\text{Br} = 1.0$ . For the *A* and the *H* cases, where there is a negative heat source, the energy conservation is such that  $q_w = \tau_w U_w - |\varphi \delta|$ , and  $|\text{Br}| < 1$ . Hence, aerodynamic heating is non-negligible in all our DNSs.

## B. WMLES

We use the finite-volume code CharLES for our WMLESs. The code solves the filtered full compressible NS equations with a nominally fourth-order accurate discretization in space and a third-order accurate discretization in time. The subgrid stress is modeled via the dynamic Vreman model [65]. Further details of the code could be found in Refs. [66,67] and the references cited therein. The code has been extensively used for wall-bounded flow calculations. Recent applications of this code could be found in Refs. [68,69]. Flow in the wall layer is modeled via the equilibrium wall model detailed in Secs. I and II. We use three wall models, i.e., the EWM in Ref. [26], the semilocal EWM in Ref. [42], and the wall model detailed in Sec. II, whose results are denoted as  $\text{WMLES}^+$ ,  $\text{WMLES}_{\text{YL18}}$ , and  $\text{WMLES}^*$ . Here  $\text{WMLES}_{\text{YL18}}$  differs from the EWM in that it employs a semilocal scaled damping function,  $\text{WMLES}^*$  differs from  $\text{WMLES}_{\text{YL18}}$  in that it employs the turbulent Prandtl number in Eq. (17). The grid is of size  $N_x \times N_y \times N_z = 120 \times 40 \times 40$  and resolves a domain of size  $L_x \times L_y \times L_z = 6\pi \times 2 \times 2\pi(\delta)$ . The wall model/LES matching location is at the third off-wall grid, i.e.,  $h_{\text{wm}}/\delta = 0.125$ , or the first off-wall grid, i.e.,  $h_{\text{wm}}/\delta = 0.025$ . While not shown here, we get the same mean flow by doubling the grid numbers in all three directions while keeping  $h_{\text{wm}}/\delta = 0.05$ . This suggests that the errors due to the subgrid scales are negligible [26,28]. The setups of the WMLESs are otherwise the same as the DNSs.

## IV. RESULTS

We compare the DNSs, the EWM, where we solve the wall model equations directly, and WMLESs, where we employ the EWM for near-wall turbulence modeling in LES.  $\text{EWM}^+$  denotes the conventional viscous units scaled wall model, and  $\text{EWM}^*$  denotes the semilocal units scaled wall model. We focus on the temperature statistics. A detailed discussion of the velocity results falls out of the scope of this work.

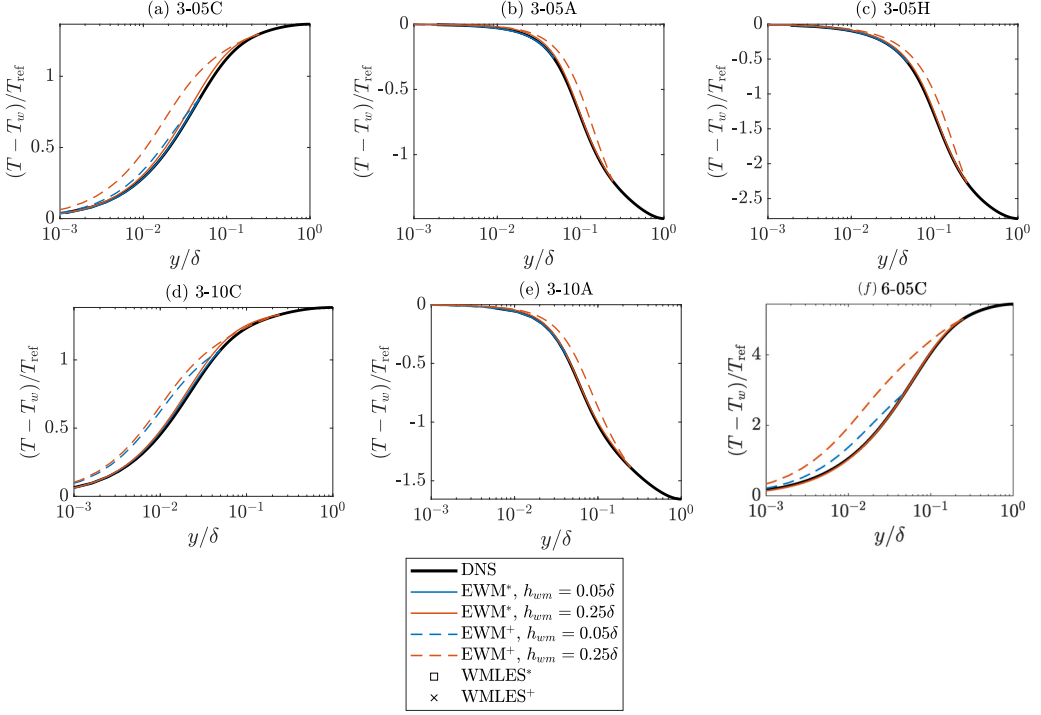


FIG. 7. Wall model results. The temperature profiles are symmetric with respect to the channel centerline, and we show only the results in the lower half channel.

### A. Wall model results

We solve the wall model equations from the wall, i.e.,  $y/\delta = 0$  up to a distance  $y/\delta = 0.05$  and  $0.25$  and compare the results to our DNSs. This is an *a priori* test of our wall model. Figure 7 compare the results to the DNSs. EWM\* agrees fairly well with the DNSs for both  $h_{wm}/\delta = 0.05$  and  $0.25$ . Large errors are found in EWM<sup>+</sup> when  $h_{wm}/\delta = 0.25$ . Except for 6-05C and 3-10C, EWM<sup>+</sup> agree well with the DNSs when  $h_{wm}/\delta = 0.05$ .

Figure 8 shows EWM\*'s errors for  $h_{wm}/\delta = 0.01, 0.05, 0.10$ , and  $0.25$ . Before we discuss how EWM\* does, we answer the following question: what is an acceptable error in a wall model. To answer this question, we consider the error due to the von Karman constant  $\kappa$ . A conservative estimate of  $\kappa$ 's uncertainty is 2.5%, i.e.,  $\kappa = 0.4 \pm 0.01$  [70]. This 2.5% uncertainty in  $\kappa$ 's value translates to about a 5% uncertainty in the wall flux [27]. If the von Karman constant is not a significant source of error, a  $\pm 2.5\%$  error should be considered acceptable. Now, we examine the results in Fig. 8. The error at a given wall-normal height is generally an increasing function of  $h_{wm}$ , but limiting  $h_{wm} < 0.25\delta$ , the error is, by and large, within  $\pm 2.5\%$ . Hence, the results here suggest that, aside from the log layer mismatch,  $h_{wm}$  should not be a leading source of error in channel flow calculations.

### B. WMLES results

We compare WMLESs directly to DNSs. This is an *a posteriori* test. The discussion in this section is limited to the channel flow configuration. While boundary-layer flow statistics at one (a few) streamwise locations in supersonic boundary layers is publicly available [41,46], a fair comparison between WMLESs and these DNSs is highly nontrivial because of a lack of knowledge of boundary layer's Reynolds number scalings at high speeds as well as a lack of knowledge of the

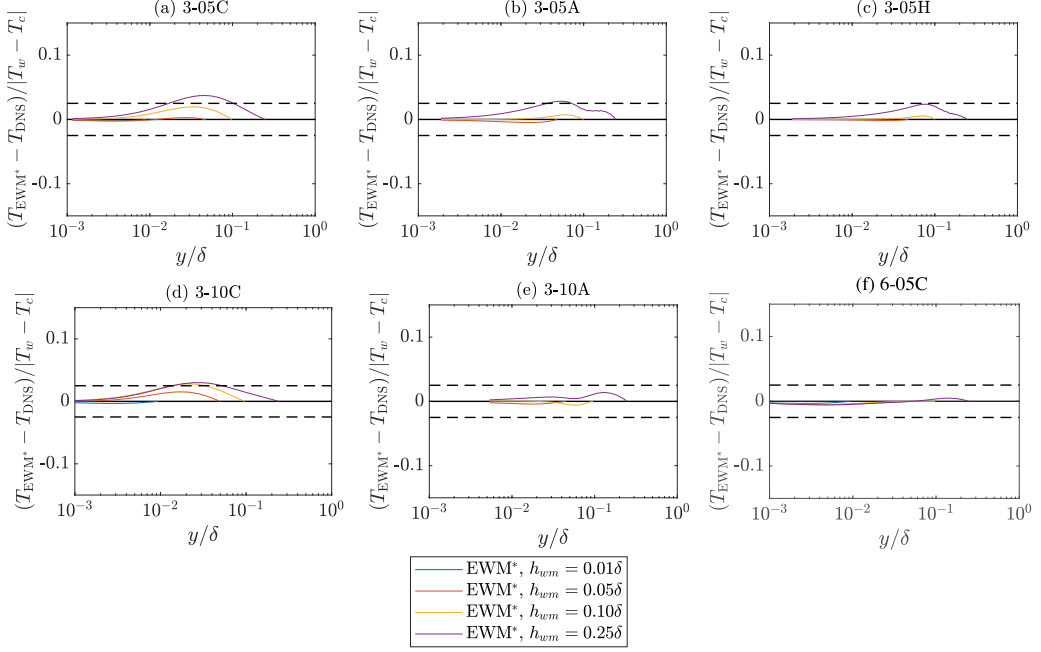


FIG. 8. Error in EWM\* for all DNS conditions and for  $h_{wm} = 0.01\delta$ ,  $0.05\delta$ ,  $0.10\delta$ , and  $0.25\delta$ . The two dashed lines are at  $\pm 0.025$ .

inflow condition. The above difficulty has also motivated the H and A cases in Table I, where we include a heat source in a channel.

Figure 9 compares WMLESs and the two channel flow DNSs in Ref. [43]. The two DNSs are at a bulk Reynolds number  $Re_b = 34\,000$ , and the bulk Mach numbers are  $M = 0.8$ ,  $1.5$ . For DNSs of supersonic channel flow,  $Re_b = 34\,000$  is a high Reynolds number. Figure 9 compares both  $U$  and  $T$ . We make the following observations. First and foremost, we see a logarithmic layer in Figs. 9(b) and 9(d). Examining the velocity  $U/a_w$ , the conventional viscous scaled wall model, i.e., WMLES<sup>+</sup> gives an accurate estimate of the velocity at the subsonic speed  $M = 0.8$  but underestimates the velocity at the supersonic speed  $M = 1.5$ ; the model in Ref. [42], i.e., WMLES<sub>YL18</sub>, gives an accurate estimate of the velocity at both  $M = 0.8$  and  $1.5$ . This is consistent with Ref. [42]. The present model, i.e., WMLES\*, does not seem to have an effect on the velocity. Examining the temperature  $T/T_w$ , WMLES<sub>YL18</sub> follows the DNSs more closely than WMLES<sup>+</sup>, WMLES\* more closely than WMLES<sub>YL18</sub>, and WMLES\* agree well with the DNSs at both the subsonic speed  $M = 0.8$  and the supersonic speed  $M = 1.5$ . The errors in  $T - T_w$  are 10%, 6%, and 2% in WMLES<sup>+</sup>, WMLES<sub>YL18</sub>, and WMLES\* at the channel centerline for the  $M = 0.8$  case. For the  $M = 1.5$  case, the errors are 18%, 7%, and 2%. These results validate our model. Meanwhile, they explain why Yang and Lv were able to get accurate velocity estimates without the present eddy conductivity [42]. Figure 10 compares WMLESs and DNSs for the C cases in Table I. For brevity, we show only the thermal field results. The results are similar to these in Fig. 9—although the difference between different wall models is less apparent. While not shown here, the first grid point implementation [27] and the third grid point implementation [26] give the same results.

Next, we compare WMLESs and DNSs for the A (nearly adiabatic wall) and the H (hot wall) cases. WMLES<sup>+</sup>, WMLES<sub>YL18</sub>, and WMLES\* differ in a similar way as in the C cases, and we do not repeat the discussion for brevity. Here, the first grid point implementation and the third grid point implementation lead to different results for the A and H cases. Figure 11 shows the results. We show WMLES\* with the first grid point implementation, i.e., FGI\*, and WMLES\* with the

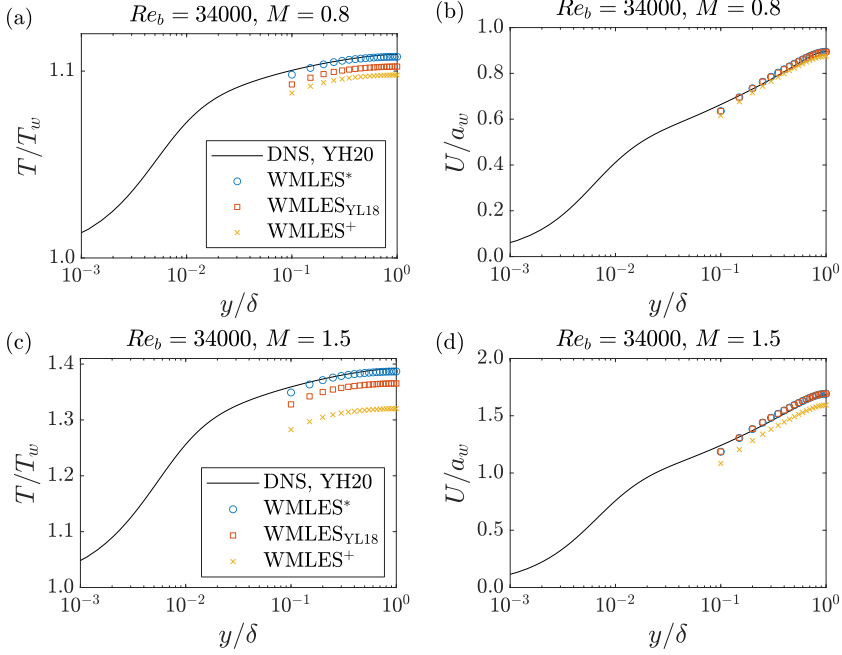


FIG. 9. (a)  $T/T_w$  as a function of  $y/\delta$  for the channel flow DNS in Ref. [43] and for WMLES\*, WMLES<sub>YL18</sub>, and WMLES<sup>+</sup>. The flow condition is  $Re_b = 34\,000$ ,  $M = 0.8$ . (b) Same as (a) but for  $U/a_w$ . (c) Same as (a) but at the flow condition  $Re_b = 34\,000$ ,  $M = 1.5$ . (d) Same as (c) but for  $U/a_w$ .

third grid point implementation, i.e., TGI\*. WMLES<sup>+</sup> with the first grid point implementation, i.e., FGI<sup>+</sup>, is included for comparison. Examining the velocity  $(U - U_w)/a_w$  in Figs. 11(b) and 11(d), FGI\* and TGI\* give similar velocity estimates, and they both agree reasonably well with the DNSs. Examining the temperature  $T/T_{ref}$  in Figs. 11(a) and 11(c), FGI\* gives reasonably accurate estimates of  $T$  at both wall conditions, but TGI\* overestimates  $T$ . This directly contradicts the results in Fig. 8. The exact mechanism is not clear to us and is a topic for future research.

Further details of the WMLESs, including the wall heat flux and the wall heat transfer rate, are reported in the Appendix. In all, we see that the eddy conductivity in Eq. (16) works very well in the context of WMLES. This is in direct contrast to the results in Fig. 3, where the semilocal scaling fails catastrophically.

### C. A caveat

In this section, the DNS data are used for *a priori* tests of the wall model only. It is tempting to compare DNS to, e.g., the turbulent Prandtl number in Eq. (17), particular in the wall layer.

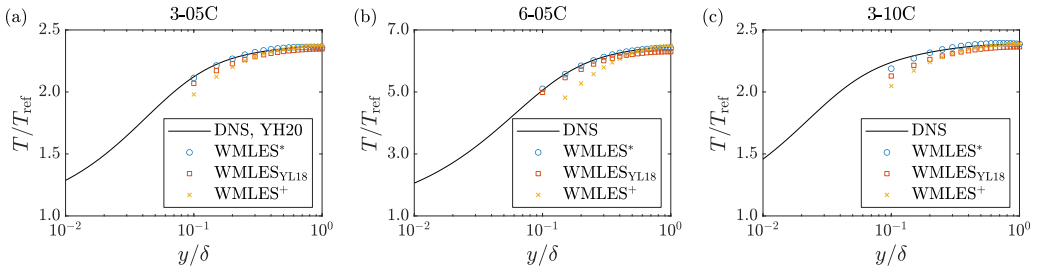


FIG. 10.  $T/T_{ref}$  as a function of  $y/\delta$  at the flow condition (a) 3-05C, (b) 6-05C, and (c) 3-10C.

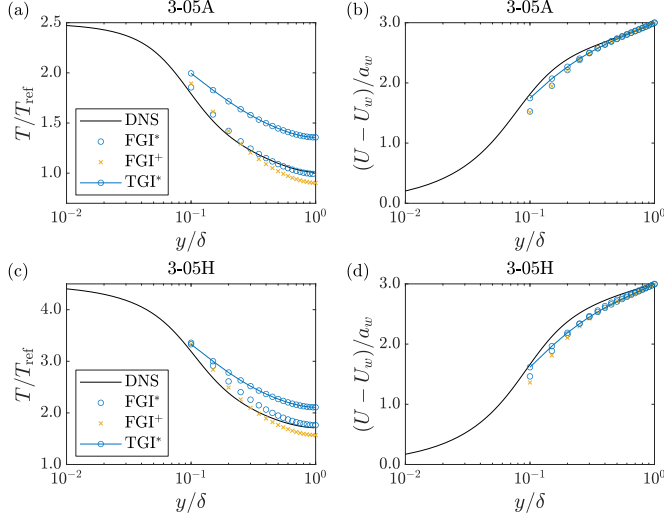


FIG. 11.  $T/T_{\text{ref}}$  as a function of  $y/\delta$  at the flow condition (a) 3-05A and (c) 3-05H.  $(U - U_w)/a_w$  as a function of  $y/\delta$  at the flow condition (b) 3-05A and (d) 3-05H.

But before we do that, we run through a simple analytical analysis. Consider the following limit:

$$\lim_{y \rightarrow 0} \text{Pr}_t = \lim_{y \rightarrow 0} \frac{\langle u'v' \rangle^+}{\langle v'T' \rangle^+} \frac{dT^+/dy^+}{dU^+/dy^+}. \quad (22)$$

For simplicity, let us limit the discussion to low-speed flows. The law of the wall gives

$$\lim_{y \rightarrow 0} \frac{dT^+/dy^+}{dU^+/dy^+} = \text{Pr}, \quad (23)$$

i.e., a nonzero constant. Meanwhile, because  $\langle u'v' \rangle = C_1 y^3 + O(y^4)$  and  $\langle v'T' \rangle = C_2 y^3 + O(y^4)$ , L'Hospital's rule gives

$$\lim_{y \rightarrow 0} \frac{\langle u'v' \rangle}{\langle v'T' \rangle} = \lim_{y \rightarrow 0} \frac{d^3 \langle u'v' \rangle / dy^3}{d^3 \langle v'T' \rangle / dy^3} = C_1 / C_2 (\neq 0). \quad (24)$$

It is interesting to note that we need to take  $y$  derivative three times. Hence, turbulent Prandtl number is a high-order statistics. It follows from Eqs. (23) and (24) that the turbulent Prandtl number should be some nonzero constant at  $y = 0$ , i.e.,

$$\text{Pr}_t(y = 0) = \text{Const} (\neq 0). \quad (25)$$

A direct consequence of Eq. (25) is that, when we plot  $\text{Pr}_t$  as a function of  $y$  on a logarithmic scale, we should have

$$\lim_{y^+ \rightarrow 0} \frac{d\text{Pr}_t}{d \log(y^+)} = \lim_{y^+ \rightarrow 0} y^+ \frac{d\text{Pr}_t}{dy^+} = 0. \quad (26)$$

That is,  $\text{Pr}_t[\log(y^+)]$  must have a 0 gradient in a semilog plot at the wall. In fact, any flow statistics, as long as it is not singular at the wall, must have a 0 slope in a semilog plot. Figure 12(b) shows the velocity, the temperature, and the Reynolds shear stress in a semilog plot, and they all have 0 slope at the wall. We now examine  $\text{Pr}_t$ . Figure 12(a) shows the turbulent Prandtl number in  $\text{Re}_\tau = 4000$  channel [45]. The Reynolds number is sufficiently high that the viscous sublayer is very well separated from the outer layer. Aside from some statistical convergence issues in the bulk, we notice

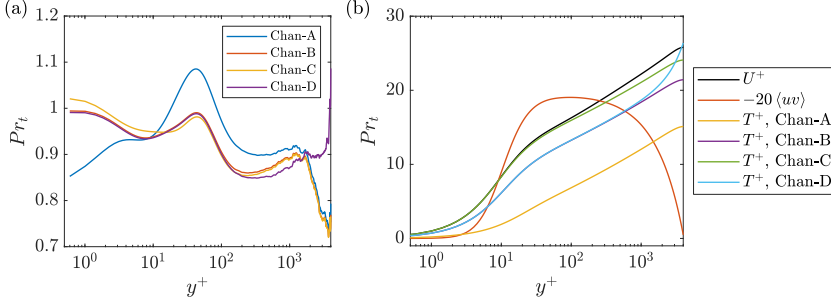


FIG. 12. (a) Turbulent Prandtl number (b) mean velocity, temperature, Reynolds shear stress in  $Re_\tau = 4000$  channel at different thermal conditions [45]. The reader is directed to Ref. [45] for more details of the data.

that  $Pr_t[\log(y)]$  in Chan-A does not have a zero slope. These DNSs are well resolved in the wall layer, and we do not know exactly what these DNSs miss, but it is safe to say that DNS is not entirely reliable for high-order statistics like  $Pr_t$  in the viscous sublayer. That said, DNS is still reliable for low-order statistics [6], which is why we have limited the comparison to low-order statistics.

## V. DISCUSSION

In this section, we answer the following question: Why does the semilocal scaling fail in collapsing the high-speed data but the resulting wall model works well. We will show that some of the assumptions in Ref. [37] do not necessarily apply to high-speed flows.

First, we examine the Reynolds averaged energy equation:

$$\frac{d}{dy} \left( \overline{\sigma_{yij} u_j} - \rho v \overline{\frac{1}{2} u_j u_j} - \rho v c_p \overline{T} + \lambda \frac{\partial \overline{T}}{\partial y} \right) + \varphi = 0, \quad (27)$$

where  $u$ ,  $v$ ,  $w$  are the instantaneous velocity in the three Cartesian directions and  $\sigma_{ij}$  is the instantaneous viscous stress tensor. The reader is directed to Ref. [71] for more details of the energy equation. Integrating Eq. (27) in  $y$  from the channel centerline to a distance  $y$  from the wall, we get

$$\epsilon + \epsilon_t + D + D_t = (\delta - y)\varphi, \quad (28)$$

where  $\delta\varphi = q_w - U_w \tau_w$ . Here  $\epsilon = \overline{\sigma_{yij} u_j}$ ,  $\epsilon_t = -\rho v \overline{\frac{1}{2} u_j u_j}$ ,  $D = \lambda \partial \overline{T} / \partial y$ , and  $D_t = -c_p \rho v \overline{T}$ . Figure 13 shows the terms in the energy equation for the cases D3-05H and D3-10C, where the source term  $\varphi$  is nonzero and zero in Figs. 13(a) and 13(b), respectively. The sums of the four terms

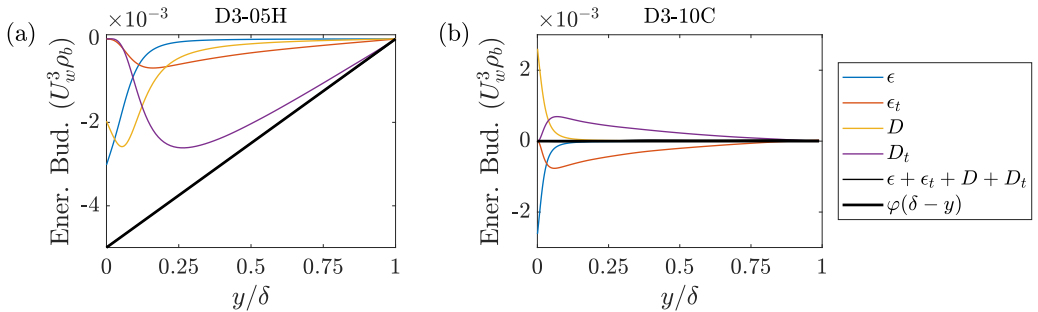


FIG. 13. Energy budget terms in (a) D3-05H, where the heat source term is nonzero, and (b) D3-10C, where the heat source term is  $\varphi = 0$ .

are  $(\delta - y)\varphi$  in the case D3-05H and 0 in the case D3-10C. We see from Fig. 13 that the viscous heating, i.e.,  $\epsilon$  and  $\epsilon_t$ , are non-negligible in wall layer for  $y/\delta \lesssim 0.2$ .

We take a reference frame such that  $U_w = 0$ . In the constant stress layer,  $y \ll \delta$  and Eq. (28) becomes

$$\epsilon + \epsilon_t + D + D_t = \delta\varphi = q_w. \quad (29)$$

Comparing Eq. (29) and the energy equation in Ref. [37], Patel *et al.* did not account for aerodynamic heating, i.e.,  $\epsilon$  and  $\epsilon_t$ . Now, comparing Eq. (29) and the wall model equation Eq. (1b), the EWM models the four terms in Eq. (29) as follows:

$$\epsilon = \mu \frac{du_{||}}{dy} u_{||}, \quad \epsilon_t = \mu_t \frac{du_{||}}{dy} u_{||}, \quad D = \lambda \frac{dT}{dy}, \quad D_t = \frac{c_p \mu_t}{\text{Pr}_t} \frac{dT}{dy}. \quad (30)$$

Considering that both  $\epsilon$  and  $\epsilon_t$  are non-negligible, the fact that the EWM models aerodynamic heating but Patel *et al.* did not partly explains why the EWM works well but the semilocal scaling fails.

To further verify our conclusion, we incorporate aerodynamic heating in the semilocal scaling and examine if the resulting scaling does a better job collapsing the high-speed data. We follow Ref. [37] and rewrite Eq. (29):

$$\frac{\epsilon + \epsilon_t}{q_w} + \left( \frac{\alpha_t}{\mu} + \frac{1}{\text{Pr}^*} \right) \frac{\delta}{\text{Re}_\tau^*} \sqrt{\frac{\rho}{\rho_w}} \frac{d\theta^+}{dy} = 1. \quad (31)$$

Rearranging the above equation, we have

$$\frac{h}{\text{Re}_\tau^*} \sqrt{\frac{\rho}{\rho_w}} \frac{dy^*}{dy} \frac{d\theta^+}{dy^*} \left/ \left( 1 - \frac{\epsilon + \epsilon_t}{q_w} \right) \right. = 1 \left/ \left( \frac{\alpha_t}{\mu} + \frac{1}{\text{Pr}^*} \right) \right. . \quad (32)$$

Patel *et al.* argue that the right-hand side of Eq. (32) is a function of  $y^*$  when given the molecular Prandtl number. This is the universality assumption, and we retain this assumption. It follows that we may define  $\theta^*$  such that

$$\frac{d\theta^*}{dy^*} = 1 \left/ \left( \frac{\alpha_t}{\mu} + \frac{1}{\text{Pr}^*} \right) \right. . \quad (33)$$

Equations (32) and (33) give

$$\frac{d\theta^*}{d\theta^+} = \frac{d\theta^*}{dy^*} \frac{dy^*}{d\theta^+} = \frac{\delta}{\text{Re}_\tau^*} \sqrt{\frac{\rho}{\rho_w}} \frac{dy^*}{dy} \left/ \left( 1 - \frac{\epsilon + \epsilon_t}{q_w} \right) \right. = \left( 1 + \frac{y^*}{\text{Re}_\tau^*} \frac{d\text{Re}_\tau^*}{dy} \right) \sqrt{\frac{\rho}{\rho_w}} \left/ \left( 1 - \frac{\epsilon + \epsilon_t}{q_w} \right) \right. , \quad (34)$$

Equations (34) and (6b) together define a new semilocal scaling. The new scaling contains a new term  $1 - (\epsilon + \epsilon_t)/q_w$  that accounts for aerodynamic heating. To accurately evaluate these terms, the energy equation must be statistically converged. This requirement prevents us from utilizing some of the publicly available data. We apply this new semilocal scaling to the data in Fig. 3. Figure 14 shows the results. Comparing Figs. 3 and 14, the new semilocal scaling collapses the high-speed data considerably better than the one in Ref. [37]. The new scaling degenerates to Patel *et al.*'s scaling when there is no aerodynamic heating, and therefore the data in Ref. [37] are not included for brevity. In all, we can conclude that the semilocal scaling in Ref. [37] fails at collapsing the high-speed data because it does not account for aerodynamic heating. (Again, this is not a criticism. In Ref. [37], the flow is at low speeds, and therefore aerodynamic heating is indeed negligible.)

Having answered why the semilocal scaling in Ref. [37] fails, we are yet to answer why our wall model works. In the following, we show that the new semilocal scaling that incorporates aerodynamic heating leads to the same eddy conductivity as the semilocal scaling in Ref. [37].

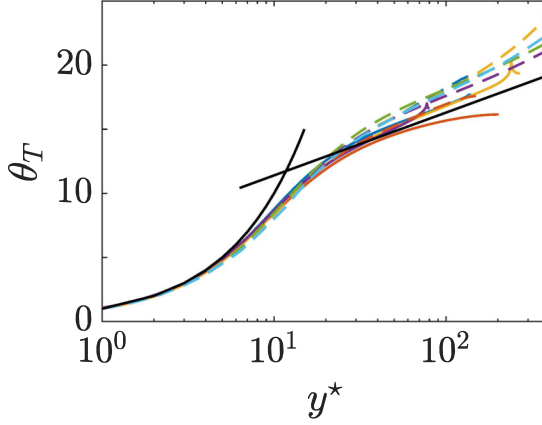


FIG. 14. Same as Fig. 3 but for the scaling in Eqs. (34) and (6b), which accounts for aerodynamic heating.

We follow the same steps as in Sec. II:

$$\begin{aligned} \frac{\alpha_t}{\mu} &= \frac{-\rho \overline{v''\theta''}}{\mu \frac{d\theta}{dy}} = \frac{q_w \left(1 - \frac{\epsilon + \epsilon_t}{q_w}\right)}{\frac{d\theta}{d\theta^*} \frac{d\theta^*}{d\theta_T} \frac{d\theta_T}{dy^*} \frac{dy^*}{dy}} - \frac{1}{\text{Pr}^*} = \frac{q_w}{q_w \frac{d\theta_T}{dy^*} \frac{1 + \alpha_t/\mu}{1/\text{Pr}^* + \alpha_t/\mu} \frac{1}{\mu}} - \frac{1}{\text{Pr}^*} \\ &= \frac{(1 + \kappa_T y^* D_T)(\mu/\text{Pr}^* + \alpha_t)}{\mu + \alpha_t} - \frac{1}{\text{Pr}^*}. \end{aligned} \quad (35)$$

Equation (35) is again a quadratic equation of  $\alpha_t/\mu$ . Its solution is just Eq. (16). Here  $-\rho \overline{v''\theta''} = q_w - \epsilon - \epsilon_t - D$  according to Eq. (29), and  $d\theta^*/d\theta^+$  is given by Eq. (34). In conclusion, incorporating aerodynamic heating or not, the semilocal scaling gives the same eddy conductivity, and this is why the wall model had worked.

## VI. CONCLUSIONS

We exploit the semilocal scaling and derive an eddy conductivity closure for the EWM. The wall distance scaling in the damping functions depends on the universal laws of the wall, in this case, the semilocal scalings. The derivation leads to the following turbulent Prandtl number in the sublayer and the logarithmic layer

$$\text{Pr}_t = \frac{\kappa}{\kappa_T} \left[ \frac{1 - \exp(-y^*/A)}{1 - \exp(-y^*/A_T)} \right]^2,$$

where  $y^* = \sqrt{\rho \tau_w} y / \mu$  is the semilocal scaled distance from the wall. Comparing WMLESs to DNSs, we conclude as follows: First, the thermal field is more challenging to model and compute than the momentum field; second, *a priori* tests and *a posteriori* tests do not necessarily agree with each other; third, the present model gives similar velocity predictions as the model in Ref. [42]; fourth, the present wall model gives more accurate results than both the model in Ref. [42] and the conventional viscous scaled model; and, fifth, the first grid point implementation in Ref. [27] gives more accurate results than the  $n$ th ( $n > 1$ ) grid point implementation in Ref. [26] for adiabatic and hot walls (but it is not clear why).

The success of semilocal scaling in the WMLES and the poor collapse of the high-speed DNS data begs the question. Detailed analysis of the energy budget shows that the EWM works because it models, albeit very crudely, aerodynamic heating, a term that is missing in the semilocal scaling [37]. By accounting for aerodynamic heating in semilocal scaling, we show that the new semilocal scaling collapses the DNS data considerably better. We also show that the new semilocal scaling

TABLE III. Wall shear stresses  $\tau_w \times 10^2 / (\rho_b U_w^2)$  and the wall heat fluxes  $q_w \times 10^2 / \rho_b U_w^3$ . A positive heat flux is a flux from the fluid to the wall, and a negative heat flux is a flux from the wall to the fluid. For all the numbers reported in the table, we keep two digits behind the decimal point.

|                    | 03-05C     | 03-05A      | 03-05H      | 03-10C     | 06-05C     |
|--------------------|------------|-------------|-------------|------------|------------|
| DNS                | 0.28, 0.28 | 0.26, -0.03 | 0.31, -0.20 | 0.25, 0.25 | 0.32, 0.32 |
| WMLES*             | 0.29, 0.29 | 0.26, -0.04 | 0.32, -0.19 | 0.25, 0.25 | 0.33, 0.33 |
| WMLES <sup>+</sup> | 0.42, 0.42 | 0.25, -0.04 | 0.29, -0.21 | 0.35, 0.35 | 0.51, 0.51 |

gives rise to exactly the same eddy conductivity as the semilocal scaling in Ref. [37], thereby answering the above question.

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### APPENDIX: FURTHER WMLES DETAILS

Table III shows the wall heat flux  $q_w / \rho_b U_w^3$  and  $\tau_w / \rho_b U_w^2$  for DNSs, and WMLESs. Here  $\tau_w / \rho_b U_w^2$  is the skin friction coefficient  $C_f$ ; we report the heat flux  $q_w / \rho_b U_w^3$  instead of the heat transfer rate  $B_q = q_w / \rho_w c_{pw} u_\tau T_w$  because the errors in  $\tau_w$  and  $q_w$  may cancel.

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