Mass transfer from a cylindrical body in a linear ambient velocity distribution

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We consider the two-dimensional problem of mass transport from a cylindrical body of circular cross section which is immersed in a fluid whose ambient velocity varies linearly with position. Such a flow is quantified by a single parameter Ω representing the ratio of its associated vorticity to its characteristic rate-of-strain magnitude, $\Omega = \pm 1$ corresponding to simple shear. Using matched asymptotic expansions to analyze the limit of small Péclet numbers, $\mathrm{Pe} \ll 1$, we find that the leading-order Nusselt number is $2/[\log(1/\mathrm{Pe}) + \lambda(\Omega)]$, wherein the function $\lambda(\Omega)$ is provided in terms of simple quadratures. No steady solutions exist for $|\Omega| > 1$, where the streamlines of the ambient flow are closed. The case of simple shear, analyzed by Frankel and Acrivos [Phys. Fluids **11**, 1913 (1968)], is accordingly a borderline one. Using conformal mappings, the more general problem of arbitrary cross-sectional shape is recast as the above transport problem about a circle, with the Péclet number appropriately modified. While the more general problem is unsteady in the case of a freely suspended cylinder, the associated Nusselt number is independent of time.

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I. INTRODUCTION

There are several well-known physical problems, notably heat and mass transport in a fluid, where the pertinent scalar quantity is governed by an advection-diffusion equation in which the velocity field is effectively prescribed [1,2]. A canonical problem in that context involves the specification of the intensity of the transported quantity at the boundary of a particle, say at a fixed uniform value above the equilibrium value at infinity [3]. A common measure of the associated transport to infinity is the net flux emanating from that particle, which, when rendered dimensionless, defines the Nusselt number Nu. The combined effect of advection and diffusion is reflected in the dependence of Nu upon the Péclet number Pe, the latter representing the ratio of the characteristic magnitudes of these two processes. In the present contribution we shall refer for convenience to mass transport, where the local intensity is the solute concentration.

Our interest is in the limit of small Péclet numbers. This limit often exhibits a singular behavior resembling that appearing in flows at low Reynolds numbers [4], its resolution thus requiring the use of matched asymptotic expansions between an "inner" region, at the scale of the particle, and an "outer" region at large distances away from the particle, where advection and diffusion are comparable in magnitude. At these distances, the particle appears as a point singularity.

The first analysis of that type was carried out by Acrivos and Taylor [5] who considered a sphere in a uniform stream under creeping-flow conditions. At small Pe, the spherically symmetric leading-order concentration imparts an $O(1)$ contribution to Nu, and the interest lies in the leading-order correction which is induced by advection. Earlier attempts to evaluate that correction have failed due to an inability to satisfy a decay condition at infinity. Acrivos and Taylor [5] attributed that failure to the algebraic decay of the concentration as $1/r$, r being the radial coordinate normalized by particle size. At large r, the ratio of the (presumably subdominant) advection to the (presumably dominant) diffusion is of order Pe r, implying that the neglect of advection breaks down at distances

$O(1/\mathrm{Pe})$. Using asymptotic matching between the outer solution at these distances and an inner solution at $r = O(1)$, Acrivos and Taylor [5] obtained the $O(\mathrm{Pe})$ correction to Nu. Brenner [6] showed that the scheme of Acrivos and Taylor [5] may be extended so as to analyze the transport from a fixed particle of arbitrary shape.

Later on, Frankel and Acrivos [7] considered a freely suspended sphere in a simple shear flow, again under Stokes-flow conditions. In this case, the linear growth of the ambient velocity implies that the dominance of diffusion breaks down at closer (though still large) distances, namely, $r = O(\mathrm{Pe}^{-1/2})$, with the consequent correction to Nu being $O(\mathrm{Pe}^{1/2})$. Since the Stokes-flow problem of a circular cylinder under shear is well posed (as opposed to that of a cylinder under a uniform streaming flow), Frankel and Acrivos [7] were also able to analyze the two-dimensional configuration of such a freely suspended cylinder. In the latter problem, the small-Pe singularity is already manifested at leading order; this has to do with the logarithmic growth associated with Laplace's equation in two dimensions, which is incompatible with the conditions at infinity.

The three-dimensional analysis of Frankel and Acrivos [7] has been generalized by Batchelor [8] to arbitrary linear distributions of the ambient velocity. Making use of Brenner's ideas [6], Batchelor [8] considered a particle of arbitrary shape. Since the concentration distributions associated with different linear flows cannot be superimposed, it is necessary to solve the general case of arbitrary linear flow. For a spherical particle, where the Nusselt number cannot depend upon orientation, the specification of such an ambient flow requires four parameters. Batchelor [8] showed that this is also true for arbitrary particle shape, provided attention is restricted to the $O(\mathrm{Pe}^{1/2})$ correction to the Nusselt number.

Our goal here is to perform the comparable generalization of the two-dimensional problem. With advection having a pronounced effect in two-dimensional transport, it plays there a critical role in the determination of the leading-order Nusselt number. The two-dimensional problem is attractive for its simplicity, with the specification of the ambient flow requiring only a single parameter, say the ratio Ω between the ambient vorticity and the characteristic rate-of-strain magnitude. We are particularly motivated by the role of the ambient vorticity: In the limit $\Omega \to \infty$ the ambient flow constitutes rigid-body rotation; with circular streamlines, that flow cannot remedy the logarithmic growth associated with diffusive transport. The question then arises as to the suppressing effect of vorticity at finite values of Ω. In the three-dimensional case it was found that at asymptotically large Ω, the correction to the Nusselt number is strongly inhibited [8]. In the present two-dimensional problem, one expects an even more dramatic outcome.

While this work has been motivated by its fundamental interest, it may have some relevance to realistic transport problems. Moreover, with microfluidic realizations it is possible to generate the requisite linear flows in microchannel junctions—a convenient alternative to the classical four-roll-mill apparatus.

In what follows we begin with a circular cross-section geometry. The exact problem is formulated in Sec. II. The small-Pe limit is introduced in Sec. III, with the outer problem being solved in Sec. IV and the asymptotic matching being performed in Sec. V. The dependence upon Ω of the resulting Nusselt-number approximation is determined in Sec. VI. The extension to a cylinder of arbitrary cross-sectional shape is provided in Sec. VII for a stationary cylinder and in Sec. VIII for a freely suspended cylinder. We conclude in Sec. IX.

II. PROBLEM FORMULATION

A cylindrical body of circular cross section is immersed in an unbounded fluid domain which undergoes an ambient flow where the velocity varies linearly with position. The cylinder may be either stationary or freely suspended, and we assume Stokes-flow conditions. Our interest is in the net mass transport of solute due to a prescribed fixed difference $\mathcal{C}$ between the (uniform) solute concentration at the cylinder boundary and its "equilibrium" concentration far from the particle. This transport takes place by a combination of diffusion (with diffusivity κ) and advection.

In what follows, we normalize length variables by the cylinder radius a. With no loss of generality we may employ Cartesian coordinates with the x and y axes chosen such that the diagonal entries of the ambient rate-of-strain tensor vanish and the origin coincides with the cylinder center. We normalize the ambient-flow gradient (namely the vorticity and rate-of-strain components) by G, which is defined by the requirement that the rate-of-strain tensor adopts the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

The associated ambient vorticity is denoted by 2Ω. Normalizing the fluid velocity field $\mathbf{u}$ by aG, it accordingly satisfies the far-field condition

$$\mathbf{u} \sim (1-\Omega)y\hat{\mathbf{e}}_x + (1+\Omega)x\hat{\mathbf{e}}_y \quad \text{as} \quad r \to \infty, \tag{2}$$

wherein $r = \sqrt{x^2+y^2}$. For $\Omega = 0$ Eq. (2) represents pure straining motion; for $\Omega = \pm 1$ it stands for simple shearing motion; and for $|\Omega| \to \infty$ it constitutes rigid-body rotation.

The velocity field $\mathbf{u}$ is governed by the Stokes equations, subject to the far-field condition (2) and the specification of rigid-body rotation—say with an angular velocity ω—as a condition at the cylinder boundary, $\mathbf{u} = \omega(x\hat{\mathbf{e}}_y - y\hat{\mathbf{e}}_x)$. The velocity ω is determined by the specific mechanical description of the problem. Thus, for a stationary cylinder $\omega = 0$, while for a freely suspended cylinder ω is set by the condition that the cylinder is torque free (giving $\omega = \Omega$ in the present case of a circular cylinder). As will become evident, the leading-order solute transport is indifferent to the details of the velocity field; there is accordingly no need to provide here the exact form of $\mathbf{u}$ in these two cases.

The mass-transport problem is formulated in terms of the excess solute concentration relative to its far-field value. Normalizing it by $\mathcal{C}$, the steady problem governing the dimensionless excess concentration c consists of (i) the advection-diffusion equation,

$$\mathrm{Pe}\,\mathbf{u}\cdot\nabla c = \nabla^2 c, \tag{3}$$

in which

$$\mathrm{Pe} = \frac{a^2 G}{\kappa} \tag{4}$$

is the Péclet number associated with the ambient flow; (ii) the imposed concentration condition,

$$c = 1 \quad \text{at} \quad r = 1; \tag{5}$$

and (iii) far-field decay,

$$c \to 0 \quad \text{as} \quad r \to \infty. \tag{6}$$

The preceding problem uniquely determines c, which depends upon both the position in the plane and the two parameters Ω and Pe.

Our interest lies not in the detailed transport process but rather in the Nusselt number, representing the net transport of solute. Following Batchelor [8], that number is defined as

$$\mathrm{Nu} = -\frac{1}{2\pi}\oint \frac{\partial c}{\partial n}\,dl. \tag{7}$$

Here, the integration is carried out over the cylinder boundary, wherein dl is a differential length element and $\partial/\partial n = \hat{\mathbf{n}}\cdot\nabla$, $\hat{\mathbf{n}}$ being a normal unit vector pointing into the fluid. In the present case of a circular cylinder it is convenient to employ the polar coordinates r and θ, giving

$$\mathrm{Nu} = -\frac{1}{2\pi}\int_0^{2\pi} \left.\frac{\partial c}{\partial r}\right|_{r=1} d\theta. \tag{8}$$

The Nusselt number is a function of both Pe and Ω. In what follows we consider the singular limit $\mathrm{Pe} \ll 1$.

III. SMALL PÉCLET NUMBERS

In analyzing the small Péclet-number limit, we seek to calculate a leading-order approximation to Nu with an "algebraically small" (i.e. smaller than some positive power of Pe) asymptotic error. At leading order the advection-diffusion equation (3) degenerates to Laplace's equation,

$$\nabla^2 c = 0. \tag{9}$$

The solution of that equation which satisfies (5) and is consistent with (8) is

$$c = -\mathrm{Nu} \log r + 1, \tag{10}$$

wherein the parameter Nu (which remains to be determined) is hereafter understood as the leading-order Nusselt number. Consistent with the above-mentioned intent to derive an algebraically accurate approximation, we allow (as in Ref. [7]) for a weak (i.e. logarithmic) dependence of Nu upon Pe [see indeed (38)].

Since the solution (10) cannot satisfy the decay condition (6) it must be regarded as an inner solution, valid at $O(1)$ distances from the cylinder, where (6) does not apply. An outer region, at which solute advection enters at the leading-order balance, must accordingly be analyzed separately. The Nusselt number, which appears as a constant of integration in (10), can only be determined via asymptotic matching between the concentration distributions in the two regions.

It follows from (2) and (3) that advection becomes comparable to diffusion at distances $r = O(\mathrm{Pe}^{-1/2})$. We accordingly define the outer coordinates

$$X = \mathrm{Pe}^{1/2} x, \quad Y = \mathrm{Pe}^{1/2} y, \quad R = \mathrm{Pe}^{1/2} r, \tag{11}$$

and consider the limit $\mathrm{Pe} \to 0$ with (X, Y) fixed. Making use of (2) we find that, at leading order, the advection-diffusion equation (3) becomes

$$(1-\Omega) Y \frac{\partial C}{\partial X} + (1+\Omega) X \frac{\partial C}{\partial Y} = \frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2}. \tag{12}$$

This equation must be solved subject to the decay requirement [cf. Eq. (6)]

$$C \to 0 \quad \text{for} \quad R \to \infty. \tag{13}$$

In the outer region, condition (5) does not apply. Rather, we require that the outer solution matches the inner one. In particular, it follows from (10) that

$$C \sim -\mathrm{Nu} \log R \quad \text{for} \quad R \ll 1. \tag{14}$$

The cylinder thus appears as a point singularity.

IV. SOLUTION OF THE OUTER PROBLEM

In solving the outer problem we observe that Eq. (14) implies the maintained release of substance at the origin with a rate $2\pi\mathrm{Nu}$. Following Batchelor [8] we represent the requisite concentration as a "history integral,"

$$C(X, Y) = \int_0^\infty \grave{C}(X, Y; t)\, dt, \tag{15}$$

wherein $\grave{C}(X, Y; t)$ is the associated *unsteady* concentration distribution due to an instantaneous release of solute, of amount $2\pi\mathrm{Nu}$, at time zero $t = 0$. The latter distribution satisfies [cf. (12)]

$$\frac{\partial \grave{C}}{\partial t} + (1-\Omega) Y \frac{\partial \grave{C}}{\partial X} + (1+\Omega) X \frac{\partial \grave{C}}{\partial Y} = \frac{\partial^2 \grave{C}}{\partial X^2} + \frac{\partial^2 \grave{C}}{\partial Y^2} \quad \text{for} \quad t > 0 \tag{16}$$

and is subject to the initial condition

$$\grave{C} = 2\pi\mathrm{Nu}\, \delta(X)\delta(Y) \quad \text{at} \quad t = 0, \tag{17}$$

as well as far-field decay [see (13)].

We employ the Fourier transform,

$$\hat{C}(\alpha,\beta;t) = \iint_{-\infty}^{\infty} dX\,dY\,\grave{C}(X,Y;t)e^{-i(\alpha X+\beta Y)}, \tag{18}$$

and its inverse,

$$\grave{C}(X,Y;t) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} d\alpha\,d\beta\,\hat{C}(\alpha,\beta;t)e^{i(\alpha X+\beta Y)}. \tag{19}$$

Forming the transform of (16) and (17) and making use of the large-R decay yields the differential equation

$$\frac{\partial\hat{C}}{\partial t} - (1+\Omega)\beta\frac{\partial\hat{C}}{\partial\alpha} - (1-\Omega)\alpha\frac{\partial\hat{C}}{\partial\beta} = -(\alpha^2+\beta^2)\hat{C} \quad \text{for} \quad t>0 \tag{20}$$

and initial condition

$$\hat{C} = 2\pi\text{Nu} \quad \text{at} \quad t=0. \tag{21}$$

Following Batchelor [8] we seek a solution of the Gaussian form

$$\hat{C} = 2\pi\text{Nu}\,\exp\{-B_{\alpha\alpha}(t)\alpha^2 - 2B_{\alpha\beta}(t)\alpha\beta - B_{\beta\beta}(t)\beta^2\}. \tag{22}$$

Condition (21) is accordingly satisfied provided the three functions appearing in (22) satisfy

$$B_{\alpha\alpha} = B_{\alpha\beta} = B_{\beta\beta} = 0 \quad \text{at} \quad t=0. \tag{23}$$

It is convenient to temporarily postpone the explicit calculation of these functions. Forming the inverse transform of (22) readily yields

$$\grave{C}(X,Y;t) = \frac{\text{Nu}}{\sqrt{4D(t)}}\exp\left\{-\frac{B_{\alpha\alpha}(t)X^2 - 2B_{\alpha\beta}(t)XY + B_{\beta\beta}(t)Y^2}{4D(t)}\right\}, \tag{24}$$

wherein

$$D(t) = B_{\alpha\alpha}(t)B_{\beta\beta}(t) - B_{\alpha\beta}^2(t). \tag{25}$$

Since at small t advection is negligible compared to diffusion, $\grave{C}$ must approach then the familiar Green's function of the diffusion equation. It follows that [see indeed (41) and (42)]

$$B_{\alpha\alpha} \sim t, \quad B_{\beta\beta} \sim t, \quad B_{\alpha\beta} \ll t \quad \text{for} \quad t\to 0, \tag{26}$$

whereby

$$D \sim t^2 \quad \text{for} \quad t\to 0. \tag{27}$$

Substitution of (24) into (15) furnishes the outer concentration

$$C(X,Y) = \frac{\text{Nu}}{2}\int_0^{\infty}\frac{dt}{\sqrt{D(t)}}\exp\left\{-\frac{B_{\alpha\alpha}(t)X^2 - 2B_{\alpha\beta}(t)XY + B_{\beta\beta}(t)Y^2}{4D(t)}\right\}. \tag{28}$$

Note that convergence at the upper integration limit requires that

$$D \gg t^2 \quad \text{for} \quad t\to\infty. \tag{29}$$

That limit represents the effect of contributions due to the release of substance a long time prior to the moment of observation.

V. ASYMPTOTIC MATCHING

Toward asymptotic matching with Eq. (10) we need the small-R behavior of (28). In view of (27), a naïve substitution of $R = 0$ into (28) would result in a $1/t$ divergence of the integrand at small t. Since that behavior is just short of being integrable, it follows that the asymptotic evaluation of the integral for $R \ll 1$ falls under the intermediate case where the dominant contribution is neither local nor global [9]. To handle that case we split the integration interval, writing the integral appearing in (28) as

$$\left(\int_0^\eta + \int_\eta^1 + \int_1^\infty\right) \cdots dt, \tag{30}$$

wherein $0 < \eta < 1$ is an auxiliary parameter, whose value is at our disposal. In what follows we consider the limit $R \ll 1$ with $R^2 \ll \eta \ll 1$.

Since $\eta \ll 1$ we may employ (26) and (27) in the evaluation of the first integral, giving at leading order

$$\int_0^\eta \exp\left(-\frac{R^2}{4t}\right)\frac{dt}{t} = \Gamma\left(0, \frac{R^2}{4\eta}\right), \tag{31}$$

wherein $\Gamma(s, z)$ is the incomplete gamma function. Making use of the small-argument approximation,

$$\Gamma(0, z) \sim -\gamma - \log z + O(s) \quad \text{for} \quad 0 < z \ll 1, \tag{32}$$

wherein γ is the Euler-Mascheroni constant, we eventually obtain for the first integral

$$-\gamma - 2\log R + \log 4 + \log \eta, \tag{33}$$

with an algebraically small error.

In the leading-order evaluation of the second integral it is legitimate to set $R = 0$. Subtraction and addition of $1/t$ to the integrand then gives

$$\int_\eta^1 \left(\frac{1}{\sqrt{D}} - \frac{1}{t}\right) dt - \log \eta. \tag{34}$$

In view of (27) the integrand in the expression above is $o(1/t)$ at small t. It follows that, to leading order, it is legitimate to set the lower integration limit to zero. Finally, when calculating the third integral we obtain, upon setting $R = 0$,

$$\int_1^\infty \frac{dt}{\sqrt{D}}; \tag{35}$$

if the necessary condition (29) is satisfied, that integral is ensured to converge.

Adding up the three contributions (33)–(35) and substitution into (28) yields

$$C(X, Y) \sim \mathrm{Nu}(-\ln R + \lambda/2) \quad \text{for} \quad R \ll 1, \tag{36}$$

where the dependence upon the arbitrary parameter η has canceled out and the constant

$$\lambda = \log 4 - \gamma + \int_0^1 \left(\frac{1}{\sqrt{D}} - \frac{1}{t}\right) dt + \int_1^\infty \frac{dt}{\sqrt{D}} \tag{37}$$

depends only upon Ω.

Note that Eq. (36) trivially satisfies (14). More precisely, applying the 1–1 van-Dyke matching rule (wherein logarithmically small terms are considered on par with $O(1)$ terms [9,10]) using (10) and (36) yields the requisite Nusselt number as

$$\mathrm{Nu} = \frac{2}{\log(1/\mathrm{Pe}) + \lambda(\Omega)}. \tag{38}$$

Expression (38) constitutes our key result. In what follows, we evaluate the function $\lambda(\Omega)$. To this end we calculate $D(t)$ for the relevant values of the parameter Ω and then affect the quadratures in (37).

VI. EVALUATION OF $\lambda(\Omega)$

To calculate $D(t)$ we need to obtain the functions $B_{\alpha\alpha}$, $B_{\beta\beta}$, and $B_{\alpha\beta}$. Substitution of (22) into (20) yields

$$\frac{dB_{\alpha\alpha}}{dt} - 2(1-\Omega)B_{\alpha\beta} = 1, \quad \frac{dB_{\beta\beta}}{dt} - 2(1+\Omega)B_{\alpha\beta} = 1, \tag{39}$$

and

$$\frac{dB_{\alpha\beta}}{dt} - (1+\Omega)B_{\alpha\alpha} - (1-\Omega)B_{\beta\beta} = 0. \tag{40}$$

These ordinary differential equations need to be solved subject to conditions (23).

For $|\Omega| < 1$ we obtain (cf. Ref. [8])

$$(1+\Omega)B_{\alpha\alpha} - \Omega t = (1-\Omega)B_{\beta\beta} + \Omega t = \frac{\sinh[2(1-\Omega^2)^{1/2}t]}{2(1-\Omega^2)^{1/2}} \tag{41}$$

and

$$B_{\alpha\beta} = \frac{\sinh^2[(1-\Omega^2)^{1/2}t]}{1-\Omega^2}. \tag{42}$$

Substitution into (25) then gives

$$D = \frac{\sinh^2[(1-\Omega^2)^{1/2}t]}{(1-\Omega^2)^2} - \frac{\Omega^2 t^2}{1-\Omega^2}, \tag{43}$$

which indeed satisfies (29). As expected from symmetry arguments, D is an even function of Ω. By forming the limit $|\Omega| \to 1$ of (43) we readily obtain

$$D = t^2 + \tfrac{1}{3}t^4 \quad \text{for} \quad \Omega = \pm 1, \tag{44}$$

which also satisfies (29).

It readily follows from (43) that for $|\Omega| > 1$

$$D = \frac{\sin^2[(\Omega^2-1)^{1/2}t]}{(\Omega^2-1)^2} + \frac{\Omega^2 t^2}{\Omega^2 - 1}. \tag{45}$$

In that case $D = O(t^2)$ for large t, whereby condition (29) is *not* satisfied. Recall that this condition has originated from the requirement that the integrand appearing in (28) is integrable at $t = \infty$. The failure to satisfy that requirement implies that the effect of substance release a long time previously does not decay sufficiently rapidly: a steady solution does not exist. This is indeed suggested by the form of the stream function associated with the ambient flow (2),

$$\psi = \tfrac{1}{2}(1-\Omega)y^2 - \tfrac{1}{2}(1+\Omega)x^2. \tag{46}$$

For $|\Omega| < 1$ the streamlines $\psi =$ constant are hyperbolae, convecting solute to infinity; for $|\Omega| > 1$, however, the streamlines are ellipses. It turns out that the latter closed streamlines are insufficient to arrest the logarithmic growth (10). The lack of a steady solution is evident in the limit $|\Omega| \to \infty$ where the circular streamlines are compatible with (10), which accordingly becomes an *exact* solution of (3).

We conclude that the function $\lambda(\Omega)$ (and hence a meaningful Nusselt number) can only be calculated for $|\Omega| \leqslant 1$. This function is obtained by substitution of D into (37). For the case of

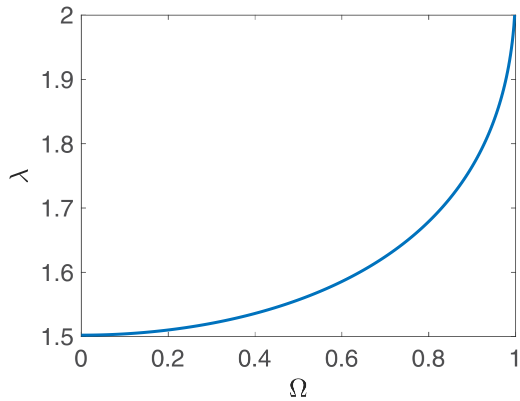

FIG. 1. Variation with Ω of λ.

pure straining motion, $\Omega = 0$, we obtain from (43) $D = \sinh^2 t$, whereby

$$\int_0^1 \left(\frac{1}{\sqrt{D}} - \frac{1}{t} \right) dt = \log 2 - 2 \operatorname{arccoth} e, \qquad \int_1^\infty \frac{dt}{\sqrt{D}} = 2 \operatorname{arccoth} e. \tag{47}$$

Substituting these values into (37) gives

$$\lambda = 1.50223 \quad \text{for} \quad \Omega = 0. \tag{48}$$

For $0 < |\Omega| < 1$, where D is given by (43), the quadratures appearing in (37) are evaluated numerically; the resulting variation with Ω of λ is portrayed in Fig. 1. For the case of simple shear, $\Omega = \pm 1$, we obtain using (44)

$$\int_0^1 \left(\frac{1}{\sqrt{D}} - \frac{1}{t} \right) dt = \log(4\sqrt{3} - 6), \qquad \int_1^\infty \frac{dt}{\sqrt{D}} = \operatorname{arcsinh} \sqrt{3}. \tag{49}$$

Substituting these values into (37) gives

$$\lambda = 2.05153 \quad \text{for} \quad \Omega = \pm 1. \tag{50}$$

Now, with no loss of generality, the cases $\Omega = \pm 1$ may be consolidated as $\Omega = -1$, where the ambient flow (2) reduces to the pure shear $2y\hat{\mathbf{e}}_x$. The pure-shear problem was solved by Frankel and Acrivos [7]; the combination of (38) with (50) agrees with the expression they provide for the outer solution [11].

VII. STATIONARY CYLINDER OF ARBITRARY CROSS-SECTIONAL SHAPE

Consider now the generalization to an arbitrary cross-sectional shape. Consistent with our analysis of a circular shape, the length scale a is now chosen such that the dimensional cross-sectional area of the cylinder is πa^2, while the origin is chosen to coincide with the cross-section centroid. We start with the case of a stationary cylinder.

In the problem formulation, condition (5) now applies over the cylinder boundary. The Nusselt number is still defined by (7). In addition to its dependence upon Pe and Ω, it may now also depend upon both the cross-sectional shape and its orientation relative to the the principal axes of the ambient rate-of-strain tensor.

Consider again the limit $\mathrm{Pe} \ll 1$. Since the Dirichlet condition (5) is now imposed at a possibly complicated boundary, we do not have an explicit (or even implicit) knowledge of the inner-region solute concentration c. Nonetheless, it follows from (9) in conjunction with the two-dimensional variant of Gauss's theorem that the integral in (7) may be deformed to any closed contour which encircles the cylinder. By considering a circle of large radius (still in the inner region) it then follows that

$$c \sim -\mathrm{Nu} \log r \quad \text{for} \quad r \gg 1. \tag{51}$$

More generally [cf. (10)],

$$c \sim -\mathrm{Nu} \log r + \Delta + o(1) \quad \text{for} \quad r \gg 1. \tag{52}$$

As is evident from the analysis of a circular cross section, asymptotic matching requires the value of the constant term Δ in (52). To determine it, we write the harmonic function c as the real part of a complex potential, say $\Phi(z)$, wherein $z = x + iy$. We now employ a conformal mapping $F(\zeta)$ [see e.g. (59)] which (i) maps a circle in the complex ζ plane, say $|\zeta| = \rho$, to the cylinder boundary and (ii) does not deform the far field,

$$F(\zeta) \sim \zeta \quad \text{for} \quad \zeta \to \infty. \tag{53}$$

We additionally use the mapping $\zeta = \rho\varpi$, which transforms the circle in the complex ζ plane to the unit circle $|\varpi| = 1$ in the complex ϖ plane. [In general it is not possible to map the unit circle to the cylinder boundary and simultaneously satisfy (53), whence the need for two subsequent maps.] Denoting the complex potential in the ϖ plane by $\Psi(\varpi)$ we have

$$\Phi(z) = \Psi\left[\frac{F^{-1}(z)}{\rho}\right]. \tag{54}$$

Since Dirichlet conditions are preserved under conformal maps [12] it is evident that, up to an (irrelevant) additive imaginary constant,

$$\Psi(\varpi) = -\mathrm{Nu} \log \varpi + 1. \tag{55}$$

It follows from (53)–(55) that, up to an (irrelevant) additive imaginary constant,

$$\Phi(z) \sim -\mathrm{Nu} \log z + \mathrm{Nu} \log \rho + 1 + o(1) \quad \text{for} \quad z \to \infty. \tag{56}$$

Forming the real part of Φ and comparing with (52) we find

$$\Delta = 1 + \mathrm{Nu} \log \rho. \tag{57}$$

Since the cylinder shape does not appear in the specification of the outer problem [see (14)], the analysis leading to (36) and (37) remains valid. With (10) and (52) having the same form aside from the value of the constant term, asymptotic matching results in the following extension of (38)

$$\mathrm{Nu} = \frac{2}{\log(1/\mathrm{Pe}) + \log(1/\rho^2) + \lambda(\Omega)}, \tag{58}$$

which amounts to the replacement of Pe by ρ^2 Pe. Since the orientation of a circle is irrelevant, it follows that ρ, and hence Nu, are unaffected by the orientation of the ambient flow relative to the cylindrical cross section.

As an example, consider the case of elliptic cross section. Given our use of a dimensionless length, and in view of the independence upon orientation, this shape is described by a single parameter, say the eccentricity $\mathcal{E}$. Specifically, with our choice of length normalization, the semimajor and semiminor axes are given by $(1 - \mathcal{E}^2)^{-1/4}$ and $(1 - \mathcal{E}^2)^{1/4}$, respectively. It is well known [13] that, in the case of an ellipse, the transformation F which satisfies conditions (i) and (ii) above is that of Joukowski. In terms of $\mathcal{E}$, it reads

$$F(\zeta) = \zeta + \frac{\chi^2}{\zeta^2}, \quad \text{wherein} \quad \chi = \frac{\mathcal{E}}{2(1 - \mathcal{E}^2)^{1/4}}. \tag{59}$$

With that transformation, the radius of the circle that is mapped to the ellipse is

$$\rho = \frac{(1-\mathcal{E}^2)^{-1/4} + (1-\mathcal{E}^2)^{1/4}}{2}. \tag{60}$$

Substitution into (58) thus provides the Nusselt number as a function of Pe, Ω, and $\mathcal{E}$.

VIII. FREELY SUSPENDED CYLINDER OF ARBITRARY CROSS SECTION

When a noncircular cylinder is free to rotate about its axis both the geometry and flow are time dependent; it follows that so must be the mass-transport problem. The relevant dimensional time scale is that associated with the cylinder rotation; since we restrict our attention to $|\Omega| < 1$, that scale is provided by $1/G$. Denoting the associated dimensionless time by τ, the advection-diffusion equation (3) is replaced by the conservation equation

$$\text{Pe}\left(\frac{\partial c}{\partial \tau} + \mathbf{u}\cdot\nabla c\right) = \nabla^2 c. \tag{61}$$

In the limit Pe $\ll 1$ the leading-order inner concentration is still governed by Laplace's equation (9). It follows that the inner-region transport is quasisteady: Eqs. (52) and (57) still hold, with the parameters appearing therein possibly being time dependent.

In the outer region, Eq. (12) is modified to

$$\frac{\partial C}{\partial \tau} + (1-\Omega)Y\frac{\partial C}{\partial X} + (1+\Omega)X\frac{\partial C}{\partial Y} = \frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2}. \tag{62}$$

While the time dependence explicitly appears in that leading-order equation, there is no mechanism in the outer region which generates any such dependence. Moreover, as we have already seen, ρ is independent of the cylinder orientation and whence of time. We therefore postulate a steady leading-order outer solution, which is accordingly given by (28). With Nu being independent of time, that solution clearly satisfies (62). We conclude that (58) holds for both a stationary and freely rotating cylinder.

The preceding results apply even if the cylinder is also free to translate (i.e. a freely suspended cylinder). In that more general case, however, care must be taken to ensure that the problem is well posed in the Stokes regime. In the most general case of arbitrary geometry, an imposed linear flow may result in a net hydrodynamic force (per unit length) acting on the body. Had the body been three dimensional, such a force would imply an appropriate rectilinear motion which would keep the particle force free without violating the far-field condition (2); see e.g. [14]. In the present two-dimensional context, no solution exists to such rectilinear motion within the creeping-flow framework. A class of geometries for which a well-posed problem is guaranteed consists of those shapes which are symmetric about two perpendicular lines (e.g. ellipses). For such shapes, the symmetry properties of Stokes flows [15,16] ensure that an imposed linear flow would only result in rigid-body rotation about the centroid, which necessarily coincides with the intersection of these two lines.

IX. CONCLUDING REMARKS

We have analyzed the two-dimensional mass transport from a circular cylinder under a linear ambient velocity distribution. The Nusselt number depends only upon two parameters, namely the Péclet number Pe and the representative ratio Ω of ambient vorticity to the rate of strain. Utilizing the method of matched asymptotic expansions, we have calculated the leading-order Nusselt number in the limit of small Péclet number. That number is independent of the flow details in the vicinity of the cylinder, and is accordingly insensitive to the manner by which it is immersed in the fluid—whether it is held stationary or is free to rotate. Using a conformal mapping that transforms that shape to a circle without deforming the far field, we have also analyzed the problem of a fixed cylinder of arbitrary cross-sectional shape. While the specification of that mapping may require an infinite number of parameters, the resulting Nusselt number is only affected by the radius of the circle. The same Nusselt number also quantifies the mass transport which prevails in the unsteady problem of a freely rotating cylinder of arbitrary cross-sectional shape, and, for sufficiently symmetric shapes, even for a freely suspended cylinder.

No steady solution exists beyond the critical value $|\Omega| = 1$, corresponding to simple shear. In that aspect, the two-dimensional case is fundamentally different from the three-dimensional one, where a solution exists for any finite value of the ambient vorticity. To explain the lack of solution in the present problem, it is expedient to consider the diametric limit of large Pe. Since all the streamlines surrounding the cylinder are closed when $|\Omega| > 1$, the Nusselt number must vanish in that limit [7,17]. If we take it for granted that, for a fixed value of Ω, the Nusselt number is a monotonically increasing function of Pe, it follows that it must vanish for all Pe when $|\Omega| > 1$, suggesting that no steady solution actually exists.

In the present contribution we have assumed the transport to occur due to the explicit specification of the solute concentration value at the cylinder boundary. While this is a fairly standard model [18,19], one can imagine more realistic conditions, e.g. those involving the specification of chemical reactions. In the case of a circular cross section, where the inner solute concentration is isotropic, this would merely result in the modification of the unity constant appearing in (10). For a cylinder of noncircular cross section, however, it is unlikely that the conformal mapping method would be applicable, since the boundary conditions associated with specified chemical reactions are generally not conformally invariant. The analysis of small Péclet numbers would therefore require an explicit (closed-form or numerical) solution of the inner problem which satisfies the far-field behavior (51).

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[11] In making the comparison, we note that the shear flow in Ref. [7] is $y\hat{\mathbf{e}}_x$ and accordingly corresponds to the choice of a representative shear rate which is twice the present G. Taking into account the linear

dependence of Pe upon G [see Eq. (4)], it is then verified that our results indeed reduce to those in Ref. [7].

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Correction: A misprint introduced during the production process has been fixed in the equation appearing in the abstract.