

Mathematical framework for analysis of internal energy dynamics and spectral distribution in compressible turbulent flows

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The objective of this work is to develop the framework and governing equations needed to analyze the interactions and spectral distribution of internal energy in a manner similar to that of a kinetic energy analysis. To emulate the role of velocity in a kinetic energy analysis, a variable ($\phi \sim \sqrt{p}$, where p is pressure) is introduced to enable the examination of internal energy dynamics. Evolution equations for the mean and fluctuating components of ϕ are derived. These equations enable the precise examination of mean-turbulent flow internal energy interactions, internal-kinetic energy exchange, and the spectral distribution of internal energy.

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Kinetic energy dynamics such as interscale transfer and spectral distribution are key features of turbulence and have been the subject of several investigations over many decades [1]. In incompressible flows, wherein the density is constant and uniform, the velocity (u_i) evolution equation, the Navier-Stokes or momentum conservation equation, forms the basis of the kinetic energy analysis. For example, scale-to-scale energy transfer characteristics are dictated by the triadic interactions incumbent in the advective term of the spectral Navier-Stokes equation [2,3]. The kinetic energy spectrum is computed from Fourier transforming the autocovariance function of the velocity field [4].

The advent of compressibility renders the kinetic energy dynamics more complicated due to two factors: (i) spatiotemporal variations in density, and (ii) interactions with internal energy. Some of the complexities in kinetic energy dynamics due to density variations can be adequately addressed by considering the density-weighted velocity field ρu_i leading to Favre averaging [5]. But investigations of scale-to-scale transfer and spectral distribution require additional considerations. A new variable $\sqrt{\rho} u_i$, first proposed by Kida and Orszag [6], has been utilized in many works for computing the kinetic energy spectrum in compressible turbulence [6–9].

The focus of this Rapid Communication is on the investigation of the internal energy dynamics and its interaction with kinetic energy. It is desirable to investigate and characterize internal energy interactions in turbulence in a manner similar to that of a kinetic energy analysis. Many studies perform insightful investigations of turbulent fluctuations of pressure [10–13]. However, such studies do not address the internal energy spectral distribution or transfer in a manner similar to that of kinetic energy. For example, the scale-to-scale transfer of internal energy cannot be computed from the advection term in the spectral pressure equation. Further, the turbulent internal energy spectrum is not the same as the turbulent pressure spectrum. To overcome these limitations, Miura

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and Kida [14] propose a new internal energy variable ($\phi \sim \sqrt{p}$) corresponding to velocity for investigating the internal energy spectrum.

The objective of this Rapid Communication is to extend the proposal of Miura and Kida [14] and develop the mathematical framework and evolution equations required for performing a comprehensive and rigorous analysis of turbulent internal energy dynamics based on ϕ . Similar to a kinetic energy analysis, we aim at partitioning the internal energy into two parts corresponding to the contribution of the mean field and the fluctuating field. The equations based on ϕ will enable rigorous investigations of (i) internal-kinetic energy interactions at each scale, (ii) mean-turbulent internal energy exchange, and (iii) the spectral distribution of internal and total energies in a turbulent flow field. The proposed framework is very important for a comprehensive understanding of the total energy dynamics and spectral energy distribution in high-speed compressible transition and turbulent flows.

The compressible Navier-Stokes equations for a calorically perfect ideal gas form the basis of this analysis,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \quad (1a)$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}, \quad (1b)$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} \right) + \frac{\partial}{\partial x_j} \left(\frac{p u_j}{\gamma - 1} \right) = \frac{\partial}{\partial x_j} \left(\kappa \frac{\partial T}{\partial x_j} \right) - p \frac{\partial u_k}{\partial x_k} + \tau_{ij} \frac{\partial u_i}{\partial x_j}, \quad (1c)$$

$$p = \rho RT, \quad (1d)$$

where ρ is the fluid density, u_i is the velocity component, τ_{ij} is the viscous stress tensor, p is the gas pressure, γ is the ratio of specific heats at constant pressure and volume, κ is the coefficient of thermal conductivity, T is the temperature, x_i is the spatial coordinate, and t is time. The viscous stress tensor is given by

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij}, \quad (2)$$

where μ is the coefficient of dynamic viscosity, λ is the coefficient of second viscosity, and δ_{ij} is the Kronecker delta.

We will first present the kinetic (K), internal (e), and total (E) energy equations to identify the key interactions. The energy equations can be derived using Eqs. (1b) and (1c),

$$\frac{\partial K}{\partial t} + \frac{\partial(K u_k)}{\partial x_k} + \frac{\partial}{\partial x_k} [p u_k - \tau_{ik} u_i] = p \frac{\partial u_k}{\partial x_k} - \tau_{ij} \frac{\partial u_i}{\partial x_j}, \quad (3a)$$

$$\frac{\partial e}{\partial t} + \frac{\partial(e u_k)}{\partial x_k} + \frac{\partial q_k}{\partial x_k} = -p \frac{\partial u_k}{\partial x_k} + \tau_{ij} \frac{\partial u_i}{\partial x_j}, \quad (3b)$$

$$\frac{\partial E}{\partial t} + \frac{\partial(E u_k)}{\partial x_k} + \frac{\partial}{\partial x_k} [p u_k - \tau_{ik} u_i + q_k] = 0, \quad (3c)$$

where

$$K = \frac{1}{2} \rho u_i u_i, \quad e = \frac{p}{\gamma - 1}, \quad q_k = -\kappa \frac{\partial T}{\partial x_k}, \quad \text{and} \quad E = K + e. \quad (4)$$

Equations (3a) and (3b) indicate interactions between kinetic and internal energy via pressure dilatation and viscous action. Pressure dilatation permits a two-way exchange while viscous action can only lead to a one-way transfer from kinetic to internal energy.

The velocity field is decomposed into a Favre average and a corresponding fluctuation field [5],

$$u_i = \tilde{U}_i + u_i'', \quad (5)$$

while other variables (ψ) are decomposed using Reynolds averaging,

$$\psi = \bar{\psi} + \psi'. \quad (6)$$

For some interactions, Reynolds averaging of the velocity field is also considered. Throughout this Rapid Communication, $(\bar{\cdot})$ corresponds to Favre averaging and $(\cdot)$ to Reynolds averaging. The evolution of the mean kinetic ($\bar{K}$) and internal ($\bar{e}$) energies is given by

$$\frac{\partial \bar{K}}{\partial t} + \frac{\partial (\tilde{U}_k \bar{K})}{\partial x_k} + \frac{\partial}{\partial x_k} [\overline{K u_k''} - \bar{\tau}_{ik} u_i + \bar{p} u_k] = \bar{p} \frac{\partial \bar{U}_k}{\partial x_k} + \overline{p' \frac{\partial u_k'}{\partial x_k}} - \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} - \overline{\tau'_{ij} \frac{\partial u_i'}{\partial x_j}}, \quad (7a)$$

$$\frac{\partial \bar{e}}{\partial t} + \frac{\partial (\tilde{U}_k \bar{e})}{\partial x_k} + \frac{\partial}{\partial x_k} [\overline{e u_k''} + \bar{q}_k] = -\bar{p} \frac{\partial \bar{U}_k}{\partial x_k} - \overline{p' \frac{\partial u_k'}{\partial x_k}} + \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} + \overline{\tau'_{ij} \frac{\partial u_i'}{\partial x_j}}. \quad (7b)$$

It is important to recognize that the mean kinetic energy of the flow has two contributions, the kinetic energy associated with the mean flow field and that associated with the fluctuations,

$$\bar{K} = K_m + k, \quad (8)$$

where

$$K_m = \frac{1}{2} \bar{\rho} \tilde{U}_i \tilde{U}_i \quad \text{and} \quad k = \frac{1}{2} \overline{\rho u_i'' u_i''}.$$

The mean field ($\tilde{U}_i$) kinetic energy is K_m and turbulent field (u_i'') kinetic energy is k in Eq. (8).

The governing equation for K_m can be derived using the Favre-averaged Navier-Stokes equation [15],

$$\bar{\rho} \frac{\partial \tilde{U}_i}{\partial t} + \bar{\rho} \tilde{U}_j \frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \overline{\rho u_i'' u_j''}}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ij}}{\partial x_j}. \quad (9)$$

The equation for the mean-flow kinetic energy (K_m) can be obtained by multiplying Eq. (9) with $\tilde{U}_i$,

$$\begin{aligned} \frac{\partial K_m}{\partial t} + \frac{\partial (K_m \tilde{U}_j)}{\partial x_j} + \frac{\partial}{\partial x_j} [\overline{\rho u_i'' u_j''} \tilde{U}_i + \bar{p} \bar{U}_j - \bar{\tau}_{ij} \bar{U}_i] \\ = \overline{\rho u_i'' u_j''} \frac{\partial \tilde{U}_i}{\partial x_j} + \bar{p} \frac{\partial \bar{U}_k}{\partial x_k} - \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} + \overline{u_k'' \frac{\partial \bar{p}}{\partial x_k}} - \overline{u_k'' \frac{\partial \bar{\tau}_{kj}}{\partial x_j}}. \end{aligned} \quad (10)$$

Here, we have used the relation $\bar{U}_k = \tilde{U}_k + \overline{u_k''}$ to simplify some terms. The evolution of turbulent kinetic energy (k) in compressible flows follows from Refs. [16,17],

$$\begin{aligned} \frac{\partial k}{\partial t} + \frac{\partial (k \tilde{U}_j)}{\partial x_j} + \frac{\partial}{\partial x_j} \left[\frac{1}{2} \overline{\rho u_i'' u_j'' u_j''} + \overline{p' u_j''} - \overline{\tau'_{ij} u_i''} \right] \\ = -\overline{\rho u_i'' u_j''} \frac{\partial \tilde{U}_i}{\partial x_j} + \overline{p' \frac{\partial u_k''}{\partial x_k}} - \overline{\tau'_{ij} \frac{\partial u_i''}{\partial x_j}} - \overline{u_k'' \frac{\partial \bar{p}}{\partial x_k}} + \overline{u_k'' \frac{\partial \bar{\tau}_{kj}}{\partial x_j}}. \end{aligned} \quad (11)$$

Utilizing $u_k'' = \bar{U}_k - \tilde{U}_k + u_k'$ and $\bar{\psi}' = 0$, we can establish that

$$\overline{\psi' \frac{\partial u_k''}{\partial x_k}} = \overline{\psi' \frac{\partial u_k'}{\partial x_k}}. \quad (12)$$

Therefore, the equation for turbulent kinetic energy (k) can be rewritten as

$$\begin{aligned} \frac{\partial k}{\partial t} + \frac{\partial(k\tilde{U}_j)}{\partial x_j} + \frac{\partial}{\partial x_j} \left[\frac{1}{2} \overline{\rho u_i'' u_j'' u_i''} + \overline{p' u_j'} - \overline{\tau_{ij}' u_i'} \right] \\ = -\overline{\rho u_i'' u_j''} \frac{\partial \tilde{U}_i}{\partial x_j} + \overline{p' \frac{\partial u_k'}{\partial x_k}} - \overline{\tau_{ij}' \frac{\partial u_i'}{\partial x_j}} - \overline{u_k'' \frac{\partial \bar{p}}{\partial x_k}} + \overline{u_k'' \frac{\partial \tau_{kj}}{\partial x_j}}. \end{aligned} \quad (13)$$

The various terms in these equations and the incumbent physics will be discussed further below. In incompressible flows, the spectral transfer and distribution of kinetic energy across different scales of motion can be examined by performing a Fourier transform of u_i' . As proposed by Kida and Orszag [6], a similar spectral energy analysis of compressible flows requires a new variable,

$$w_i = \sqrt{\rho} u_i'', \quad (14)$$

to account for spatiotemporal variations in density. The Fourier transform of the autocorrelation function of w_i yields the kinetic energy spectrum. The quartic interactions for a turbulent kinetic energy cascade are also reduced to triadic interactions as

$$\frac{\partial}{\partial x_j} \left[\frac{1}{2} \rho u_i'' u_j'' u_i'' \right] = \frac{\partial}{\partial x_j} \left[\frac{1}{2} w_i w_j u_i'' \right]. \quad (15)$$

Unlike kinetic energy, it is more complicated to clearly identify the internal energy associated with the mean (or background field) and the turbulent internal energy. In the linear regime, when the density fluctuations are small, Sarkar *et al.* [10] identified that the potential energy incumbent in pressure fluctuations can be approximated as $\overline{p' p'} / (2\gamma \bar{p})$. However, this representation is inadequate when the density fluctuations are large. Miura and Kida [14] propose a more general approach which is valid for strongly compressible flows. They propose a new variable,

$$\phi \equiv \sqrt{e} = \sqrt{\frac{p}{\gamma - 1}}, \quad (16)$$

for examining the internal energy spectrum in simple isotropic flows. The justification is that ϕ^2 corresponds to internal energy.

In this study we develop the framework of equations based on ϕ to investigate flows with strong mean fields and more importantly to examine internal-kinetic energy interactions. First, the choice of ϕ as the internal energy equivalent of velocity is further justified on the following physics-based argument. Using the relation $\gamma p = \rho a^2$, where a is the speed of sound, ϕ can be written as

$$\phi = \frac{1}{\sqrt{\gamma(\gamma - 1)}} \sqrt{\rho} a. \quad (17)$$

Equation (17) shows that ϕ includes density-weighted acoustic speed information which relates to the internal energy of the system. The state variable ϕ for internal energy is thus analogous to w_i for turbulent kinetic energy. ϕ can now be decomposed using Reynolds averaging,

$$\phi = \bar{\phi} + \phi', \quad (18)$$

where

$$\bar{\phi} = \sqrt{\frac{\bar{p}}{\gamma - 1}} \quad \text{and} \quad \phi' = \sqrt{\frac{p}{\gamma - 1}} - \sqrt{\frac{\bar{p}}{\gamma - 1}}.$$

With this formulation, it is possible to clearly separate the mean internal energy of the flow into a mean field and a fluctuating field contribution,

$$\bar{e} = \bar{\phi}^2 = \bar{\phi} \bar{\phi} + \overline{\phi' \phi'}, \quad (19)$$

therefore,

$$e_m = \overline{\phi \phi} \quad \text{and} \quad e_t = \overline{\phi' \phi'}. \quad (20)$$

The strong similarity of internal energy decomposition in Eq. (19) and kinetic energy partitioning in Eq. (8) is clearly evident. The spectral transfer and spectral distribution of turbulent internal energy can be characterized from the behavior ϕ' much in the same manner as the turbulent kinetic energy transfer/distribution can be analyzed using $w_i = \sqrt{\rho} u'_i$. The Fourier transform of the autocorrelation of ϕ' yields the turbulent internal energy spectrum. Thus, $\phi' - w_i$ interactions hold the key to examining internal-kinetic energy exchange.

In the linear limit, as indicated by a small turbulent Mach number $M_t \equiv u'/a \ll 1$ [10], the pressure fluctuations can be considered small relative to the mean pressure. Therefore, higher-order moments are negligible ($\overline{p'p'} \approx 0$, $\overline{p'p'p'} \approx 0$, ...) and can be ignored. $\overline{\phi}$ and ϕ' can now be approximated as

$$\overline{\phi} = \sqrt{\frac{\overline{p}}{\gamma - 1}} \left[1 + \frac{p'}{\overline{p}} \right]^{1/2} \approx \sqrt{\frac{\overline{p}}{\gamma - 1}} \quad \text{and} \quad \phi' \approx \frac{p'}{2\sqrt{\overline{p}(\gamma - 1)}} + O\left(\frac{p'p'}{\overline{p}^{3/2}}\right). \quad (21)$$

Then the turbulent internal energy in the linear limit can be shown to be

$$e_t = \overline{\phi' \phi'} \approx \frac{1}{4(\gamma - 1)} \left[\overline{\frac{p'p'}{\overline{p}}} \right]. \quad (22)$$

The above expression is similar to the linear limit potential energy of pressure fluctuations as identified in the literature [10,18].

The mean and the fluctuation pressure field can be written in terms of the internal energy velocity variable ϕ as

$$\overline{p} = (\gamma - 1)[\overline{\phi \phi} + \overline{\phi' \phi'}] \quad \text{and} \quad p' = (\gamma - 1)[\phi' \phi' + 2\overline{\phi \phi'} - \overline{\phi' \phi'}]. \quad (23)$$

We can clearly see that mean and fluctuating pressure have a linear and nonlinear component,

$$\overline{p} = \overline{p}_m + \overline{p}_t \quad \text{and} \quad p' = p'_m + p'_t, \quad (24)$$

where

$$\begin{aligned} \overline{p}_m &= (\gamma - 1)\overline{\phi \phi} \quad \text{and} \quad \overline{p}_t = (\gamma - 1)\overline{\phi' \phi'}, \\ p'_m &= 2(\gamma - 1)\overline{\phi \phi'} \quad \text{and} \quad p'_t = (\gamma - 1)[\phi' \phi' - \overline{\phi' \phi'}]. \end{aligned} \quad (25)$$

Here, $\overline{p}_m$ and $\overline{p}_t$ are the components of mean pressure associated with the mean field and the perturbation field, respectively. Similarly, p'_m and p'_t are the components of fluctuating pressure associated with the mean field and the perturbation field. $\overline{p}_m$ and p'_m can also be understood as linear (or rapid) components while $\overline{p}_t$ and p'_t as nonlinear (or slow) components.

We can now derive the evolution equations for $e_m = \overline{\phi \phi}$ and $e_t = \overline{\phi' \phi'}$. We start with the internal energy equation, Eq. (3b), and use definitions $e = \phi^2$ and $p = (\gamma - 1)\phi^2$ to obtain an equation for ϕ ,

$$\frac{\partial \phi}{\partial t} + u_k \frac{\partial \phi}{\partial x_k} = -\frac{\gamma}{2} \phi \frac{\partial u_k}{\partial x_k} + \frac{f}{2\phi}, \quad (26)$$

where

$$f = \left[-\frac{\partial q_k}{\partial x_k} + \tau_{ij} \frac{\partial u_i}{\partial x_j} \right].$$

The Reynolds average of the thermal flux and viscous terms (f) can be written as

$$\bar{f} = \left[-\frac{\partial \bar{q}_k}{\partial x_k} + \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} + \overline{\tau'_{ij} \frac{\partial u'_i}{\partial x_j}} \right]. \quad (27)$$

The last term in Eq. (26) can be simplified as

$$\frac{f}{2\phi} = \frac{\bar{f}}{2\bar{\phi}} - \frac{f\phi'}{2\bar{\phi}\phi}, \quad (28)$$

which gives

$$\overline{\left(\frac{f}{2\phi} \right)} = \frac{\bar{f}}{2\bar{\phi}} - \frac{1}{2\bar{\phi}} \overline{\left(\frac{f\phi'}{\phi} \right)}.$$

Reynolds-averaging Eq. (26) gives an equation for $\bar{\phi}$ as

$$\frac{\partial \bar{\phi}}{\partial t} + \tilde{U}_k \frac{\partial \bar{\phi}}{\partial x_k} + \overline{u''_k \frac{\partial \bar{\phi}}{\partial x_k}} + \overline{u''_k \frac{\partial \bar{\phi}}{\partial x_k}} = -\frac{\gamma}{2} \left[\bar{\phi} \frac{\partial \tilde{U}_k}{\partial x_k} + \bar{\phi} \frac{\partial u''_k}{\partial x_k} + \overline{\phi' \frac{\partial u''_k}{\partial x_k}} \right] + \frac{\bar{f}}{2\bar{\phi}} - \frac{1}{2\bar{\phi}} \overline{\left(\frac{f\phi'}{\phi} \right)}. \quad (29)$$

Multiplying Eq. (29) by $2\bar{\phi}$ yields the governing equation for the mean field contribution of mean internal energy (e_m). Using $\bar{U}_k = \tilde{U}_k + \overline{u''_k}$ and Eq. (27), the simplified equation for e_m becomes

$$\begin{aligned} \frac{\partial e_m}{\partial t} + \frac{\partial (e_m \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k} [\bar{\phi} \bar{\phi} \overline{u''_k} + 2\bar{\phi} \overline{\phi' u''_k} + \bar{q}_k] \\ = \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} + \overline{\tau'_{ij} \frac{\partial u'_i}{\partial x_j}} - (\gamma - 1) \bar{\phi} \bar{\phi} \frac{\partial \bar{U}_k}{\partial x_k} + 2\overline{u''_k \phi'} \frac{\partial \bar{\phi}}{\partial x_k} + (2 - \gamma) \bar{\phi} \overline{\phi' \frac{\partial u''_k}{\partial x_k}} - \overline{\left(\frac{f\phi'}{\phi} \right)}. \end{aligned} \quad (30)$$

The above equation is further simplified using the expression in Eq. (12) and the definitions in Eq. (25),

$$\begin{aligned} \frac{\partial e_m}{\partial t} + \frac{\partial (e_m \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k} [\bar{\phi} \bar{\phi} \overline{u''_k} + 2\bar{\phi} \overline{\phi' u''_k} + \bar{q}_k] \\ = \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} + \overline{\tau'_{ij} \frac{\partial u'_i}{\partial x_j}} - \bar{p}_m \frac{\partial \bar{U}_k}{\partial x_k} + 2\overline{u''_k \phi'} \frac{\partial \bar{\phi}}{\partial x_k} + \frac{(2 - \gamma)}{2(\gamma - 1)} \bar{p}'_m \frac{\partial \overline{u''_k}}{\partial x_k} - \overline{\left(\frac{f\phi'}{\phi} \right)}. \end{aligned} \quad (31)$$

We can now determine the ϕ' equation by subtracting Eq. (29) from Eq. (26),

$$\begin{aligned} \frac{\partial \phi'}{\partial t} + \tilde{U}_k \frac{\partial \phi'}{\partial x_k} + u''_k \frac{\partial \bar{\phi}}{\partial x_k} + u''_k \frac{\partial \phi'}{\partial x_k} - \overline{u''_k \frac{\partial \bar{\phi}}{\partial x_k}} - \overline{u''_k \frac{\partial \phi'}{\partial x_k}} \\ = -\frac{\gamma}{2} \left[\phi' \frac{\partial \tilde{U}_k}{\partial x_k} + \bar{\phi} \frac{\partial u''_k}{\partial x_k} + \phi' \frac{\partial u''_k}{\partial x_k} - \bar{\phi} \frac{\partial \overline{u''_k}}{\partial x_k} - \phi' \frac{\partial \overline{u''_k}}{\partial x_k} \right] + \left(\frac{f}{2\phi} \right)'. \end{aligned} \quad (32)$$

To obtain the governing equation for turbulent internal energy (e_t), we multiply Eq. (32) with $2\phi'$ and take the average,

$$\frac{\partial e_t}{\partial t} + \tilde{U}_k \frac{\partial e_t}{\partial x_k} + 2\overline{\phi' u''_k \frac{\partial \bar{\phi}}{\partial x_k}} + \overline{u''_k \frac{\partial \phi' \phi'}{\partial x_k}} = -\gamma \left[\overline{\phi' \phi' \frac{\partial \tilde{U}_k}{\partial x_k}} + \bar{\phi} \overline{\phi' \frac{\partial u''_k}{\partial x_k}} + \overline{\phi' \phi' \frac{\partial u''_k}{\partial x_k}} \right] + \overline{\left(\frac{f}{\phi} \right)' \phi'}. \quad (33)$$

The last term in Eq. (33) can be simplified as

$$\overline{\left(\frac{f}{\phi} \right)' \phi'} = \overline{\left[\left(\frac{f}{\phi} \right) \phi' + \left(\frac{f}{\phi} \right)' \phi \right]} = \overline{\left(\frac{f\phi'}{\phi} \right)}. \quad (34)$$

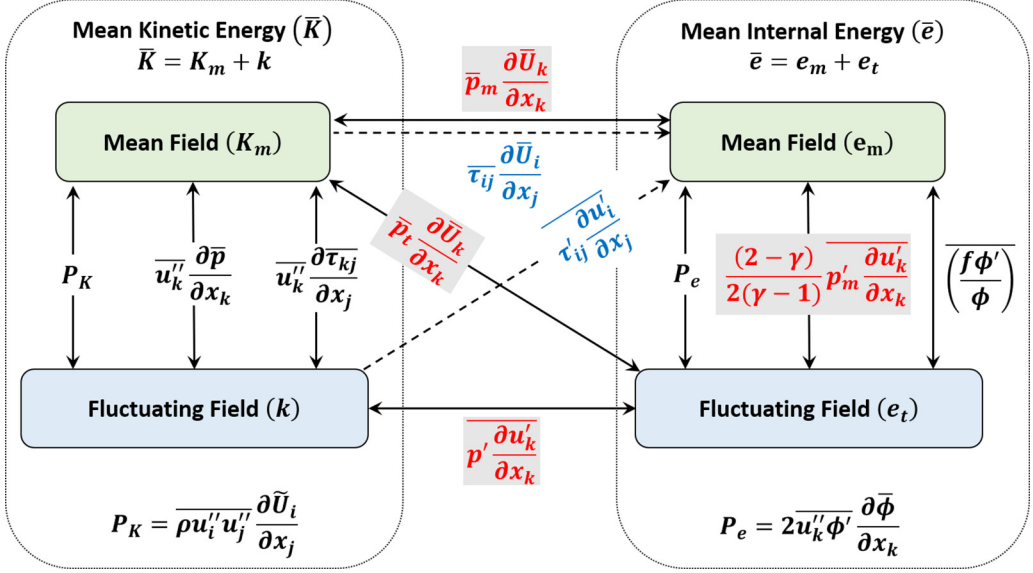


FIG. 1. Interactions between kinetic and internal energies in the nonlinear regime using Eqs. (10), (13), (31), and (36).

Simplifying Eq. (33) and using the relation $\tilde{U}_k = \bar{U}_k - \overline{u''_k}$, we get

$$\begin{aligned} \frac{\partial e_t}{\partial t} + \frac{\partial (e_t \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k} [\overline{\phi' \phi' u''_k}] = & -(\gamma - 1) \overline{\phi' \phi'} \frac{\partial \bar{U}_k}{\partial x_k} - (\gamma - 1) (\phi' \phi' + 2 \bar{\phi} \phi' - \overline{\phi' \phi'}) \frac{\partial u''_k}{\partial x_k} \\ & - 2 \overline{u''_k \phi'} \frac{\partial \bar{\phi}}{\partial x_k} - (2 - \gamma) \overline{\phi \phi'} \frac{\partial u''_k}{\partial x_k} + \overline{\left(\frac{f \phi'}{\phi}\right)}. \end{aligned} \quad (35)$$

Now we can use the relations in Eqs. (12), (23), and (25) to obtain the simplified form of the above equation,

$$\frac{\partial e_t}{\partial t} + \frac{\partial (e_t \tilde{U}_k)}{\partial x_k} + \frac{\partial}{\partial x_k} [\overline{\phi' \phi' u''_k}] = -\bar{p}_t \frac{\partial \bar{U}_k}{\partial x_k} - \overline{p'_m} \frac{\partial \bar{U}_k}{\partial x_k} - 2 \overline{u''_k \phi'} \frac{\partial \bar{\phi}}{\partial x_k} - \frac{(2 - \gamma)}{2(\gamma - 1)} \overline{p'_m} \frac{\partial u''_k}{\partial x_k} + \overline{\left(\frac{f \phi'}{\phi}\right)}. \quad (36)$$

The triadic interactions of the turbulent internal energy cascade can now be analyzed similarly to the turbulent kinetic energy via $\frac{\partial}{\partial x_k} (\phi' \phi' u''_k)$. The equations for $\bar{\phi}$, e_m , ϕ' , and e_t in Eqs. (29), (31), (32), and (36) constitute the framework required to describe internal energy interactions.

The mean total energy ($\bar{E} = K_m + k + e_m + e_t$) obtained through this framework can be written as

$$\frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x_k} \left[\bar{E} \tilde{U}_k + \overline{\rho u''_i u''_j} \tilde{U}_i + \frac{1}{2} \overline{\rho u''_i u''_i u''_k} + \overline{p u_k} - \overline{\tau_{ik} u_k} + \overline{q_k} + \frac{\overline{p u''_k}}{\gamma - 1} \right] = 0. \quad (37)$$

The mean total energy ($\bar{E}$) of a system changes only due to energy flux from the outside. Therefore, $\bar{E}$ for an isolated system remains constant. This is an important property of the chosen variables.

We now discuss important aspects of kinetic-internal energy interactions and summarize the exchanges among K_m , k , e_m , and e_t (Fig. 1).

($K_m - k$) interactions. The most important interaction between mean-turbulent kinetic energy is the production of turbulent kinetic energy (k) via $\overline{\rho u''_i u''_j} \frac{\partial \bar{U}_i}{\partial x_j}$ as seen in Eqs. (10) and (13). Production draws energy from the mean flow and deposits it into the fluctuating field. Other interactions appear

as a consequence of compressibility and Favre averaging. Both pressure work ($\overline{u'_k \frac{\partial \bar{p}}{\partial x_k}}$) and viscous action ($\overline{u'_k \frac{\partial \tau_{kj}}{\partial x_j}}$) can lead to a two-way exchange between the mean and fluctuating fields. For an incompressible flow, only production remains active among all ($K_m - k$) interactions.

($e_m - e_t$) interactions. The terms $2\overline{u'_k \phi' \frac{\partial \phi}{\partial x_k}}$, $\frac{(2-\gamma)}{2(\gamma-1)} \overline{p'_m \frac{\partial u'_k}{\partial x_k}}$, and $(\frac{f\phi'}{\phi})$ in Eqs. (31) and (36) lead to an exchange between the mean field internal energy (e_m) and the turbulent internal energy (e_t). The first term is analogous to production in the kinetic energy equations. The second term causes an exchange due to the linear or rapid component of the fluctuating field pressure-dilatation mechanism while the last term is an exchange due to viscous and thermal flux action.

($K_m - e_m$) interactions. The mean field kinetic-energy internal energies interact via the linear or rapid component of mean field pressure dilatation ($\overline{\bar{p}_m \frac{\partial \bar{U}_k}{\partial x_k}}$) and viscous action ($\overline{(\tau'_{ij}) \frac{\partial \bar{U}_i}{\partial x_j}}$) as evident in Eqs. (10) and (31). The pressure-dilatation action causes a two-way exchange and is represented with a double solid arrow in Fig. 1. On the other hand, viscous action dissipates mean field kinetic energy to mean field internal energy and is shown with a dashed one-way arrow. As mentioned by Huang *et al.* [16], the Reynolds-averaged velocity field ($\bar{U}_i$) contributes towards the internal-kinetic exchange and not the Favre-averaged mean field ($\bar{U}_i$).

($K_m - e_t$) interactions. The mean-field kinetic energy and turbulent internal energy only interact via the nonlinear or slow component of mean field pressure dilatation ($\overline{\bar{p}_t \frac{\partial \bar{U}_k}{\partial x_k}}$). This exchange is identified using Eqs. (10) and (36).

($k - e_m$) interactions. The turbulent kinetic energy dissipates energy into the mean field internal energy via the fluctuating field viscous action ($\overline{(\tau'_{ij}) \frac{\partial u'_k}{\partial x_j}}$) as seen from Eqs. (13) and (31).

($k - e_t$) interactions. The exchange between turbulent kinetic and turbulent internal energy is caused via the fluctuating field pressure-dilatation mechanism ($\overline{p' \frac{\partial u'_k}{\partial x_k}}$) as evident in Eqs. (13) and (36). This exchange includes a contribution of both linear (or rapid) and nonlinear (or slow) pressure fluctuations. It is also synchronous with ($e_m - e_t$) exchange via the linear or rapid component of fluctuating field pressure dilatation.

In conclusion, we have established a framework for examining the internal energy of any fluid flow governed by the compressible Navier-Stokes equations. Analogous to fluid velocity, a suitable thermodynamic state variable for internal energy is identified ($\phi \sim \sqrt{p}$). Defining the internal energy in terms of ϕ allows partitioning of the mean internal energy of the flow into the mean field and perturbation field contributions, similar to mean kinetic energy. It is also established that the spectral behavior of turbulent internal energy can be rigorously examined using ϕ' . Governing equations for e_m and e_t are derived to identify key internal-kinetic energy interactions. It is anticipated that this framework will be important for the analysis and modeling of energy dynamics in high-speed compressible transition and turbulent flows.

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