

Viscous damping of gravity-capillary waves: Dispersion relations and nonlinear corrections

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We discuss the impact of viscosity on nonlinear propagation of surface waves at the interface of air and a fluid of large depth. After a survey of the available approximations of the dispersion relation, we propose to modify the hydrodynamic boundary conditions to model both short and long waves. From them, we derive a nonlinear Schrödinger equation where both linear and nonlinear parts are modified by dissipation and show that the former plays the main role in both gravity and capillary-gravity waves while, in most situations, the latter represents only small corrections. This provides a justification of the conventional approaches to damped propagation found in the literature.

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I. INTRODUCTION

Deep-water surface waves are among the most studied examples of nonlinear physical systems. The nonlinearity stems from kinematic and dynamic boundary conditions at the air-fluid interface. Triplet [1] and quadruplet [2] interactions can be included and a general integrodifferential equation can be derived. A suitable expansion of the integral kernel [3,4] was shown to connect it to simple propagation models, like the nonlinear Schrödinger equation (NLS) [5] and its generalizations (modified NLS, MNLS), among which the best known is the Dysthe equation [6]. The universal NLS possesses many remarkable properties and solutions, such as the modulation instability (MI, also known as Benjamin-Feir instability, BFI) [7–9], solitons, and breathers, the clearest examples of the balance of dispersion and nonlinearity.

In the propagation of deep-water waves, gravity effects dominate for long waves (small wavenumbers) and surface tension for short waves. Moreover, viscosity is the ubiquitous damping mechanism [10,11] (other routes to dissipation include impurities, obstacles, wave breaking, etc.) and plays a role in fundamental studies of analogue gravity [12]. It induces a vorticity in the fluid and therefore requires solving the full Navier-Stokes (NS) equations. For small viscosity, the vorticity is significant only in a small boundary layer close to the surface [10,11,13,14].

For small-amplitude waves, i.e., when nonlinearities are negligible, the velocity potential formulation valid for an inviscid flow can be corrected to include the effects of viscosity [15–17]. Different loss rates and corrections to the group velocity can be found in Ref. [18]. The availability of a quasipotential formulation greatly simplifies the numerical solution of the hydrodynamic problem [19,20]. Moreover, it is often assumed without justification that the linear dissipation is so small that no nonlinear correction is required for the formulation of a modified NLS including damping [21,22]. Alternatively, nonlinear damping mechanisms were proposed in the form of a Landau damping [23] or in a phenomenological way [24,25].

In this work, we quantify the nonlinear corrections to propagation equations stemming from kinematic viscosity and justify therefore the conventional assumption of Refs. [21,22]. Our approach

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is based on the dissipation method detailed in Refs. [18,26], but can be generalized to any expression for the dispersion relations and damping mechanisms. After recalling some simple considerations about dispersion relations and justifying the choice of a specific form (Sec. II), we propose a modified set of hydrodynamic equations consistent with our approach (Sec. III). These are straightforwardly adapted in Sec. IV to generalize the 1D MNLS [3]

$$\frac{\partial \hat{B}}{\partial t} + i[\omega(k_0 + \kappa) - \omega_0]\hat{B} + i\frac{\omega_0 k_0^2}{2}\mathcal{F}_\kappa[|B|^2 B] = 0, \quad (1)$$

with $\hat{B}(\kappa, t) \equiv \mathcal{F}_\kappa[B(x, t)] = \frac{1}{\sqrt{2\pi}} \int dx B(x, t) e^{-i\kappa x}$ being the Fourier transform from real space of x (in the comoving frame) to the relative wave-number space of κ . In fact, the envelope $B(x, t)$ of the surface elevation is centered around the carrier wave number k_0 ; $\omega(k)$ is the real-valued dispersion relation [$\omega_0 \equiv \omega(k_0)$]. We quantify the nonlinear corrections to the recurrence period and spectral-mean downshift to be less than 1% in typical experimental conditions of a water tank. We also verify that the full dispersion relation plays a key role in the nonlinear evolution of the MI, e.g., in explaining the frequency downshift observed in Ref. [27]. Finally (Sec. V), we show that the considered dispersion relation can be inverted so that not only the spacelike but also the timelike formulation of the MNLS naturally generalizes to the dissipative case. The same approach can be adapted to high-order NLS, such as the Dysthe [6] or the recently proposed compact [28] and supercompact [29] equations. Conclusions and outlook complete our paper.

II. VISCOUS DISPERSION RELATIONS

We first briefly review how to express the dispersion relation for deep-water capillary-gravity waves at the surface of an incompressible viscous fluid. The solution of the linearized hydrodynamic equations for 1D propagation in the x direction is a plane wave $\exp[ikx - i\omega(k)t]$. We decompose $\omega(k) = \omega_R(k) + i\omega_I(k)$, where $\omega_R > 0$ (resp., $\omega_R < 0$) represents forward-propagating (resp., backward-propagating) waves (for $k > 0$; otherwise the opposite applies) and $\omega_I < 0$ is the damping rate.

In the limit of infinite depth, the dispersion relation can be shown to be the solution $\omega(k)$ of the implicit equation [10 (Sec. 349), 11 (Sec. 25)]

$$\left(2 - i\frac{\omega}{\nu k^2}\right)^2 + \frac{|k|(g + sk^2)}{\nu^2 k^4} = 4\left(1 - i\frac{\omega}{\nu k^2}\right)^{\frac{1}{2}}, \quad (2)$$

where g is the standard acceleration due to gravity, $s \equiv T/\rho_f$ with T the surface tension (in Nm^{-1}) of the fluid-air interface and ρ_f is the density of the fluid. The gravity and capillary contributions dominate, respectively, for small and large k , i.e. for large and small wavelength. Finally, ν denotes the kinematic viscosity of the fluid (in m^2/s) and is the physical origin of ω_I .

The detailed derivation of Eq. (2) from the linearized Euler equations for an incompressible fluid, along with their kinematic and boundary conditions, consists in including a vorticity field and assuming that the mass transport occurs only in a boundary layer close to the air-fluid interface [10, 11, 13–15, 22, 30]. In Eq. (2), being quartic, two forward-traveling wave branches exist: One is unphysical (it corresponds to a velocity potential diverging for infinite depth $z \rightarrow -\infty$; see also Sec. III). The physical solution is shown in Fig. 1 for a gravity-capillary wave propagating at the interface of air and water ($s = 7.28 \times 10^{-5} \text{ m}^3 \text{ s}^{-2}$) in the presence of kinematic viscosity $\nu = 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$. For large k , in the μm^{-1} range, ω_R rises and then drops abruptly. At $k = k_c = 1.25 \times 10^8 \text{ m}^{-1}$ it cuts off; i.e., the viscous damping is so strong that $\omega_R = 0$: The mode is a standing wave. For $k > k_c$, two branches with different damping are admitted; see the right axis of Fig. 1. The least-damped branch (solid line) is expected to take the leading physical role at those wave numbers; the other (dashed line) is dissipated much more rapidly [11] and does not behave as a surface wave [31].

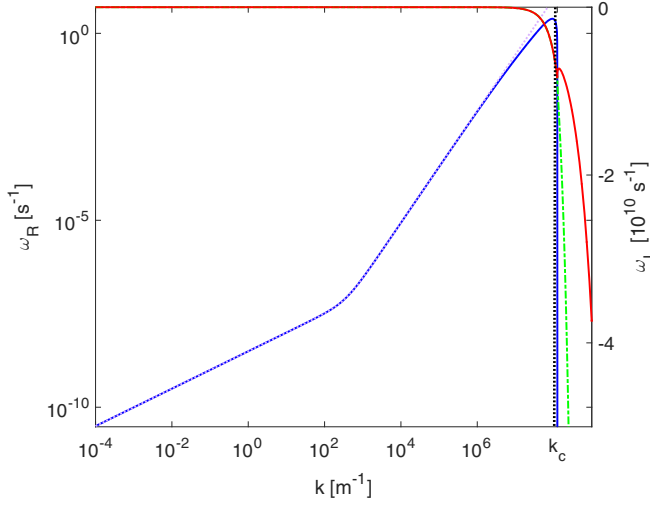


FIG. 1. Dispersion relation for gravity-capillary waves propagating at the interface of air and water in the presence of kinematic viscosity. Numerical solution of Eq. (2) for forward-propagating waves ($\omega_R \geq 0$). On the left axis (in logarithmic scale), the solid blue and dotted lines refer to $\text{Re } \omega$ and its inviscid counterpart $\tilde{\omega}$. The red dotted and green dash-dotted lines pertain to the right axis (linear scale) and show the two branches of $\text{Im } \omega$. The vertical dotted line marks to the cutoff k_c beyond which $\omega_R = 0$ and ω_I bifurcates in two standing-wave branches, yielding different damping rates.

Relying on physical arguments [15–18,32], it was shown that simpler explicit formulas can be obtained by expressing the vorticity in terms of the velocity potential and surface elevation by assuming linear propagation and small viscosity. Nevertheless, a simple algebraic manipulation of Eq. (2) allows us to rederive them easily. Let us define $\theta \equiv -i \frac{\omega}{\nu k^2}$ and $\tilde{\theta} \equiv -i \frac{\tilde{\omega}}{\nu k^2}$ (related to the Reynolds number defined for water waves), with $\tilde{\omega}(k) \equiv \sqrt{|k|(g + s k^2)}$ (i.e., the inviscid dispersion relation). Equation (2) is written compactly as

$$\left(\frac{2 + \theta}{\tilde{\theta}} \right)^2 - 1 = \frac{4}{\tilde{\theta}^{\frac{3}{2}}} \left(\frac{1 + \theta}{\tilde{\theta}} \right)^{\frac{1}{2}}. \quad (3)$$

Notice that for $\nu k^2 \ll 1$, $\tilde{\theta} \gg 1$. Moreover, for small k it can be safely assumed that the quantity between parentheses in the left-hand side (LHS) is of order one, i.e., $\theta \approx \tilde{\theta}$, while the right-hand side (RHS) can be considered of higher order, by virtue of its prefactor. The standard dispersion relation is indeed obtained by neglecting the RHS of Eq. (3): We write $\theta = \pm \tilde{\theta} - 2$, i.e.,

$$\omega = \pm \tilde{\omega} - 2i \nu k^2, \quad (4)$$

which we refer to as (small- k) Lamb approximation [10]. It was shown that its physical justification can be traced back to the smallness of the vortical contribution to pressure and the fact that the surface deformation and boundary layer (where the vorticity is non-negligible) are small [15,22]. However, the viscosity enters only in the imaginary part, so no cutoff for traveling waves can possibly appear. Lamb derived also the two branches of the damping rate in the opposite case of small $\tilde{\theta}$, which read respectively $\omega_I = -\frac{\tilde{\omega}^2}{2\nu k^2}$ and $\omega_I = -0.91 \nu k^2$, the former being the most physically important. The cutoff is estimated from Eq. (3), by looking for a real solution θ of double multiplicity. We solve the system composed by Eq. (3) and its derivative with respect to θ . We obtain $\theta = \beta$, with β the real root of $\beta^3 + 5\beta^2 + 8\beta + 3 = 0$. Thus, θ is eliminated and k_c is the solution of $\tilde{\omega}^2(k_c) = (4/\alpha - \alpha^2)(\nu k_c^2)^2 \approx 0.58(\nu k_c^2)^2$, where $\alpha \equiv \beta + 2$.

TABLE I. Comparison of the asymptotic behavior of the implicit and approximated dispersion relations. The classical estimations of Lamb on the implicit formula, the DM, VPF, and MVPF, are simply derived in the text from the full dispersion relation; SDM is the simplified DM found by neglecting the viscous terms under the square root in Eq. (9). The first two are also discussed in Ref. [18].

	$\omega_R(k \ll 1)$	$\omega_I(k \ll 1)$	Cutoff	$\omega_I(k \gg 1)$
Implicit (3)	$\tilde{\omega}$	$-2\nu k^2$	$\tilde{\omega}^2 = 0.58(\nu k^2)^2$	$-\frac{\tilde{\omega}^2}{2\nu k^2}$ $-0.91\nu k^2$
DM (5)	$\tilde{\omega}$	$-2\nu k^2$	$\tilde{\omega}^2 = 4(\nu k^2)^2$	$-\frac{\tilde{\omega}^2}{4\nu k^2}$ $-4\nu k^2$
VPF (6)	$\tilde{\omega}$	$-\nu k^2$	$\tilde{\omega}^2 = (\nu k^2)^2$	$-\frac{\tilde{\omega}^2}{2\nu k^2}$ $-2\nu k^2$
MVPF (7)	$\sqrt{\frac{3}{2}}\tilde{\omega}$	$-\frac{2\nu k^2}{3}$	$\tilde{\omega}^2 = \frac{2}{3}(\nu k^2)^2$	$-\frac{\tilde{\omega}^2}{2\nu k^2}$ $-\frac{4}{3}\nu k^2$
SDM (10)	$\tilde{\omega}$	$-2\nu k^2$	n/a	$-\frac{\tilde{\omega}^2}{2\nu k^2}$

In order to have a single expression for small and large k , different approximations of the RHS of Eq. (2) are shown to behave better than Eq. (4). Indeed, if we let $1 + \theta \approx 1$ on the RHS of Eq. (3), we reobtain the result of the dissipation method (DM), [18,26], i.e.,

$$\theta = -2 \pm \sqrt{\tilde{\theta}^2 + 4}. \quad (5)$$

In contrast to Eq. (4), this relation exhibits a cutoff in the real part of ω , for $\tilde{\theta} = -2i$. Beyond, the two signs represent the two branches of dissipation of the standing mode.

Further, we notice that in the alternative approach proposed in Ref. [16] (viscous potential flow, VPF), only the dynamic boundary condition (Bernoulli equation) is modified, by introducing viscosity as an external pressure perturbation, analogously to Ref. [21]. It can be reproduced by the expansion $(1 + \theta)^{\frac{1}{2}} \approx 1 + \theta/2$ and its solution is written, in our notation,

$$\theta = -1 \pm \sqrt{\tilde{\theta}^2 + 1}. \quad (6)$$

Finally, we may ask ourselves what is the result of expanding the RHS of Eq. (3) to the second order: We write a third approximation,

$$\theta = -\frac{2}{3} \pm \frac{2}{3}\sqrt{\frac{3}{2}\tilde{\theta}^2 + 1}, \quad (7)$$

which we will refer to as modified VPF (MVPF).

The results are summarized in Table I. Among the dispersion relations (5)–(7), only DM corresponds asymptotically to the Lamb solution as far as damping at small k is concerned; k_c is not accurate, though. The real part of the VPF is a better approximation at the cutoff and matches very well with the conventional dispersion relation, but the damping is half of the expected one at small k , while standing waves ($k > k_c$) of lesser damping have the correct asymptotic form (compared to the estimation made by Lamb; see above). Finally, the MVPF behaves better at the cutoff but does not reproduce the behavior of either ω_R or ω_I for small values of k , so it is of little practical use in the most accessible oceanic regimes. In Ref. [17], it was shown that irrotational theories fail to provide a good approximation around k_c . The MVPF shows instead that a good approximation of the cutoff is incompatible to a satisfactory asymptotic behavior at small and large k simultaneously.

The numerical values of cutoff for the fluids considered in Ref. [18] are reported in Table II, where we include also the MVPF results, for the sake of completeness. In order to visually confirm

TABLE II. Cutoff values (in m^{-1}) for the three examples of Ref. [18] and their comparison to the MVPF result. SO10000 stands for silicone oil of viscosity 10000 cSt, for which in SI units $\rho_f = 9.69 \times 10^2 \text{ kg m}^{-3}$, $\nu = 1.02 \times 10^{-2} \text{ m}^2 \text{ s}^{-1}$, $s = 2.10 \times 10^{-5} \text{ m}^3 \text{ s}^{-2}$.

	Water	Glycerin	SO10000
Implicit (3)	1.25×10^8	445.18	54.30
DM (5)	1.82×10^7	196.81	28.50
VPF (6)	7.28×10^7	344.64	45.29
MVPF (7)	1.09×10^8	416.10	51.86

the formulas of Table I, we compare the different approximations in Fig. 2 for a surface between glycerin and air ($\rho_f = 1.26 \times 10^3 \text{ kg m}^{-3}$, $\nu = 6.21 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$, and $s = 5.03 \times 10^{-5} \text{ m}^3 \text{ s}^{-2}$). The larger viscosity allows us to have a cutoff for relatively small k and observe both short and long wave ranges in linear scale. The same behavior applies to other fluids.

We include also a further simplification. Suppose we choose the DM relation of Eq. (5), which provides the best approximation of damping and group velocity for long waves. We rewrite it as

$$\omega(k) = \frac{\tilde{\omega}}{\hat{D}(k)} \quad (8)$$

with

$$\hat{D}(k) \equiv \pm \sqrt{1 - \left(\frac{2\nu k^2}{\tilde{\omega}} \right)^2} + \frac{2i\nu k^2}{\tilde{\omega}}. \quad (9)$$

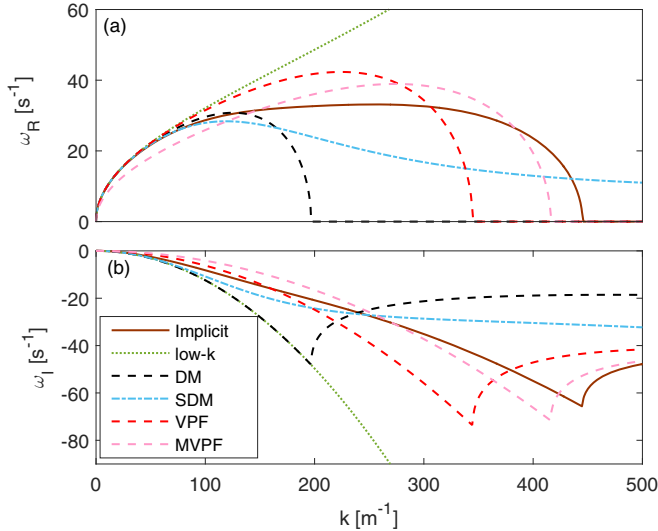


FIG. 2. Dispersion relation for gravity-capillary waves propagating at the interface of air and glycerin. (a) Real part; (b) imaginary part. Numerical solutions of Eq. (2) for forward-propagating waves ($\omega_R \geq 0$) are shown as a solid dark yellow line. The conventional approximation is shown by a green dotted line. The different approximations are represented by dashed lines (black, DM; red, VPF; pink, MVPF). Finally, the dash-dotted lines correspond to the SDM: It does not exhibit a clear cutoff, but provides a good approximation of damping at both k limits.

For small νk^2 , we can neglect the term proportional to ν^2 under the square root in Eq. (9) and obtain the simplified DM (SDM),

$$\omega \approx \frac{\tilde{\omega}}{\pm 1 + \frac{2i\nu k^2}{\tilde{\omega}}}. \quad (10)$$

It is trivial to verify—see also Table I—that Eq. (10) provides a better asymptotic behavior than Eq. (5) for damping at large k , at the expense of a smooth transition around the cutoff, as shown in Fig. 2 as a cyan dash-dotted line. That is, the series expansion of $\tilde{\omega}/\omega$ provides a more robust approximation than the conventional direct Taylor expansion of ω (i.e., the low- k Lamb approach, green dotted lines in Fig. 2), which predicts ever-increasing phase velocity $v_p \equiv \omega_R/k$ and damping. This is similar to the fit of a dispersion relation by means of a Padé approximant, well known in optics [33].

III. HYDRODYNAMIC EQUATIONS

Viscosity induces vorticity, thus making the solution of the nonlinear hydrodynamic problem (NS equations) extremely complicated. The most practical workaround is to extend the use of a velocity potential $\phi(x, z, t)$, where z is the depth coordinate and x is the longitudinal propagation direction, to the viscous case.

By denoting the free surface elevation $\eta(x, t)$, the system of hydrodynamic equations in an inviscid and infinitely deep fluid reads as

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 & \text{for } -\infty < z < \eta, \\ \nabla \phi &\rightarrow 0 & \text{for } z \rightarrow -\infty, \\ \eta_t + \phi_x \eta_x - \phi_z &= 0 & \text{for } z = \eta, \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta - s \frac{\eta_{xx}}{(1 + \eta_x^2)^{\frac{3}{2}}} &= 0 & \text{for } z = \eta. \end{aligned} \quad (11)$$

These are respectively the Laplace equation in the fluid, the rigid bottom condition, and the kinematic and the dynamic boundary conditions at the free surface. In the linear limit, the solutions of Eq. (11) are plane waves, $\eta = \eta_0 \exp(ikx - i\omega t)$ and $\phi = \phi_0 \exp(ikx - i\omega t + |k|z)$. The inviscid dispersion relation $\omega = \tilde{\omega}(k)$ is the compatibility condition of the homogeneous system

$$\begin{bmatrix} i\omega & |k| \\ -(g + sk^2) & i\omega \end{bmatrix} \begin{bmatrix} \eta_0 \\ \phi_0 \end{bmatrix} = 0. \quad (12)$$

Equation (4) is obtained by the substitution $i\omega \rightarrow i\omega - 2\nu k^2$ in Eq. (12). In the wave-number domain, we obtain a correction $-2\nu k^2 \eta_0$ to the kinematic boundary condition and $-2\nu k^2 \phi_0$ to the dynamic one. To transform them back to the spatial domain, we notice that η does not depend on z and ϕ is the solution of the Laplace equation. The operator correspondence $ik \leftrightarrow \frac{\partial}{\partial x}$ (established by the plane-wave definition) is correct for both terms and allows us to obtain the well-known weakly viscous hydrodynamic system [15,22]:

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 & \text{for } -\infty < z < \eta, \\ \nabla \phi &\rightarrow 0 & \text{for } z \rightarrow -\infty, \\ \eta_t + \phi_x \eta_x - \phi_z &= 2\nu \eta_{xx} & \text{for } z = \eta, \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta - s \frac{\eta_{xx}}{(1 + \eta_x^2)^{\frac{3}{2}}} &= 2\nu \phi_{xx} = -2\nu \phi_{zz} & \text{for } z = \eta. \end{aligned} \quad (13)$$

Its physical motivation is that the velocity field can be decomposed in a potential and a vorticity contribution, $(u, w) = (\phi_x - \Omega_z^y, \phi_z + \Omega_x^y)$. The vorticity pseudovector has only the y component,

$\Omega \equiv (0, \Omega^y, 0)$, and is expressed as a function of ϕ and η by using the linearized boundary conditions and by assuming $\Omega_z^y \approx 0$ [15,22]. Strictly speaking, neglecting this term violates the conservation of mass. This will be discussed in details in a future publication. The fully nonlinear NS equations couple different velocity components and is hard to write in as simple terms as Eqs. (11) and (13).

We propose a solution to this difficulty based on the approximation presented in the previous section. The principle behind our choice is that, in Eq. (11) as well as in the unidirectional models derived from it (e.g., the NLS or the Dysthe equation [6]), the energy is the sum of a kinetic part, which depends on dispersion, and a potential part, which is ascribed to nonlinear interaction. A periodic exchange between the two parts characterizes, e.g., the nonlinear stage of MI. That both are damped by kinematic viscosity is thus plausible. Since the Hamiltonian density which encompasses the different energy terms is associated to the evolution in time, the associated operator must be modified as a whole. This allows us to better quantify the role of dissipation in the nonlinear propagation of waves. Consider Eq. (8): It is the solution of

$$\begin{vmatrix} i\omega\hat{D}(k) & |k| \\ -(g + sk^2) & i\omega\hat{D}(k) \end{vmatrix} = 0.$$

Thus, in Eq. (11), we make the substitution $\frac{\partial}{\partial t} \rightarrow \bar{\partial}_t \equiv \mathcal{D}(-i\frac{\partial}{\partial x})\frac{\partial}{\partial t}$, where $\mathcal{D}(-i\frac{\partial}{\partial x})$ is the operator in the physical space associated in the Fourier space to $\hat{D}(k)$ in Eq. (9).

This allows us to formally obtain an alternative form for the hydrodynamic equations:

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 & \text{for } -\infty < z < \eta, \\ \nabla\phi &\rightarrow 0 & \text{for } z \rightarrow -\infty, \\ \bar{\partial}_t\eta + \phi_x\eta_x - \phi_z &= 0 & \text{for } z = \eta, \\ \bar{\partial}_t\phi + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta - s\frac{\eta_{xx}}{(1 + \eta_x^2)^{\frac{3}{2}}} &= 0 & \text{for } z = \eta. \end{aligned} \quad (14)$$

The proposed system reduces to Eq. (13) for $\nu k^2 \ll \tilde{\omega}$, by replacing $-1/i\tilde{\omega} \leftrightarrow \int dt$ and $-k^2 \leftrightarrow \frac{\partial^2}{\partial x^2}$.

Equation (14) provides, however, an alternative set of equations to be employed in full hydrodynamic solvers, like the high-order spectral method (HOSM) [19,20,34,35].

We stress that the choice of DM or SDM is not binding. A similar approach could be used with the other approximations discussed in Sec. II, the numerical solution of Eq. (2), as well as a different fit based on experimental data.

In the next section, we will discuss how to derive a nonlinear NLS-like propagation equation and show that small albeit measurable differences between the linear and nonlinear dissipation terms exist.

IV. A SPACELIKE PROPAGATION EQUATION

In order to generalize the MNLS of Eq. (1) to the dissipative case, we can follow the approach of Ref. [3], based on Zakharov's method [2], to provide the justification for an NLS in which a general dispersion relation and not a polynomial truncation thereof is included. The free-surface elevation is reconstructed from the envelope $B(x, t)$ as $\eta = \frac{1}{2}[b(x, t) + \text{c.c.}]$, where $b \equiv Be^{-i(k_0x - \omega_0t)}$ is the free-wave component and c.c. denotes the complex conjugate. Notice that the plane wave factors are defined to be real: $\omega_0 \equiv \omega_R(k_0)$.

Equation (1), where the nonlinearity is assumed small and viscosity is neglected ($\omega_R = \tilde{\omega}$), is written for $\hat{b}(k, t) \equiv \mathcal{F}_k[b(x, t)]$ as

$$\hat{b}_t + i\tilde{\omega}(k)\hat{b} + i\frac{\omega_0 k_0^2}{2}\mathcal{F}_k[|b|^2b] = 0 \quad (15)$$

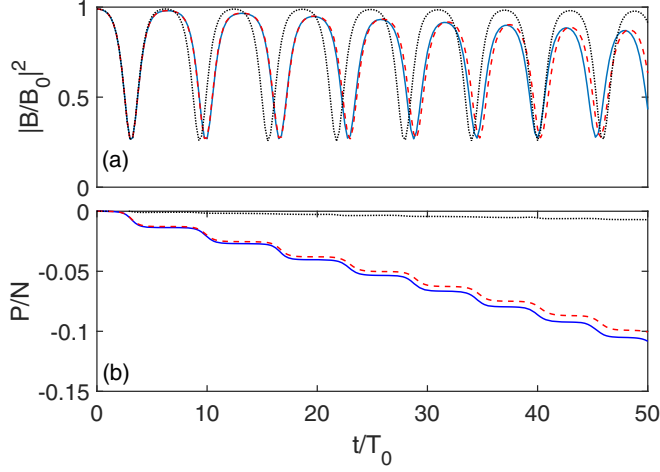


FIG. 3. Comparison of the nonlinear evolution of MI with linear and nonlinear damping. Time is normalized by nonlinear time $T_0 = (2\varepsilon^2 k_0)^{-1}$. $k_0 = 10$, $\varepsilon = 0.1$, $\nu = 1 \times 10^{-6}$, $\alpha = 0.01$, and surface tension is neglected. (a) Oscillations of the central peak (Stokes wave); (b) spectral mean κ_m . The solid blue lines represent the full model [Eq. (17)], and the dashed red (resp., black dotted) lines are obtained by neglecting $\hat{D}(k)$ in the denominators of the nonlinear (resp., linear) part.

from which, by means of the substitution $\partial_t \rightarrow \bar{\partial}_t$, we derive the equation of motion for

$$\frac{\partial \hat{b}}{\partial t} + i \frac{\tilde{\omega}(k)}{\hat{D}(k)} \hat{b} + i \frac{\omega_0 k_0^2}{2 \hat{D}(k)} \mathcal{F}_k[|b|^2 b] = 0. \quad (16)$$

Back to the slowly varying variable B , we can write, for $\hat{B}(\kappa, t) = \mathcal{F}_\kappa[B(x, t)]$,

$$\frac{\partial \hat{B}}{\partial t} + i \left[\frac{\tilde{\omega}(k_0 + \kappa)}{\hat{D}(k_0 + \kappa)} - \omega_0 \right] \hat{B} + i \frac{\omega_0 k_0^2}{2 \hat{D}(k_0 + \kappa)} \mathcal{F}_\kappa[|B|^2 B] = 0. \quad (17)$$

This represents our generalization of Eq. (1) in the dissipative case.

By Taylor expanding it to fourth order in κ and writing the resulting terms in the physical space by replacing powers of κ by derivatives in x , we obtain the linear terms of Ref. [36]. This approach may prove convenient also for generalizing a forced MNLS model [37].

Below, in Figs. 3 and 4, we show that the dispersive contribution explains alone most of the frequency downshift observed in experiments. Our approach provides also a nonlinear viscous damping (the imaginary part of the nonlinearity), valid for small k where the bound modes (small corrections to η oscillating at integer multiples of k_0 and enslaved, for pure gravity waves, to the free modes B) are not resonantly excited. It also introduces a wave-number-dependent correction to nonlinearity (proportional to ν^2 under the square root in $\hat{D}$; see Eq. (8). We expand the nonlinear damping coefficients as $\frac{2\nu k^2}{\omega} \frac{\omega_0 k_0^2}{2} \approx \nu k_0^3 [k_0 + (2 - \frac{v_g^0}{v_p^0})\kappa]$, where $v_p^0 \equiv \tilde{\omega}(k_0)/k_0$ and $v_g^0 \equiv \tilde{\omega}'(k_0)$ are the phase and group velocities at k_0 , respectively, neglecting viscosity. The first contribution is a homogeneous nonlinear damping (which can be obtained independently by the method of multiple scales [38]), while the second is a derivative damping term, i.e., the dissipative counterpart of the self-steepening in nonlinear optics [39]. Its effect is small, because energy dissipation caused by linear attenuation limits the bandwidth of B .

As an example, we show in Fig. 3 the solution for a harmonically perturbed Stokes wave propagating at the water-air interface ($B(x, 0) = B_0[\sqrt{1 - \alpha} + \sqrt{2\alpha} \cos \kappa_0 x]$), with initial steepness $\varepsilon \equiv \frac{B_0 k_0}{\sqrt{2}} = 0.1$, $k_0 = 10$, $\alpha = 1 \times 10^{-2}$, and neglecting surface tension. The perturbation wave number is $\kappa_0 = 2\varepsilon k_0 \sqrt{2B_0}$, i.e., the maximally unstable mode predicted by the NLS. We notice that

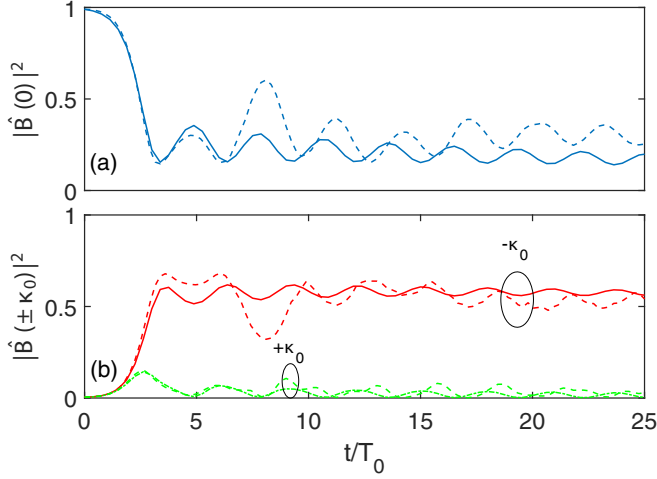


FIG. 4. Comparison of results with purely linear (solid lines) and purely nonlinear (dashed lines) damping. Parameters are $k_0 = 75$, $\varepsilon = 0.1$, $\nu = 1 \times 10^{-6}$, $s = 7.28 \times 10^{-5}$, and $\alpha = 0.01$. (a) Central mode $\kappa = 0$; (b) main unstable modes $\kappa = \pm\kappa_0$. We notice that the downshifted mode at $-\kappa_0$ soon acquires most of the energy. While the linear damping irreversibly stops recurrence, the nonlinear damping is not sufficient.

the linear and nonlinear dissipation scale as $2\nu k_0^2$ and $2\nu k_0^4 |B|^2 \approx 2\nu k_0^2 \varepsilon^2$, respectively. We thus expect that only at extreme steepness and breather peaks is the nonlinear damping non-negligible with respect to the linear one.

In Fig. 3, we compare the dynamics of Eq. (17) (full model) with the results of neglecting $\hat{D}(k)$ in either the linear or nonlinear part (we refer to them as nonlinear and linear damping, respectively). The comparison shows both the energy attenuation and downshift of the spectral mean $\kappa_m \equiv P/N$, with

$$N = \int dx |B|^2, \quad P = \text{Im} \int dx B_x B^*, \quad (18)$$

respectively, the norm and the momentum of the field. κ_m depends mainly on the linear damping; see the Appendix. As shown in Ref. [40], the slightest amount of dissipation causes the recurrence period to stretch and a period-1 orbit to be attracted to a period-2 one.

This behavior is found in results of the full model (blue solid lines) and in the linearly damped (dashed red lines) simulations. The discrepancy in recurrence periods is just a contraction of about 1% (per period). In Fig. 3(a), we notice also that the nonlinear damping alone (black dotted line) leads instead to a behavior more similar to the undamped result. As far as the downshift of κ_m is concerned [Fig. 3(b)], the full model exhibits about 1% more shift than the linearly damped one. Notice that the difference between the two nearly equals the pure nonlinear contribution (black dotted line).

The blue solid lines exhibits the same behavior if Eq. (10) is used instead of Eq. (8) (not shown). The discrepancies between the solid and dashed lines, observed in Fig. 3, are mainly explained by the nonlinear damping $\nu k_0^4 |B|^2 B$: Viscous corrections to group velocity have an even smaller impact on nonlinear coefficients.

This behavior does not qualitatively change if we include Dysthe nonlinear terms in Eq. (17), as in, e.g., Ref. [4]: The downshift is associated to a transient upshift at each recurrence cycle [40].

A different dynamics is observed for shorter wavelengths even though stronger damping limits the impact of nonlinear phenomena. We thus simulated the same initial conditions as above, with $k_0 = 75$, i.e., near the transition from gravity to capillary waves. We show in Fig. 4 the results of the purely linear and purely nonlinear dissipation models. In Fig. 4(a), we plot the central mode ($\kappa = 0$),

and in Fig. 4(b), we plot the two main unstable modes at each side of it ($\kappa = \pm\kappa_0$). Even during the initial MI phase the energy is converted into a pair of unstable sidebands in an asymmetric fashion, which favors the lower wave number. As soon as most of the energy is located at $-\kappa_0$ (at $t/T_0 \approx 2.5$), we observe a remarkable difference: The central mode cannot recover, not even partially, its initial condition and stabilizes below 0.5 for pure linear damping (solid lines), while a regime of more erratic recurrence, in which every five to six cycles the energy is transferred back to the central mode (above 0.6), is observed for pure nonlinear damping (dashed lines). The full model (with linear and nonlinear damping, not shown) exhibits almost the same behavior as the purely linear case. This is consistent with the interplay of surface tension and viscosity to favor a permanent downshift of the spectral peak, observed in several full numerical simulations of the Euler system [41,42].

We finally remark that the present nonlinear viscous damping does not exclude the existence of other nonlinear loss mechanisms. We would like to mention the Landau damping [23], which plays a role at large k , where the bound modes are resonantly excited and dissipated by viscous damping. Its even symmetry prevents, however, a downshift of either the spectral peak or mean. Alternatively, a loss mechanism like the β term introduced in Refs. [24,25] explains the frequency downshift observed in the nonlinear stage of MI, by virtue of its odd symmetry, but has no clear physical origin: It may represent a model of wave breaking, not included here. Nevertheless, we have shown that the spectral mean shifts permanently simply due to the variation of damping with frequency.

Both examples, Figs. 3 and 4, show that an MNLS where the full dispersion (with dissipation) included in the linear part is sufficient to explain at least qualitatively the partial recurrence and spectral downshift in the nonlinear behavior of MI. Thus the implicit assumption of Ref. [22] that a nonlinear correction due to viscosity plays a minor role in deep-water wave propagation is confirmed in its physical soundness. In the Appendix, we derive the expressions for the rate of change of the spectral mean and motivate more rigorously the downshifting trend.

V. A TIMELIKE PROPAGATION EQUATION

Laboratory conditions usually imply the measurement of the temporal profiles at different positions along a wave tank by means of wave gauges. The timelike formulation of the NLS and its generalizations are thus more practical than their spacelike counterparts for describing and interpreting measurements.

In order to obtain such a formulation, we need to invert the dispersion relation and derive an explicit expression for $k(\omega)$. The method we used above to derive the DM approximation provides a straightforward solution, at least for $s = 0$, that reads

$$k(\omega) = \frac{\tilde{k}}{\hat{D}_k(\omega)} \quad (19)$$

with $\tilde{k}(\omega) \equiv \frac{\omega^2}{g}$, the conventional dispersion of deep-water gravity waves, and $\hat{D}_k(\omega) \equiv \sqrt{1 + 16i \frac{\nu \omega^3}{g^2}}$, the correction due to viscosity, which contributes to both the real and imaginary parts of $k(\omega)$. Equation (19) can be simplified as $k(\omega) \approx \frac{\omega^2}{g} [1 - 4i \frac{\nu \omega^3}{g^2}]^{-1}$.

As shown in Sec. IV, the nonlinear damping is negligible in most cases. We write the dissipative timelike MNLS in the frequency domain—with respect to the detuning $\Omega \equiv \omega - \omega_0$ (here $\hat{B}(x, \Omega) \equiv \mathcal{F}_\Omega[B(x, t)]$)—as

$$\frac{\partial \hat{B}}{\partial x} - i \left[\frac{\tilde{k}(\omega_0 + \Omega)}{\hat{D}_k(\omega_0 + \Omega)} - k_0 \right] \hat{B} - i \frac{k_0^3}{2} \mathcal{F}_\Omega[|B|^2 B] = 0. \quad (20)$$

This model can be applied to assess the effects of the dispersive damping in a wave-tank experiment.

VI. CONCLUSIONS

After having recalled the different forms of dispersion relation for deep-water gravity-capillary waves in the presence of viscosity and unified their derivation, we discussed their physical validity all over the wavenumber and frequency ranges. We exploited these results to reformulate the hydrodynamic equations and quantify the impact of kinematic viscosity on nonlinear damping. We showed that the simplest NLS model with full dispersion (in both the real and imaginary parts) provides most of the justification for the downshift of the spectral mean during the nonlinear stage of evolution of the MI: Corrections in the nonlinear behavior have only a small effect. This provides an *a posteriori* justification of the choice of using dispersion relations only in the linear part of a nonlinear propagation equation [3,22]. However, this does not forbid consideration of other damping mechanisms, e.g., wave breaking. Experiments are needed to determine the best form for a rational dispersion relation, which could be used even in fully nonlinear hydrodynamic simulations.

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APPENDIX: DOWNSHIFT OF THE SPECTRAL MEAN

In this Appendix, we will derive from Eq. (17) the rate of change of N and P defined above in the main text, Eq. (18), and of the spectral mean P/N .

We consider only the linear term of Eq. (17),

$$\frac{\partial \hat{B}}{\partial t} + i \left[\frac{\tilde{\omega}(k_0 + \kappa)}{\hat{D}(k_0 + \kappa)} - \omega_0 \right] \hat{B} = 0. \quad (\text{A1})$$

First, we rewrite Eq. (18) as

$$N = \frac{1}{2\pi} \int d\kappa |\hat{B}|^2, \quad P = \frac{1}{2\pi} \int d\kappa \kappa |\hat{B}|^2, \quad (\text{A2})$$

the first by Parseval theorem and the second by simple manipulations of Fourier transforms.

From Eqs. (A1) and (A2), by elementary substitution, we derive

$$\frac{dN}{dt} = \chi \int d\kappa (k_0 + \kappa)^2 |\hat{B}|^2, \quad \frac{dP}{dt} = \chi \int d\kappa (k_0 + \kappa)^2 \kappa |\hat{B}|^2, \quad (\text{A3})$$

with $\chi = -\frac{2\nu}{\pi}$.

Now, expanding each integral of Eq. (A3), we write

$$\frac{d\kappa_m}{dt} = -\frac{4\nu}{N^2} [2k_0(QN - P^2) + (SN - PQ)], \quad (\text{A4})$$

where

$$Q = \frac{1}{2\pi} \int d\kappa \kappa^2 |\hat{B}|^2 = \int dx |B_x|^2, \quad (\text{A5})$$

$$S = \frac{1}{2\pi} \int d\kappa \kappa^3 |\hat{B}|^2 = \frac{1}{2i} \int dx [B_x^* B_{xx} - \text{c.c.}]. \quad (\text{A6})$$

Notice that Q corresponds to the same quantity defined in Ref. [36].

We find that $QN - P^2 \geq 0$ directly from Cauchy-Schwarz inequality, $|\langle \hat{f}, \hat{g} \rangle| \leq \|f\| \|g\|$, with the two square-integrable functions $\hat{f}(\kappa) \equiv \hat{B}(\kappa)$ and $\hat{g}(\kappa) \equiv \kappa^{\frac{1}{2}} \hat{B}(\kappa)$.

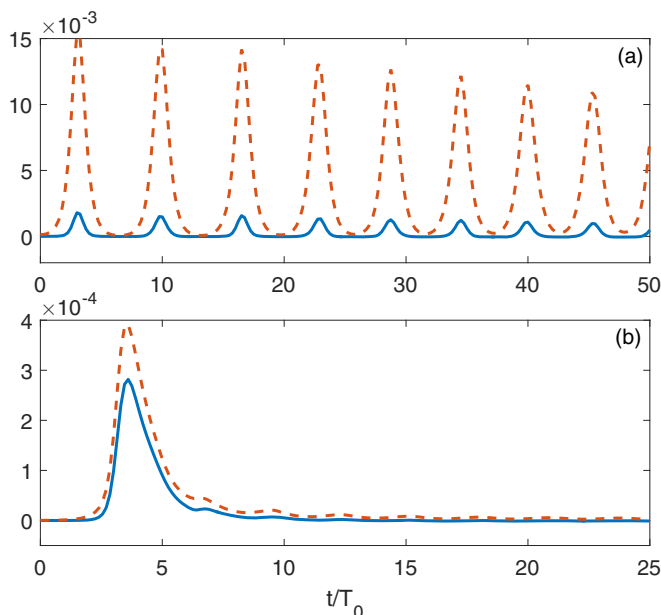


FIG. 5. Evolution in time of $SN - PQ$ (blue solid line) and $2k_0(QN - P^2) + (SN - PQ)$ (red dashed line) for surface waves at the air-water interface. Panel (a) [resp., (b)] corresponds to the gravity (resp., gravity-capillary) case of Fig. 3 [resp., Fig. 4] in the main text. While $SN - PQ$ is not always rigorously positive, so it is in most of the situations and the net mean-frequency shift, as predicted by Eq. (A4) is always positive.

As far as $SN - PQ$ is concerned, nothing more can be stated mathematically. Numerical simulations show that its sign is positive during most of the evolution of surface waves at the air-water interface. This is displayed as solid blue lines in Fig. 5, where we consider the examples of Figs. 3 and 4. Moreover, the overall sign of the terms in square brackets of Eq. (A4) is positive in both examples (red dashed lines in Fig. 5); thus, according to Eq. (A4), the mean frequency shifts downward, for both gravity and gravity-capillary regimes. These results hold also for larger values of steepness up to $\varepsilon \approx 0.3$, for which the use of NLS-like nonlinearity becomes questionable, and for smaller k_0 (now shown).

As far as the nonlinear terms are concerned, no rigorous explicit relation can be derived, but numerical simulations hint at their contribution to *increase* the said downshift.

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