Effect of shear thinning on superhydrophobic slip: Perturbative corrections to the effective slip length

Darren Crowdy*
Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom
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Analytical expressions are derived for the first-order correction to the effective slip length of a weakly shear-thinning Carreau-Yasuda fluid in both longitudinal and transverse semi-infinite shear flow over a unidirectional superhydrophobic surface of flat no-shear slots. The formulas, which are derived using suitably generalized forms of the standard reciprocal theorem for Stokes flow, are given by explicit integrals which require only numerical quadrature for their evaluation. For both longitudinal and transverse flow we find that for a given no-shear fraction of the superhydrophobic surface and a given power-law index characterizing the Carreau-Yasuda fluid, there is a critical imposed strain rate of the shear at which the enhancement of effective slip is maximal. The theoretical results are qualitatively consistent with recent numerical work by other authors for the transverse case.

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I. INTRODUCTION

The study of complex non-Newtonian liquids flowing over superhydrophobic surfaces has received little attention in the literature, at least compared to the Newtonian fluid case [1–3], even though many real fluids used in microfluidics applications have significant shear-thinning or viscoelastic properties. Many diagnostic lab-on-a-chip applications [4], for example, involve the manipulation of blood or other biological samples whose dynamics are often far from that of a Newtonian fluid [5]. Assessing the effect on the hydrodynamic slip of these more general non-Newtonian flows over superhydrophobic surfaces is therefore important, especially since shear-thinning fluids might afford even greater slip advantages than their Newtonian counterparts because the high variability of the shear rates close to highly heterogeneous surface patterning will reduce the local fluid viscosity there. This may facilitate higher flow rates for a given pressure gradient or driving shear stress.

There has been recent work in this direction. Vagner and Patlazhan [6] have performed a numerical study of the motion of Carreau-Yasuda power-law fluids [7] in a microchannel with a striped superhydrophobic wall. Those authors did not focus on the friction-reducing properties of the surfaces but on the modified helical streamline structure that determines the mixing efficiency in such channel flows. In a similar vein, Pereira [8] performed a perturbation analysis, within a continuum theory model, to study non-Newtonian effects on mixing efficiency and found that it is enhanced for shear-thickening fluids and suppressed for shear-thinning fluids. Dhondi *et al.* [9] used molecular dynamics simulations to examine the mixing efficiency of polymeric fluids over patterned substrates and the slip length associated with such flows. Haase *et al.* [10] have carried out a numerical study of the slip properties of transverse pressure-driven flow of the class of Carreau-Yasuda fluids in a channel whose lower wall comprises a unidirectional bubble mattress of protruding bubbles [10]. They find that, in the range of flow rates at which shear-thinning is significant, slip lengths can increase more than threefold; they also find that the critical protrusion angle at which the obstruction formed by the protruding bubble negates any slip advantage afforded by the shear-free bubbles is decreased relative to the Newtonian case [11,12].

*d.crowdy@imperial.ac.uk

The present theoretical paper complements this recent work by deriving closed-form integral expressions for the first-order slip length correction for weakly nonlinear Carreau-Yasuda fluids for both transverse and longitudinal flow over a unidirectional superhydrophobic surface of periodic flat no-shear slots. In the longitudinal flow scenario the flow direction is now parallel to the unidirectional grooves over which the menisci are suspended. The flows are weakly nonlinear in the sense that the viscosity contrast in the zero- and infinite-shear-rate limits of the Carreau-Yasuda generalized Newtonian fluid model [7] is taken to be small.

This work explores the effect on longitudinal slip of shear thinning in the working fluid. The numerical study of Haase *et al.* [10] treats only the transverse flow scenario and focuses on the particular no-shear fraction of 0.5 for menisci with differing protrusion angles. The present work, while restricted to weakly shear-thinning fluids and flat menisci, has the advantage of providing explicit integral expressions for the slip length modification for both the transverse and longitudinal flows, thereby allowing ready investigation into the effects of changing the geometry of the surface (the no-shear fraction), the Carreau number (which is a ratio of the imposed strain rate of the shear and the crossover strain rate defined by the relaxation time of the non-Newtonian fluid), and the index n of the shear-thinning working fluid. The formulas are of theoretical interest and provide valuable quantitative checks on numerical approaches to the fully nonlinear problem.

The integral formulas are derived by exploiting generalized reciprocal theorems relevant to the low-Reynolds-number flows of weakly shear-thinning fluids in this setting. Similar ideas have been well explored in the context of particle motion in Stokes flows [13,14] and in understanding the dynamics of swimming microorganisms in these complex fluid environments [15–17]. In recent work [18], the present author has demonstrated the advantages of combining perturbation theory with integral relations derived from reciprocal theorems to study superhydrophobic surfaces. For the longitudinal flow problem along unidirectional surfaces he has derived explicit formulas for the first-order slip length corrections due to weak meniscus curvature and the effect of a subphase fluid (of both small and large viscosity contrast with the upper working fluid). Such integral formulas for the slip length correction due to weak meniscus curvature had been missed by previous authors [19] who derived the same results by direct methods involving the numerical solution to an infinite system of dual series equations for the first-order flow correction. Squires [20] employed reciprocal theorems to examine electrokinetic effects on flat slipping surfaces; Baier *et al.* [21] used them to study thermal Marangoni flow over a superhydrophobic array of fins oriented parallel or perpendicular to an applied temperature gradient. This paper generalizes these reciprocal theorem ideas and shows how to apply them to study the flow of shear-thinning fluids over superhydrophobic surfaces.

II. NEWTONIAN FLOW SOLUTIONS

Philip [22] found solutions to the problems of both longitudinal and transverse semi-infinite shear flow of a Newtonian fluid over a periodic array of flat no-shear slots in a no-slip surface. Figure 1 shows a schematic of a single period window D of the two flow scenarios. A no-shear slot of width $2b$ sits in each period window of width $2a$ with $b < a$; in between the no-shear slots the surface is no slip. It has been shown [22,23] that the solution to the longitudinal flow problem with far-field shear rate $\dot{\gamma}$ has the form $\mathbf{u} = (0,0,w(x,y))$ with

$$w(x,y) = \dot{\gamma}\,\mathrm{Im}[h(z)], \tag{1}$$

where $z = x + iy$ (z is not used here to label the axis going into the page) and the transverse problem, with the same far-field shear rate, has the stream function

$$\psi(x,y) = \mathrm{Im}[\overline{z} f(z) + g(z)], \quad z = x + iy, \tag{2}$$

where

$$f(z) = \frac{i\dot{\gamma}}{4}h(z), \quad g(z) = -zf(z) = -\frac{i\dot{\gamma}}{4}zh(z). \tag{3}$$

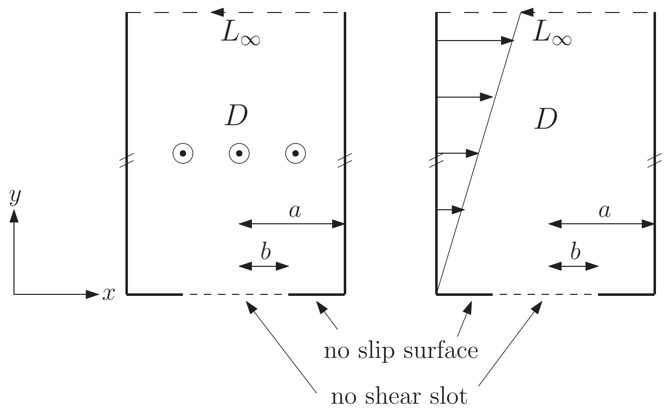

FIG. 1. A single period window D for longitudinal shear (left, flow into the page) and transverse shear (right) over a periodic array of flat no-shear slots in a no-slip surface. The width of the period window is $2a$; the width of each no-shear slot is $2b$. The boundary ∂D of D is closed, as $y \to \infty$, by the line segment L_∞.

The analytic function $h(z)$, which appears in both solutions, is

$$h(z) = \frac{2a}{\pi} \cos^{-1} \left[\frac{\cos\left(\frac{\pi z}{2a}\right)}{\cos\left(\frac{\pi b}{2a}\right)} \right]. \tag{4}$$

As $b \to 0$, $h(z) \to (2a/\pi)z$, which is simple shear over a no-slip surface at $y = 0$. Philip [22] found the longitudinal slip length $\lambda_\parallel$ and transverse slip length $\lambda_\perp$ to be

$$\lambda_\parallel = \frac{2a}{\pi} \ln \sec\left(\frac{\pi b}{2a}\right) = 2\lambda_\perp. \tag{5}$$

We ask how these quantities change when the fluid is weakly shear thinning.

III. GENERALIZED RECIPROCAL THEOREM

The approach is by an extension of the reciprocal theorem of Stokes flow [14]. The three-dimensional Stokes equations are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_j} = 0, \tag{6}$$

where u_i denotes the components of the velocity field and

$$\sigma_{ij} = -p\delta_{ij} + 2\eta e_{ij}, \quad e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]. \tag{7}$$

Here p is the fluid pressure and η is the viscosity. Let two Stokes flows taking place in the same domain D be given by $\{\sigma_{ij}, u_i, \eta\}$ and $\{\sigma'_{ij}, u'_i, \eta'\}$, where the first is a Newtonian flow with viscosity η and the second is the flow of a Carreau-Yasuda shear-thinning fluid of viscosity η'. The generalized viscosity of the Carreau-Yasuda fluid is given as a function of the shear rate as [7]

$$\eta'(\Pi') = \eta_\infty + (\eta_0 - \eta_\infty)(1 + \Lambda^2\Pi'/2)^{(n-1)/2}, \tag{8}$$

where Λ is a typical relaxation time of the non-Newtonian fluid, η_0 and η_∞ denote the zero- and infinite-shear-rate viscosities, respectively, and $\Pi' = \dot{\gamma}'_{ij}\dot{\gamma}'_{ij}$ with

$$\dot{\gamma}'_{ij} = \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}. \tag{9}$$

The notation $\Pi = \dot{\gamma}_{ij}\dot{\gamma}_{ij}$ is used to denote the analogous quantity for the Newtonian flow (the Einstein summation convention is used). It is of interest to study the possible slip advantages associated with any shear thinning of the working fluid, so we focus on the case $n < 1$; smaller values of n correspond to more strongly shear-thinning fluids.

For a longitudinal flow along the grooves, so that $\mathbf{u} = (0,0,w(x,y))$ and $\mathbf{u}' = (0,0,w'(x,y))$, the far-field boundary conditions for simple shear over the surface are

$$w \to \dot{\gamma}(y + \lambda_\parallel), \quad w' \to \dot{\gamma}'(y + \lambda'_\parallel), \tag{10}$$

where $\lambda'_\parallel$ is to be determined. For transverse flow,

$$\mathbf{u} \to \begin{pmatrix} \dot{\gamma}(y+\lambda_\perp) \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}' \to \begin{pmatrix} \dot{\gamma}'(y+\lambda'_\perp) \\ 0 \\ 0 \end{pmatrix}, \tag{11}$$

where $\lambda'_\perp$ is to be found. The far-field strain rates for the Newtonian and non-Newtonian flows are denoted by $\dot{\gamma}$ and $\dot{\gamma}'$, respectively. Later on, in the perturbation analysis, it is assumed that the difference between these two far-field strain rates is either zero or small.

The usual reciprocal theorem for the Stokes flow [14] of a Newtonian fluid must be modified in order to give information concerning the non-Newtonian flows under consideration. This is because it fails to give useful results as a consequence of the unbounded nature of the velocities as $y \to \infty$ and the unboundedness of the shear rates at the edges $x = \pm b$ of the no-shear slot. The following generalized reciprocal theorem is similar in spirit to, but different from, those generalizations used in other contexts such as low-Reynolds-number locomotion [15–17]. To proceed we consider

$$\begin{aligned}
\frac{\partial(\eta u_i \sigma'_{ij})}{\partial x_j} - \frac{\partial(\eta' u'_i \sigma_{ij})}{\partial x_j} &= \eta \frac{\partial u_i}{\partial x_j}\sigma'_{ij} + \eta u_i \frac{\partial \sigma'_{ij}}{\partial x_j} - \frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij} - \eta' \frac{\partial u'_i}{\partial x_j}\sigma_{ij} - \eta' u'_i \frac{\partial \sigma_{ij}}{\partial x_j} \\
&= \eta \frac{\partial u_i}{\partial x_j}\sigma'_{ij} - \eta' \frac{\partial u'_i}{\partial x_j}\sigma_{ij} - \frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij} \\
&= \frac{\eta}{2}\left[\frac{\partial u_i}{\partial x_j}\sigma'_{ij} + \frac{\partial u_j}{\partial x_i}\sigma'_{ji}\right] - \frac{\eta'}{2}\left[\frac{\partial u'_i}{\partial x_j}\sigma_{ij} + \frac{\partial u'_j}{\partial x_i}\sigma_{ji}\right] - \frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij}.
\end{aligned} \tag{12}$$

On use of the symmetry of σ_{ij} and σ'_{ij},

$$\begin{aligned}
\frac{\partial(\eta u_i \sigma'_{ij})}{\partial x_j} - \frac{\partial(\eta' u'_i \sigma_{ij})}{\partial x_j} &= \eta e_{ij}\sigma'_{ij} - \eta' e'_{ij}\sigma_{ij} - \frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij} \\
&= \eta e_{ij}[-p'\delta_{ij} + 2\eta' e'_{ij}] - \eta' e'_{ij}[-p\delta_{ij} + 2\eta e_{ij}] - \frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij} \\
&= -\frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij},
\end{aligned} \tag{13}$$

where the last step follows from the incompressibility of the flows:

$$e_{ii} = \frac{\partial u_i}{\partial x_i} = 0, \quad e'_{ii} = \frac{\partial u'_i}{\partial x_i} = 0. \tag{14}$$

The integral of (13) over the period window D gives

$$\iint_D \left\{ \frac{\partial(\eta u_i \sigma'_{ij})}{\partial x_j} - \frac{\partial(\eta' u'_i \sigma_{ij})}{\partial x_j} \right\} dA = \iint_D -\frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij} dA. \tag{15}$$

By the divergence theorem applied to the left-hand side of this equation it follows that

$$\oint_{\partial D} \eta u_i \sigma'_{ij} n_j ds - \oint_{\partial D} \eta' u'_i \sigma_{ij} n_j ds = \iint_D -\frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij} dA. \tag{16}$$

Focusing on the boundary integral, since there is no imposed pressure gradient the periodicity of both velocity and stress fields implies that the contributions from the two edges of the period window cancel out. On the no-slip regions of the surface we have

$$u_i = u'_i = 0, \quad i = 1,2, \tag{17}$$

resulting in no contribution to the boundary integral. On the no-shear slots, we have

$$u_i n_i = u'_i n_i = 0, \quad e_{ij} n_j = e'_{ij} n_j = 0, \quad i = 1,2, \tag{18}$$

which together mean that the no-shear portions of the boundary integral also give no contribution. The only contribution therefore comes from the boundary at infinity, called L_∞ in Fig. 1:

$$\oint_{L_\infty} (\eta u_i \sigma'_{ij} n_j - \eta' u'_i \sigma_{ij} n_j) ds = \iint_D -\frac{\partial \eta'}{\partial x_j} u'_i \sigma_{ij} dA. \tag{19}$$

It is this useful equation that furnishes formulas for the first-order slip length correction for both longitudinal and transverse flow within the framework of a perturbation analysis of the problem. We now consider each flow separately.

For a longitudinal flow,

$$\dot{\gamma}_{ij} = \begin{bmatrix} 0 & 0 & \partial w/\partial x \\ 0 & 0 & \partial w/\partial y \\ \partial w/\partial x & \partial w/\partial y & 0 \end{bmatrix}_{ij}, \quad \dot{\gamma}'_{ij} = \begin{bmatrix} 0 & 0 & \partial w'/\partial x \\ 0 & 0 & \partial w'/\partial y \\ \partial w'/\partial x & \partial w'/\partial y & 0 \end{bmatrix}_{ij} \tag{20}$$

and

$$\Pi = 2|\nabla w|^2, \quad \Pi' = 2|\nabla w'|^2. \tag{21}$$

It is then easy to establish that

$$\oint_{L_\infty} (\eta u_i \sigma'_{ij} n_j - \eta' u'_i \sigma_{ij} n_j) ds = 2a\dot{\gamma}\dot{\gamma}'(\lambda_\parallel - \lambda'_\parallel)\eta\eta'(2\dot{\gamma}'^2) \tag{22}$$

and

$$u'_i \sigma_{ij} \frac{\partial \eta'}{\partial x_j} = \eta \frac{d\eta'(\Pi')}{d\Pi'} w' \nabla w \cdot \nabla \Pi'. \tag{23}$$

On substitution into (19) we find

$$2a\dot{\gamma}\dot{\gamma}'(\lambda'_\parallel - \lambda_\parallel)\eta\eta'(2\dot{\gamma}'^2) = \eta \iint_D \frac{d\eta'(\Pi')}{d\Pi'} w' \nabla w \cdot \nabla \Pi' dA. \tag{24}$$

We now introduce the nondimensional quantity $\beta = \eta_\infty/\eta_0$, so (8) becomes

$$\eta' = \eta_0[\beta + (1-\beta)[1 + \mathrm{Cu}^2 \Pi'/2]^{(n-1)/2}]. \tag{25}$$

Time is nondimensionalized with respect to the time scale $T = 1/\dot{\gamma}$ and lengths with respect to the half-period a so that velocities are rescaled with $a\dot{\gamma}$; we also choose to nondimensionalize viscosities with respect to the value η_0. The Carreau number $\mathrm{Cu} = \Lambda/T$ is the ratio of the characteristic strain rate of the flow to the crossover strain rate $1/\Lambda$ defined by the rheology of the fluid. Under the assumption of weak nonlinearity, with

$$\epsilon \equiv 1 - \beta \ll 1, \tag{26}$$

the nondimensionalized version of (25) is

$$\eta' = 1 + \epsilon F(\Pi'), \quad F(\Pi') \equiv (1 + \mathrm{Cu}^2\Pi'/2)^{(n-1)/2} - 1. \tag{27}$$

The (nondimensionalized) non-Newtonian flow quantities are developed as regular perturbation expansions about the Newtonian flow:

$$w' = w + \epsilon w^{(1)} + \cdots, \quad \lambda'_{\parallel} = \lambda_{\parallel} + \epsilon\lambda_{\parallel}^{(1)} + \cdots, \quad \Pi' = \Pi + \epsilon\Pi^{(1)} + \cdots, \tag{28}$$

where it is $\lambda_{\parallel}^{(1)}$ that is of primary interest. This means that, in all situations, the far-field strain rates satisfy

$$\dot{\gamma}' = 1 + O(\epsilon). \tag{29}$$

We are free to impose that the far-field strain rates of the Newtonian and non-Newtonian flows are equal, meaning that $\dot{\gamma}' = 1$ and (29) certainly holds. On the other hand, we might insist that both flows are driven by the same far-field stress and, in view of (27), this also implies that (29) pertains. It turns out that the result for the first-order slip length correction is the same for both choices of these far-field boundary conditions. Indeed, the nondimensionalized version of (24) implies that, to first order in ϵ,

$$\lambda_{\parallel}^{(1)} = \frac{1}{2}\iint_D w\frac{dF(\Pi)}{d\Pi}\nabla\Pi\cdot\nabla w\, dA. \tag{30}$$

For given n and Cu, the right-hand side of expression (30) depends only on the Newtonian flow solution, which is known explicitly from Sec. II. It is therefore unnecessary to solve for the first-order correction to the non-Newtonian flow to determine the first-order correction to the slip length.

It is convenient to rewrite the area integral in (30) in terms of the complex potential $h(z)$ introduced in Sec. II (now with $\dot{\gamma} = 1$ and $a = 1$ to accord with the nondimensionalizations chosen above). This is done in Appendix A with the result that the first-order correction to the slip length is

$$\lambda_{\parallel}^{(1)} = \frac{(n-1)\mathrm{Cu}^2}{2}\iint_D \frac{\mathrm{Im}[h(z)]\mathrm{Im}[h'(z)^2\overline{h''(z)}]}{[1 + \mathrm{Cu}^2|h'(z)|^2]]^{(3-n)/2}}dA, \tag{31}$$

where, when applied to the analytic function $h(z)$, the prime notation denotes differentiation with respect to z, i.e., $h'(z) \equiv dh/dz$. Equation (31) is our main result for longitudinal shear flow. Given that $h(z)$ is known from (4), it is an explicit expression for the first-order slip length correction due to weak shear thinning. As $b \to 0$, we know that $h''(z) \to 0$, so the slip length correction clearly vanishes, as expected.

For transverse flow we develop the (nondimensionalized) flow quantities as

$$u'_i = u_i + \epsilon u_i^{(1)} + \cdots, \quad \lambda'_{\perp} = \lambda_{\perp} + \epsilon\lambda_{\perp}^{(1)} + \cdots,$$
$$\sigma'_{ij} = \sigma_{ij} + \epsilon\sigma_{ij}^{(1)} + \cdots, \quad \Pi' = \Pi + \epsilon\Pi^{(1)} + \cdots. \tag{32}$$

Again, $\dot{\gamma}' = 1 + O(\epsilon)$ irrespective of whether the far-field shear rates or shear stresses are taken to be equal, and the nondimensionalized version of (19) then reduces, at first order in ϵ, to

$$\lambda_{\perp}^{(1)} = \frac{1}{2}\iint_D \frac{dF(\Pi)}{d\Pi}u_i\sigma_{ij}\frac{\partial\Pi}{\partial x_j}dA. \tag{33}$$

On use of (27), Eq. (33) can be rewritten

$$\lambda_{\perp}^{(1)} = \frac{(n-1)\mathrm{Cu}^2}{8}\iint_D u_i\sigma_{ij}\frac{\partial\Pi}{\partial x_j}\frac{dA}{[1 + \mathrm{Cu}^2\Pi/2]^{(3-n)/2}}. \tag{34}$$

It is shown in Appendix B that, in terms of the $h(z)$ introduced in Sec. II,

$$\Pi = -\frac{1}{2}[(z-\bar{z})\overline{h''} - 2\overline{h'}][(z-\bar{z})h'' + 2h'],$$
$$\frac{\partial \Pi}{\partial z} = -2(z-\bar{z})|h''|^2 - \frac{(z-\bar{z})^2}{2}\overline{h''}h''' - h'\overline{h''} + 3\overline{h'}h'' + \overline{h'}h'''(z-\bar{z}), \tag{35}$$

and

$$u_i\sigma_{ij}\frac{\partial \Pi}{\partial x_j} = \frac{1}{4}\mathrm{Re}\bigg[\{-(h-\bar{h})(z-\bar{z})\overline{h''} - (h-\bar{h})(h'-3\overline{h'}) - (z-\bar{z})^2 h'\overline{h''} + (z-\bar{z})(h'+\overline{h'})\overline{h'}\}\frac{\partial \Pi}{\partial z}\bigg]. \tag{36}$$

Formulas (34)–(36) constitute our main result for transverse shear. All elements of the integrand in (34) are known in terms of $h(z)$, which is itself given in (4). As $b \to 0$, $h''(z) \to 0$, which implies that $\partial\Pi/\partial z \to 0$ and the slip length correction vanishes, as expected.

IV. RESULTS

The foregoing analysis assumes that the weakly shear-thinning motion is a regular perturbation of the Newtonian flow. In the latter, the local shear rates near the edges of the no-shear slots, where the boundary condition changes type, become singular. It is therefore of concern that a regular perturbation expansion might not be valid in these circumstances. However, the perturbation expansion parameter we have chosen is the difference between the infinite-shear-rate and zero-shear-rate viscosities of the Carreau-Yasuda fluid and there is therefore no restriction on the shear rates imposed. Indeed, it is precisely because of the large shear rates near the edges of the no-shear slot where the boundary condition changes type that we might expect enhanced slip in a shear-thinning fluid. Another possibility [16] to consider is an expansion in small Carreau number $\mathrm{Cu} \ll 1$, with no restriction on the viscosity difference at zero- and infinite-shear rates, but such an expansion is not uniformly valid across strain rates, so it is not a sensible expansion to make for this superhydrophobic surface problem.

The integrals (31) and (34) require truncation of the domain D at some $y = H$, where H is sufficiently large; the integrands decrease rapidly with y, so moderate values of $H \gg b$ are found to be adequate. To maintain accuracy a graded mesh in the numerical integration is used where integration over the rectangle $-b < x < b$ and $0 < y < b$ is performed with a smaller step size in each direction than the rest of the truncated domain (where larger step sizes can be employed). An alternative approach is to make use of a parametric form of the solution over an upper-half unit disk in a complex ζ plane derived elsewhere by the author [23]. This second evaluation method was used here to check the computations. It has the advantage that the domain of integration is bounded.

Figure 2 shows the normalized first-order slip length corrections, for both the longitudinal and transverse flow, as a function of Carreau number Cu for no-shear fractions $b/a = 0.5$, 0.75, and 0.95 and power-law index $n = 0.5$; Fig. 3 shows the same graphs for $n = 0.1$. As expected, weak shear thinning enhances the slip, and this is found to be the case for all surface geometries. The more-shear-thinning fluid, with $n = 0.1$, shows greater enhancement in slip. Higher values of no-shear fraction b/a exhibit larger increases in the effective slip length as the fluid becomes weakly shear thinning. Most interesting is that, for each no-shear fraction b/a and for each power-law index n, there is a critical Carreau number Cu at which the enhancement of slip is maximal. This maximal value occurs for Carreau numbers of order unity—its precise value depends on both b/a and n—which is where the crossover occurs at which relaxation rates associated with the rheology of the shear-thinning fluid are commensurate with the shear rate of the flow. For a fixed n this critical Carreau number decreases with increasing no-shear fraction b/a. Figure 4 shows the normalized first-order slip length modifications as a function of b/a for three fixed values $\mathrm{Cu} = 0.1$, 1, and

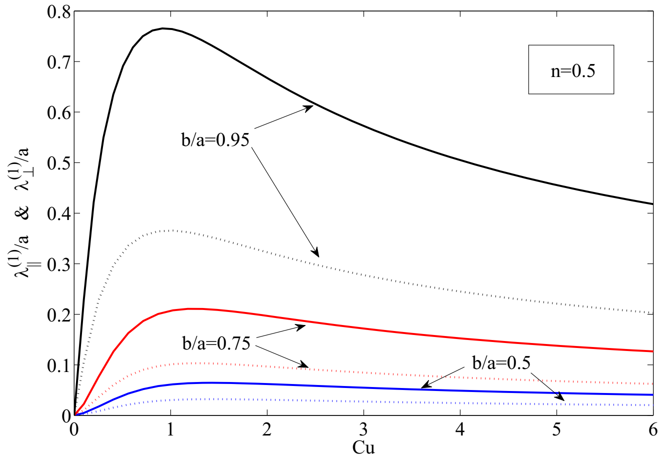

FIG. 2. Normalized first-order slip length corrections $\lambda_{\parallel}^{(1)}/a$ (solid lines) and $\lambda_{\perp}^{(1)}/a$ (dashed lines) as a function of Cu for a weakly nonlinear Carreau-Yasuda fluid with power-law index $n = 0.5$ and $b/a = 0.5$, 0.75, and 0.95.

10 with $n = 0.5$. The slip length modification increases monotonically with increasing no-shear fraction. This is found to be the case for other choices of n.

V. DISCUSSION

Explicit integral formulas for the slip length correction due to weak shear thinning of the working fluid in both longitudinal and transverse semi-infinite shear flow over a superhydrophobic surface of a periodic array of flat no-shear slots have been derived. Prior theoretical investigation of the longitudinal flow scenario for shear-thinning fluids appears to be lacking; even for transverse flow, the study of semi-infinite shear appears to be new and should be relevant to a channel flow where the height to pitch ratio is large.

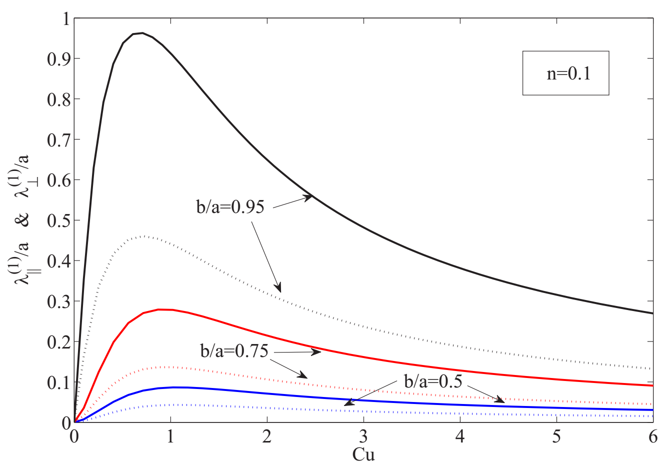

FIG. 3. Normalized first-order slip length corrections $\lambda_{\parallel}^{(1)}/a$ (solid lines) and $\lambda_{\perp}^{(1)}/a$ (dashed lines) as a function of Cu for a weakly nonlinear Carreau-Yasuda fluid with power-law index $n = 0.1$ and $b/a = 0.5$, 0.75, and 0.95.

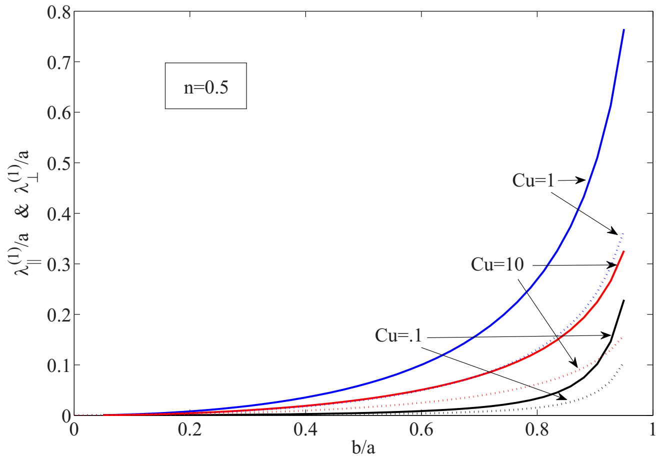

FIG. 4. Normalized first-order slip length corrections $\lambda_{\parallel}^{(1)}/a$ (solid lines) and $\lambda_{\perp}^{(1)}/a$ (dashed lines) as a function of b/a for Cu $= 0.1$, 1, and 10 and $n = 0.5$. The slip enhancement increases with no-shear fraction and, for fixed b/a, is larger for intermediate values of Cu.

The analysis shows that the shear-thinning correction to the slip is nonmonotonic in Cu, peaking around Cu ≈ 1 for all superhydrophobic surface geometries. This is where the imposed far-field shear rate is commensurate with the natural relaxation rate of the shear-thinning fluid. Physically, this enhanced response is reminiscent of a "resonance" effect in a system driven at frequencies close to its natural frequencies. The slip-length correction decreases with the no-shear fraction b/a as might be expected given that the leading-order slip length itself decreases with the width of the no-shear slots.

Given the interest in tuning the slip properties of superhydrophobic surfaces for specific areas of application [24], the results highlight how the non-Newtonian nature of the working fluid can also be used to provide additional freedoms in optimizing surface design. For a given Carreau-Yasuda fluid, with a given power-law exponent n, our results indicate that for each surface there is an optimal operating flow rate at which the hydrodynamic slip is maximized. This has been shown explicitly here for weakly nonlinear fluids, but it is reasonable to expect that this qualitative feature will persist in general. In the context of the aforementioned problem of assessing the effects of shear thinning on the locomotion of microorganisms, Datt *et al.* [16] have used methods analogous to those presented here and have found that the perturbation results are qualitatively representative of the behavior of the system in the fully nonlinear regime. We expect the same to be true for the superhydrophobic surface problem; indeed, we already know that for transverse flow our weakly nonlinear results are qualitatively consistent with numerical work on the fully nonlinear problem [9,10].

Concerning generalizations, and given that previous authors have studied transverse flow of shear-thinning fluids with curved menisci [10], it is worth pointing out that the present author [25,26] has recently derived explicit expressions (as an asymptotic expansion in the no-shear fraction) for the complex potentials associated with semi-infinite longitudinal (Newtonian) shear flow over protruding bubble mattresses with curved menisci of arbitrary protrusion angle. The slip correction integral formulas derived here can also be used in conjunction with those complex potentials to examine how interface curvature affects slip in longitudinal flow of weakly shear-thinning fluids.

The approach via integral identities can be adapted to study longitudinal and transverse pressure-driven flow in a channel of finite height. For the longitudinal problem Philip [22] solves for the Newtonian flow in terms of Jacobi cnoidal functions; use of this leads to an analogous explicit integral expression for the first-order slip length correction by extension of the steps presented herein for semi-infinite shear. There is no known analytical solution for the transverse flow problem in a channel of finite height for a Newtonian fluid, so while integral expressions for the first-order slip length correction for a weakly shear-thinning fluid can be derived, the Newtonian solution appearing in the integrand will likely have to be computed numerically. However, for confined channel geometries where the channel height is large compared to its width, as is often the case in practice, matched asymptotic methods can be used where a one-dimensional outer flow solution in the main bulk of the flow is matched to an inner solution that takes account of the surface heterogeneity. Those inner solutions are precisely the semi-infinite shear flow scenarios studied in this paper.

A similar approach, via the generalized reciprocal theorem identities derived here coupled with perturbation theory, can be used to study the flows of weakly viscoelastic fluids over superhydrophobic surfaces and will be left for future work.

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APPENDIX A: INTEGRAND FOR LONGITUDINAL FLOW

It is convenient to rewrite the area integral in (30) in terms of the complex potential $h(z)$ introduced in Sec. II (with $\dot{\gamma} = 1$ and $a = 1$ to accord with our chosen nondimensionalizations). This integrand involves only the Newtonian flow. Using subscripts to denote partial derivatives,

$$\nabla \Pi = 2\begin{pmatrix} 2w_x w_{xx} + 2w_y w_{xy} \\ 2w_x w_{xy} + 2w_y w_{yy} \end{pmatrix} = 4\begin{pmatrix} w_x w_{xx} + w_y w_{xy} \\ w_x w_{xy} - w_y w_{xx} \end{pmatrix}, \tag{A1}$$

where we have used the fact that

$$\frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 w}{\partial y^2}. \tag{A2}$$

Therefore,

$$\nabla w \cdot \nabla \Pi = 4\big[\big(w_x^2 - w_y^2\big)w_{xx} + 2w_x w_y w_{xy}\big]. \tag{A3}$$

However, since (now employing primes to denote differentiation),

$$h'(z) = \frac{\partial w}{\partial y} + i\frac{\partial w}{\partial x}, \quad h''(z) = \frac{\partial^2 w}{\partial x \partial y} + i\frac{\partial^2 w}{\partial x^2}, \tag{A4}$$

then it follows, after some algebra, that

$$\mathrm{Im}[h'(z)^2\overline{h''(z)}] = \big(w_x^2 - w_y^2\big)w_{xx} + 2w_x w_y w_{xy}. \tag{A5}$$

Hence,

$$\Pi = 2|h'(z)|^2, \quad \nabla w \cdot \nabla \Pi = 4\,\mathrm{Im}[h'(z)^2\overline{h''(z)}]. \tag{A6}$$

Therefore, in terms of the complex potential $h(z)$ for the Newtonian base state flow, the first-order correction to the slip length is given by (31).

APPENDIX B: INTEGRAND FOR TRANSVERSE FLOW

We want to rewrite the integral on the right-hand side of (34) in terms of the analytic function $h(z)$ of Sec. II (again, with nondimensionalizations consistent with those in the main body of the paper). This integrand involves only the Newtonian flow. Consider first the quantity $\sigma_{ji}u_i = u_i\sigma_{ij}$, which has components

$$\begin{pmatrix} u\sigma_{11} + v\sigma_{21} \\ u\sigma_{12} + v\sigma_{22} \end{pmatrix} = \begin{pmatrix} -pu \\ -pv \end{pmatrix} + 2\begin{pmatrix} ue_{11} + ve_{12} \\ ue_{21} + ve_{22} \end{pmatrix}. \tag{B1}$$

We will use the notation $\mapsto$ to denote the process of taking a vector $\mathbf{a} = (a_x, a_y)$ and forming its complex analog $a = a_x + ia_y$. Hence,

$$u_i\sigma_{ij} \mapsto -p(u + iv) + 2(u - iv)(e_{11} + ie_{12}) \tag{B2}$$

since

$$(u - iv)(e_{11} + ie_{12}) = (ue_{11} + ve_{12}) + i(ue_{12} - ve_{11}) = (ue_{11} + ve_{12}) + i(ue_{21} + ve_{22}). \tag{B3}$$

Now we have (employing primes to denote differentiation)

$$g(z) = -zf(z), \quad g'(z) = -zf'(z) - f(z), \quad g''(z) = -zf''(z) - 2f'(z). \tag{B4}$$

Hence,

$$\begin{aligned} p &= 2[f'(z) + \overline{f'(z)}], \\ u + iv &= -[f(z) + \overline{f(z)}] + (z - \bar{z})\overline{f'(z)}, \\ u - iv &= -[f(z) + \overline{f(z)}] - (z - \bar{z})f'(z), \\ e_{11} + ie_{12} &= z\overline{f''(z)} + \overline{g''(z)} = (z - \bar{z})\overline{f''(z)} - 2\overline{f'(z)}. \end{aligned} \tag{B5}$$

On substitution into (B2),

$$u_i\sigma_{ij} \mapsto 2[-(f + \bar{f})(z - \bar{z})\overline{f''} + (f + \bar{f})(f' + 3\overline{f'}) - (z - \bar{z})^2 f'\overline{f''} + (z - \bar{z})(f' - \overline{f'})\overline{f'}]. \tag{B6}$$

Also,

$$\frac{\partial \Pi}{\partial x_j} \mapsto 2\frac{\partial}{\partial \bar{z}}\Pi. \tag{B7}$$

It follows that

$$\begin{aligned} u_i\sigma_{ij}\frac{\partial \Pi}{\partial x_j} \mapsto 4\,\mathrm{Re}\Big[&\{-(f + \bar{f})(z - \bar{z})\overline{f''} + (f + \bar{f})(f' + 3\overline{f'}) \\ &-(z - \bar{z})^2 f'\overline{f''} + (z - \bar{z})(f' - \overline{f'})\overline{f'}\}\frac{\partial \Pi}{\partial z}\Big], \end{aligned} \tag{B8}$$

where we have used the fact that if $\mathbf{a} = (a_x, a_y) \mapsto a_x + ia_y$ and $\mathbf{b} = (b_x, b_y) \mapsto b_x + ib_y$ then $\mathbf{a} \cdot \mathbf{b} \mapsto \mathrm{Re}[a\bar{b}]$. We also have

$$\Pi = 4e_{mn}^2 = 8\left(e_{11}^2 + e_{12}^2\right), \quad e_{11}^2 + e_{12}^2 = -[(z - \bar{z})\overline{f''} - 2\overline{f'}][(z - \bar{z})f'' + 2f']. \tag{B9}$$

On use of (3), which relates $f(z)$ to $h(z)$, we find

$$\begin{aligned} \Pi &= -\frac{1}{2}[(z - \bar{z})\overline{h''} - 2\overline{h'}][(z - \bar{z})h'' + 2h'], \\ \frac{\partial \Pi}{\partial z} &= -2(z - \bar{z})|h''|^2 - \frac{(z - \bar{z})^2}{2}\overline{h''}h''' - h'\overline{h''} + 3\overline{h'}h'' + \overline{h'}h'''(z - \bar{z}). \end{aligned} \tag{B10}$$

We also find expression (36) for the quantity $u_i\sigma_{ij}\partial\Pi/\partial x_j$.

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