Coexistence of two equilibrium configurations in two-dimensional turbulence

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In the past, two families of statistical mechanics approaches have been applied to the two-dimensional Euler equations. The first one is formulated in Fourier space and considers the Galerkin-truncated dynamics. The other one is formulated in physical space and considers either point vortices or coarse-grained vorticity. We show that in a Galerkin-truncated system, both methods describe a part of the flow. A condensate can be identified, assuming that it can be characterized by an unspecified functional relation between the vorticity and the stream function. It is shown, *a posteriori*, that this function is a hyperbolic sine relation, as predicted by point-vortex statistical mechanics. The energy spectrum associated with the condensate is well described by an exponential function and the tails of the probability density function of the vorticity follow a power law. Analytical arguments to explain these observations are proposed. After removing the condensate, the remaining field can be described by Fourier-statistical mechanics.

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I. INTRODUCTION

An impressive feature of two-dimensional turbulence is its tendency to self-organize, forming large-scale structures [1]. These structures are observed in freely decaying two-dimensional turbulent flows [2] but they also survive in the presence of forcing [3] and in thin layer turbulence (e.g., Refs. [4–6]). In the presence of forcing, the formation of condensates at the largest size of the domain can be associated with the inverse energy cascade [7]. For freely decaying turbulence, this explanation is somewhat more tenuous.

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A heuristic explanation of the tendency to self-organize in freely evolving two-dimensional flow was given by Onsager in 1949 [8], who suggested applying statistical mechanics to a point-vortex system, introduced by von Helmholtz [9], and predicted that point vortices, under certain constraints, should have the tendency to clump together, forming large clusters of like-signed vortices.

An alternative statistical mechanics approach in turbulence to predict the most probable state consists of considering, instead of vortices in physical space, the dynamics of modes, after a convenient decomposition [7]. The most common decomposition in turbulence research is the Fourier decomposition, and once a flow field is decomposed onto a large but finite number of modes, the statistically most probable state can be determined by maximizing a well-chosen Boltzmann entropy, accounting for physical constraints, such as energy conservation.

These two approaches (vorticity based versus Fourier-mode based) rely on two different finite-dimensional approximations and only partially describe the full system, albeit in a complementary way. Indeed, as we shall discuss, the structure-based statistical mechanics corresponds to a system of finitely many point vortices where no truncation is applied on the number of modes. In contrast, in Fourier-statistical mechanics a truncation is imposed and ergodicity is assumed. Ergodicity means, in this case, that all positions in phase space are visited with an equal probability. In the presence of long-living structures, it seems that ergodicity is violated. In fact, our numerical results, also supported by previous numerical experiments [10,11], suggest that in fact a superposition of the two equilibria predicted from both statistical approaches is present. This coexistence was investigated previously in the more general case of long-range interacting systems with varying interaction range [12]. In that study, the violent relaxation toward a space-filling structure was studied in a more general system of which the Euler equations are a special case. In the present paper, we reproduce certain results and further investigate the 2D Euler case. In particular, we determine the shape of the spectral energy density of the condensate and the probability density function (PDF) of its vorticity. We explain, by analytical reasoning why these quantities have this shape.

In the next section, we will briefly recall some major results of both statistical mechanics approaches. Subsequently, in Sec. III, we will show numerical experiments where both statistical mechanics approaches fail, but with an appropriate decomposition of the flow, each method allows us to explain parts of the observations. In Sec. IV, we argue, as a consequence of the functional relation between the stream function and the vorticity, that the condensate energy spectrum in the freely decaying case can be expected to have exponential decay with respect to wave number and that its associated vorticity is described by a PDF with power-law tails.

In Sec. V, before concluding, we propose to describe the phenomenology of the system on the level of the Fourier spectra.

II. TWO RESULTS OF STATISTICAL MECHANICS APPLIED TO TWO-DIMENSIONAL TURBULENCE

The equations we study here are the 2D Euler equations describing an incompressible inviscid fluid flow,

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1b}$$

with $\boldsymbol{u}$ the velocity and p the pressure. We restrict this study to flows in a periodic box. Let $\omega = \nabla \times \boldsymbol{u}$ be the vorticity field and define ψ to be the stream function given by $-\Delta\psi = \omega$, where Δ denotes the Laplacian operator. We have that $\boldsymbol{u} = \nabla \times \psi \boldsymbol{e}_3$, where $\boldsymbol{e}_3$ is the unit vector normal to the plane. Then, taking the curl of (1) we can write the vorticity equation,

$$\partial_t \omega + \{\omega, \psi\} = 0, \tag{2}$$

where the Poisson bracket is given by

$$\{\omega, \psi\} = \frac{\partial \omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y}\frac{\partial \psi}{\partial x} = ((\nabla \times \psi \boldsymbol{e}_3) \cdot \nabla)\omega. \tag{3}$$

We will briefly review two analytical predictions that were obtained by applying statistical mechanics to the above equations. We will first discuss the statistical mechanics of a truncated set of Fourier modes. After that, we recall an important result associated with point-vortex statistical mechanics.

In the following, we will distinguish the wave-number spectrum of the kinetic energy $E(k)$ from the kinetic energy E by the wave-number argument k. Similarly, we distinguish the enstrophy spectrum $W(k)$ from the enstrophy W.

A. Fourier statistical mechanics

For three-dimensional Euler turbulence, statistical mechanics on a truncated set of Fourier modes yields, if the only constraint of the system is energy conservation, an equipartition of energy between all the modes. This means that, statistically, every Fourier mode contains the same amount of energy. This result was predicted by Lee [13], who observed that the Fourier-transformed truncated Euler equations satisfy a Liouville theorem.

Let us now consider the two-dimensional case of the Euler equations, governed by the weak form

$$(\partial_t \omega + \{\omega, \psi\}, \phi)_{L^2} = 0 \quad \text{for all } \phi \in V_K, \tag{4}$$

where V_K is the space of test functions ϕ consisting of all Fourier modes with wave vectors below a truncation frequency K. By construction, the finite-dimensional system conserves total circulation, enstrophy, and energy. However, Casimir conservation (integrals of functions of the vorticity other than total circulation and enstrophy) is lost, indicating divergence from the full system as the true solution starts to populate wave numbers higher than K.

In his seminal paper on two-dimensional turbulence, Kraichnan [7] applied statistical mechanics to the system (4) and predicted that the equilibrium energy spectrum should behave as

$$E(k) = \frac{2\pi k}{\alpha + \beta k^2}. \tag{5}$$

The parameters α and β can be determined by considering the conservation of energy and enstrophy of the system (see, for instance, Ref. [14] for analytical expressions relating α and β to $E, W, k_0, k_{\max}$). For further details on this prediction, we refer to Ref. [2] or a more recent manuscript [15].

The prediction (5) was verified in early simulations of the two-dimensional Euler equations. Indeed, the development of pseudospectral methods in the 1970s allowed us to solve nondissipative simulations of truncated Euler dynamics. It was shown by Refs. [16,17] that the theory of Kraichnan roughly predicts the correct energy spectrum (5) of the equilibrium state. It was furthermore observed that "substantial deviations from equilibrium are present at low wavenumbers. These are probably explained by the long dynamical time scale of large-scale turbulent eddies, so that continuing the calculation to much longer times may give relaxation" [16]. However, more recent numerical investigations showed that these deviations persist even at very long times with no hint of relaxation toward the supposed equilibrium [12,18,19]. In particular, when the ratio of energy to enstrophy is large, noticeable discrepancy is observed for small wave numbers. To understand the origin of this discrepancy, we need to review certain key results of an alternative statistical mechanics approach.

B. Point-vortex statistical mechanics

The structure-based statistical mechanics proposed by Onsager [8] was further explored by Joyce and Montgomery [20,21], and more recently by the works of Miller [22,23] and Robert and Sommeria [24]. A concise review can be found in the article of Eyink and Sreenivasan [25] and more comprehensive reviews in Refs. [2,26]. The point-vortex and coarse-grained vorticity statistical mechanics are approaches based on the observation that enstrophy, and small vorticity patches in general, are advected conservatively in the two-dimensional Euler equations.

In the point-vortex approach used by Joyce and Montgomery and studied numerically by Lundgren and Pointin [27], the flow is phenomenologically modeled by an ensemble of a large number of N point vortices of circulation Γ_i (with $\sum_{i=1}^{N} \Gamma_i = 0$ in our case) at positions $\boldsymbol{x}_i(t)$ at time t. The vorticity distribution is then given by

$$\omega_N(\boldsymbol{x}, t) = \sum_{i=1}^{N} \Gamma_i \delta_{\boldsymbol{x}_i(t)}(\boldsymbol{x}). \tag{6}$$

The point vortices are advected under the influence of the velocity field induced by all the other point vortices, which conserves by construction the total circulation. Furthermore, the so-defined system is Hamiltonian, conserving, in addition, energy. We focus on the case of zero circulation. The limit of infinitely many point vortices may be compared with the geometric formulation of Arnold [28], where the configuration space for the Euler equations is the space of all volume-preserving diffeomorphisms.

Joyce and Montgomery postulated that the most likely point-vortex configuration arising from the Hamiltonian system are microstates, which maximize the Gibbs-Boltzmann entropy $S(\omega) = \int_\Omega -\omega \ln \omega \, d\boldsymbol{x}$. Therefore, given an energy level E, the most likely vorticity microstate can be obtained by applying a Lagrangian multiplier method for the constrained optimization problem (maximizing entropy, subject to given energy level). This procedure yields a nontrivial relation between the vorticity and the stream function ψ,

$$\omega = \beta \sinh(\lambda \psi), \tag{7}$$

for some β and λ positive ($\beta\lambda < 0$ is not possible in a periodic box). We note that due to the functional dependence between ω and ψ, the advection term in Eqs. (1) vanishes identically, making these solutions steady states of the Euler equations. This equilibrium corresponds to a state where like-signed vortices have clustered together, forming two blobs, or clusters of vortices [21,27].

C. Are Fourier and point-vortex approaches compatible?

It is interesting that the result (7), which strictly applies to the Hamiltonian system of point vortices, is also observed to a very good approximation as a transient for the two-dimensional Navier-Stokes equation, i.e., a dissipative system [29]. Even though different results are observed for certain initial conditions [30] and the data can in certain cases be equally well fitted by functions different from a sinh [31], the result (7) is widely observed, even in three-dimensional systems in the absence of vortex stretching [32].

One explanation of this somewhat surprising result is that it corresponds to a physical space structure, consisting of two counter-rotating vortices which retain the energy at large scales, limiting the amount of dissipation. Indeed, if the third-order structure functions satisfy $\langle |u(x+r) - u(x)|^3 \rangle^{1/3} \sim C|r|^\alpha$ for any $\alpha > 1/3$, then by the first half of Onsager's conjecture shown by Constantin *et al.* [33] (and also previous works by Eyink [34]), total viscous dissipation goes to 0 in the inviscid limit. This is the case for solution of the Euler equations in 2D with smooth initial condition as the regularity is known to be preserved by the theorem of Beale *et al.* [35], so for small enough ν, the dynamics of large-scale structures is practically nondissipative.

Therefore, assuming that we may compare the $k_{\max} \to \infty$ limit of the Galerkin-truncated system with the $\nu \to 0$ limit of the nontruncated Navier-Stokes equations, it is then plausible to postulate that the large-scale structure is stable in the truncated system, as long as this truncation is applied at large-enough wave number. If this assumption is valid, and a large-scale stable structure is present in the flow, ergodicity is most certainly lost since the large structure will impede the system to equally sample all possible positions in phase space.

Note that the limit $k_{\max} \to \infty$ is discussed in Ref. [14]. It is argued there that, if the vorticity distribution is averaged over small distances, the relaxation of systems with infinite $k_{\max}$, and with large finite $k_{\max}$, should be equivalent. It is also suggested that states in which the vorticity field is dominated by a few large eddies cannot be reached from an initial state with energy at large scales, but sufficiently smaller than the domain size, for $k_{\max}$ tending to infinity. Indeed, enstrophy equipartition implies that for large $k_{\max}$ enstrophy is dominated by contributions around $k_{\max}$.

This section contains a number of unresolved issues. First, would a large-scale structure, formed in a truncated system, be regular enough? What is the corresponding meaning of regularity for truncated solutions? And how to qualify the notions large-enough $k_{\max}$, or small enough ν? To advance the understanding of these notions, we will carry out numerical experiments. A previous numerical study of the system was carried out by Venaille *et al.* [12] and we refer to their paper for results on a more general model of long-range interaction and additional explanations.

III. NUMERICAL EXPERIMENTS OF TRUNCATED 2D EULER DYNAMICS

We carry out pseudospectral numerical simulations of the Galerkin-truncated Euler equations. We anticipate that the results are not fully described by either of the two statistical methods, rather, for the same flow, one part can be described by point-vortex results and the rest by a Fourier equilibrium, satisfying Eq. (5). We then introduce a physical-space procedure to disentangle the two coexisting equilibrium flows and give analytic evidence for the shape of the energy spectrum of the condensate and the PDF of its vorticity.

A. Numerical simulations

We perform numerical experiments for the Galerkin-truncated Euler equations (4). Numerical simulations are performed with the parallel pseudospectral code GHOST [36] on a 2π-periodic square box, so the largest wavelength corresponds to the wave number $k_{\min} = 1$. The initial condition for the velocity field is given in Fourier space by a Gaussian-shaped spectrum distribution

$$E(k) = Ce^{-(k-k_0)^2/2\sigma^2}, \tag{8}$$

for some real numbers C and σ to be chosen to determine the shape of the initial spectrum.

The system we consider has two important control parameters. The first one is the spatial resolution. We will present results for $k_{\max}/k_{\min} = 169$, corresponding to a resolution of 512^2 grid points, since the 2/3–method is used to dealiase.

The other parameter defining the final state is the ratio of enstrophy to energy, which can be expressed as a characteristic wave number:

$$k_u = \sqrt{\frac{W}{E}}. \tag{9}$$

The minimum value of this wave number is $k_u = 1$, corresponding to the case where the energy spectrum is concentrated on the first wave number. The largest value of k_u is analogously obtained when all energy is concentrated on $k_{\max}$, in which case $k_u/k_{\max} = 1$. In Eq. (8), we set $k_0 = 4$, $\sigma = 2$, and C is chosen so $\int_1^{k_{\max}} E(k)\ dk\ \equiv E = 1$. All Fourier phases are chosen randomly so the initial vorticity field is, in general, structureless.

In Fig. 1, we show steady-state spectra obtained for simulations for initial conditions with typical wave number k_u in the range $k_u \in [10, 100]$. We observe that the long-time shape is in agreement with the theoretical predictions at large wave numbers for all cases. For small wave numbers, discrepancies are observed. Whereas for the case $k_u = 100$, this discrepancy might be attributed to the discrete lattice effect associated with the limited number of Fourier modes in the lowest wave-number shells, for the cases with the smaller values of k_u this discrepancy becomes more important. In particular, for the case $k_u = 10$, it is difficult to attribute the difference between prediction and observation to nonconverged statistics.

These observations confirm previous results [12,16,18,19]: the large wave numbers are well predicted by Kraichnan's theory, but the deviation is clearly noticeable in the small wave numbers, in particular, in the case where $k_u/k_{\max}$ is small. Indeed, it seems that with enough separation between

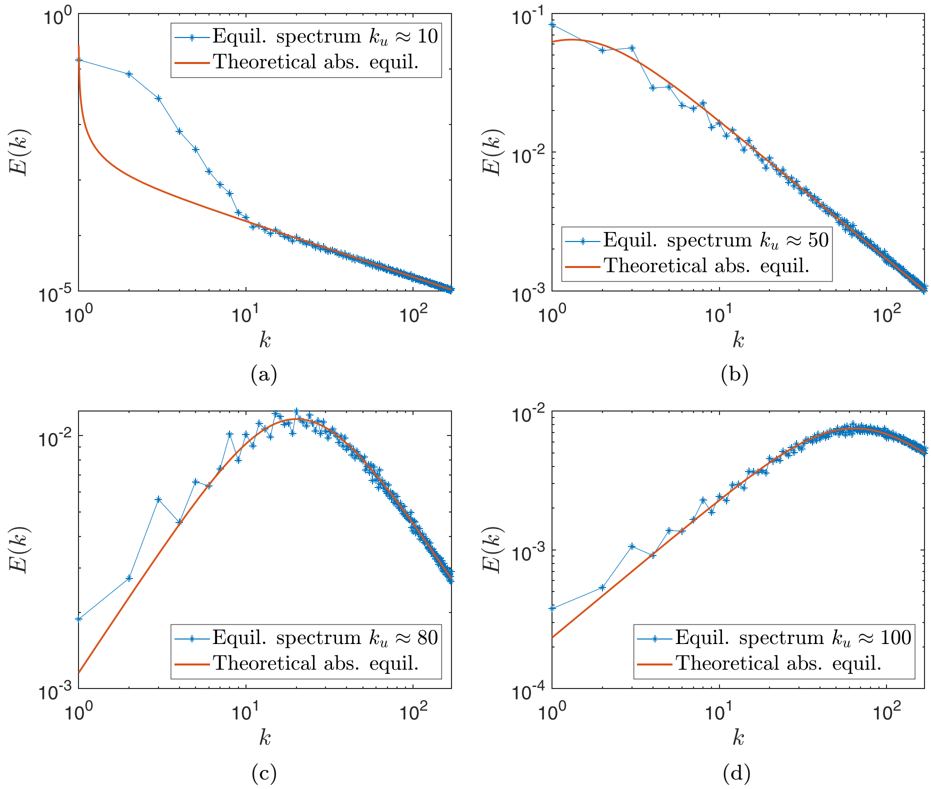


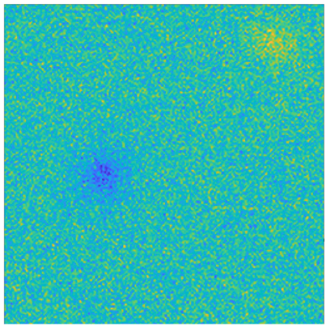
FIG. 1. Equilibrium energy spectrum for different values of k_u with Gaussian initial fields.

the box size and the truncation scale, a robust and persistent condensate can form and that this structure leads to violation of ergodicity in Fourier modes. In the next section, we will focus on the case where $k_u = 10$ [Fig. 1(a)], where the deviation from Eq. (5) is most significant.

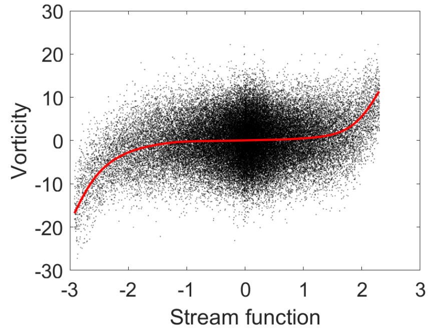
B. Identification of the condensate

Visualization of the vorticity field in 2D provides a good hint to the underlying reason for the deviation of numerical simulations from the statistical predictions. In Fig. 2(a), we observe that a well-defined dipole is present in the vorticity field. This dipole is persistent and observed at all time instants during the statistically steady state of this simulation. We therefore hypothesize that the flow can be decomposed into a condensate and an incoherent part. This leads to the questions of how to define this decomposition and whether the results of statistical mechanics discussed in Secs. II A and II B may be relevant to the decomposition.

We first decompose the flow and extract the condensate. Since the dipole is persistent, a first attempt could consist of time averaging the flow to identify the condensate. Since the position and orientation of the dipole is not necessarily stationary, this would necessitate a Lagrangian averaging, introducing the need to determine the position of the dipole *a priori*. This is not the way we proceed. Alternatively, it is possible to isolate large-scale features of the solution through a low-pass filter, but it is not immediately clear how this filter should be chosen and whether this choice artificially determines the statistics of the remaining small scales. A more adapted way would be coherent-vortex extraction [37,38], which still relies on a filtering threshold and the shape and properties of the wavelets.



(a)



(b)

FIG. 2. Results of direct numerical simulations of the Galerkin-truncated 2D Euler equations. (a) A visualization of the vorticity field for the case $k_{\max} = 170$, $k_u = 28.4$. (b) Scatter plot between vorticity $\omega(x, y, t)$ and stream function $\psi(x, y, t)$ for a given time instant. The solid line is a time average over a long-enough (approximately 26 turnover times) time interval during the statistically steady equilibrium state, yielding a functional relation $\omega = \bar{f}(\psi)$.

In the present paper, we use a more physical way, using the assumption that the condensate is close to a steady solution of the Euler equations. If we assume that the condensates arise from a maximum entropy configuration as discussed in Sec. II B, we can further assume a functional relation between the vorticity and stream function, without specifying the shape of this function. We therefore attempt the following decomposition: Given a proposed function f for the functional relationship $\omega = f(\psi)$, we define the condensate vorticity as $\bar{\omega} = f(\psi)$ and an oscillation part $\omega' = \omega - \bar{\omega}$:

$$\bar{\omega} = f(-\Delta^{-1}\omega), \tag{10}$$

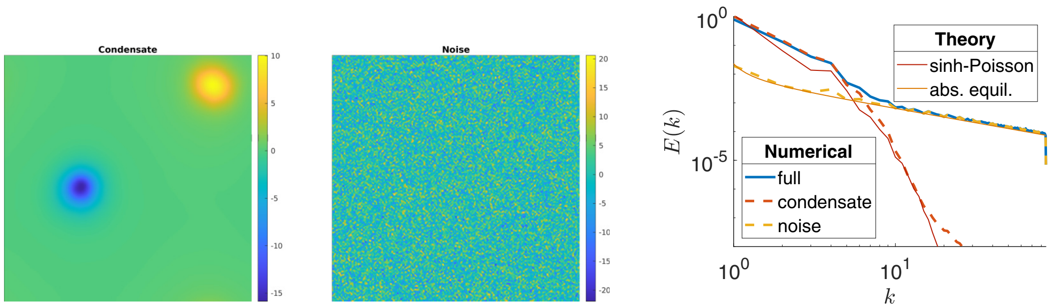
$$\omega' = \omega - f(-\Delta^{-1}\omega). \tag{11}$$

We stress that we do not prescribe any particular function f. However, our assumption is that this function is constant in time and that the temporal fluctuations of ω' are of zero average. We then obtain our estimate of f by assessing from the data

$$\langle f(\psi)\rangle = \langle \Delta\psi\rangle. \tag{12}$$

We apply this method to the data obtained from our numerical experiment shown in Fig. 2(a). In Fig. 2(b), we show a scatter plot of the data, i.e., every black dot corresponds to a data point $(\psi(x, y, t), \omega(x, y, t))$. In this representation, the information on the spatial dependence in (x, y) is lost. Instead, only the distribution of the ω values along level sets of ψ is represented. Averaging ω conditional on the value of ψ for a sufficiently long time yields the red solid curve, which is our estimate of f. We note that for this particular realization, the range of the stream function is wider in the negative values. This could be due to the initial condition, or, more plausibly, due to insufficiently long simulation time. We also note that, although the Joyce-Montgomery statistical theory predicts a sinh relation between the vorticity and the stream function, to the author's knowledge, the range of the positive and negative branches remains unknown. In light of the estimates in Sec. IV, it is not excluded that the positive and negative vortex blobs forming the condensate have different radii, resulting in an uneven range for the stream function.

The fitted function f now, in turn, provides a self-consistently determined and physically motivated low-pass filter by defining the condensate $\bar{\omega}(x, y, t)$ at every point in space and time through the relation (10). The incoherent part is the remaining part of the flow, $\omega'(x, y, t) = \omega(x, y, t) - \bar{\omega}(x, y, t)$. The resulting condensate and incoherent fields are shown in Figs. 3(a) and 3(b). We show that the condensate is a well-defined smooth dipole structure. Furthermore, the noise



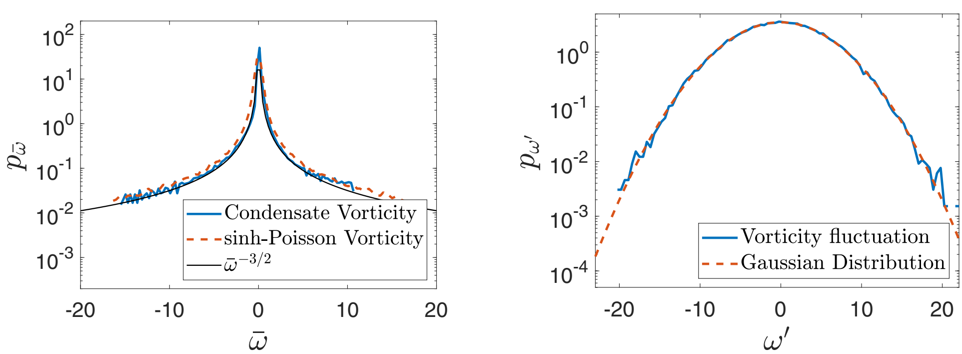
(a) (b)

FIG. 3. (a) The condensate vorticity (left) decomposed from the remaining vorticity incoherent (right). (b) The time-averaged energy spectrum associated with both contributions. The predictions from Kraichnan's equilibrium theory for the incoherent part and from the point-vortex system for the sinh-Poisson condensate are also shown in full lines (see Sec. IV for a closed form approximation).

part seems to be completely structureless. In Fig. 3(b), we show the energy spectra associated with the two contributions. The incoherent part, containing less energy, is well described by a Kraichnan equilibrium Eq. (5), with a close to k^{-1} scaling over most of the wave-number range, associated with enstrophy equipartition. The noise is also homogeneous; the distribution of the fluctuating vorticity ω', as shown in Fig. 4(b), is close to Gaussian, indicative of a lack of coherent structures. On the other hand, the spectrum of the condensate $E_0(k)$ in Fig. 3(b) falls off rapidly close to exponentially. Indeed, the energy spectrum associated with the condensate is well approximated by

$$E_0(k) \sim \exp(-ck), \tag{13}$$

with $c = \mathcal{O}(1)$ (see Sec. IV for an explanation of the origin of this type of spectrum associated with the sinh-Poisson condensate). This part of the flow is associated with large-scale coherent structures whose non-Gaussian distribution [see Fig. 4(a)] follows closely the approximations for the sinh-Poisson condensate derived in Sec. IV and the corresponding $\bar{\omega}^{-3/2}$ scaling analytically derived in Sec. IV B. We note that in Fig. 3(b), the condensate spectrum deviates somewhat from the sinh-Poisson spectrum. Indeed, the condensate isolated from the realization does not exactly satisfy the sinh-Poisson relation, which can lead to small differences between the theoretical spectrum and



(a) (b)

FIG. 4. The distribution of the vorticity field compared against theoretical predictions. (a) Distribution of the vorticity condensate $\bar{\omega}$ and distribution of a sinh-Poisson condensate obtained from Eq. (18). The probability densities follow the predicted $\bar{\omega}^{-3/2}$ scaling predicted in Sec. IV B. (b) Distribution of the incoherent ω' showing a Gaussian distribution.

the analytical estimate. It remains, however, that the spectrum decays exponentially, indicative of the smoothness of the large-scale dipole structure.

It is interesting to note the similarities and differences of the system compared to forced two-dimensional turbulence where a condensate forms. The incoherent part is in these flows also characterized by a k^{-1} spectrum [3] (also observed in the nonforced case [39]). The spectrum associated with the condensate is generally observed to exhibit a scaling proportional to k^{-3} (see, e.g., Refs. [3,40,41]), in contrast to the exponential spectrum of the condensate in the present case. The difference in these latter cases is that there is a continuous flux of energy toward the condensate, which is absent in the present system.

The decomposition carried out shows that we can describe the condensate part of the flow by a sinh relation between stream function and (filtered) vorticity. It is therefore in the point-vortex system, in decaying 2D Navier-Stokes turbulence [29], and, on average, in truncated Euler at high resolution for large values of E/W that we observe this phenomenology. Simultaneously, the remaining, noiselike vorticity field can be described by the Kraichnan Fourier equilibrium.

IV. ESTIMATES FOR THE ENERGY SPECTRUM AND PROBABILITY DENSITY FUNCTION OF THE CONDENSATE

In experiments and simulations of two-dimensional or thin-layer turbulence, the energy spectrum of a flow in the presence of a condensate was previously assessed. In these studies [3–5,42,43], the condensate was argued to be associated with an energy distribution proportional to k^{-3}, in contrast to the observations in the present investigation. We will illustrate now that in the present case, in the unforced Euler equations, when the condensate is well described by a hyperbolic sine relation, the energy spectrum should be of exponential shape.

The sinh relation requires solutions of the nonlinear Boltzmann-Poisson equation $-\Delta\psi = \sinh(\Lambda\psi)$. However, for $\Lambda > 0$, the typical maximum principle and energy estimates cannot be applied to guarantee uniqueness of the solution. Motivated by numerical observations of vortex pairs, if we assume that ψ is axisymmetric around its local extrema, we write a model ordinary differential equation (ODE) satisfying the sinh-Poisson relation in a small circular domain:

$$\partial_r^2\psi + \frac{1}{r}\partial_r\psi + \exp(\psi) = 0. \tag{14}$$

There exist families of solutions of this ODE with the general form

$$\psi(r) = \ln\left(\frac{2c_1^2 c_2 r^{c_1-2}}{(c_2 + r^{c_1})^2}\right), \tag{15}$$

where c_1, c_2 are arbitrary positive constants.

These solutions are generally not regular, implying a singularity at the vortex core. We argue that these singular cases should be ruled out as limiting states of the unforced Euler equations, as the extrema of the vorticity field are preserved in time. If the condensate is thought of as the infinite-time coarse-grained limit of the vorticity field, then the range of the vorticity values will shrink, ruling out condensate states with singular point vortices. In Eq. (15), taking $c_1 = 2$ reveals a possible existence of a regular stream function of the form

$$\psi(r) = -2\ln(1 + (ar)^2) + \ln(8a^2) \tag{16}$$

to leading order as $r \to 0$, for $a = c_2^{-1/2}$, with vorticity of the form

$$\omega(r) = 8a^2(1 + (ar)^2)^{-2}. \tag{17}$$

The parameter a essentially fixes the radius of the vortex blobs.

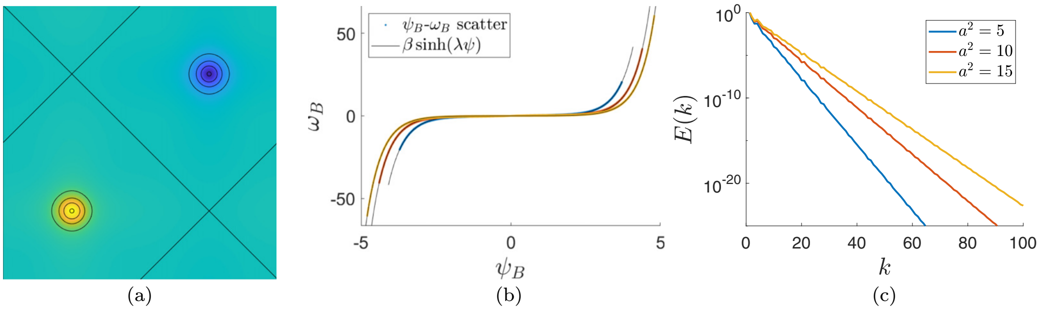


FIG. 5. sinh-Poisson approximation (18). (a) Visualization of the vorticity field for $a^2 = 5$. (b) Scatter plots of ψ_B vs ω_B for $a^2 = 5$, 10, and 15 reveal a close agreement with the hyperbolic sine relation. (c) The energy spectrum of the sinh-Poisson approximation shows a clear exponential decay [$\approx \exp(-ck)$] with rate of decay $c = 0.88$, 0.63, and 0.51, respectively .

A. Estimate of the energy spectrum

We use Eq. (17) as an ansatz to propose a sinh-Poisson approximation for the stream function:

$$\psi_B(x, y) = \ln\left(\frac{1 + 2a^2(2 - \cos(x) - \cos(y))}{1 + 2a^2(2 + \cos(x) + \cos(y))}\right), \tag{18}$$

which essentially consists of two vortex blobs of opposite signs at $(0, 0)$ and (π, π), respectively [see Fig. 5(a)]. The idea being that $\cos(x) + \cos(y) \approx 2 - (x^2 + y^2)/2$ near $(0, 0)$ so $2 - (\cos(x) + \cos(y))$ approximates $r^2/2$ near $(0, 0)$ while being strictly positive elsewhere. Similarly, $2 + (\cos(x) + \cos(y))$ approximates $r^2/2$ near (π, π) and is strictly positive elsewhere. Expression (18) therefore consists of compositions of the logarithm in Eq. (16), with trigonometric functions bounded away from zero. Hence ψ_B is a periodic real-analytic function on the torus for any a, with $\pm(1 + (ar)^2)^{-2}$ behavior near $(0, 0)$ and (π, π) and a smooth transition between these two vortex cores. It can be shown that for periodic real analytic functions, the coefficients of the Fourier series have at least exponential decay [44,45] as $k \to \infty$, with the rate of decay depending on the distance of the closest complex pole to the real axis. Therefore, in the absence of singular point vortices, our ansatz (18) shows that there exists condensate states consisting of two counter-rotating vortices approximating the sinh-Poisson vorticity-stream relation and whose energy spectrum decays (at least) exponentially. In fact, we expect this to hold for more general functions f, since the regularity of the condensate relies only on the boundedness of the vorticity (maximum condensate vorticity should be less than the maximum initial vorticity) and the smoothness of f.

We check numerically that the ψ_B-ω_B relation approximates well a hyperbolic sine in Fig. 5(b) and the energy spectrum has exponential decay in frequency space as shown in Fig. 5(c). We stress that these results do not constitute a mathematical proof but rather a plausible explanation that, if the condensate is described by a sinh relation, its spectrum should be close to exponential. We also note the discrepancy of these results with existing findings on equilibrium statistics of the forced two-dimensional Navier-Stokes or similar dissipative systems [3] where a k^{-3} spectrum is observed, possibly due to the presence of forcing.

B. Analytical estimate of the probability density function

Equation (17) also allows us to determine the shape of the PDF of the vorticity. Indeed, considering a single axisymmetric vorticity distribution of the form Eq. (17), in a circular domain of surface S_{tot}. The probability to find a value of the vorticity between ω and $\omega + \delta\omega$ is proportional to

$$\delta\omega p(\omega) = \delta S(\omega)/S_{\text{tot}}, \tag{19}$$

where $\delta S(\omega)$ is the part of the surface where the vorticity takes a value between ω and $\omega + \delta\omega$. We have therefore that

$$p(\omega) \sim \frac{dS(\omega)}{d\omega}. \tag{20}$$

Equation (17) can be inverted to express r as a function of ω,

$$r = a^{-1}\left(\sqrt{\frac{8a^2}{\omega}} - 1\right)^{1/2}. \tag{21}$$

Since in our axisymmetric approximation $S(r) = \pi r^2$ this allows to compute $dS(\omega)/d\omega$

$$\frac{dS(\omega)}{d\omega} = \frac{\partial S}{\partial r}\frac{\partial r}{\partial \omega} = -\frac{\sqrt{2}\pi}{a}\omega^{-3/2}. \tag{22}$$

We find therefore that

$$p(\omega) \sim \omega^{-3/2}, \tag{23}$$

with a normalization factor which depends on the domain size. Furthermore, $p(\omega)$ is bounded by the values $|\omega| \leqslant 8a^2$. This expression is also shown in Fig. 4(a).

V. PHENOMENOLOGY OF THE COEXISTING EQUILIBRIUM FLOWS

We describe here the observed phenomenology from the point of view of energy spectra and make a hypothesis on the dynamics of the Galerkin truncated Euler equations in the limit of infinite truncation wave number. The observations from the numerical experiments can be summarized as follows:

(1) For a given conserved total energy E and total enstrophy W, the long-time solution approaches a steady energy distribution. For large enough $k_{\max}$, the solution admits a splitting into a condensate part with total energy and enstrophy E_0, W_0 and a incoherent part with total energy and enstrophy E_1, W_1.

(2) The energy spectrum of the condensate is a truncation of some spectrum $g(k)$ with finite second moment: $\int_1^\infty k^2 g(k)dk < \infty$. The exponential spectrum observed for the condensate in our numerical simulation satisfies this requirement.

(3) The incoherent part satisfies an enstrophy equidistribution.

Assuming these observations hold in the limit of $k_{\max} \to \infty$, we see that the enstrophy spectrum for the incoherent part is

$$W_1(k) = \frac{2W_1}{k_{\max}^2}k, \tag{24}$$

with an associated energy spectrum

$$E_1(k) = \frac{2W_1}{k_{\max}^2}k^{-1}. \tag{25}$$

The incoherent energy is then given by

$$E_1 = 2W_1\frac{\ln(k_{\max}) - 1}{k_{\max}^2} \leqslant 2W\frac{\ln(k_{\max}) - 1}{k_{\max}^2}, \tag{26}$$

which tends to 0 as $k_{\max}$ tends to infinity. Therefore, the conservation (or boundedness) of total enstrophy implies that in the limit of $k_{\max} \to \infty$, all the energy is contained in the condensate.

This is consistent with the picture [46,47] where solutions of the full Euler equations in 2D approach a stable asymptotic state through the effect of mixing. The asymptotic steady state should have the same energy as the initial state but part of the enstrophy will be lost to mixing so the vorticity only converges in an averaged coarse-grained sense.

VI. CONCLUSION

In this paper, we investigated the observation that in certain Galerkin-truncated pseudospectral simulations of the Euler equations, the solution disagrees with the classical predictions from Fourier statistical mechanics. The cause of this discrepancy is the tendency of 2D turbulence to form large-scale structures. The formation of these structures is well observed in the viscous system and can be predicted using the statistical mechanics applied to the inviscid point-vortex system. It seems that for the truncated equations, with large enough scale separation between the flow's energetic scales and the truncation scale, the system approximates well the full nontruncated system, allowing a condensate to form.

From our simulations, it seems that the mean flow created by the condensate does not change statistical properties of the superposed Kraichnan equilibrium state, which, with sufficient scale separation, is not able to perturb the condensate. Thereby, both parts of the flow can coexist.

We note that our procedure to extract the condensate is still fundamentally a filtering operation where the filter is provided by the shape of f. Indeed, linearizing the operator $f \circ -\Delta^{-1}$: $\omega \mapsto f(-\Delta^{-1}\omega) = \bar{\omega}$, we see that modes with high wave numbers k are damped by a factor of $\max |f'|k^{-2}$. On the other hand, lower wave numbers can be amplified, however, assuming good fit of the vorticity-stream data at lower wave numbers by the function f, large-scale structures are mostly unaffected by this filtering. The filter used in the present system differs from scale-based filters in that no shape or threshold is prescribed; it is uniquely based on the assumptions that there exists a functional relation f and that a large-scale condensate has already formed. In fact, we do not expect this procedure to work without either of these two assumptions.

We believe that the main cause for differences between 2D and 3D Galerkin-truncated Euler equations is the vanishing energy transfer at large wave numbers for the 2D case as opposed to the 3D case. This means that the amount of energy reinjected to the large scales by the Galerkin system becomes limited as $k_{\max} \to \infty$ and so the incoherent energy can be controlled. From the point of view of energy spectra, the conservation of both energy and enstrophy in 2D controls a characteristic wave number $\sqrt{W/E}$, preventing its escape to infinity in frequency space. It would therefore be interesting to investigate this problem on more complicated systems such as band-forced 2D Navier-Stokes equations, where structures are observed at scales different from the domain size (e.g., Refs. [48–50]) or the turbulence without vortex stretching models in 3D [51]. Lastly, one interesting direction to examine is the link to mixing. Indeed, one may see the persistence of the vortex structures as some sort of immunity to mixing: as mixing breaks down large-scale structures by transferring them to small scales, a stable persistent structure at large scales must experience less mixing by the flow. In this regard, the study of long-time behaviors in 2D flows is related to the study of scalar mixing and the influence thereon of the correlation with the vorticity or stream functions [52].

Note added. Recently, we became aware of Ref. [12], where some of our results were already obtained and discussed.

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