

Correlated disorder suppresses solitary-wave attenuation in Hertzian chains

Leopoldo R. Gómez ^{*}

Departamento de Física, Universidad Nacional del Sur – IFISUR – CONICET, 8000 Bahía Blanca, Argentina



(Received 13 April 2026; revised 15 July 2026; accepted 18 August 2026; published 2 September 2026)

Mass disorder attenuates solitary waves traveling through Hertzian chains in a way that resembles viscous damping, even though the microscopic dynamics remains Hamiltonian and conserves energy. We show that this attenuation depends not only on the strength of the disorder, but also on its spatial organization. While the leading pulse remains identifiable, simulations reveal an approximately exponential decay, with an attenuation rate proportional to the mass variance in the weak-disorder regime. At fixed variance, spatial correlations are found to considerably reduce the attenuation rate. We describe this suppression using a spectral-overlap framework, in which attenuation depends on how the disorder spectrum couples to the response of the solitary wave. This response is measured directly using sinusoidally graded chains: long-wavelength variations interact weakly with the pulse, whereas variations at shorter spatial scales scatter it more strongly. The apparent damping therefore reflects a redistribution of energy from the coherent pulse into secondary excitations and an irregular wake, rather than true dissipation.

DOI: [10.1103/w2fh-pkld](https://doi.org/10.1103/w2fh-pkld)

I. INTRODUCTION

Solitary waves in granular chains are one of the best-known realizations of strongly nonlinear wave transport in an experimentally accessible many-body system (see schematic drawing in Fig. 1). In the modern theory of nonlinear waves, the conceptual background begins with the Fermi-Pasta-Ulam problem and the subsequent work of Zabusky and Kruskal, which showed that nonlinearity and dispersion can combine to produce localized, particlelike excitations rather than irreversible thermalization [1,2]. Granular chains offer a particularly striking setting because they are both discrete and strongly nonlinear at the microscopic level: neighboring beads interact through the Hertz contact law, and at vanishing precompression the linear sound speed goes to zero. Nesterenko identified this regime as a *sonic vacuum* and showed that impacts generate compact, strongly nonlinear compression pulses rather than ordinary phonons [3–5]. Experiments and simulations later confirmed the existence, width, and amplitude-dependent speed of these pulses, establishing them as the canonical solitary waves of granular media [5–8].

The same systems also support shocklike propagation regimes. In dissipative chains, sustained or sufficiently strong compression generates traveling shocks rather than isolated solitary pulses [9]. In jammed and weakly compressed granular matter, the dominant carriers of energy can become nonlinear shock fronts instead of linear sound waves [10]. Disorder adds a further layer of complexity: random masses, contacts, or geometry scatter the coherent pulse and redistribute its energy into a fluctuating wake [11,12]. In the weak-disorder regime the pulse still forms, but it attenuates as it propagates.

The present work is motivated by a particularly suggestive observation relating disorder and dissipation [12]. In disordered fragile granular packings it was found that the leading solitary wave decays exponentially at early times, and that the resulting profiles look remarkably similar to those obtained from genuine viscous damping. This is puzzling because static disorder is Hamiltonian and time-reversal invariant: it does not remove energy from the chain. The weak-disorder quasiparticle picture of Ref. [12] explains the exponential decay and its dependence on the disorder variance, but it depends only on the variance and therefore cannot distinguish between white and correlated disorder.

Our numerical results show that this is not sufficient. At fixed mass variance, increasing the spatial correlation length of the disorder suppresses the attenuation dramatically. This immediately suggests that the key quantity is the *spectrum* of the disorder. The central aim of the paper is to formulate that idea for the observable actually measured in simulations: the amplitude of the *leading coherent solitary pulse*. We use a general weak-disorder quadratic-response representation in which the decay rate is a spectral overlap between the disorder power spectrum and an effective pulse kernel. A collective-coordinate projection identifies the clean acceleration profile as the leading drive and motivates a bare approximation to the kernel; it does not determine the complete adjoint-, radiation-, and memory-resolved response. We therefore test the effective response operationally, both with correlated random disorder and with sinusoidally graded chains carrying a single spectral component.

This framing helps clarify the apparent viscosity paradox. Disorder does not dissipate energy microscopically. Instead, it induces a progressive loss of coherence of the leading traveling pulse, while the energy is redistributed into fluctuations and secondary excitations. When the dynamics is projected onto the coherent-pulse sector, that redistribution appears as an effective damping. In this sense disorder acts

^{*}Contact author: lgomez@uns.edu.ar

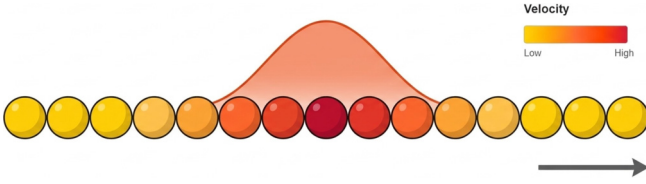


FIG. 1. Schematic drawing of a Nesterenko solitary wave traveling through a granular chain.

like viscosity for the leading solitary wave, although the full Hamiltonian dynamics remains time-reversal invariant and conserves total energy.

This work is organized as follows. Section II introduces the Hertzian chain, the clean solitary pulse, and the measured observable. Section III develops the weak-disorder effective-response framework, explains the collective-coordinate motivation for the acceleration proxy, and states the limits of the approximation. Section IV describes the numerical protocol and baseline tests. Section V presents the random-correlation and single-mode results. Section VI discusses what is established, what remains approximate, and the experimental implications. Details of the projection attempt and the regularized inverse estimate are retained in the Appendixes as motivation and diagnostics, respectively.

II. HERTZIAN CHAIN, SOLITARY WAVES, AND SHOCKS

A. Discrete chain

We consider a one-dimensional chain of spherical beads with displacements $q_i(t)$ and masses

$$m_i = m_0(1 + \epsilon\mu_i), \quad \langle \mu_i \rangle = 0, \quad \langle \mu_i \mu_j \rangle = C_{ij}. \quad (1)$$

The equations of motion are

$$m_i \ddot{q}_i = F_{i-1,i} - F_{i,i+1}, \quad F_{i,i+1} = K(q_i - q_{i+1})_+^{\alpha-1}, \quad (2)$$

with $(x)_+ = \max(x, 0)$ and $\alpha = 5/2$ for Hertzian contacts. We will work at zero precompression, where the chain is a sonic vacuum in Nesterenko's sense [3,4]: the linear sound speed vanishes, and finite-amplitude impacts generate strongly nonlinear compression pulses.

For uncorrelated disorder the μ_i are independent unit-variance Gaussian variables. For correlated disorder we generate white Gaussian noise, convolve it with a Gaussian kernel of width ξ , subtract the finite-chain mean, and renormalize to unit variance. Every mass array used in the analysis was verified to satisfy $m_i > 0$; the revised generator rejects and redraws any field that fails this check. Since the autocorrelation of a Gaussian kernel is again Gaussian, with the variances adding under convolution, the resulting lattice correlator is well approximated by [13]

$$C(j) \equiv \langle \mu_i \mu_{i+j} \rangle \simeq \exp\left(-\frac{j^2}{4\xi^2}\right), \quad (3)$$

and the corresponding disorder spectrum is the discrete Fourier transform of the correlator,

$$S_\mu(k; \xi) = \sum_{j=-\infty}^{\infty} C(j) e^{-ikj}, \quad k \in [-\pi, \pi], \quad (4)$$

which is the lattice version of the Wiener-Khinchin relation between the two-point correlator and the power spectrum [14]. In the white-noise limit $\xi = 0$ one simply has $S_\mu(k; 0) = 1$.

B. Clean Nesterenko pulse

The solitary pulse of the Hertzian chain is most naturally introduced in terms of the compressive strain $\delta(x, t)$ or overlap field. In the long-wavelength approximation one seeks traveling solutions of the form $\delta(x, t) = \delta(\zeta)$ with $\zeta = x - v_s t$. For Hertzian interactions the resulting continuum equation supports the well-known Nesterenko pulse, which provides an excellent description of the strongly nonlinear solitary waves observed in simulations and experiments [4,5]

$$\delta(\zeta) = \begin{cases} \delta_m \cos^4\left(\frac{\zeta}{R\sqrt{10}}\right), & |\zeta| \leq \frac{\pi R\sqrt{10}}{2}, \\ 0, & \text{otherwise} \end{cases}, \quad (5)$$

for the Hertz exponent $\alpha = 5/2$. Here R is the bead radius and δ_m is the peak overlap. The support of Eq. (5) spans

$$L_{\text{sw}} = \pi\sqrt{10}R \approx 5a, \quad a = 2R, \quad (6)$$

which is why the pulse occupies about five bead diameters in experiments and simulations. The contact force scales as $F \propto \delta^{3/2} \propto \cos^6$, and the pulse speed obeys the characteristic amplitude law,

$$v_s \propto \delta_m^{1/4} \propto F_m^{1/6}, \quad (7)$$

with F_m the peak force [4,5]. These relations are the basic signatures of strongly nonlinear solitary-wave transport in Hertzian chains.

Equation (5) is an accurate continuum approximation, not an exact discrete solution of Eq. (2). In the calculations below, whenever we need the clean pulse profile we extract it directly from a clean simulation of the discrete chain. We nevertheless write Eq. (5) explicitly because it remains the most compact statement of what the Hertzian solitary wave looks like and how its width and speed scale.

C. From solitary waves to shocks

Granular chains and packings can display both solitary-wave and shocklike propagation, depending on the type of driving, the amount of dissipation, and the confining pressure. Nesterenko showed that, in the sonic-vacuum regime, elastic energy is transported by strongly nonlinear solitary pulses [3,4]. Later experiments confirmed several characteristic properties of these waves, such as their nearly constant width, their amplitude-dependent speed, and their scattering at interfaces and boundaries [6–8]. When dissipation is present, chains can also support traveling shocklike fronts [9]. Likewise, under continuous compression, weakly confined granular matter develops nonlinear propagating fronts that are better understood as shocks than as ordinary linear sound waves [10]. In disordered media, the evolution is richer: the leading solitary pulse attenuates exponentially at early times, but at longer times the dynamics crosses over to a broader triangular envelope with a different, power-law-like attenuation mechanism [12,15]. Here we focus exclusively on the early regime, where the leading solitary pulse remains identifiable and its attenuation rate can be measured cleanly.

D. Measured observable

The observable whose attenuation we analyze is the amplitude of the *leading coherent pulse*. In practice we locate the rightmost identifiable velocity maximum in each time slice, refine its position and height by parabolic interpolation, and average the resulting peak amplitudes over disorder realizations. When the pulse remains well separated from the trailing fluctuations, the ensemble-mean amplitude obeys

$$\langle A(n) \rangle \simeq A_0 e^{-\Gamma n}, \quad (8)$$

where n is the distance traversed in bead units and Γ is the attenuation rate per bead. The late-time triangular envelope discussed in Refs. [12,15] is a different object and is not described by Eq. (8).

III. SPECTRAL RESPONSE OF THE COHERENT PULSE

The main physical idea is simple. Two mass landscapes can have the same variance but distribute that variance over different spatial wavelengths. Because a solitary pulse samples several neighboring beads at once, those wavelengths need not attenuate it equally. We therefore describe the attenuation as an overlap between the disorder spectrum and the spectral response of the pulse.

A. Weak-disorder spectral response

We assume that the mass disorder is weak, has zero mean, and is statistically stationary along the chain. We also restrict attention to the propagation window in which the ensemble-mean leading-pulse amplitude is well described by Eq. (8). Because the disorder has zero mean, its contribution to the averaged rate has no term linear in the disorder amplitude. The leading contribution is quadratic and can therefore be written in terms of the covariance $C(r)$ as

$$\Gamma[C] = \epsilon^2 \sum_r K_{\text{eff}}(r) C(r) + R_\Gamma, \quad (9)$$

$$\frac{R_\Gamma}{\epsilon^2} \longrightarrow 0 \quad (\epsilon \longrightarrow 0),$$

where $K_{\text{eff}}(r)$ describes how correlations separated by r beads contribute to the attenuation of the leading coherent pulse. For stationary disorder, Eq. (9) is equivalently expressed in Fourier space as

$$\Gamma(\xi) = \epsilon^2 \int_{-\pi}^{\pi} \frac{dk}{2\pi} S_\mu(k; \xi) \mathcal{W}_{\text{eff}}(k) + R_\Gamma. \quad (10)$$

Here $S_\mu(k; \xi)$ is the disorder power spectrum and $\mathcal{W}_{\text{eff}}(k)$ is the effective spectral response of the measured pulse amplitude. Only its real, even part contributes for the covariances considered here.

Equation (10) is the central representation used below. It states that attenuation is controlled not only by the total disorder variance, but also by where that variance lies in wave-number space. The kernel $\mathcal{W}_{\text{eff}}(k)$ measures how efficiently each spatial wave number removes amplitude from the leading pulse.

One may ask why Eqs. (9) and (10) pair the two spectra inside an integral instead of multiplying them at each k , as a linear-response relation between two fields would. The reason is that Γ is not a field: it is a single number attached to a

moving object and measured in the frame that follows the pulse maximum, so the free spatial argument that a convolution theorem would convert has already been removed by the measurement. The convolution is present one order earlier. In the comoving frame the first-order disorder-induced drive on the coherent amplitude is

$$\dot{A}^{(1)}(t) = \epsilon \sum_j g_{\text{sol}}[j - X(t)] \mu_j = \epsilon (g_{\text{sol}} * \mu)[X(t)],$$

where $X(t)$ is the pulse position in bead units and g_{sol} is the projection-weighted filter of Appendix A, localized on the scale of the pulse width. In the bare constant-projection approximation the filter reduces to the clean acceleration profile of the pulse, $g_{\text{sol}}(j) \propto h_{\text{sol}}(j) = \ddot{q}_j^{\text{sol}}$; its Fourier transform defines the bare kernel $\mathcal{W}_{\text{bare}}$ of Eq. (12) below.

The pulse therefore acts as a linear spatial filter, weighting the landscape across its own support, but the measured rate is a second moment of that filtered signal: the drive at one instant must be propagated and reprojected against the drive at another (Appendix A). Averaging over stationary disorder contracts the two lag indices, and Parseval turns the resulting real-space pairing into Eq. (10). The pointwise product one expects is in fact inside the integral: $S_\mu(k) \mathcal{W}_{\text{eff}}(k)$ is the spectral contribution to the quadratic drift of the coherent amplitude, and Γ is its integral over the zone. In the bare approximation, where $\mathcal{W}_{\text{bare}}(k) \propto |\widetilde{g}_{\text{sol}}(k)|^2$, that contribution is literally the power spectrum of the drive and the relation is the standard transfer-function statement; for the full memory- and radiation-resolved kernel the same reading is not automatic. The overlap structure itself is familiar: it appears in the Born approximation for a wave in a random medium, where an inverse mean free path is an overlap of the disorder spectrum with a form factor set by the scatterer [16,17], and in a Green-Kubo transport coefficient. In both, a spectrally resolved input produces a scalar output.

This representation does not, by itself, provide a microscopic calculation of $\mathcal{W}_{\text{eff}}$. The kernel includes the response of the pulse position and shape, as well as energy transferred to secondary pulses and to the fluctuating wake. The measured scaling $\Gamma \propto \epsilon^2$ supports the leading quadratic approximation, while the approximately exponential decay supports the use of a fitted local rate over the reported propagation window. Neither observation determines the complete kernel or proves that the full chain has a memoryless reduced dynamics.

B. Acceleration-spectrum proxy

To obtain a simple prediction for the shape of the response, we ask how mass disorder first enters the dynamics. Expanding about the clean solitary-wave trajectory shows that the leading disorder-induced force on bead i is

$$f_i^{(1)}(t) = -\epsilon \mu_i m_0 \ddot{q}_i^{\text{sol}}(t). \quad (11)$$

Thus, at first order, the local mass variation is weighted by the acceleration of the clean pulse. This suggests that the squared Fourier amplitude of the clean acceleration profile provides the simplest bare approximation to the spectral

response:

$$\mathcal{W}_{\text{bare}}(k) \propto |\widetilde{h}_{\text{sol}}(k)|^2, \quad \widetilde{h}_{\text{sol}}(k) = \sum_j h_{\text{sol}}(j) e^{-ikj}. \quad (12)$$

Here $h_{\text{sol}}(j) = \ddot{q}_j^{\text{sol}}$ is the *spatial* acceleration profile of the clean pulse at a fixed reference time, with j labeling beads relative to the pulse center and support of about five beads, and $\widetilde{h}_{\text{sol}}(k)$ is its discrete Fourier transform; i is the imaginary unit. Because the clean pulse translates with essentially fixed shape, $h_{\text{sol}}(j)$ is equivalent to the acceleration history of a single bead under $j \leftrightarrow -v_s t/a$. h_{sol} is extracted from a clean discrete-chain simulation at the corresponding impact speed. We use Eq. (12) only as a parameter-free prediction for the *normalized shape* of the response. It is not identified with the complete kernel $\mathcal{W}_{\text{eff}}$. Projecting the forcing onto the amplitude of a moving pulse requires the appropriate phase-space response mode, and the result can also be modified by pulse-shape relaxation, secondary-wave emission, and delayed feedback from the wake. These contributions are summarized by $\mathcal{W}_{\text{eff}}$ and are measured operationally below using sinusoidally graded chains. The corresponding collective-coordinate and linear-response formulation is given in Appendix A.

C. Uniform and long-wavelength mass variations

The zero-wave-number limit provides an exact constraint. A $k = 0$ perturbation changes all bead masses by the same amount and therefore leaves the chain spatially homogeneous. Such a change can modify the pulse speed, launch transient, or reference amplitude, but it cannot produce cumulative attenuation by scattering from mass inhomogeneity. Consequently,

$$\mathcal{W}_{\text{eff}}(0) = 0. \quad (13)$$

The bare acceleration proxy has the same property because the internal contact forces telescope when summed over the chain, giving $\sum_j h_{\text{sol}}(j) = 0$ and hence $\widetilde{h}_{\text{sol}}(0) = 0$.

Equation (13) explains why sufficiently smooth mass variations produce little scattering. It does not, however, prove that attenuation must decrease monotonically with correlation length: the behavior at finite ξ depends on the full finite- k structure of $\mathcal{W}_{\text{eff}}$. The monotonic suppression observed over $0 \leq \xi \leq 6$ is therefore a numerical result, tested below against the independently measured spectral response.

Finally, the normalized clean-acceleration spectrum changes only weakly over the simulated impact-amplitude range because the solitary pulse remains about five beads wide. This gives a qualitative explanation for the weak amplitude dependence of the measured rate, without claiming a complete microscopic prediction.

D. Equivalent viscous damping

The relation between disorder-induced attenuation and viscous damping is easiest to see in a continuum traveling-wave description of a Hertzian chain, written in the style commonly used for nonlinear waves and shocks in granular media. Let $\delta(x, t)$ denote the local strain (or overlap) field. In the absence of precompression, the continuum equation for δ is nonlinear

and dispersive, and it supports localized traveling pulses. If one adds a linear viscous contact damping with coefficient η , and then looks for a traveling-wave solution $\delta(x, t) = \delta(\tilde{t})$ with

$$\tilde{t} = t - \frac{x}{v_s}, \quad (14)$$

the partial differential equation reduces, after integration, to an ordinary differential equation of the form [10]

$$\frac{d^2 \delta}{d\tilde{t}^2} = -\frac{dW_0}{d\delta} - 12\eta \frac{d\delta}{d\tilde{t}}. \quad (15)$$

The first term on the right is conservative and can be written in terms of an effective potential $W_0(\delta)$. The second is a linear friction term. Thus, in the comoving coordinate $\tilde{t}$, the traveling pulse is mathematically equivalent to an effective particle moving in the potential W_0 and subject to viscous damping. In the undamped case ($\eta = 0$), the solitary wave corresponds to a homoclinic orbit of this effective-particle problem; when $\eta > 0$, the same reduced equation predicts an exponential attenuation of the coherent pulse.

This provides a natural way to define an *equivalent* viscosity for the disordered chain. Our disorder calculation yields an exponential decay of the coherent-pulse amplitude with rate $\Gamma(\xi)$ per bead. Converting this to a decay rate per unit time gives $(v_s/a)\Gamma(\xi)$, since the pulse advances by one lattice spacing a in a time a/v_s . Matching that decay rate to the friction term in Eq. (15) defines

$$\eta_{\text{eq}}(\xi) = \frac{v_s}{12a} \Gamma(\xi). \quad (16)$$

Equation (16) is the precise sense in which “disorder acts as viscosity” here. The underlying chain remains Hamiltonian and does not lose energy microscopically. Rather, η_{eq} is the viscous coefficient that, in the reduced traveling-wave description, would produce the same attenuation rate of the *coherent leading pulse* as the Hamiltonian scattering and redistribution caused by disorder.

Because the pulse moves through the quenched landscape, a spatial Fourier component k appears in the comoving frame as a temporal forcing of frequency $\omega = v_s k/a$. This gives Eqs. (10) and (16) a Green-Kubo-like spectral structure, with the disorder spectrum in the role of a forcing spectrum and $\mathcal{W}_{\text{eff}}$ in that of a dissipative response. The analogy stops short of a fluctuation-dissipation theorem: the disorder is quenched, the wake is generated dynamically by the passage of the pulse, and there is neither an equilibrium bath nor detailed balance nor a temperature fixing a fluctuation-response ratio, as would be required for an equilibrium or generalized steady-state relation [18]. The disorder does, however, produce at the same $\mathcal{O}(\epsilon^2)$ both the mean drift reported as Γ and a variance of the leading-pulse amplitude across realizations, visible in Fig. 3(b). Whether the two are related by a spectrum-independent ratio remains an open question.

E. Range of validity and status of the approximations

Four distinct assumptions delimit the present description. First, the leading quadratic response in Eq. (10) requires weak, zero-mean disorder. The measured linearity of Γ with ϵ^2 is an empirical consistency check; it does not bound every

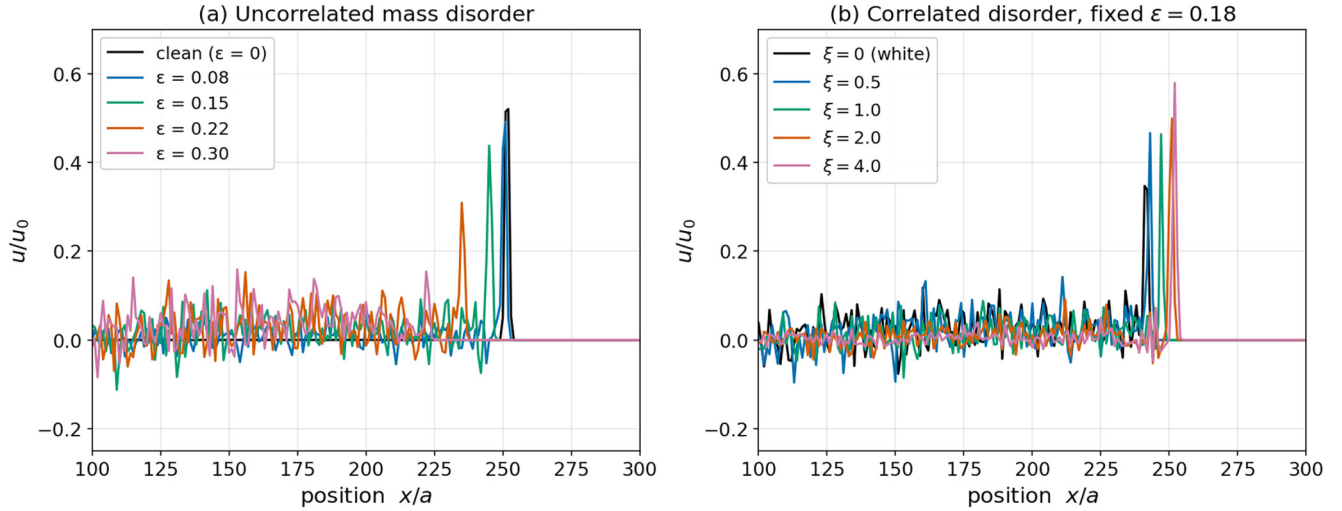


FIG. 2. Snapshots of the longitudinal velocity field in the one-dimensional disordered Hertzian chain. (a) Uncorrelated mass disorder with increasing strength ϵ . The clean pulse propagates as a narrow solitary wave; increasing disorder lowers the leading peak and leaves behind a fluctuating trail. (b) Fixed disorder strength $\epsilon = 0.18$ and increasing correlation length ξ . The leading peak recovers strongly as the disorder is spatially smoothed, even though the variance of the masses is unchanged. This is the main qualitative message of the paper: attenuation depends on the spectrum of the disorder, not only on its variance.

higher-order contribution. Second, the description assumes that the leading coherent pulse remains identifiable and separated from its wake. This fails at strong disorder and after the crossover to the late triangular envelope. Third, the phase-space projection weight discussed in Appendix A has not been obtained from the full adjoint problem; replacing it by a slowly varying envelope is one approximation behind $\mathcal{W}_{\text{bare}}$. Fourth, radiation and multiplicative feedback generate an $\mathcal{O}(\epsilon^2)$ retarded contribution. A local exponential rate is therefore an empirical closure over the reported fitting windows, not a derivation that makes the omitted memory disappear.

For random correlated disorder, a useful scale estimate is $\tau_c \sim (L_{\text{sw}} + \xi)/v_s$ for the pulse to traverse the forcing landscape, compared with $\tau_A \sim a/(v_s \Gamma)$ for attenuation. The product $\Gamma(L_{\text{sw}} + \xi)$ is below roughly 10^{-1} for the quoted rates and sizes, which makes a short-memory treatment plausible but does not calculate the radiation Green function. For the sinusoidally graded chain, the temporal correlation is instead persistent and periodic. Those simulations are consequently used as an empirical single-frequency probe and cross-ensemble test of Eq. (10); they are not a validation of the short-memory argument.

The exact content is narrower: a uniform mass component produces no cumulative scattering-induced attenuation, Eq. (13). The finite- k kernel, the observed monotonic variation with ξ , and the accuracy of the acceleration proxy are numerical results. The description also requires $\xi \ll N$ and propagation windows that end before boundary reflections or loss of the leading peak.

IV. NUMERICAL METHODS AND RESULTS

The discrete chain, Eq. (2), is integrated with a velocity-Verlet scheme using $K = m_0 = a = 1$ and $dt = 0.02$. The pulse is launched by assigning the leftmost bead an impact velocity v_0 . At each recorded time we identify the rightmost

velocity maximum that remains separated from the trailing field, refine its position and height parabolically, and stop using a trace once that leading maximum can no longer be identified by the same rule. Rates are obtained by a linear fit of $\log(A(n))$ over the stated propagation window; the same estimator is used when comparing spectra.

For the correlation-length sweep in Fig. 5(a), $\epsilon = 0.18$, the chains contain 600–1300 beads and the ensemble contains 14–28 disorder realizations, with the longer chains and larger ensembles used for the smaller rates. We fit the ensemble-mean amplitude rather than average rates fitted separately to individual traces. Error bars are obtained by resampling complete realizations and repeating the averaging and fit. Because every normalized rate shares the fitted $\Gamma(0)$ denominator, their uncertainties are correlated; the plot shows marginal bootstrap intervals and we do not assign a diagonal-error-reduced χ^2 . The single-mode study uses eight uniformly spaced launch phases at each k_0 . The older baseline figures use chains of 400–700 beads and three to ten realizations, as stated with the corresponding results.

Phenomenology. Figure 2 shows the basic effect of disorder on the velocity profiles. For uncorrelated disorder, increasing the mass variance lowers the leading peak and leaves behind a fluctuating wake. Note that the lost coherent amplitude does not disappear: it is redistributed into smaller excitations and irregular fluctuations trailing the main front. When the variance is held fixed and the disorder is spatially smoothed, the leading peak recovers strongly. This already shows that the attenuation cannot be a variance-only effect.

Variance scaling. Figure 3 shows the attenuation for six disorder strengths in the white-noise case. Each curve is an ensemble mean over ten independent mass configurations. While the pulse remains identifiable, the decay is approximately exponential and the fitted rates are linear in ϵ^2 over the displayed range. The dispersion in panel (b) is obtained from the ten configurations and is used as a regime check

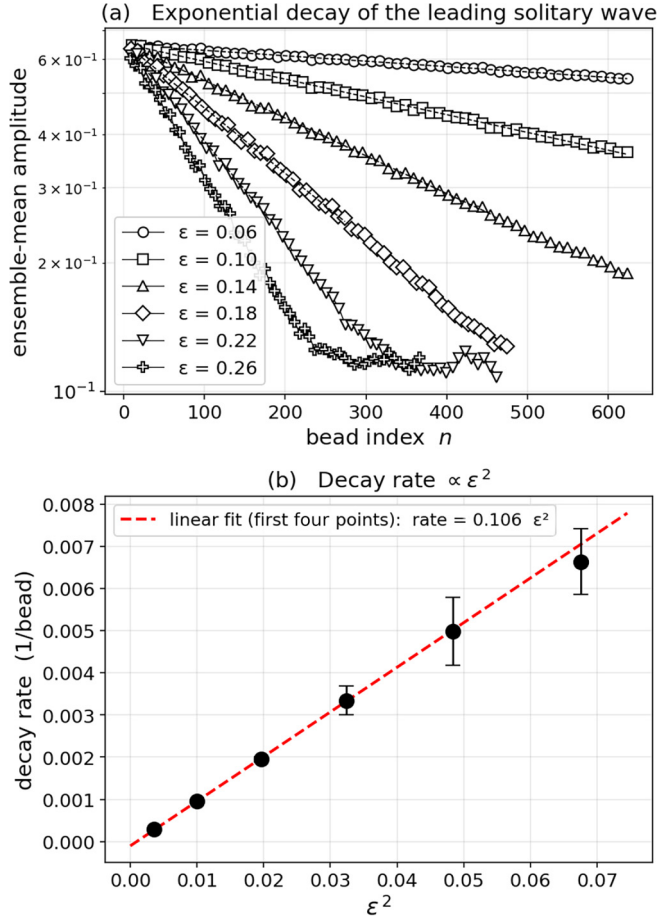


FIG. 3. (a) Leading-pulse amplitude averaged over ten independent uncorrelated mass configurations for each disorder strength ϵ ; dashed lines are exponential fits over the range in which the pulse is identifiable. (b) Corresponding fitted rate vs ϵ^2 , with dispersion estimated from the ten configurations. The linear trend is consistent with a leading quadratic response.

rather than as proof of the collective-coordinate truncation. This behavior is consistent with retaining the leading term of Eq. (10).

Amplitude dependence. Figure 4 tests the pulse-amplitude dependence at fixed $\epsilon = 0.15$. Varying v_0 from 0.4 to 1.8 changes the observed pulse amplitude by roughly a factor of 4.5, while Γ/ϵ^2 varies comparatively weakly. This is an empirical result over the displayed range. The nearly amplitude-independent width of the Hertzian pulse makes it qualitatively compatible with the bare spectral proxy, but a quantitative amplitude dependence would require the missing effective projection. This near-independence is a separate matter from the additivity over wave numbers discussed in Sec. V A and does not bear on it; a fixed pulse width constrains the normalized shape of the response, not its absolute scale.

Energy conservation and the meaning of effective damping. We checked that the total kinetic plus potential energy is conserved to numerical precision over windows in which the leading-pulse amplitude decreases by more than 90%. The decrease of the measured coherent component is accompanied by energy in secondary pulses and the irregular wake. It is

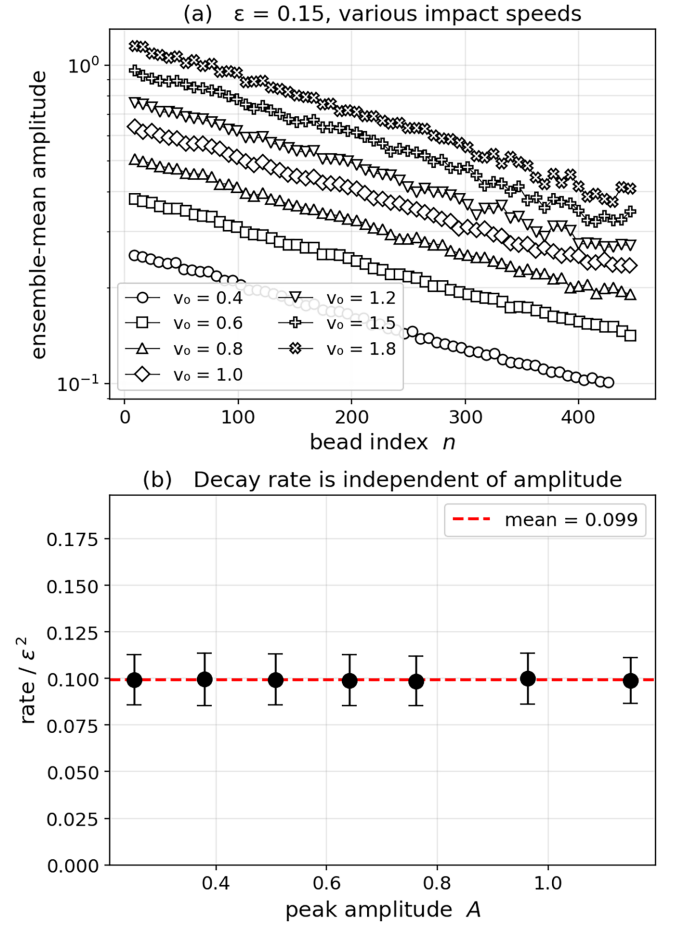


FIG. 4. (a) Ensemble-mean attenuation at $\epsilon = 0.15$ for several impact speeds; the curves are nearly parallel on the semilogarithmic plot. (b) Normalized rate Γ/ϵ^2 vs observed pulse amplitude. The variation is weak over the range tested.

therefore an effective attenuation of a projected observable, not microscopic dissipation.

V. CORRELATION-LENGTH DEPENDENCE AND SPECTRAL PROBES

At fixed variance ϵ^2 with $\epsilon = 0.18$, the normalized rate decreases from 1 at $\xi = 0$ to 0.0239 at $\xi = 6$ [Fig. 5(a)], a factor of about 42 (approximately 1.62 decades). This is the principal numerical result: smoothing the same amount of mass disorder strongly protects the coherent leading pulse.

For comparison, we extract the acceleration profile of a clean discrete pulse at the same impact speed and insert $\mathcal{W}_{\text{bare}}$ from Eq. (12) into Eq. (10). The red curve in Fig. 5(a) is normalized by its value at $\xi = 0$. It captures the qualitative fall of the rate but shows systematic quantitative deviations. Because the prediction and data are each divided by their own $\xi = 0$ value, this panel tests the *normalized shape*, not the absolute response prefactor; the absolute comparison is reported separately below.

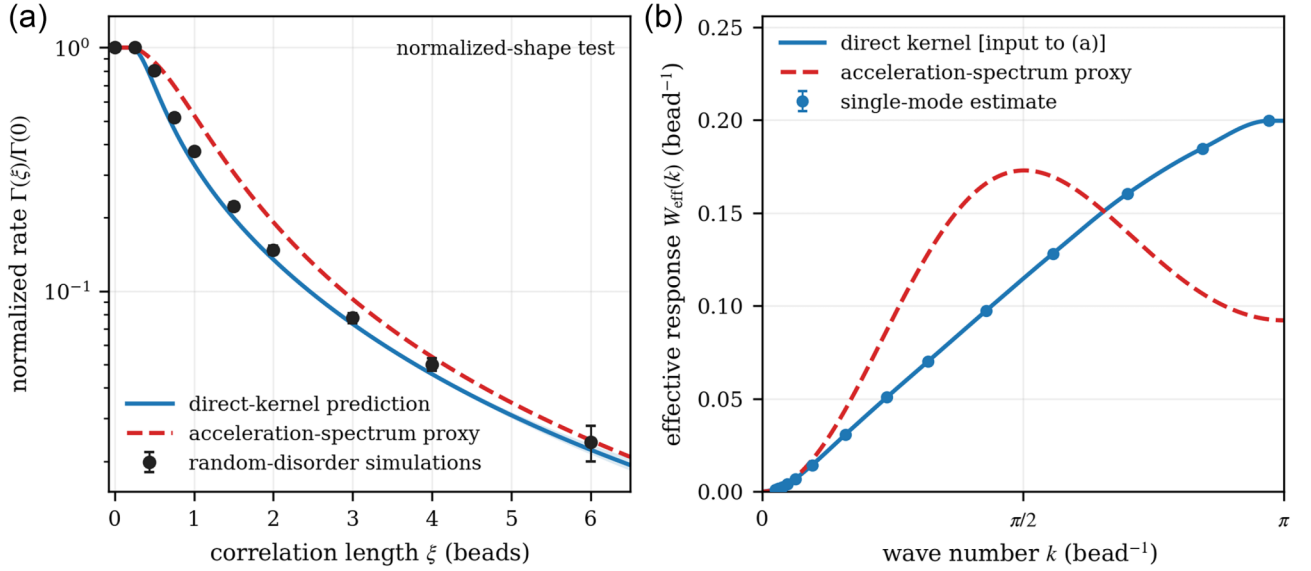


FIG. 5. Correlation suppression and effective spectral response. (a) Normalized attenuation rate $\Gamma(\xi)/\Gamma(0)$ for Gaussian-correlated mass disorder at fixed variance. Black circles are simulation data with marginal bootstrap error bars. The blue curve and pointwise 95% band are the prediction obtained from the independently measured single-mode response in the spectral overlap; the band propagates the phase-bootstrap standard errors of the single-mode measurements but excludes systematic and end-point uncertainties. The red curve uses the clean-acceleration bare proxy. Every curve is normalized separately at $\xi = 0$, so this is a normalized-shape comparison. The measured decrease from $\xi = 0$ to 6 is a factor of about 42. (b) Single-mode estimates $\mathcal{W}_{\text{eff}}(k_0) = \Gamma(k_0)/\epsilon^2$ from the sinusoidally graded chains for $k \geq 0$, together with the direct kernel (solid blue): a shape-preserving interpolation of these points that imposes $\mathcal{W}_{\text{eff}}(0) = 0$ and uses the stated constant continuation from the last measured point to the Brillouin-zone edge. This kernel is the input to the prediction in (a). The acceleration-spectrum proxy (dashed red) is scaled to the same Brillouin-zone average as the direct kernel for comparison; it is not a fit of a microscopic adjoint theory.

Single-mode response in sinusoidally graded chains

To probe the response without inverting the random-correlation data, we simulate an ordered chain whose masses vary sinusoidally. For each finite chain and launch phase, define

$$c_i = \cos(k_0 i + \phi), \quad \bar{c} = \frac{1}{N} \sum_{j=1}^N c_j, \quad s_c^2 = \frac{1}{N} \sum_{j=1}^N (c_j - \bar{c})^2, \quad (17)$$

and set

$$m_i = m_0[1 + \epsilon g_i], \quad g_i = \frac{c_i - \bar{c}}{s_c}. \quad (18)$$

Thus every simulated phase has exactly zero mean and unit variance on its finite chain, and all masses remain positive. In the infinite or commensurate-chain limit the spectrum consists of the pair $\pm k_0$; the two peaks share unit total spectral weight. Finite-window centering removes the residual zero mode. For noncommensurate finite chains, the window produces weak spectral leakage around the two peaks; this finite-size effect is part of the simulated response, and Eq. (19) is its leading large-window interpretation.

Applying the leading quadratic representation to this two-line spectrum gives

$$\frac{\Gamma(k_0)}{\epsilon^2} = \mathcal{W}_{\text{eff}}(k_0) + o(1), \quad (19)$$

where evenness combines the two equal spectral weights. Accordingly, $\Gamma(k_0)/\epsilon^2$ is a direct, nonparametric estimate of the effective response *within the weak-disorder quadratic framework*. It does not assume the acceleration shape and requires neither deconvolution nor regularization, but it does assume that a secular exponential rate can be separated from the phase-dependent oscillation.

The leading pulse is located at each output time as the last velocity maximum exceeding a fixed detection threshold (10% of the impact speed), refined by parabolic interpolation, and subject to a monotonicity constraint on the tracked position: tracking terminates at the first backward jump. This safeguard is essential for the strongly attenuated modes. Without it, once the pulse amplitude approaches the threshold the tracker latches onto slower trailing oscillations, which produces a spurious amplitude plateau pinned at the threshold and biases the fitted rate low; the bias grows with Γ and is largest near the Brillouin-zone edge. For the same reason, the exponential fits are restricted to phase-mean amplitudes at least twice the detection threshold. Tim-estep convergence was verified directly: halving and quartering the integration step changes the fitted rates by less than 0.1%.

We average eight uniformly spaced phases at each k_0 , fit the logarithm of the phase mean over the range shared by all phases, and estimate uncertainties by bootstrapping whole phases through the complete estimator. Figure 6(a) shows the phase-mean trace and its uncertainty band only over the range shared by all phases. The long-wavelength modulation ($k_0 = 0.10$) produces little cumulative decay, whereas the finite-

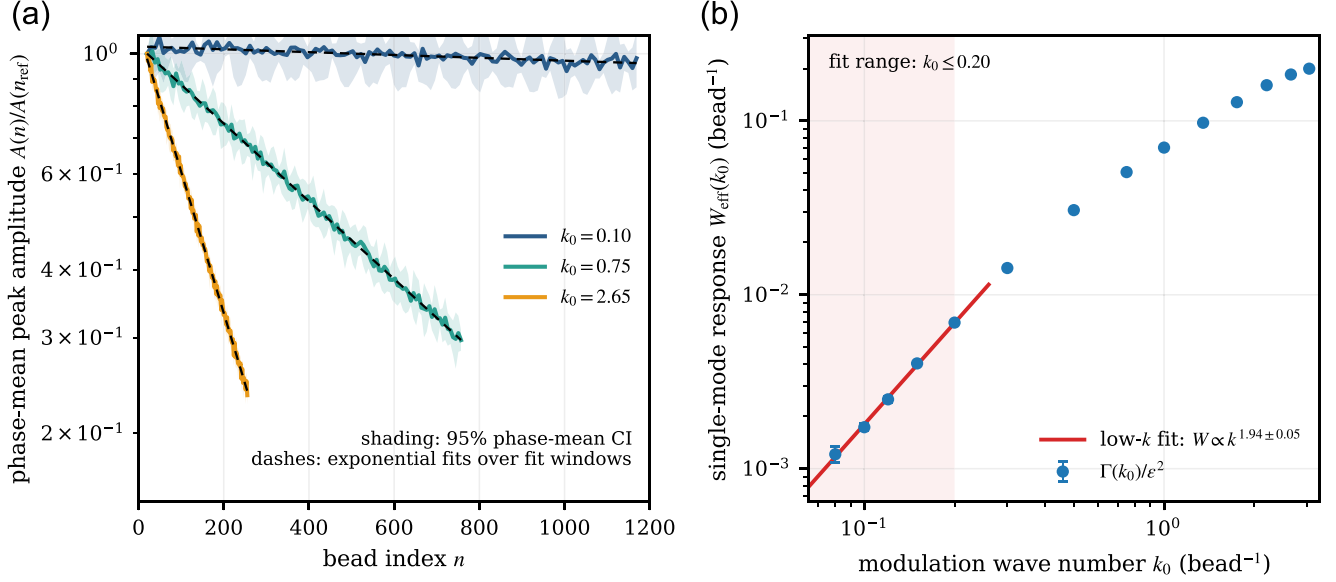


FIG. 6. Sinusoidally graded-chain probe. (a) Representative leading-pulse amplitudes for $k_0 = 0.10, 0.75$, and 2.65 , normalized at the start of the common fitting range and averaged over eight launch phases. Bands are 95% intervals for the phase mean; black dashed lines are exponential fits, truncated where the phase-mean amplitude approaches the peak-tracking threshold. Each curve ends at the last position for which all phases contribute a tracked leading pulse. (b) Direct quadratic-response estimate $\Gamma(k_0)/\epsilon^2$ with phase-bootstrap uncertainties. The low- k_0 line is a fitted power law over $k_0 \leq 0.20$, rather than a guide anchored to one point; its exponent is reported in the panel. The response rises monotonically toward the Brillouin-zone edge.

wave-number cases attenuate much more strongly at the same mass variance; the tracked range shortens correspondingly, and the $k_0 = 2.65$ trace terminates once the pulse reaches the truncation floor. Within these windows the phase-mean decay is well described by a single exponential, but Γ remains an operational one-rate summary over the stated window. The individual periodic chains retain a persistent oscillatory correlation, so they do not satisfy the short-memory premise used to motivate a Markov closure for random disorder. They are instead an empirical frequency-selective probe. Moreover, although the uncompressed chain has no conventional linear phonon band, nonlinear scattering and secondary-pulse production remain possible and are included in the measured W_{eff} .

The full estimate in Fig. 6(b) approaches zero at long wavelength. An unweighted ordinary-least-squares fit of $\log W_{\text{eff}} = \log a + p \log k_0$ over the five values $k_0 \leq 0.20$ gives $p = 1.94 \pm 0.05$ (one standard error from the residual scatter) and $a = 0.155$. This is consistent with a quadratic node over the resolved range, but does not by itself prove the $k = 0$ limit. At higher k_0 the measured response rises monotonically toward the Brillouin-zone edge, reaching $W_{\text{eff}} \simeq 0.20$ at $k_0 = 3.05$; no monotonicity is imposed on these points, and the acceleration-spectrum proxy, which instead peaks near $k \approx \pi/2$ and decreases toward the edge, fails to reproduce this behavior [Fig. 5(b)].

Finally, we insert a shape-preserving interpolation of the measured single-mode response—the direct kernel shown in Fig. 5(b)—into Eq. (10) for the independent Gaussian-correlated spectra. The blue curve and Monte Carlo band in Fig. 5(a) reproduce the main correlation-induced suppression and improve the shape relative to the bare acceleration proxy, with a mean absolute deviation from the random-

disorder points of about 9% and visible residual discrepancies at intermediate ξ . The displayed band propagates the phase-bootstrap standard errors of the single-mode measurements pointwise; it does not include systematic fit-window or endpoint uncertainty. The interpolation imposes the exact endpoint $W_{\text{eff}}(0) = 0$ and, because measurements stop at $k_0 = 3.05$, uses an explicit constant continuation from that point to π . On an absolute scale, its Brillouin-zone average is approximately 0.110, compared with the random-chain value $\Gamma(0)/\epsilon^2 \simeq 0.107$: the two independent ensembles agree to about 2.5% without any adjustable parameter. Because the displayed panel normalizes each curve at $\xi = 0$, Fig. 5(a) remains a normalized-shape test; the absolute agreement is an additional cross-ensemble consistency check, quoted without a full systematic error budget. This comparison supports the effective spectral-response picture, but it is not an assumption-free derivation of the full microscopic kernel.

That single-mode responses combine in this way is not accidental within the leading quadratic framework. After ensemble or phase averaging, statistical stationarity diagonalizes the two-point covariance in Fourier space, so distinct wave numbers contribute additively to Γ at $\mathcal{O}(\epsilon^2)$. This is linearity in the disorder power spectrum, not linearity of the Hertzian dynamics; the nonlinear pulse response remains encoded in $W_{\text{eff}}(k)$. The agreement in Fig. 5(a) therefore tests the applicability of the leading quadratic representation at the disorder strength studied, rather than establishing exact superposition. It also indicates insensitivity to the higher cumulants of the disorder over the range probed: the two ensembles have very different one-point statistics—Gaussian in the random chains, bounded and bimodal in the graded ones—yet the same two-point information reproduces the measured rate.

Both are symmetric, so all odd cumulants vanish and the first non-Gaussian corrections can arise through four-point terms at $\mathcal{O}(\epsilon^4)$; for a generic skewed zero-mean disorder field, third-cumulant contributions may already appear at $\mathcal{O}(\epsilon^3)$. Departures from additivity should therefore be sought at larger ϵ or for skewed disorder. A narrow-band spectrum at small k is a different matter: its slowly decaying correlations threaten the local exponential-rate and short-memory interpretation of Sec. III E rather than the additivity itself. We attribute the 9% residual shape discrepancy to fit-window and interpolation systematics.

VI. DISCUSSION

A. Relation to the collision picture

In the white-noise limit $\xi = 0$, the disorder spectrum is flat and Eq. (10) reduces, at leading order, to

$$\Gamma(0) = \epsilon^2 \int_{-\pi}^{\pi} \frac{dk}{2\pi} \mathcal{W}_{\text{eff}}(k) + R_{\Gamma}. \quad (20)$$

Only the integrated response is sampled, so spectral information is lost. This is the sense in which the framework reduces to a variance-only attenuation law, consistent with the quasiparticle collision picture of Ref. [12]. Our normalized white-noise rate is close to, though somewhat below, the $1/8$ amplitude-decay value quoted there. The two approaches use different coherent-pulse estimators and approximations; their common content is the early approximately exponential regime and leading variance scaling.

B. What is established and what remains approximate

The strongest conclusions are numerical. Over the resolved early propagation window the leading-pulse amplitude is approximately exponential; the baseline sweep is consistent with $\Gamma \propto \epsilon^2$; at $\epsilon = 0.18$ the rate falls by a factor of about 42 between $\xi = 0$ and 6, and ordered single-mode chains exhibit a strongly wave-number-dependent rate at the same variance. Using the single-mode response in the random-disorder overlap captures the principal suppression across a different disorder ensemble.

Equation (10) is a general leading quadratic, translation-invariant effective-response representation, conditional on the existence of the fitted local rate. We have not derived $\mathcal{W}_{\text{eff}}$ from the full phase-space propagator. The collective-coordinate construction motivates the acceleration drive, but the adjoint choice, radiation, memory, and replacement of the exact response by Eq. (12) remain approximations. In particular, a Markov approximation cannot include a radiation term that has not first been derived; those effects are represented in the operationally measured $\mathcal{W}_{\text{eff}}$, not calculated by the bare kernel. A complete microscopic theory would compute the left modes and Green function about the moving discrete pulse and only then assess a local-memory limit.

Finally, “reversible” refers to the time-reversal-invariant, energy-conserving equations of the full chain. We have demonstrated redistribution out of the chosen coherent observable, not spontaneous recovery of the leading pulse in a forward run.

C. Experimental implications

The results suggest a direct experiment using chains with the same mean mass and mass variance but different spectral organization. A particularly clean implementation would keep bead diameter and contact material fixed while varying inertia through bead density, hollow interiors, or securely attached internal masses. Changing bead radius alone is not equivalent, because it also changes geometry and the Hertz contact coefficient. Periodic sequences can approximate selected k_0 values, while shuffled and smoothly correlated sequences provide broad- and low-wave-number comparisons. The simulations predict little cumulative attenuation for a long-wavelength mass grading and a larger response when the grading lies in the measured finite- k band. Measuring several k_0 values would map the operational response point by point, with the same caveat that a periodic chain probes persistent rather than short-memory forcing.

This idea fits naturally within the broader granular-chain literature on pulse mitigation by structured heterogeneity. Tapered and decorated chains were shown early on to be efficient impulse absorbers, precisely because a deliberately engineered sequence of inertial mismatches redistributes and weakens the incident strongly nonlinear pulse [19]. More recently, inverse tapered chains have been shown to produce solitary-wave expansion and impact decimation in short systems [20]. Our result is complementary to those studies. Rather than changing the mean mass profile or tapering the chain, we isolate the effect of spectral organization at fixed disorder variance: the data are consistent with attenuation governed by the overlap between the disorder spectrum and an effective pulse response. In this sense, correlated disorder offers a route to engineering apparent damping in granular metamaterials without introducing microscopic dissipation.

VII. CONCLUSION

We have shown numerically that equal-variance mass landscapes can attenuate a Hertzian solitary pulse very differently when their spectral content differs. Gaussian correlations reduce the measured rate by a factor of about 42 over the studied range, and sinusoidally graded chains provide a direct finite-wave-number response estimate within the weak-disorder quadratic framework. A low- k fit gives $\mathcal{W}_{\text{eff}} \propto k^{1.94 \pm 0.05}$ over the resolved interval, consistent with the exact absence of cumulative attenuation for a uniform mass component. Reusing the measured single-mode response for the random correlated spectra captures the main suppression without assuming the acceleration-kernel shape.

The theoretical status is deliberately limited. Translation invariance and leading quadratic response give the spectral representation in Eq. (10); the clean acceleration spectrum is a heuristic bare proxy. A complete adjoint-, radiation-, and memory-resolved kernel has not been derived. The main open problem is precisely that microscopic calculation, together with wider phase and disorder ensembles and controlled narrow-band random tests. The established physical message does not depend on calling the bare approximation exact: coherent attenuation in these Hamiltonian chains is strongly sensitive to where the disorder variance lies in wave-number space.

ACKNOWLEDGMENTS

L.R.G. thanks V. Vitelli for useful discussions at early stages of this project. This work was supported by the National Research Council of Argentina, CONICET (Grant No. PIP 11220200103059), the Fondo para la Investigación Científica y Tecnológica (FONCYT, Grant No. PICT-2021-1272), the Universidad Nacional del Sur, and the Humboldt Foundation through the Georg Forster fellowship.

DATA AVAILABILITY

Data supporting this study are openly available in Zenodo at [21].

APPENDIX A: WHAT A COMPLETE PROJECTION CALCULATION WOULD REQUIRE

This Appendix records the collective-coordinate motivation without presenting it as a completed derivation of $\mathcal{W}_{\text{eff}}$. To display the origin of the bare forcing used in the main text, first write

$$q_i(t) = q_i^{\text{sol}}(t) + w_i(t). \quad (\text{A1})$$

Expanding Eq. (2) about the moving clean pulse gives

$$m_0 \ddot{w}_i + \sum_j L_{ij}(t) w_j = -\epsilon \mu_i m_0 \ddot{q}_i^{\text{sol}}(t) - \epsilon \mu_i m_0 \ddot{w}_i, \quad (\text{A2})$$

where $L_{ij}(t)$ is the Hessian evaluated along the clean trajectory. The first term on the right gives Eq. (11); the second is multiplicative feedback through the disorder-induced field.

In phase space, let $\mathbf{z}_{\text{sol}}(A, X)$ denote the moving-pulse manifold and write

$$\mathbf{z}(t) = \mathbf{z}_{\text{sol}}[A(t), X(t)] + \mathbf{r}(t). \quad (\text{A3})$$

The tangent vectors $\Phi_A = \partial_A \mathbf{z}_{\text{sol}}$ and $\Phi_X = \partial_X \mathbf{z}_{\text{sol}}$ span the retained directions. Covectors ψ_A and ψ_X may be normalized by

$$\langle \psi_\alpha, \Phi_\beta \rangle = \delta_{\alpha\beta}, \quad \alpha, \beta \in \{A, X\}, \quad (\text{A4})$$

and used to impose $\langle \psi_\alpha, \mathbf{r} \rangle = 0$. These scalar conditions define coordinates once a projection convention is supplied; they do not uniquely construct the covectors in the full $2N$ -dimensional phase space. That construction requires the left linearized evolution, boundary conditions, and normalization along the moving orbit.

Projecting the additive part of the mass-disorder forcing onto an amplitude covector produces

$$\dot{A} = \epsilon \sum_i \chi_i(A) \mu_i \ddot{q}_i^{\text{sol}}(t) + \mathcal{O}(\epsilon^2, \epsilon r), \quad (\text{A5})$$

which motivates the acceleration observable in the main text. It also identifies the filter of Sec. III A: g_{sol} is the comoving-frame form of $\chi_i(A) \ddot{q}_i^{\text{sol}}$, which reduces to the clean acceleration profile h_{sol} when χ_i is replaced by a constant. Equation (A5) alone does not imply a negative drift of $\langle A \rangle$. At second order, a van Kampen or response calculation must propagate the first perturbation before it is reprojected.

Schematically, the required contribution has the structure

$$\mathcal{D}_A^{(2)}(t) = \epsilon^2 \int_0^\infty d\tau \langle \langle \psi_A(t), \mathbf{B}_\mu(t) \mathbf{U}(t, t - \tau) \mathbf{B}_\mu(t - \tau) \mathbf{z}_{\text{sol}}(t - \tau) \rangle \rangle, \quad (\text{A6})$$

where $\mathbf{U}$ is the propagator of the full linearization about the moving pulse and $\mathbf{B}_\mu$ denotes the disorder coupling. Derivatives of the projection with respect to A and X also enter. The component of $\mathbf{U}$ outside the two-coordinate manifold is precisely the radiation channel.

The multiplicative term $-\epsilon \mu_i m_0 \ddot{w}_i$ in Eq. (A2) must also be retained: since $w_i = \mathcal{O}(\epsilon)$, its reprojected contribution at $\mathcal{O}(\epsilon^2)$ to the rate and is generally retarded. A Markov approximation may be considered only after the kernel in Eq. (A6), including this radiation contribution, has been evaluated. It cannot be used to claim that an omitted term has automatically been included.

For stationary disorder, averaging the complete bilinear response over translations would yield a real-space functional of $C(i - j)$ and hence the Fourier representation in Eq. (10). In the present paper that representation is introduced directly as the leading translation-invariant effective response. Replacing $\mathbf{U}$ and the covectors by an instantaneous constant-weight projection gives the acceleration spectrum in Eq. (12); this is why we label it $\mathcal{W}_{\text{bare}}$. No $\mathcal{O}(\epsilon^4)$ remainder is assigned to that approximation, because the omitted radiation and memory contributions can already modify the coefficient at $\mathcal{O}(\epsilon^2)$.

APPENDIX B: REGULARIZED INVERSE ESTIMATE AS A DIAGNOSTIC

For completeness, we also examined whether the ten random-disorder measurements could be inverted to estimate a kernel. On a discrete wave-number grid the diagnostic minimizes

$$\mathcal{J}(\mathbf{W}) = \sum_i \frac{[F_i - \hat{F}_i(\mathbf{W})]^2}{\sigma_i^2} + \lambda \|\mathbf{D}_2 \mathbf{W}\|^2, \quad (\text{B1})$$

$$W_j \geq 0, \quad W_j = W_{-j},$$

where $F_i = \Gamma(\xi_i)/\Gamma(0)$, D_2 is a second-difference matrix, and $\hat{F}_i$ is the normalized overlap predicted from $\mathbf{W}$. Evenness and non-negativity are imposed constraints rather than outcomes. Moreover, ten normalized measurements cannot determine a fine wave-number grid without substantial regularization, and all F_i share the noisy denominator $\Gamma(0)$. An analysis using only marginal errors neglects that covariance.

The inversion therefore depends visibly on grid spacing, regularization strength, and end-point treatment, and can retain spiky structure. It does suggest low response near $k = 0$ and weight at finite k , but those features are not used as an independent validation because Eq. (10) is assumed in performing the inversion. For these reasons the deconvolution has been removed from the main figures. The single-mode simulations in Sec. V A provide the more transparent operational estimate used in the main comparison.

- [1] E. Fermi, J. Pasta, and S. Ulam, Studies of nonlinear problems, Los Alamos Scientific Laboratory Report LA-1940, 1955.
- [2] N. J. Zabusky and M. D. Kruskal, Interaction of “solitons” in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* **15**, 240 (1965).
- [3] V. F. Nesterenko, Propagation of nonlinear compression pulses in granular media, *J. Appl. Mech. Tech. Phys.* **24**, 733 (1983).
- [4] V. F. Nesterenko, *Dynamics of Heterogeneous Materials* (Springer, New York, 2001).
- [5] S. Sen, J. Hong, J. Bang, E. Avalos, and R. Doney, Solitary waves in the granular chain, *Phys. Rep.* **462**, 21 (2008).
- [6] C. Coste, E. Falcon, and S. Fauve, Solitary waves in a chain of beads under Hertz contact, *Phys. Rev. E* **56**, 6104 (1997).
- [7] C. Daraio, V. F. Nesterenko, E. B. Herbold, and S. Jin, Strongly nonlinear waves in a chain of Teflon beads, *Phys. Rev. E* **72**, 016603 (2005).
- [8] S. Job, F. Melo, A. Sokolow, and S. Sen, How Hertzian solitary waves interact with boundaries in a 1D granular medium, *Phys. Rev. Lett.* **94**, 178002 (2005).
- [9] V. Garzó, J. W. Dufty, and C. M. Hrenya, Enskog theory for polydisperse granular mixtures. I. Navier-Stokes order transport, *Phys. Rev. E* **76**, 031303 (2007).
- [10] L. R. Gómez, A. M. Turner, and V. Vitelli, Uniform shock waves in disordered granular matter, *Phys. Rev. E* **86**, 041302 (2012).
- [11] M. Manjunath, A. P. Awasthi, and P. H. Geubelle, Wave propagation in random granular chains, *Phys. Rev. E* **85**, 031308 (2012).
- [12] N. Upadhyaya, L. R. Gómez, and V. Vitelli, Soliton attenuation and emergent hydrodynamics in fragile matter, *Phys. Rev. X* **4**, 011045 (2014).
- [13] T. Lindeberg, *Scale-Space Theory in Computer Vision*, The Springer International Series in Engineering and Computer Science (Kluwer Academic Publishers, Boston, 1994), Vol. 256.
- [14] N. Wiener, Generalized harmonic analysis, *Acta Math.* **55**, 117 (1930).
- [15] L. R. Gómez, Finite amplitude waves in jammed matter, *Soft Matter* **19**, 1749 (2023).
- [16] F. G. Bass, Y. S. Kivshar, V. V. Konotop, and Y. A. Sinitsyn, Dynamics of solitons under random perturbations, *Phys. Rep.* **157**, 63 (1988).
- [17] Y. S. Kivshar and B. A. Malomed, Dynamics of solitons in nearly integrable systems, *Rev. Mod. Phys.* **61**, 763 (1989).
- [18] J. Prost, J.-F. Joanny, and J. M. R. Parrondo, Generalized fluctuation-dissipation theorem for steady-state systems, *Phys. Rev. Lett.* **103**, 090601 (2009).
- [19] R. L. Doney and S. Sen, Decorated, tapered, and highly nonlinear granular chain, *Phys. Rev. Lett.* **97**, 155502 (2006).
- [20] E. Bogale and S. Sen, Solitary wave expansion and impact decimation in short inverse tapered granular chains, *Granular Matter* **28**, 11 (2026).
- [21] L. Gomez, Dataset: Correlated disorder suppresses solitary-wave attenuation in Hertzian chains, Zenodo, 2026, <https://doi.org/10.5281/zenodo.22003113>.