Only the ambidextrous can flock: Two-dimensional chiral Malthusian flocks, time cholesterics, and the Kardar-Parisi-Zhang equation

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We study the hydrodynamic behavior of two-dimensional chiral dry Malthusian flocks, that is, chiral polar-ordered active matter with neither number nor momentum conservation. We show that, in the absence of fluctuations, such systems generically form a "time cholesteric," in which the velocity of the entire system rotates uniformly at a fixed frequency b. Fluctuations about this state belong to the universality class of $(2+1)$-dimensional Kardar-Parisi-Zhang (KPZ) equation, which implies short-ranged orientational order in the hydrodynamic limit. We then show that, in the limit of weak chirality, the hydrodynamics of a system with reasonable size is expected to be governed by the linear regime of the KPZ equation, exhibiting quasi-long-ranged orientational order. Our predictions for the velocity and number density correlations are testable in both simulations and experiments.

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I. INTRODUCTION

Active matter [1–14] differs from equilibrium systems due to a variety of intrinsically nonequilibrium effects, including self-propulsion [1,4–7] and birth and death [15–17]. These nonequilibrium effects, which are most commonly found in biological systems, lead to many surprising phenomena, including long-ranged order in two-dimensional (2D) systems with spontaneously broken continuous symmetries [1,4–7] (i.e., 2D flocks), and hydrodynamic instabilities in very viscous systems [18,19].

While it is not an intrinsically nonequilibrium phenomenon, chirality [20] is also a ubiquitous feature of biological systems. It has recently been shown [21–23] that chirality in a collection of self-propelled particles (hereafter "flockers") moving in two dimensions—i.e., a 2D "flock"—can lead to a very unusual "steady state," in which the flockers move coherently in circles. That is, at any instant of time, all of the flockers are moving coherently in the same direction, but that coherent direction rotates at a constant angular velocity. As a result, each flocker moves in a circle around a center unique to itself, in synchrony with all of the other flockers.

Mathematically, the coarse-grained velocity field $\mathbf{v}(\mathbf{r},t)$ of a chiral flock in this state is spatially uniform (that is, independent of 2D position $\mathbf{r}$), and periodic in time:

$$\mathbf{v} = v_0[\cos(bt+\phi)\hat{\mathbf{x}} - \sin(bt+\phi)\hat{\mathbf{y}}], \tag{1.1}$$

where b is an intrinsic frequency characteristic of the system. An achiral system must have $b = 0$, since, without chirality, it has no way to decide whether to rotate right or left. Hence, we can use b as a measure of the chirality of the system [24].

Note that any "snapshot" of such a flock [i.e., a flock with its velocity field given by (1.1)] has perfect infinite ranged orientational order; that is, at every instant of time, all of the particles are moving in precisely the same direction.

One very simple and natural way to simulate a chiral flock is to modify the "Vicsek" algorithm by altering the direction selecting step of the algorithm to make the flockers select, not the average direction of their neighbors, but a direction that is, at all times and for all flockers, some fixed angle δ to the right of that mean direction (obviously, by choosing δ negative, one can make the flockers turn consistently to the left). Indeed, such simulations have been done [21] and do, indeed, find a state like (1.1).

In this paper, we investigate whether this long-ranged order is stable in the presence of noise, and furthermore determine the nature, size, and scaling of the fluctuations induced by the noise. In particular, we focus on 2D chiral "dry Malthusian" flocks, that is, flocks in which neither momentum nor flocker number is conserved. The absence of conservation of

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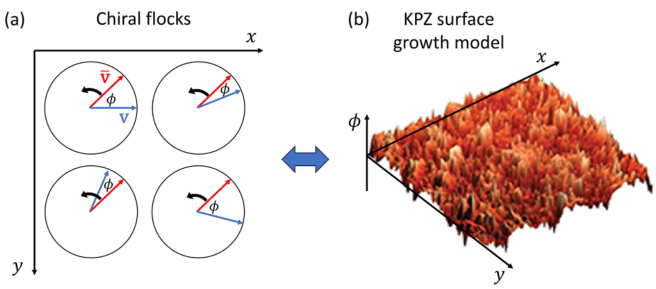

FIG. 1. (a) In a generic 2D chiral Malthusian flock, the mean velocity of the whole flock is denoted by $\bar{\mathbf{v}}$ (red), where $\bar{\mathbf{v}} = v_0[\cos(bt)\hat{\mathbf{x}} - \sin(bt)\hat{\mathbf{y}}]$, while the spatiotemporally fluctuating velocity field is denoted by $\mathbf{v}$ (blue) where $\mathbf{v} = v_0[\cos(bt+\phi)\hat{\mathbf{x}} - \sin(bt+\phi)\hat{\mathbf{y}}]$. Ignoring variations in $|\mathbf{v}|$, all fluctuations can be captured by the angular field $\phi(x, y, t)$. (b) We show that the hydrodynamic behavior of ϕ can generically be mapped onto the KPZ surface growth model [25] in $(2+1)$ dimensions. The image in (b) is reproduced with permission from EPL **109**, 46003 (2015) [33].

momentum is due to the presence of a frictional substrate over which the self-propelled flockers move, while the absence of number conservation is a consequence of birth and death of flockers while in motion.

To study the effects of noise, we allow the phase ϕ in (1.1) to vary slowly in space and time; i.e., we take the velocity to be

$$\mathbf{v}(\mathbf{r}, t) = v_0\{\cos[bt + \phi(\mathbf{r}, t)]\hat{\mathbf{x}} - \sin[bt + \phi(\mathbf{r}, t)]\hat{\mathbf{y}}\}. \quad (1.2)$$

We find that $\phi(\mathbf{r}, t)$ is the Goldstone mode of a 2D chiral flock. Its equation of motion (EOM) proves to be the $(2+1)$-dimensional Kardar-Parisi-Zhang (KPZ) equation [25]:

$$\partial_t\phi = \nu\nabla^2\phi + \frac{\lambda_\kappa}{2}(\nabla\phi)^2 + f_\phi, \quad (1.3)$$

where f_ϕ is a Gaussian, zero-mean white noise with the following statistics:

$$\langle f_\phi(\mathbf{r}, t) f_\phi(\mathbf{r}', t')\rangle = 2D_\phi\delta^2(\mathbf{r} - \mathbf{r}')\delta(t - t'). \quad (1.4)$$

This mapping onto the KPZ equation implies universal scaling relations between length scales, timescales, and phase fluctuations. More specifically, characteristic timescales $t(L)$ on length scales L grow like L^z, while phase fluctuations scale like L^χ. The current estimate of the values of χ and z based on simulations are [26–32] $\chi = 0.388 \pm 0.002$, $z = 1.622 \pm 0.002$. The analogy between a 2D chiral flock and the KPZ surface growth model is illustrated in Fig. 1.

We note that the KPZ universality class (UC) encompasses an amazing array of diverse systems: recent additions to this include incompressible polar active fluids [34], driven Bose-Einstein condensates [35], and nonreciprocal systems [36,37]. Furthermore, our active-matter realization of the compact $(2+1)$-dimensional KPZ equation provides a complementary and experimentally accessible counterpart to the exciton-polariton platforms recently proposed and analyzed within the same universality class [38]. Whereas the exciton-polariton systems realize compact KPZ dynamics in driven quantum condensates, our framework demonstrates how analogous spatiotemporal scaling behavior can emerge in a classical active-matter setting. This connection highlights the broad physical relevance of compact $(2+1)$-dimensional KPZ dynamics across both quantum and classical nonequilibrium systems.

The parameters ν and λ_κ respectively diverge and vanish in the limit of small chirality b; specifically, we find, for the "racemic" case [24]

$$\nu(b) \propto b^{-\eta_\nu}, \quad \lambda_\kappa(b) \propto b, \quad (1.5)$$

where the universal exponent η_ν is related to the dynamical exponent [15,16] z_a of "achiral" Malthusian flocks via the exact relation

$$\eta_\nu = \frac{2}{z_a} - 1 \approx \frac{3}{5}. \quad (1.6)$$

Here z_a is defined much as z for the KPZ equation; specifically, characteristic timescales $t(L)$ on length scales y in the direction orthogonal to the mean flock motion direction (which remains fixed in the achiral case) grow like $|y|^{z_a}$ [15–17]. The numerical value given here is based on numerical simulations of the noisy hydrodynamic equation for achiral Malthusian flocks by [39]. The other parameter in the KPZ equation, which is the noise strength D_ϕ, is independent of chirality b in the limit of small chirality.

As is well known [25], the fluctuations of a field ϕ described by the $(2+1)$-dimensional KPZ equation diverge in the limit of large distances. More quantitatively, the mean squared difference between the phases at widely separated spatiotemporal positions $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$ diverges algebraically as $|\mathbf{r} - \mathbf{r}'|$ or $|t - t'| \to \infty$:

$$\langle[\phi(\mathbf{r}, t) - \phi(\mathbf{r}', t')]^2\rangle = A_\phi|\mathbf{r} - \mathbf{r}'|^{2\chi}\mathcal{G}_\phi\left(\frac{B_\phi|t - t'|}{|\mathbf{r} - \mathbf{r}'|^z}\right), \quad (1.7)$$

where A_ϕ and B_ϕ are nonuniversal positive constants, and the universal scaling function $\mathcal{G}_\phi(X)$ has the following scaling behavior:

$$\mathcal{G}_\phi(X) = \begin{cases} 1, & X \ll 1, \\ X^{\frac{2\chi}{z}}, & X \gg 1. \end{cases} \quad (1.8)$$

This form of the scaling function implies

$$\langle[\phi(\mathbf{r}, t) - \phi(\mathbf{r}', t)]^2\rangle \propto \begin{cases} |\mathbf{r} - \mathbf{r}'|^{2\chi}, & \frac{B_\phi|t-t'|}{|\mathbf{r}-\mathbf{r}'|^z} \ll 1, \\ |t - t'|^{\frac{2\chi}{z}}, & \frac{B_\phi|t-t'|}{|\mathbf{r}-\mathbf{r}'|^z} \gg 1. \end{cases} \quad (1.9)$$

The current estimate of the values of χ and z based on simulations are [26–29,31,32] $\chi = 0.388 \pm 0.002$, and $z = 1.622 \pm 0.002$.

This divergence (1.7) implies that equal-time velocity correlations will be short-ranged. That is, chirality destroys the long-ranged orientational order present [15–17] in an achiral Malthusian flock.

In fact, 2D chiral Malthusian flocks are probably even more disordered than this would suggest. This is because, in contrast to the usual KPZ equation, the achiral Malthusian flock maps onto the *compact* $(2+1)$-dimensional KPZ equation.

What we mean by "compact" is that, unlike, say, a growing interface, in which the KPZ variable ϕ is the height of the interface, and every value of the height represents a physically distinct local state, our KPZ variable is a *phase*. That is, a state with a given local value $\phi(\mathbf{r},t)$ is locally physically indistinguishable from a state with $\phi(\mathbf{r},t)+2\pi n$, for any integer n, as can be seen directly from Eq. (1.2).

This is the same symmetry that is present in the equilibrium 2D XY model. Here, as there, it allows topological defects, i.e., vortices [40]. These become important on length scales larger than the mean intervortex distance L_v. These destroy the order even more thoroughly than the "spin-wave" (i.e., vortex-free) fluctuations that lead to the growth of ϕ fluctuations embodied in Eq. (1.7); indeed, they make it impossible [40] to even *define* a single-valued phase field $\phi(\mathbf{r},t)$ over all space $\mathbf{r}$.

The exact behavior of vortices in the KPZ equation remains an open question [38]. Indeed, it is an open question for many active systems. For example, it has recently been shown that active smectics [41], which, at the spin wave level, look exactly like an equilibrium XY model, exhibit completely different behavior of topological defects [42]. Our results here for the vortex length L_v and the behavior of a compact KPZ equation on length scales longer than L_v are based upon our speculation that this does *not* happen in KPZ like systems. We believe this is reasonable, since the surprising behavior of topological defects in active smectics is intimately connected with the existence of *two* spontaneously broken symmetries in those systems (translation *and* rotation invariance), while in our system only one continuous symmetry (time translation invariance) is broken. But our results for the behavior in the presence of vortices are somewhat speculative.

Despite our fundamental conclusion that chirality destroys long-ranged order in 2D chiral flocks, for *weak* chirality, order persists out to quite large distances. In fact, we find that for weak chirality, there is a hierarchy of length and time scales, illustrated in Fig. 2, separating regimes of quite different scaling behavior of the velocity correlations $\langle \mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t')\rangle$. Some of these regions will be effectively ordered.

All of these length and time scales diverge in the limit of small chirality b, according to the following scaling laws:

$$L_c(b) \propto b^{-1/z_a}, \quad 1/z_a \approx 4/5, \tag{1.10}$$

$$L_{\rm NL}(b) \propto \exp(C_{\rm NL} b^{-\eta_g}), \quad \eta_g \approx 19/5, \tag{1.11}$$

$$L_v(b) \propto \exp(C_{vL} b^{-\eta_y}), \quad \eta_y \approx 22/5, \tag{1.12}$$

$$\tau_c(b) \propto b^{-1}, \tag{1.13}$$

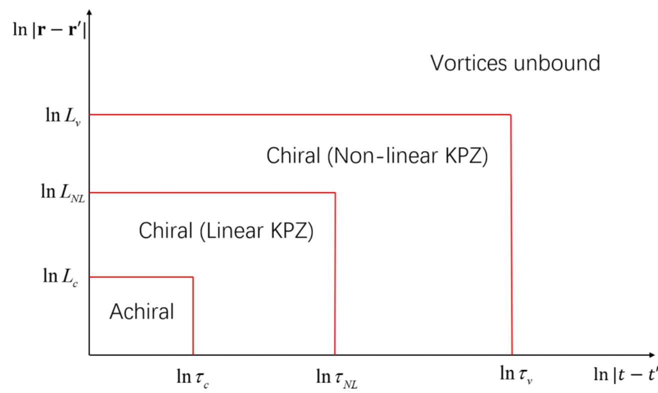

FIG. 2. Regimes of different behavior in the limit of weak chirality. For small chirality (even only *moderately* small chirality), the longest length scales $L_{\rm NL}$ and L_v and timescales $\tau_{\rm NL}$ and τ_v will be long enough to allow several decades of length and time scale to lie in each region, making our scaling predictions for each region experimentally accessible. See the text for more details.

$$\tau_{\rm NL}(b) \propto \exp(2C_{\rm NL} b^{-\eta_g}), \tag{1.14}$$

$$\tau_v(b) \propto \exp(C_{v\tau} b^{-\eta_y}), \tag{1.15}$$

where $C_{\rm NL}$, C_{vL}, and $C_{v\tau}$ are all nonuniversal constants.

The universal exponents η_v, η_g, and η_y are related to the equally universal exponent z_a by the relations given in Table I. Here z_a is the "dynamical exponent" for 2D *achiral* Malthusian flocks, which is defined by the scaling of characteristic times t with length scale y in the direction perpendicular to the mean flock motion; specifically $t \propto y^{z_a}$.

The velocity correlations $\langle \mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t')\rangle$ behave as follows in each of these regimes of length and time scales:

(1) Achiral regime: $|\mathbf{r}-\mathbf{r}'| \ll L_c$ and $|t-t'| \ll \tau_c$. In this regime, the velocity correlations are simply those of an achiral Malthusian flock, as given in [15–17], exhibiting long-ranged order.

(2) Linear KPZ or Edwards-Wilkinson (EW) regime: $L_c \ll |\mathbf{r}-\mathbf{r}'| \ll L_{\rm NL}$ and $|t-t'| \ll \tau_{\rm NL}$, or $|\mathbf{r}-\mathbf{r}'| \ll L_{\rm NL}$ and $\tau_c \ll |t-t'| \ll \tau_{\rm NL}$. In this regime, the effects of the λ_κ nonlinearity in the KPZ equation are negligible. As a result, that term can be dropped, leaving the dynamics of the ϕ field to be described by the linearized version of the KPZ equation, which is also known [43] as the EW equation:

$$\partial_t \phi = \nu \nabla^2 \phi + f_\phi. \tag{1.16}$$

Being linear, this EOM is easily solved to obtain, *inter alia*, the mean-squared phase fluctuations:

$$\langle[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t')]^2\rangle = \frac{D_\phi}{2\pi\nu}\left[-\mathrm{Ei}\left(-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4(\nu|t-t'|+L_c^2)}\right)+2\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{2L_c}\right)+\gamma\right]$$

$$\approx \begin{cases} \frac{D_\phi}{\pi\nu}\left[\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)+\gamma/2\right], & \frac{|t-t'|}{\tau_c} \ll \left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)^2, \\ \frac{D_\phi}{2\pi\nu}\left[\ln\left(\frac{|t-t'|}{\tau_c}\right)\right], & \frac{|t-t'|}{\tau_c} \gg \left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)^2, \end{cases} \tag{1.17}$$

TABLE I. Exponents that appear in this paper, expressions for them, and estimates of their numerical values. The achiral exponents are those that give the relations between length and time scales in an *achiral* flock; their numerical values were obtained by [39] from their numerical solution of the noisy hydrodynamic equation for achiral flocks. The numerical values of the KPZ exponents z and χ are taken from the numerical work of Refs. [26–29,31,32].

Exponent	Definition and/or defining equation	Expression and/or numerical value
Achiral dynamical exponent z_a	$t \propto \|y\|^{z_a}$	$\approx 5/4$
Achiral anisotropy exponent ζ_a	$\|x\| \propto \|y\|^{\zeta_a}$	$\approx 3/4$
η_ν	$\nu(b) \propto b^{-\eta_\nu}$, (1.5)	$\eta_\nu = 2/z_a - 1 \approx 3/5$
χ	(2+1)-dimensional KPZ "roughness" exponent	$\approx 0.388 \pm 0.002$
z	(2+1)-dimensional KPZ "dynamical" exponent	$\approx 1.622 \pm 0.002$
z_L	Linearized KPZ (Edwards-Wilkinson) "dynamical" exponent	2
η_g	$g_0(b) \propto b^{\eta_g}$, (7.34)	$\eta_g = 6/z_a - 1 \approx 19/5$
η_y	$L_v \propto \exp[-\text{const}\, b^{-\eta_y}]$, (11.6)	$\eta_y = \eta_g + \eta_\nu \approx 22/5$
α	(1.18), (1.20)	$\alpha = \frac{D_\phi}{2\pi\nu}$

where $\gamma = 0.57721566\ldots$ is the Euler-Mascheroni constant [44], and $\mathrm{Ei}(x) \equiv -\int_{-x}^{\infty}(\frac{e^{-u}}{u})\mathrm{d}u$ is the exponential integral function, whose well-known [45] asymptotic limits imply the limiting behaviors on the last two lines.

It follows from (1.2), (1.17), the Gaussian nature of the noise f_ϕ, and the linearity of the EW equation, that the velocity correlations oscillate rapidly in time and decay algebraically in space and time:

$$\begin{aligned}\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t')\rangle &= v_0^2\cos[b(t-t')]\langle\exp\{\mathrm{i}[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t')]\}\rangle\\ &= v_0^2\cos[b(t-t')]\exp\left\{-\frac{1}{2}\langle[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t')]^2\rangle\right\}\\ &= v_0^2 e^{-\alpha\gamma/2}\cos[b(t-t')]\left(\frac{|\mathbf{r}-\mathbf{r}'|}{2L_c}\right)^{-\alpha}\exp\left[\frac{\alpha}{2}\mathrm{Ei}\left(-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4(\nu|t-t'|+L_c^2)}\right)\right]\\ &\approx v_0^2\cos[b(t-t')]\times\begin{cases} e^{-\alpha\gamma/2}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{2L_c}\right)^{-\alpha}, & \frac{|t-t'|}{\tau_c}\ll\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)^2,\\ \left(\frac{|t-t'|}{\tau_c}\right)^{-\frac{\alpha}{2}}, & \frac{|t-t'|}{\tau_c}\gg\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)^2,\end{cases}\end{aligned}\tag{1.18}$$

where the exponent α is nonuniversal and given by

$$\alpha = \frac{D_\phi}{2\pi\nu} = C_\alpha b^{\eta_\nu}, \quad \eta_\nu \approx \frac{3}{5}, \tag{1.19}$$

with C_α being a nonuniversal constant.

To investigate the order of the system, we take a snapshot of the system at any time and focus on the velocity correlations on that snapshot. These velocity correlations are equal-time correlations, which can be easily gotten by setting $t = t'$ in (1.18). Doing this we find

$$\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t)\rangle = v_0^2\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)^{-\alpha}\times O(1). \tag{1.20}$$

Note that for small chirality b, the exponent α will be small. Hence, the algebraic decay of velocity correlations (1.20) will be extremely slow; this is what we meant by our earlier cryptic comment that "some regimes are effectively ordered." Strictly speaking, the order in this regime is "quasi-long-ranged."

The behavior of the purely temporal Fourier transform of the velocity-velocity correlation function, that is,

$$I(\mathbf{R},\omega) \equiv \int_{-\infty}^{\infty}\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}+\mathbf{R},t+T)\rangle\, e^{\mathrm{i}\omega T}\,\mathrm{d}T, \tag{1.21}$$

is also interesting in this regime (and experimentally measurable). Unlike the equal-time case, the non-equal-time velocity correlations carry a frequency $\omega = b$ [see the cosine factor in (1.18) and (1.25)]. This is reminiscent of the density correlations in smectics which oscillate in real space with period a (the layer spacing) along the layer normal $\hat{n}$. The Fourier component of this density correlation function in the vicinity of the "Bragg peaks" $\mathbf{q} = (2m\pi/a)\hat{n}$ ($m = 1, 2, \ldots$) is proportional to the x-ray scattering intensity. Here we make a similar prediction for the Fourier component of the velocity correlation function near the "Bragg peaks" at $\omega = \pm b$ in frequency. We find that, in the weak chirality limit, this Fourier component has a scaling form over a substantial range of frequencies:

$$I(\mathbf{R},\omega) = \frac{v_0^2}{4\pi}\left(\frac{e^{\gamma/2}}{2L_c}\right)^{-\alpha}\left(\frac{R^{2-\alpha}}{4\nu}\right)F_h\left(\frac{\delta\omega R^2}{4\nu}\right), \tag{1.22}$$

where $\delta\omega \equiv \omega - b$ near the peak at $\omega = b$, and $\delta\omega \equiv \omega + b$ near the peak at $\omega = -b$. Here the scaling function $F_h(S)$ is given by

$$F_h(S) = \int_{-\infty}^{\infty}\exp\left[\mathrm{i}Su + \left(\frac{\alpha}{2}\right)\mathrm{Ei}\left(-\frac{1}{|u|}\right)\right]\mathrm{d}u. \tag{1.23}$$

Its limiting behaviors for large and small arguments are

$$F_h(S) \approx \begin{cases} 2e^{\frac{\alpha\gamma}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)\sin\left(\frac{\alpha\pi}{4}\right)|S|^{-1+\alpha/2}, & |S|\ll 1\\ \sqrt{\pi}\alpha|S|^{-5/4}e^{-\sqrt{2|S|}}\sin\left(\frac{\sqrt{2|S|}+\pi}{8}\right), & |S|\gg 1\end{cases}. \tag{1.24}$$

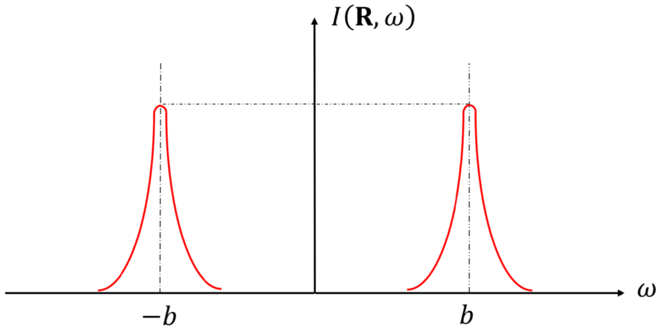

FIG. 3. Plot of the temporally Fourier transformed correlation function $I(\mathbf{R},\omega)$ versus ω for a weakly chiral system. Each peak has a power law divergence $|\delta\omega|^{\alpha/2-1}$, where $\delta\omega$ is the difference between the frequency and the peak centers, which lie at $\pm b$. This divergence is cut off sufficiently close to the peak (specifically, once $|\delta\omega| \lesssim \tau_{\rm NL}^{-1}$, leading to a finite peak height at $\delta\omega = 0$. The oscillating tails for $|\delta\omega| \gg \nu/R^2$ are invisible on the scale of this figure.

This scaling behavior is cut off for $|b-\omega| \ll \tau_{NL}^{-1}$; in this range of frequencies, the peak is rounded and featureless.

The power-law decaying "Bragg peak" for $\tau_{\rm NL}^{-1} \ll |b-\omega| \ll \frac{\nu}{R^2}$ implies quasi-long-ranged order at timescales less than $\tau_{\rm NL}$, while the broad converging "Bragg peak" for $|b-\omega| \ll \tau_{\rm NL}^{-1}$ implies short-ranged order at timescales beyond $\tau_{\rm NL}$.

The behavior of the purely temporal Fourier transform of the velocity-velocity correlation function is illustrated in Fig. 3.

(3) Nonlinear KPZ regime: $L_{\rm NL} \ll |\mathbf{r}-\mathbf{r}'| \ll L_v$ and $|t-t'| \ll \tau_v$, or $|\mathbf{r}-\mathbf{r}'| \ll L_v$ and $\tau_{\rm NL} \ll |t-t'| \ll \tau_v$. In this regime, the effects of the λ_κ nonlinearity in the KPZ equation dominate. The mean-squared difference between the phases at widely separated spatial and temporal positions $(t,\mathbf{r})$ and $(\mathbf{r}',t')$ is given by (1.7).

If the fluctuations of the field ϕ were Gaussian, then we could say that the second equality in (1.18) still applies. Inserting (1.7) into this equality we get

$$\langle \mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t')\rangle = v_0^2\cos[b(t-t')]\exp\left[-\left(\frac{A_\phi}{2}\right)|\mathbf{r}-\mathbf{r}'|^{2\chi}\mathcal{G}_\phi\left(\frac{|t-t'|}{|\mathbf{r}-\mathbf{r}'|^z}\right)\right]. \tag{1.25}$$

The parameter A_ϕ can be calculated by arguing that the phase correlations (1.7) and (1.17) must connect at the crossovers, for example, at $|\mathbf{r}-\mathbf{r}'| = L_{\rm NL}$, $|t-t'| = 0$. This argument gives

$$A_\phi = 2\alpha L_{\rm NL}^{-2\chi}\ln\left(\frac{L_{\rm NL}}{L_c}\right) \propto b^{-\frac{16}{5}}\exp[-(b^{-3.8})\times O(1)]. \tag{1.26}$$

Again focusing on the equal-time correlation, we would now have

$$\langle \mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t)\rangle = v_0^2\exp\left[-\left(\frac{A_\phi}{2}\right)|\mathbf{r}-\mathbf{r}'|^{2\chi}\right], \tag{1.27}$$

which would imply (stretched) exponentially decaying velocity correlations.

However, since the ϕ fluctuations are non-Gaussian, due to the relevant nonlinearity λ_κ in the KPZ equation, we can say very little, beyond noting that we expect velocity correlations to decay rapidly with increasing separation $|\mathbf{r}-\mathbf{r}'|$ once that separation is large enough. Naively, this would seem to imply that correlations will decay rapidly for $|\mathbf{r}-\mathbf{r}'| > \xi_v$ where

$$\xi_v \equiv A_\phi^{-\frac{1}{2\chi}} \approx A_\phi^{-1.29}. \tag{1.28}$$

While this may be true for very chiral systems, we find that, for small chirality, $\xi_v \ll L_{\rm NL}$. To see this, simply plug the first equality of (1.26) into (1.28); this gives

$$\xi_v \propto b^{\frac{8}{5\chi}} L_{\rm NL} \ll L_{\rm NL}. \tag{1.29}$$

This implies that we will not be able to see this characteristic length in the nonlinear KPZ regime. The reason for this is simply that the velocity correlations will already have decayed to essentially zero before we even reach the nonlinear regime.

For very chiral (i.e., large b) systems, however, ξ_v will be the "velocity correlation length," although it must be kept in mind that the decay of velocity correlations is probably not exponential, nor even a simple stretched exponential like Eq. (1.27).

(4) Vortices unbound regime: $L_v \ll |\mathbf{r}-\mathbf{r}'|$ or $\tau_v \ll |t-t'|$.

As we discussed earlier, the fact that this regime even exists is a consequence of the fact that, strictly speaking, the 2D chiral Malthusian flock maps onto the *compact* (2 + 1)-dimensional KPZ equation, which allows vortices in the system. We believe, based on the previously discussed analogy between 2D chiral Malthusian flocks and the 2D equilibrium XY model, that the typical spacing between vortices is L_v, and that for length scales longer than L_v it is therefore impossible [40] to even *define* a single-valued phase field $\phi(\mathbf{r},t)$ over all space $\mathbf{r}$.

Our discussion so far has focused on fluctuations of the velocity field. The local density $\rho(\mathbf{r},t)$ of flockers "inherits" some interesting behavior from the velocity fluctuations as well. In the achiral regime, fluctuations of $\rho(\mathbf{r},t)$ are "enslaved" to $\mathbf{v}(\mathbf{r},t)$ [15–17]:

$$\delta\rho \propto \nabla\cdot\mathbf{v}, \tag{1.30}$$

where $\delta\rho = \rho(\mathbf{r},t) - \rho_0$, where ρ_0 is the value of the steady state density. In the chiral regimes, we find that fluctuations of ρ are "enslaved" to those of ϕ:

$$\delta\rho = e_3\nabla^2\phi + e_4|\nabla\phi|^2, \tag{1.31}$$

where e_3 and e_4 are nonuniversal (i.e., system-specific) constants.

As a result, the density correlations exhibit the following behaviors in the various regimes:

(1) Achiral regime: $|\mathbf{r}-\mathbf{r}'| \ll L_c$ and $|t-t'| \ll \tau_c$. The density correlations are just those given in [16,17].

(2) Linear KPZ or EW regime: $L_c \ll |\mathbf{r}-\mathbf{r}'| \ll L_{\rm NL}$ and $|t-t'| \ll \tau_{\rm NL}$, or $|\mathbf{r}-\mathbf{r}'| \ll L_{\rm NL}$ and $\tau_c \ll |t-t'| \ll \tau_{\rm NL}$. Density correlations decay algebraically with separation:

$$\langle \delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t')\rangle \propto \begin{cases} \frac{1}{|\mathbf{r}-\mathbf{r}'|^4}, & \left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)^2 \gg \frac{|t-t'|}{\tau_c}, \\ \frac{1}{|t-t'|^2}, & \left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_c}\right)^2 \ll \frac{|t-t'|}{\tau_c}. \end{cases} \tag{1.32}$$

(3) Nonlinear KPZ regime: $L_{\rm NL} \ll |\mathbf{r}-\mathbf{r}'| \ll L_v$ and $|t-t'| \ll \tau_v$, or $|\mathbf{r}-\mathbf{r}'| \ll L_v$ and $\tau_{\rm NL} \ll |t-t'| \ll \tau_v$. Once again, density correlations decay algebraically, but now with a nontrivial exponent related to the "roughness" exponent χ of the $(2+1)$-dimensional KPZ equation:

$$\langle \delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\rangle \propto \begin{cases} |\mathbf{r}-\mathbf{r}'|^{4\chi-4} \approx |\mathbf{r}-\mathbf{r}'|^{-2.448}, & \left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_{\rm NL}}\right)^z \gg \frac{|t-t'|}{\tau_{\rm NL}}, \\ |t-t'|^{\frac{4\chi-4}{z}} \approx |t-t'|^{-1.509}, & \left(\frac{|\mathbf{r}-\mathbf{r}'|}{L_{\rm NL}}\right)^z \ll \frac{|t-t'|}{\tau_{\rm NL}}. \end{cases} \tag{1.33}$$

Unfortunately, in neither of these regimes is the decay of density correlations slow enough to lead to "Giant number fluctuations." That is, in Malthusian flocks, the mean-squared number fluctuations $\langle (N-\langle N\rangle)^2\rangle$, where N is the number of particles in some large volume, obey the usual "law of large numbers" scaling $\langle (N-\langle N\rangle)^2\rangle \propto \langle N\rangle$, in contrast to the anomalous "Giant number fluctuations" scaling law $\langle (N-\langle N\rangle)^2\rangle \propto \langle N\rangle^\beta$ with $\beta > 1$ which occurs in the absence of momentum conservation for both achiral polar-ordered flocks and active nematics [14,46].

(4) Vortices unbound regime: $L_v \ll |\mathbf{r}-\mathbf{r}'|$ or $\tau_v \ll |t-t'|$. In this regime, the flock is completely disordered, and so the density correlations are short-ranged in space and time, with correlation length L_v and correlation time τ_v.

The remainder of this paper is organized as follows: in Sec. II we formulate a simplified hydrodynamic model for 2D chiral Malthusian flocks. In Sec. III we identify the broken symmetry state of the model. In Sec. IV we show that the dynamics of this model maps onto the KPZ equation for the phase ϕ introduced above. In Sec. V we derive the relation (1.31) above between the density ρ and the phase ϕ. In Sec. VI we derive the correlations of the phase ϕ, the velocity $\mathbf{v}$, and the density ρ in the nonlinear KPZ regime. In Sec. VII we consider the small chirality limit, and derive the scaling laws (1.10), (1.11), (1.13), and (1.14) above relating the crossover lengths L_c, and $L_{\rm NL}$, and times τ_c and $\tau_{\rm NL}$, to the chirality b. We also derive the scaling laws (1.17) and (1.18) that hold in the linear regime in Sec. VIII. In Sec. IX we calculate the purely temporal Fourier transform of the velocity correlation function. In Sec. X we calculate the density correlations in the linear regime. In Sec. XI, we consider the effects of vortices in our compact KPZ equation, and derive the scaling (1.12) and (1.15) of the vortex length L_v and time τ_v with b. In Sec. XII we summarize our results. In Appendix A we show that generalizing our simplified model to include all relevant terms allowed by symmetry changes none of our results. We discuss "nonracemic" mixtures in Appendix B. In Appendix C we give the mathematical details of our calculation of the phase correlation function in the linear regime. Finally, Appendix D presents the details of our calculation of the crossover function $F_h(S)$ for the purely temporally Fourier transformed velocity-velocity correlation function.

II. MODEL

Our approach is to begin with the generic hydrodynamic EOM for achiral flocks in which neither momentum nor particle number is conserved, and then supplement this with the additional terms allowed when chiral symmetry is broken. The achiral hydrodynamic EOM is [15–17]

$$\partial_t \mathbf{v} + \lambda_1(\mathbf{v}\cdot\nabla)\mathbf{v} + \lambda_2(\nabla\cdot\mathbf{v})\mathbf{v} = \mu_1\nabla^2\mathbf{v} + \mu_2\nabla(\nabla\cdot\mathbf{v}) + \mu_3(\mathbf{v}\cdot\nabla)^2\mathbf{v} + L(|\mathbf{v}|)\mathbf{v} + \mathbf{f}, \tag{2.1}$$

where $\mathbf{v}(\mathbf{r},t)$ is the local velocity field, $L(|\mathbf{v}|)$ is a Lagrange multiplier that enforces a "fixed length" or "constant speed" constraint on the velocity that $|\mathbf{v}| = v_0 = \text{const} \neq 0$ everywhere [47]. The random force $\mathbf{f}$ is a zero-mean "white noise" with spatiotemporal statistics:

$$\langle f_m(\mathbf{r},t) f_n(\mathbf{r}',t')\rangle = 2D\delta_{mn}\delta^2(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{2.2}$$

Next we incorporate the chirality. Our approach to doing so is the same as that used in constructing the EOMs (2.1) for the achiral case, which can be summarized by saying that we include in the velocity EOM (2.1) every vector that can be made out of the velocity field and its spatial derivatives. In the presence of chirality, however, we can also include in the velocity EOM any vector obtained by rotating one of the vectors that was allowed in the achiral case to the right by 90°. Such rotations can be done in two dimensions using the antisymmetric rank-2 tensor:

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.3}$$

If we act with this tensor on an arbitrary vector $\mathbf{w}$, the product $\boldsymbol{\epsilon}\cdot\boldsymbol{w}$ is just the vector obtained by rotating $\boldsymbol{w}$ clockwise by 90°, which is clearly a chiral operation.

In principle the hydrodynamic EOM for the chiral case should include all the terms in the achiral EOM plus those breaking the chiral symmetry. There are infinitely many such terms. For the sake of simplicity, we will begin by keeping only enough terms to allow us to obtain the most general ultimate final hydrodynamic model, which proves to be the $(2+1)$-dimensional KPZ equation. Later we will show that those (infinitely many!) neglected terms do not alter the conclusion that the ultimate model is the KPZ equation; rather, they change only the values of two of the parameters of that equation.

Our truncated EOM for 2D chiral flocks with nonconservation of momentum and numbers is

$$\partial_t \mathbf{v} + \lambda_1(\mathbf{v}\cdot\nabla)\mathbf{v} + \lambda_1'(\mathbf{v}\cdot\boldsymbol{\epsilon}\cdot\nabla)\mathbf{v} + \lambda_2(\nabla\cdot\mathbf{v})\mathbf{v} = \mu_1\nabla^2\mathbf{v} + \mu_2\nabla(\nabla\cdot\mathbf{v}) + \mu_3(\mathbf{v}\cdot\nabla)^2\mathbf{v} + \mu'\boldsymbol{\epsilon}\cdot\nabla^2\mathbf{v} + [L(|v|) + b\boldsymbol{\epsilon}\cdot]\mathbf{v} + \mathbf{f}, \tag{2.4}$$

where $b > 0$ if the particles in our system tend to turn right, while $b < 0$ if the particles in our system tend to turn left. Note that the random force here can also be chiral in principle, but the relevant part of its two-point correlations have the same form [15] as those of its achiral counterpart, i.e.,

$$\langle f_m(\mathbf{r},t)f_n(\mathbf{r}',t')\rangle = 2D\delta_{mn}\delta^2(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{2.5}$$

As noted in the introduction, one very simple and natural way to simulate a chiral flock is to modify the "Vicsek" algorithm by altering the direction selecting step of the algorithm to make the particles select, not the average direction of their neighbors, but a direction δ clockwise of that direction, with δ an adjustable parameter of the model. For small δ, we expect $b \propto \delta$. Hence, all of the b dependence that we calculate below can be directly mapped onto identical dependence on δ.

III. THE BROKEN SYMMETRY STATE

In the absence of noise, the EOM has an infinite family of spatially uniform, temporally oscillating solutions:

$$\mathbf{v} = v_0[\cos(bt+\phi)\hat{x} - \sin(bt+\phi)\hat{y}]. \tag{3.1}$$

Note that these solutions differ trivially by a constant phase ϕ.

Any "snapshot" of such a flock has perfect infinite ranged orientational order; that is, at every instant of time, all of the particles are moving in precisely the same direction.

We now wish to investigate whether this long-ranged order is stable in the presence of noise, and to furthermore determine the nature, size, and scaling of the fluctuations induced by the noise.

Since Eq. (3.1) is a solution of the noiseless equations of motion for any *constant* value of the phase ϕ, we expect that a velocity configuration

$$\mathbf{v}(\mathbf{r},t) = v_0\{\cos[bt+\phi(\mathbf{r},t)]\hat{x} - \sin[bt+\phi(\mathbf{r},t)]\hat{y}\}, \tag{3.2}$$

with $\phi(\mathbf{r},t)$ varying slowly in *space* $\mathbf{r}$, will exhibit a slow variation of $\phi(\mathbf{r},t)$ with *time* t as well. This is, of course, just the characteristic of a Goldstone mode. We will now show that this reasoning is correct, and that $\phi(\mathbf{r},t)$ is, indeed, the Goldstone mode for our problem.

IV. THE HYDRODYNAMICS OF THE PHASE FIELD

Inserting (3.2) into the EOM (2.4), we get, in tensor form,

$$\partial_t v_i = (b+\partial_t\phi)\epsilon_{ik}v_k, \quad \partial_j v_i = \epsilon_{ik}v_k\partial_j\phi. \tag{4.1}$$

Using these in (2.4), we obtain the EOM for the phase ϕ:

$$\begin{aligned}\epsilon_{ij}v_j\partial_t\phi = {}& -\lambda_1\epsilon_{ik}v_kv_j\partial_j\phi - \lambda_1'\left[v_i(v_k\partial_k\phi) - v_0^2(\partial_i\phi)\right]\\ &-\lambda_2\epsilon_{jk}v_iv_k(\partial_j\phi) + \mu_1[\epsilon_{ik}v_k\partial_j\partial_j\phi - v_i(\partial_j\phi)(\partial_j\phi)]\\ &+\mu_2[\epsilon_{jk}v_k\partial_i\partial_j\phi - v_j(\partial_i\phi)(\partial_j\phi)]\\ &+\mu_3[\epsilon_{i\ell}v_\ell v_jv_k\partial_j\partial_k\phi - v_iv_kv_j(\partial_k\phi)(\partial_j\phi)]\\ &-\mu'[v_i\partial_k\partial_k\phi + \epsilon_{im}v_m(\partial_k\phi)(\partial_k\phi)]\\ &+L(|v|)v_i + f_i.\end{aligned} \tag{4.2}$$

Recall that we expect ϕ to vary slowly in time. This means the Fourier transform of $\phi(\mathbf{r},t)$ in time will be dominated by frequencies much less than the characteristic oscillation frequency b in (3.2). Therefore, the left-hand side of (4.2) is dominated by frequencies near b, since the velocity factor v_j on that side oscillates at that frequency, while the $\partial_t\phi$ factor oscillates much more slowly.

To explicitly display, and calculate, the slow evolution of ϕ, we need to extract the Fourier components of the right-hand side of (4.2) that also are near b. To do this, we begin by multiplying both sides of (4.2) by $\epsilon_{in}v_n$ with the usual implied summation convention on the repeated index n. This amounts in vector notation to taking the dot product of both sides of (4.2) with $\boldsymbol{\epsilon}\cdot\mathbf{v}$, which is orthogonal to $\mathbf{v}$ itself. This eliminates all terms along $\mathbf{v}$ in (4.2) (that is, in tensor notation, all terms proportional to v_i, which includes the Lagrange multiplier term $L(|v|)v_i$, along with several others). One can also see the vanishing of these terms directly in tensor notation by using the asymmetry of ϵ_{ij} under interchange of i and j. We are left with

$$\begin{aligned}v_0^2\partial_t\phi = {}& -\lambda_1v_0^2v_j\partial_j\phi + \lambda_1'v_0^2\epsilon_{in}v_n\partial_i\phi + \mu_1v_0^2\nabla^2\phi\\ &+\mu_2\left[v_0^2\nabla^2\phi - v_iv_j\partial_i\partial_j\phi - \epsilon_{in}v_nv_j(\partial_i\phi)(\partial_j\phi)\right]\\ &+\mu_3v_0^2v_jv_k\partial_j\partial_k\phi - \mu'v_0^2|\nabla\phi|^2 + \epsilon_{in}v_nf_i,\end{aligned} \tag{4.3}$$

where we have made liberal use of the following identities obeyed by the antisymmetric tensor $\boldsymbol{\epsilon}$:

$$\epsilon_{ij}\epsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \quad \epsilon_{ik}\epsilon_{jk} = \delta_{ij}, \tag{4.4}$$

the second of which clearly follows by taking a trace of the first when we remember that we are working in spatial dimension $d=2$.

Next, we time-average Eq. (4.3) over one oscillation cycle of $\mathbf{v}$. For the purposes of such time averages, $\phi(\mathbf{r},t)$ and all of its spatiotemporal derivatives can be treated as constants, since $\phi(\mathbf{r},t)$ varies slowly on the timescale b^{-1} of the oscillating velocity. They can therefore be removed from the averages over one cycle. We will denote averages over one cycle by $\langle\rangle_c$. It is then fairly straightforward to show that time averages of products of two and four components of $\mathbf{v}$ are given by

$$\langle v_iv_j\rangle_c = \frac{v_0^2\delta_{ij}}{2}, \quad \langle v_iv_jv_kv_l\rangle_c = \frac{v_0^4}{8}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). \tag{4.5}$$

Averages over a full cycle of products of any odd number of components of $\mathbf{v}$ clearly vanish. This immediately eliminates all of the λ terms, since they have an odd number of components of $\mathbf{v}$. This is one of the radical changes introduced by the chirality, since those λ terms are known [15–17] to dominate the physics of the achiral problem.

Using (4.5) on the average of (4.3) gives

$$\partial_t\phi = \nu\nabla^2\phi + \frac{\lambda_\kappa}{2}(\nabla\phi)^2 + f_\phi, \tag{4.6}$$

where

$$\nu = \mu_1 + \left(\frac{\mu_2+\mu_3v_0^2}{2}\right), \tag{4.7}$$

$$\lambda_\kappa = -\mu'. \tag{4.8}$$

The noise in (4.6) is given by

$$f_\phi = \epsilon_{in}v_nf_i/v_0^2. \tag{4.9}$$

Its statistics, after averaging over one cycle, are

$$\begin{aligned}\langle f_\phi(\mathbf{r},t) f_\phi(\mathbf{r}',t')\rangle &= \langle \epsilon_{in} v_n f_i \epsilon_{jm} v_m f_j\rangle / v_0^4 \\ &= \epsilon_{in}\epsilon_{jm}\langle f_i(\mathbf{r},t) f_j(\mathbf{r}',t')\rangle \langle v_n v_m\rangle_c / v_0^4 \\ &= 2D_\phi \delta^2(\mathbf{r}-\mathbf{r}')\delta(t-t'),\end{aligned} \tag{4.10}$$

where we have defined $D_\phi \equiv (D/v_0^2)$. In deriving (4.10), we have made use of the identities (4.4) and (4.5) for the antisymmetric tensor and the time averages of the velocity, respectively. Equation (4.6) is just the well-known $(2+1)$-dimensional KPZ equation.

This result is not a coincidence, but required by the following symmetry argument. Since the system is isotropic in a time-averaged sense in the long-time limit because of the constant rotation, the EOM for ϕ must be invariant under the rotation $\phi \to \phi + \text{const}$, which implies the EOM can involve only derivatives of ϕ. Also because of this isotropy, the EOM must be isotropic in space, which excludes all one-derivative terms, and allows only two-derivative terms like $\nabla^2\phi$ and $(\nabla\phi)^2$. Furthermore, the EOM (2.4) is not invariant under the transformation $\phi \to -\phi$, because chiral symmetry is explicitly broken; the term that violates this invariance is $(\nabla\phi)^2$. This explains why $(\nabla\phi)^2$ can be generated from only the chiral terms in the EOM for $\mathbf{v}$ (2.4).

The above symmetry argument proves that, even though we have neglected infinitely many possible terms in the starting model (2.4), (4.6) is necessarily the ultimate form for the EOM for ϕ. An alternative argument leading to the same conclusion is presented in Appendix A. This appendix shows that all the possible one-derivative terms in (2.4) have no contribution to (4.6), all the achiral two-derivative terms can contribute only to $\nabla^2\phi$, and all the chiral two-derivative terms can contribute only to $(\nabla\phi)^2$. Further, we do not need to consider higher derivative terms since their contribution to (4.6) is irrelevant. This leaves the KPZ equation as the only possible EOM for ϕ, in agreement with the symmetry argument given above.

V. THE HYDRODYNAMICS OF THE DENSITY FIELD

Now we turn to the hydrodynamics of the local number density of flockers $\rho(\mathbf{r},t)$. In the achiral Malthusian case, we can include ρ, even though it is a nonhydrodynamic field, by adding to the usual continuity equation a source term $g(\rho(\mathbf{r},t))$ which is the difference between the local birth and death rate:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = g(\rho). \tag{5.1}$$

We assume that $g(\rho_0)=0$ at some $\rho_0 \neq 0$, $g(\rho_0) < 0$ for $\rho > \rho_0$, and $g(\rho_0) > 0$ for $\rho < \rho_0$. This implies that, in the absence of noise, ρ will approach a steady-state value of ρ_0 exponentially in time, with a time constant

$$\tau_\rho = -\frac{1}{g'(\rho_0)}. \tag{5.2}$$

In Refs. [15–17] it was shown that these equations imply that the fluctuation

$$\delta\rho(\mathbf{r},t) \equiv \rho(\mathbf{r},t) - \rho_0 \tag{5.3}$$

of ρ about its steady-state value ρ_0 becomes "enslaved" to the local configuration of the velocity field $\mathbf{v}(\mathbf{r},t)$; that is, $\delta\rho$ is determined at any instant of time by the spatial configuration of $\mathbf{v}(\mathbf{r},t)$ *at the same instant of time*. As a result, the slow hydrodynamic decay of velocity fluctuations leads to a slow decay of $\delta\rho$ as well, even though $\delta\rho$ is not, strictly speaking, a hydrodynamic variable.

We will now demonstrate that the same thing happens to the density fluctuations in the chiral problem, with $\delta\rho$ becoming enslaved to our Goldstone mode ϕ.

For the density EOM, we have

$$\begin{aligned}\partial_t\rho + \nabla\cdot\mathbf{J} = g(\rho, |\nabla\mathbf{v}|^2, (\partial_i v_j)(\partial_j v_i), (\nabla\cdot\mathbf{v})^2,\\ \epsilon_{ij}\partial_i v_j(\nabla\cdot\mathbf{v}), |(\mathbf{v}\cdot\nabla)\mathbf{v}|^2),\end{aligned} \tag{5.4}$$

where the number current $\mathbf{J}$ for the chiral case is, by the same logic we used earlier, to be made up of vectors obtained from the velocity field $\mathbf{v}(\mathbf{r},t)$, the gradient operator, and, because our system is chiral, the antisymmetric tensor $\boldsymbol{\epsilon}$.

One might at first think that the gradient operator inside the current $\mathbf{J}$ is unnecessary, since a term proportional to $\mathbf{v}$ in $\mathbf{J}$ (like the $\rho\mathbf{v}$ term already present in the achiral model) would appear to dominate any terms involving gradients of the velocity in the long-wavelength limit. However, that term, and its chiral sibling $\boldsymbol{\epsilon}\cdot\mathbf{v}$, both oscillate at frequency b, while the dominant part of $\delta\rho$ only involves frequencies much less than b, since it gets enslaved to $\phi(\mathbf{r},t)$, as we'll shortly show.

In order to make a term out of velocities that has components that do not oscillate rapidly, we need to make terms with an even number of velocity vectors $\mathbf{v}$. The only way to make such terms that are not simply trivial powers of $|\mathbf{v}|^2$ (which is just a constant v_0^2) is to include an odd number of gradient operators. The leading order terms in a gradient expansion therefore prove to be terms with two velocities and one gradient operator. Including them all, we have

$$\begin{aligned}J_i = \rho_0 v_i + c_{1a} v_i\partial_j v_j + c_{2a} v_j\partial_j v_i + c_1\epsilon_{ij}v_j + c_2\epsilon_{ij}v_k\partial_k v_j\\ + c_3\epsilon_{ij}v_j\partial_k v_k + c_4\epsilon_{jk}v_i\partial_j v_k + c_5\epsilon_{jk}v_j\partial_i v_k + c_6\epsilon_{jk}v_j\partial_k v_i\\ + c_7\epsilon_{j\ell}v_i v_\ell v_k\partial_k v_j.\end{aligned} \tag{5.5}$$

The first three terms of the right hand side of (5.5) are achiral, while the remaining terms are all chiral.

We have not included terms in $\mathbf{J}$ involving spatial derivatives of $\delta\rho$, since they always lead to terms in the EOM (5.4) for $\delta\rho$ which are irrelevant compared to terms coming from the birth minus death term $g(\rho,\ldots)$, to which we now turn.

Expanding that term in powers of its arguments, we have

$$\begin{aligned}g(\rho, |\nabla\mathbf{v}|^2, (\partial_i v_j)(\partial_j v_i), (\nabla\cdot\mathbf{v})^2, \epsilon_{ij}\partial_i v_j(\nabla\cdot\mathbf{v}), |(\mathbf{v}\cdot\nabla)\mathbf{v}|^2)\\ = -\varpi\delta\rho + f_1|\nabla\mathbf{v}|^2 + f_2(\partial_i v_j)(\partial_j v_i) + f_3(\nabla\cdot\mathbf{v})^2\\ + f_4|(\mathbf{v}\cdot\nabla)\mathbf{v}|^2 + f_c\epsilon_{ij}\partial_i v_j(\nabla\cdot\mathbf{v}),\end{aligned} \tag{5.6}$$

where ϖ, and $f_{1,2,3,4,c}$ are all constant expansion coefficients. Note that we must have $\varpi > 0$ if the state with density ρ_0 is to be stable.

We have only expanded to leading order in $|\nabla\mathbf{v}|^2$, $(\partial_i v_j)(\partial_j v_i)$, $(\nabla\cdot\mathbf{v})^2$, and $\epsilon_{ij}\partial_i v_j(\nabla\cdot\mathbf{v})$, $|(\mathbf{v}\cdot\nabla)\mathbf{v}|^2$, since higher order terms involve more derivatives, and hence prove to be irrelevant.

We can also drop the time derivative term $\partial_t\rho = \partial_t\delta\rho$ in the ρ EOM (5.4), since it is irrelevant compared to the $\varpi\delta\rho$ term (which has no derivatives).

Doing so, the resultant EOM for ρ simply becomes a linear algebraic equation for $\delta\rho$, whose solution is easily found to be

$$\begin{aligned}\delta\rho &= \left(\frac{1}{\varpi}\right)[\nabla\cdot\mathbf{J} + f_1|\nabla\mathbf{v}|^2 + f_2(\partial_i v_j)(\partial_j v_i) + f_3(\nabla\cdot\mathbf{v})^2\\ &\quad + f_4|(\mathbf{v}\cdot\nabla)\mathbf{v}|^2 + f_c\epsilon_{ij}\partial_i v_j(\nabla\cdot\mathbf{v})]\\ &= \left(\frac{1}{\varpi}\right)[\partial_i(\rho_0 v_i + c_{1a}v_i\partial_j v_j + c_{2a}v_j\partial_j v_i + c_1\epsilon_{ij}v_j\\ &\quad + c_2\epsilon_{ij}v_k\partial_k v_j + c_3\epsilon_{ij}v_j\partial_k v_k + c_4\epsilon_{jk}v_i\partial_j v_k + c_5\epsilon_{jk}v_j\partial_i v_k\\ &\quad + c_6\epsilon_{jk}v_j\partial_k v_i + c_7\epsilon_{j\ell}v_i v_\ell v_k\partial_k v_j) + f_1|\nabla\mathbf{v}|^2\\ &\quad + f_2(\partial_i v_j)(\partial_j v_i) + f_3(\nabla\cdot\mathbf{v})^2 + f_4|(\mathbf{v}\cdot\nabla)\mathbf{v}|^2\\ &\quad + f_c\epsilon_{ij}\partial_i v_j(\nabla\cdot\mathbf{v})].\end{aligned} \tag{5.7}$$

Using the second equation of (4.1) to reexpress the derivatives of the velocity in terms of derivatives of ϕ and velocities, and then averaging the result (5.7) over short-time oscillation cycles using (4.5), we find that $\delta\rho$ is enslaved to ϕ:

$$\delta\rho = e_3\nabla^2\phi + e_4|\nabla\phi|^2, \tag{5.8}$$

where

$$e_3 = \frac{v_0^2}{2\varpi}\left(c_3 - c_2 + c_4 - 2c_5 - c_6 - c_7 v_0^2\right) \tag{5.9}$$

and

$$e_4 = \frac{v_0^2}{2\varpi}\left(2f_1 + f_2 + f_3 + f_4 v_0^2\right). \tag{5.10}$$

We will use this result in the next section to show that the density fluctuations develop long-ranged spatiotemporal correlations, despite the fact that $\delta\rho$ itself is *not* a hydrodynamic variable in this Malthusian flock, since birth and death violate conservation of flocker number.

VI. PHASE, VELOCITY, AND DENSITY CORRELATIONS IN THE NONLINEAR KPZ REGIME

Having successfully mapped the 2D chiral Malthusian flock onto the $(2+1)$-dimensional KPZ equation, we can now use the known [25–29,31,32] scaling laws for the $(2+1)$-dimensional KPZ equation to determine the scaling laws for our system.

The scaling laws of the KPZ equation are characterized by two universal exponents: a "dynamical exponent" z relating length scales L to timescales t via $t \propto L^z$, and a "roughness exponent" χ relating length scales to the magnitude of the fluctuations of ϕ via $\phi \propto L^\chi$.

This implies that the mean-squared fluctuations of the difference between ϕ's at widely separated points in space and time obey

$$\langle[\phi(\mathbf{r},t) - \phi(\mathbf{r}',t')]^2\rangle = A_\phi|\mathbf{r}-\mathbf{r}'|^{2\chi}\mathcal{G}_\phi\left(\frac{|t-t'|}{|\mathbf{r}-\mathbf{r}'|^z}\right), \tag{6.1}$$

where A_ϕ is a nonuniversal positive constant, and $\mathcal{G}_\phi(X)$ has the following asymptotic scaling behavior:

$$\mathcal{G}_\phi(X) = \begin{cases}1, & X \ll 1,\\ X^{\frac{2\chi}{z}}, & X \gg 1.\end{cases} \tag{6.2}$$

The current estimate of the values of χ and z based on simulations is $\chi = 0.388 \pm 0.002$, $z = 1.622 \pm 0.002$ [26–29,31,32].

Using (3.2) the spatiotemporal correlation of the velocity can be expressed as

$$\begin{aligned}&\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t')\rangle\\ &\quad = v_0^2\{\langle\cos[bt+\phi(\mathbf{r},t)]\cos[bt'+\phi(\mathbf{r}',t')]\rangle\\ &\qquad + \langle\sin[bt+\phi(\mathbf{r},t)\sin[bt'+\phi(\mathbf{r}',t')]\rangle\}\\ &\quad = v_0^2\langle\exp\{\mathrm{i}[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t')]\}\rangle\cos[b(t-t')].\end{aligned} \tag{6.3}$$

If the fluctuations of the field ϕ were Gaussian, then we could say

$$\begin{aligned}&\langle\exp\{\mathrm{i}[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t')]\}\rangle\\ &\quad = \exp\left\{-\frac{1}{2}\langle[(\phi(\mathbf{r},t)-\phi(\mathbf{r}',t))^2]\rangle\right\}\\ &\quad = \exp\left[-\left(\frac{A_\phi}{2}\right)|\mathbf{r}-\mathbf{r}'|^{2\chi}\mathcal{G}_\phi\left(\frac{|t-t'|}{|\mathbf{r}-\mathbf{r}'|^z}\right)\right].\end{aligned} \tag{6.4}$$

Inserting (6.4) into (6.3), we would (if ϕ were Gaussian) get the velocity correlations for arbitrary spatiotemporal separation:

$$\begin{aligned}&\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t')\rangle\\ &\quad = v_0^2\exp\left[-\left(\frac{A_\phi}{2}\right)|\mathbf{r}-\mathbf{r}'|^{2\chi}\mathcal{G}_\phi\left(\frac{|t-t'|}{|\mathbf{r}-\mathbf{r}'|^z}\right)\right]\\ &\qquad \times\cos[b(t-t')].\end{aligned} \tag{6.5}$$

Now we focus on the equal-time correlation. Setting $t'=t$ in (6.5) we find

$$\langle\mathbf{v}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r}',t)\rangle = v_0^2\exp\left[-\left(\frac{A_\phi}{2}\right)|\mathbf{r}-\mathbf{r}'|^{2\chi}\right], \tag{6.6}$$

which would imply (stretched) exponentially decaying velocity correlations.

However, since the ϕ fluctuations are non-Gaussian, due to the relevant nonlinearity λ_κ in the KPZ equation, we can say very little, beyond noting that we expect velocity correlations to decay rapidly with increasing separation $|\mathbf{r}-\mathbf{r}'|$ once that separation is large enough that $\langle[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t)]^2\rangle \gtrsim O(1)$. Using (6.1), we see that this implies that correlations will decay rapidly for $|\mathbf{r}-\mathbf{r}'| > \xi_v \equiv A_\phi^{-\frac{1}{2\chi}} \approx A_\phi^{-1.29}$. However, as we mentioned in the introduction, in the limit of weak chirality, $\xi_v \ll L_{\rm NL}$, and so order will already be effectively lost before we even reach the nonlinear regime. Hence, only for highly chiral systems will ξ_v be the "velocity correlation length." Even then, it must be kept in mind that the decay of velocity correlations is probably not exponential, nor even a simple stretched exponential like Eq. (6.6).

The essential point is that velocity correlations are ultimately short-ranged; that is, chirality has destroyed the long-ranged orientational order present in achiral Malthusian flocks. And it does so for arbitrarily weak noise (i.e., arbitrarily small noise strength D and arbitrarily weak chirality). That is, if one takes a snapshot of the system at any instant of time, the average velocity $\langle \mathbf{v} \rangle$, in the limit of system size going to infinity, is $\langle \mathbf{v} \rangle = \mathbf{0}$.

Combining (5.8) with (6.1) we can predict the correlations of the density fluctuations. We start from the equal-time case:

$$\langle \delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\rangle = e_3^2\nabla^2\nabla'^2\langle\phi(\mathbf{r},t)\phi(\mathbf{r}',t)\rangle + e_3e_4(\langle|\nabla\phi(\mathbf{r},t)|^2\nabla'^2\phi(\mathbf{r}',t)\rangle + \langle|\nabla'\phi(\mathbf{r}',t)|^2\nabla^2\phi(\mathbf{r},t)\rangle) + e_4^2\langle|\nabla\phi(\mathbf{r},t)|^2|\nabla'\phi(\mathbf{r}',t)|^2\rangle. \tag{6.7}$$

The first term in this expression is readily evaluated from the two-point correlation function (6.1):

$$\begin{aligned} e_3^2\nabla^2\nabla'^2\langle\phi(\mathbf{r},t)\phi(\mathbf{r}',t)\rangle &= -\frac{1}{2}e_3^2\nabla^2\nabla'^2\langle[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t)]^2\rangle \\ &= -8A_\phi e_3^2\chi^2(\chi-1)^2|\mathbf{r}-\mathbf{r}'|^{2\chi-4} \\ &\propto |\mathbf{r}-\mathbf{r}'|^{2\chi-4} \approx |\mathbf{r}-\mathbf{r}'|^{-3.224}. \end{aligned} \tag{6.8}$$

The remaining terms cannot be evaluated directly from the two-point correlation function (6.1), since they involve three and four-point correlation functions. These cannot be directly related to two-point correlation functions because the field ϕ is non-Gaussian, due to the relevant λ_κ nonlinearity in the KPZ equation. However, we *can* extract their behavior from scaling arguments: they contain either three or four ϕ's, each of which contributes a factor of $|\mathbf{r}-\mathbf{r}'|^\chi$, while each of the four gradient contributes one factor of $|\mathbf{r}-\mathbf{r}'|^{-1}$. Hence we have

$$\begin{aligned} &\langle|\nabla\phi(\mathbf{r},t)|^2\nabla'^2\phi(\mathbf{r}',t)\rangle \propto |\mathbf{r}-\mathbf{r}'|^{3\chi-4} \approx |\mathbf{r}-\mathbf{r}'|^{-2.863} \propto \langle|\nabla'\phi(\mathbf{r}',t)|^2\nabla^2\phi(\mathbf{r},t)\rangle \\ &\langle|\nabla\phi(\mathbf{r},t)|^2|\nabla\phi(\mathbf{r}',t)|^2\rangle \propto |\mathbf{r}-\mathbf{r}'|^{4\chi-4} \approx |\mathbf{r}-\mathbf{r}'|^{-2.448}, \end{aligned} \tag{6.9}$$

where in the second, approximate, equality we have used the numerically obtained value $\chi = 0.388$ for χ [26–29,31,32].

Since $\chi > 0$, the final term in (6.7) dominates, and at large $|\mathbf{r}-\mathbf{r}'|$. Thus we have

$$\langle \delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\rangle \approx e_4^2\langle|\nabla\phi(\mathbf{r},t)|^2|\nabla\phi(\mathbf{r}',t)|^2\rangle \propto |\mathbf{r}-\mathbf{r}'|^{4\chi-4} \approx |\mathbf{r}-\mathbf{r}'|^{-2.448}. \tag{6.10}$$

Combining the scaling relation between the temporal and spatial dimensions $|t-t'| \sim |\mathbf{r}-\mathbf{r}'|^z$ and (6.10) implies the following scaling behavior for the two-point density correlation for arbitrary temporal and spatial separations:

$$\begin{aligned} &\langle \delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\rangle \\ &\propto |\mathbf{r}-\mathbf{r}'|^{4\chi-4}\mathcal{G}_\rho\left(\frac{|t-t'|}{|\mathbf{r}-\mathbf{r}'|^z}\right) \\ &\approx |\mathbf{r}-\mathbf{r}'|^{-2.448}\mathcal{G}_\rho\left(\frac{|t-t'|}{|\mathbf{r}-\mathbf{r}'|^{1.622}}\right), \end{aligned} \tag{6.11}$$

where

$$\mathcal{G}_\rho(X) = \begin{cases} 1, & X \ll 1, \\ X^{\frac{4\chi-4}{z}}, & X \gg 1. \end{cases} \tag{6.12}$$

These results [(6.5), (6.11)] hold only for length scales $|\mathbf{r}-\mathbf{r}'| \gg L_{\rm NL}$ or $|t-t'| \gg \tau_{\rm NL}$, where $L_{\rm NL}$ and $\tau_{\rm NL}$ are respectively the spatial and temporal length scales at which the nonlinear λ_κ term in the KPZ equation (4.6) becomes important. We calculate these length scales in the next section.

There is also an *upper* bound on the length scales out to which these usual KPZ results hold for our problem. This is for a rather subtle but crucial reason: our mapping is not onto the standard KPZ equation, but onto the *compact* KPZ equation. What we mean by this is that, unlike, say, a growing interface, in which the KPZ variable ϕ is the height of the interface, and every value of the height represents a physically distinct local state, our KPZ variable is a *phase*. That is, a state with a given local value $\phi(\mathbf{r},t)$ is locally physically indistinguishable from a state $\phi(\mathbf{r},t)+2\pi n$, for any integer n.

This is the same symmetry that is present in the 2D XY model. Here, as there, it allows topological defects, i.e., vortices [40]. These become important on length scales larger than the mean intervortex distance L_v. We calculate this length scale in Sec. XI, in the limit of small chirality, as well.

VII. ACHIRAL TO CHIRAL CROSSOVER LENGTHS AND TIMES IN THE SMALL CHIRALITY LIMIT

When the chirality is weak, that is, when b is small, then for times

$$t \ll \tau_c \equiv b^{-1}, \tag{7.1}$$

the direction of the mean velocity hardly changes, and the system should behave like an achiral Malthusian flock for sufficiently small distances. To determine those distances, we can use the spatiotemporal scaling properties of achiral Malthusian flocks [15–17,39].

In an achiral Malthusian flock, Fourier modes with wavevector $\mathbf{q}$ decay at a rate $\Gamma(\mathbf{q})$ given by the following scaling form [15–17]:

$$\Gamma(\mathbf{q}) = \Upsilon|q_y\xi_y|^{z_a-2}q_y^2G_{\rm achiral}\left(\frac{|q_y\xi_y|^{\zeta_a}}{|q_x\xi_x|}\right), \tag{7.2}$$

where q_x and q_y are the components of $\mathbf{q}$ along and perpendicular to the mean direction $\hat{\mathbf{x}}$ of flock motion, respectively. In addition, Υ is a nonuniversal constant with the dimensions of a diffusion constant, z_a and ζ_a are the universal "dynamic" and "anisotropy" exponents of the achiral problem, respectively, $\xi_{x,y}$ are nonuniversal microscopic length scales for the achiral problem, and $G_{\rm achiral}$ is a dimensionless scaling function, with the limiting behaviors:

$$G_{\rm achiral}(u \ll 1) \propto u^{-z_a/\zeta_a}, \tag{7.3}$$

$$G_{\rm achiral}(u \gg 1) \approx 1. \tag{7.4}$$

Contrary to recent claims in the literature [39,48], there is no exact analytic expression for the universal exponents z_a and ζ_a. Fortunately, these exponents *are* known through direct simulations of the noisy hydrodynamic equations of achiral Malthusian flocks [39], which find

$$z_a \approx 5/4, \quad \zeta_a \approx 3/4. \tag{7.5}$$

In the *chiral* system, however, the direction of mean motion is rotating at a constant angular frequency b. This means that, in a time-averaged sense, all directions are equivalent. Therefore, we can without loss of generality take $\mathbf{q} = (q, 0)$. To obtain the *instantaneous* damping rate of such a mode, we need to project this wave vector onto the *rotating* axis of mean motion $\hat{\mathbf{x}}(t) = (\cos(bt), \sin(bt))$, and the rotating axis $\hat{\mathbf{y}}(t) = (-\sin(bt), \cos(bt))$ perpendicular to it, in order to obtain the q_x and q_y that enter (7.2). Thus, we effectively have *time-dependent* components

$$q_x(t) = q\cos(bt), \quad q_y(t) = -q\sin(bt). \tag{7.6}$$

Inserting these expressions for the components of $\mathbf{q}$ into our expression (7.2), we see that the damping also becomes time dependent:

$$\Gamma(\mathbf{q}) = \Upsilon q^{z_a}\xi_y^{z_a-2}|\sin(bt)|^{z_a}G_{\rm achiral}(u(q,t)), \tag{7.7}$$

where we have defined

$$u(q,t) = q^{\zeta_a-1}\frac{|\xi_y\sin(bt)|^{\zeta_a}}{|\xi_x\cos(bt)|}. \tag{7.8}$$

At long times, we are interested in the time average of this damping coefficient over a full cycle, i.e.,

$$\langle\Gamma(\mathbf{q})\rangle_{\rm cycle} = \Upsilon q^{z_a}\xi_y^{z_a-2}\langle|\sin bt|^{z_a}G_{\rm achiral}(u(q,t))\rangle_{\rm cycle}. \tag{7.9}$$

Since ζ_a has recently been estimated to be 3/4, which is less than 1, and we are interested in the small q regime, $u \gg 1$ unless

$$\frac{|\sin bt|^{\zeta_a}}{|\cos bt|} \lesssim q^{1-\zeta_a}\frac{\xi_x}{\xi_y^{\zeta_a}}, \tag{7.10}$$

which is a vanishing window within the cycle in the small q limit. Therefore, we can approximate $G_{\rm achiral}(u(q,t))$ by 1 over most of the cycle. Hence,

$$\begin{aligned}\langle\Gamma(\mathbf{q})\rangle_{\rm cycle} &= \Upsilon q^{z_a}\xi_y^{z_a-2}\langle|\sin bt|^{z_a}\rangle_{\rm cycle}\\ &= \Upsilon q^{z_a}\xi_y^{z_a-2}\frac{\Gamma\left(\frac{1+z_a}{2}\right)}{\sqrt{\pi}\,\Gamma(1+z_a/2)}.\end{aligned} \tag{7.11}$$

Using the recent estimate of 5/4 for z_a, we have

$$\langle\Gamma(\mathbf{q})\rangle_{\rm cycle} \simeq 0.593 \times \Upsilon q^{z_a}\xi_y^{z_a-2}. \tag{7.12}$$

To obtain the achiral-chiral crossover length L_c, or, equivalently, the associated wave number $q_c = L_c^{-1}$, we now equate the above expression with the corresponding one of the linearized KPZ equation, which is

$$\Gamma(\mathbf{q}) = \nu q^2. \tag{7.13}$$

At the crossover length $L_c = \Lambda_c^{-1} = q_c^{-1}$, we have

$$\nu\Lambda_c^2 = \Upsilon\Lambda_c^{z_a}\xi_y^{z_a-2} \times O(1). \tag{7.14}$$

Further, we expect the crossover happens at a timescale when the relaxation time due to damping equals the rotation period. Therefore, we equate $\nu\Lambda_c^2$ with the rotation frequency b to arrive at

$$b = \nu\Lambda_c^2 \simeq \Upsilon\Lambda_c^{z_a}\xi_y^{z_a-2}. \tag{7.15}$$

Equating the first and third terms in (7.15) leads to

$$\Lambda_c \propto b^{1/z_a} \tag{7.16}$$

or

$$L_c \propto b^{-1/z_a} \simeq b^{-4/5}, \tag{7.17}$$

where we have used the recent numerical estimate [39] in the second relation of (7.17) above.

The fact that τ_c and L_c both diverge as the chirality $b \to 0$ leads to strong b dependence of the parameters ν and λ_κ of the ultimate KPZ model as well. We have already seen the strong dependence of λ_κ on the chirality: since λ_κ is generated exclusively by the chiral terms in our model, it follows that it must vanish in the limit of zero chirality. Hence, by analyticity, it should vanish linearly with the chirality b as $b \to 0$; i.e.,

$$\lambda_K \propto b. \tag{7.18}$$

The diffusivity ν in the ultimate KPZ equation, on the other hand, *diverges* as $b \to 0$. The origin of this divergence lies in two phenomena: first, the $\tau_c \propto b^{-1}$ divergence of the timescale τ_c for the crossover from achiral to chiral behavior, and second, the fact that the achiral system exhibits "anomalous hydrodynamics": that is, a divergence of the diffusion constants $\mu_{1,2,3}$ appearing in the EOM (2.4) as a function of the length and time scales on which they are measured.

We can express this divergence in terms of the known universal exponents z_a and ζ_a by the following argument: An achiral system will reach a state in which only small fluctuations around a nonzero average velocity will occur. Taking the direction of this nonzero average velocity to be $\hat{\mathbf{x}}$, we can expand $\mathbf{v}(\mathbf{r},t)$ around the steady state:

$$\mathbf{v}(\mathbf{r},t) = [v_0 + u_x(\mathbf{r},t)]\hat{\mathbf{x}} + u_y(\mathbf{r},t)\hat{\mathbf{y}}, \tag{7.19}$$

where $\hat{\mathbf{y}}$ denotes the direction perpendicular to $\hat{\mathbf{x}}$. Inserting (7.19) into (2.1) and focusing on the u_y part of the EOM since u_y exhibits the largest fluctuations, we get

$$\partial_t u_y = \mu_x\partial_x^2 u_y + \mu_y\partial_y^2 u_y - \lambda_1 u_y\partial_y u_y + \frac{\lambda_1}{2v_0}u_y^2\partial_x u_y + f_y, \tag{7.20}$$

where

$$\mu_x = \mu_1 + \mu_3 v_0^2, \tag{7.21}$$

$$\mu_y = \mu_1 + \mu_2. \tag{7.22}$$

In deriving (7.20), we have boosted to a moving frame via a Galilean transformation $x \to x - \lambda v_0 t$ to eliminate the term $-\lambda_1 v_0 \partial_x u_y$ on the right-hand side.

Previous work [15–17] showed that the nonlinear terms in (7.20) are "relevant" in the renormalization group sense of changing the long-wavelength behavior of the system. Indeed, they are responsible for the existence of long-ranged order in the 2D system; without it, the "Mermin-Wagner" destruction of long-ranged order by fluctuations, which always occurs in equilibrium systems trying to break continuous rotation invariance in two dimensions, would occur here as well.

The effect of the nonlinearity can be incorporated into an effective "renormalization" of the diffusion constants $\mu_{x,y}$, making them length-scale- and timescale-dependent, diverging as the length and time scales under consideration go to infinity. It is this length and time scale dependence that leads to the unusual, noninteger values of the dynamical and anisotropy exponents z_a and ζ_a. To see this, note that in Eq. (7.20) we can estimate $\partial_t u_y \sim \frac{u_y}{t}$ and $\mu_y \partial_y^2 u_y \sim \mu_y \frac{u_y}{y^2}$. Balancing these terms then gives the scaling of the diffusivity $\mu_y(t)$:

$$\mu_y(t) \propto \frac{y^2}{t} \propto t^{2/z_a - 1} = t^{3/5}, \tag{7.23}$$

where in the second proportionality we have used the scaling law $y \propto t^{1/z_a}$ mentioned earlier, while in the final equality we have used the values (7.5) for ζ_a and z_a obtained from the numerical results of [39].

A similar analysis for the diffusivity $\mu_x(t)$ gives

$$\mu_x(t) \propto \frac{x^2}{t} \propto t^{2\zeta_a/z_a - 1} = t^{1/5}. \tag{7.24}$$

Comparing our expression (4.7) for the KPZ diffusivity ν with our expressions (7.21) and (7.22), we see that

$$\nu = \frac{\mu_x + \mu_y}{2}. \tag{7.25}$$

Since $\mu_{x,y}$ are length-scale- and timescale-dependent, we must decide at which timescale t we should evaluate the right-hand side of (7.25). But the answer is clear: it must be the crossover time $t = \tau_c = b^{-1}$ at which the system crosses over from achiral to chiral behavior. Thus,

$$\nu(b) = \frac{\mu_x(t = b^{-1}) + \mu_y(t = b^{-1})}{2}. \tag{7.26}$$

Using our earlier results (7.23) and (7.24), we see that

$$\mu_y(t = b^{-1}) \propto b^{1-2/z_a} \approx b^{-3/5},$$
$$\mu_x(t = b^{-1}) \propto b^{1-2\zeta_a/z_a} \approx b^{-1/5}. \tag{7.27}$$

Inserting these results in (7.25), we see that the contribution of μ_y dominates at small b, where we therefore have

$$\nu(b) \propto b^{1-2/z_a} \equiv b^{-\eta_\nu}, \tag{7.28}$$

where the universal exponent

$$\eta_\nu = 2/z_a - 1 \approx 3/5. \tag{7.29}$$

The numerical value given here is based on numerical simulations of the noisy hydrodynamic equation for achiral Malthusian flocks by [39].

Inserting (7.28) into the first equality of (7.15), we (reassuringly) recover the value of L_c obtained earlier in (7.17).

To summarize: small chirality b only manifests itself on timescales larger than $\tau_c \propto b^{-1}$, or length scales longer than $L_c \propto b^{-1/z_a} \simeq b^{-4/5}$. On those longer length and time scales, chiral Malthusian flocks are described by a KPZ equation with a very small nonlinearity $\lambda_\kappa \propto b$ and a very large diffusivity $\nu \propto b^{-\eta_\nu}$, with $\eta_\nu \approx 3/5$.

The final parameter of the KPZ equation is the noise strength D_ϕ in Eq. (4.10). This will be independent of chirality b for small chirality, because the noise strength D_ϕ of the chiral problem is related to the velocity noise strength D and the mean speed v_0 of the flockers by $D_\phi = D/v_0^2$, and neither D nor v_0 gets any large renormalization in the achiral problem [15–17,39].

We note that our result (7.28) for the b dependence of $\nu(b)$ is consistent with the expectation that the time-space correlation function depends purely on the ratio

$$\frac{t/\tau_c}{(r/L_c)^2}. \tag{7.30}$$

This is because $\tau_c \sim b^{-1}$ and thus $L_c^2/\tau_c \simeq \nu$ [using the first relation in Eq. (7.15)], leading to the ratio being of the form

$$\frac{\nu t}{r^2}, \tag{7.31}$$

which conforms to the expectation from the linearized KPZ equation.

The divergence of ν as $b \to 0$ and the vanishing of the nonlinearity λ_K together imply that the asymptotic nonlinear KPZ behavior described in the previous section, specifically, the scaling behavior of the correlations (6.1), (6.5), and (6.11) will manifest itself only on length scales $|\mathbf{r} - \mathbf{r}'| \gg L_{\rm NL}$ or timescales $|t - t'| \gg \tau_{\rm NL}$, and for small b, $L_{\rm NL} \gg L_c$, $\tau_{\rm NL} \gg \tau_c$.

We can calculate $L_{\rm NL}$ using the renormalization group recursion relations for the dimensionless coupling

$$g \equiv \frac{\lambda_\kappa^2 D_\phi}{2\pi \nu^3}, \tag{7.32}$$

which provides a measure of the importance of nonlinear effects in the KPZ equation [25].

The recursion relation for g is [25]

$$\frac{dg}{d\ell} = \frac{g^2}{4} + O(g^3). \tag{7.33}$$

Note that the "bare" value g_0 of g, which in our problem is the value of g at the length scales L_c and τ_c, vanishes quite rapidly as the chirality b decreases. Specifically, using our earlier results (7.18) and (7.28) that $\lambda_K \propto b$ and $\nu \propto b^{-3/5}$, we see that

$$g_0 \propto b^{2-3(1-2/z_a)} = b^{6/z_a - 1} \equiv b^{\eta_g}, \tag{7.34}$$

where we have defined the universal exponent

$$\eta_g \equiv 6/z_a - 1 \approx 19/5, \tag{7.35}$$

and, as with all numerical values based on the *achiral* exponents, we have used the values ζ_a and z_a obtained from the simulations of the noisy hydrodynamic equations for achiral flocks of [39] in the last, approximate, equality.

The solution of the recursion relation (7.33) for such an initially small value of g_0 is easily found:

$$g(\ell) = \frac{g_0}{1-\left(\frac{g_0}{4}\right)\ell}. \tag{7.36}$$

It follows from this that $g(\ell)$ gets to be of $O(1)$, which signals the onset of important nonlinear effects, when

$$\ell \approx 4/g_0 \equiv \ell_{\rm NL} \propto b^{-\eta_g}, \tag{7.37}$$

where we remind the reader that the universal exponent $\eta_g \approx 19/5$. As usual in the renormalization group (RG), we can associate with this RG time a "nonlinear length scale" $L_{\rm NL}$ given by

$$L_{\rm NL} = L_c e^{\ell_{\rm NL}} = L_c \exp[C_{\rm NL} b^{-\eta_g}], \tag{7.38}$$

and a "nonlinear timescale" $\tau_{\rm NL}$ given by

$$\tau_{\rm NL} = \tau_c e^{z_L \ell_{\rm NL}} = \tau_c \exp[2C_{\rm NL} b^{-\eta_g}], \tag{7.39}$$

where $C_{\rm NL}$ is a nonuniversal, chirality b independent constant, and $z_L = 2$ is the dynamic exponent for the KPZ equation in the linear regime.

An equivalent alternative approach to the calculation of $L_{\rm NL}$ and $\tau_{\rm NL}$ is to perform perturbation theory on (4.6) and calculate the lowest order correction to the noise strength D_ϕ. The length and time scales at which this correction becomes comparable to the bare value of D_ϕ are $L_{\rm NL}$ and $\tau_{\rm NL}$, respectively. We have done this calculation, and find, reassuringly, that it reproduces the results (7.38) and (7.39) that we just obtained from the RG analysis.

Clearly, these length and time scales $L_{\rm NL}$ and $\tau_{\rm NL}$ given by (7.38) and (7.39) can become much greater than the length and and time scales L_c and τ_c at which the effects of chirality first become important, even for chiralities that are not particularly small.

To illustrate this, consider a system in which we will call "system 0," in which $(L_{\rm NL})_{\text{system } 0} = 3L_c$ [which implies $\exp(C_{\rm NL} b^{-\frac{19}{5}}) = 3$] for some small $b \equiv b_0$. Now consider two other systems, in which all of the dynamical parameters are the same as those of system 0 except for the chirality b. The first system ("system 1") will have chirality $b_1 = b_0/2$; the second ("system 2") will have chirality $b_2 = b_0/3$. Then (7.38) predicts

$$\begin{aligned}(L_{\rm NL})_{\text{system } 1} &= \left[2^{\frac{4}{5}} \times 3^{(2^{\frac{19}{5}})-1}\right](L_{\rm NL})_{\text{system } 0} \\ &\approx 2.57 \times 10^6 (L_{\rm NL})_{\text{system } 0},\end{aligned} \tag{7.40}$$

$$\begin{aligned}(L_{\rm NL})_{\text{system } 2} &= 3^{(3^{\frac{19}{5}})-\frac{1}{5}}(L_{\rm NL})_{\text{system } 0} \\ &\approx 1.315 \times 10^{31} (L_{\rm NL})_{\text{system } 0}.\end{aligned} \tag{7.41}$$

If $(L_{\rm NL})_{\text{system } 0} = 1\,\mu\text{m}$, then $(L_{\rm NL})_{\text{system } 1} \sim 2.6\,\text{m}$, and $(L_{\rm NL})_{\text{system } 2} \sim 1.3 \times 10^{25}\,\text{m}$, which is a cosmological length scale (roughly 1.4 *billion light years*, or about 3% of the size of the observable universe)! Similarly, $\tau_{\rm NL}$ also exhibits a similarly dramatic divergence in the limit of weak chirality.

So for weak chirality the nonlinear crossover length $L_{\rm NL}$ and time $\tau_{\rm NL}$ are certainly much larger than the chiral crossover length L_c and time τ_c.

In any event, at small chirality there will certainly be an appreciable range of length and time scales, specifically $L_c \ll |\mathbf{r}-\mathbf{r}'| \ll L_{\rm NL}$, $|t-t'| \ll \tau_{\rm NL}$, or $\tau_c \ll |t-t'| \ll \tau_{\rm NL}$, $|\mathbf{r}-\mathbf{r}'| \ll L_{\rm NL}$, over which the phase ϕ correlations can be evaluated based on the KPZ equation with the non-linearity neglected.

It therefore behooves us to consider the behavior of our flock in this linear regime $L_c \ll L \ll L_{\rm NL}$. We do so in the next section.

VIII. PHASE AND VELOCITY CORRELATIONS IN THE LINEAR KPZ REGIME

We begin by noting that in the linear regime, we can drop the λ_κ nonlinearity in the KPZ equation. This turns the EOM for ϕ into the "Edwards-Wilkinson" equation [43]:

$$\partial_t \phi = \nu \nabla^2 \phi + f_\phi. \tag{8.1}$$

This EOM is identical to a purely relaxational model for an equilibrium *XY* model:

$$\partial_t \phi = -\Gamma \frac{\delta H}{\delta \phi} + f_\phi, \tag{8.2}$$

with the Hamiltonian

$$H = \frac{K}{2} \int d^d r |\nabla \phi(\mathbf{r})|^2. \tag{8.3}$$

In this mapping, we have

$$\nu = \Gamma K, \tag{8.4}$$

$$D_\phi = \Gamma k_B T. \tag{8.5}$$

After spatiotemporal Fourier transforming Eq. (8.1) and solving it for ϕ, we obtain

$$\phi(\mathbf{q},\omega) = \frac{f(\mathbf{q},\omega)}{-i\omega + \nu q^2}, \tag{8.6}$$

where

$$\phi(\mathbf{q},\omega) = \int \frac{\mathrm{d}t}{\sqrt{2\pi}} \int \frac{\mathrm{d}^2\mathbf{r}}{\sqrt{A}} \phi(\mathbf{r},t) \mathrm{e}^{-\mathrm{i}(\omega t - \mathbf{q}\cdot\mathbf{r})}, \tag{8.7}$$

$$f_\phi(\mathbf{q},\omega) = \int \frac{\mathrm{d}t}{\sqrt{2\pi}} \int \frac{\mathrm{d}^2\mathbf{r}}{\sqrt{A}} f_\phi(\mathbf{r},t) \mathrm{e}^{-\mathrm{i}(\omega t - \mathbf{q}\cdot\mathbf{r})}, \tag{8.8}$$

where A is the area of the system. Autocorrelating this with itself then implies

$$\begin{aligned}\langle \phi(\mathbf{q},\omega)\phi(\mathbf{q}',\omega')\rangle &= \frac{2D_\phi \delta^K_{\mathbf{q}+\mathbf{q}'} \delta(\omega+\omega')}{\omega^2 + \nu^2 q^4} \\ &\equiv C_\phi(\mathbf{q},\omega) \delta^K_{\mathbf{q}+\mathbf{q}'} \delta(\omega+\omega'),\end{aligned} \tag{8.9}$$

where we have used the fact that

$$\langle f_\phi(\mathbf{q},\omega) f_\phi(\mathbf{q}',\omega')\rangle = 2D_\phi \delta^K_{\mathbf{q}+\mathbf{q}'} \delta(\omega+\omega'), \tag{8.10}$$

which follows from (4.10) upon Fourier transforming. Here $\delta^K_{\mathbf{p}}$ is a Kronecker delta, which is equal to 1 when $\mathbf{p} = \mathbf{0}$, and equal to 0 when $\mathbf{p} \neq \mathbf{0}$. This implies that this real-space, real-time correlation function is given by

$$\begin{aligned} C_\phi(\mathbf{r},t) &\equiv \langle[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{R},T)]^2\rangle \\ &= 2\int \frac{d\omega}{2\pi}\int \frac{d^2q}{(2\pi)^2} C_\phi(\mathbf{q},\omega)(1-\cos[\omega t-\mathbf{q}\cdot\mathbf{r}])e^{-(qL_c)^2} \\ &= 4D_\phi \int \frac{d\omega}{2\pi}\int \frac{d^2q}{(2\pi)^2}\left[\frac{(1-\cos[\omega t-\mathbf{q}\cdot\mathbf{r}])}{\omega^2+\nu^2q^4}\right]e^{-(qL_c)^2}, \end{aligned} \tag{8.11}$$

where we have introduced a Gaussian ultraviolet cutoff at wavevectors comparable to L_c^{-1}, where L_c is the achiral to chiral crossover length given by Eq. (7.17). Since, as we shall see, the form of our final result depends only logarithmically on the ultraviolet cutoff, the precise form of the cutoff we use is unimportant. More precisely, changing the cutoff will introduce only changes of additive constants of $O(1)$ to $C_\phi(\mathbf{r},t)$.

The integral in (8.11) is evaluated in Appendix C; the result is

$$C_\phi(\mathbf{r},t) = \alpha\left[2\ln\left(\frac{r}{2L_c}\right) - \mathrm{Ei}\left(-\frac{r^2}{4\left(\nu|t|+L_c^2\right)}\right)+\gamma\right], \tag{8.12}$$

where we have used our earlier definition $\alpha \equiv \frac{D_\phi}{2\pi\nu}$. This recovers the result of (1.17) of the introduction; the limiting behaviors obtained there follow straightforwardly from the large and small argument limits of the exponential integral function $\mathrm{Ei}(-\frac{r^2}{4(\nu|t|+L_c^2)})$.

We can use this result to calculate the velocity correlation function from the linear theory. In the linear theory, fluctuations are Gaussian, so we can write

$$\begin{aligned} &\langle\exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle \\ &= \exp\left\{-\frac{1}{2}\langle[(\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t))^2]\rangle\right\} \\ &= \exp\left(-\frac{1}{2}C_\phi(\mathbf{R},T)\right), \end{aligned} \tag{8.13}$$

where in the first equality we have used the well-known identity for zero-mean Gaussian random variables $\langle e^{\mathrm{i}x}\rangle = e^{-\langle x^2\rangle/2}$.

Inserting (8.12) into (8.13), and using the asymptotic limiting behaviors of the exponential integral function for large and small argument, we get

$$\begin{aligned} &\langle\exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle \\ &\approx \begin{cases} e^{-\alpha\gamma/2}\left(\frac{|\mathbf{R}|}{2L_c}\right)^{-\alpha}, & \frac{|T|}{\tau_c} \ll \left(\frac{|\mathbf{R}|}{L_c}\right)^2, \\ \left(\frac{|T|}{\tau_c}\right)^{-\frac{\alpha}{2}}, & \frac{|T|}{\tau_c} \gg \left(\frac{|\mathbf{R}|}{L_c}\right)^2, \end{cases} \end{aligned} \tag{8.14}$$

Inserting (8.14) into (6.3) leads to the velocity correlation (1.18) given in the introduction.

IX. THE PURELY TEMPORAL FOURIER TRANSFORM OF THE VELOCITY CORRELATION FUNCTION

The equal-time velocity correlations are readily measurable. However, non-equal-time velocity correlations are more subtle, since they carry a frequency b, i.e., the cosine factor which oscillates in the time difference $t-t'$ with period $2\pi/b$. This temporal oscillation is reminiscent of the spatial oscillations of density correlations in smectics [49]. Those correlations in smectics also oscillate rapidly, in that case in *space*, with period a (the layer spacing). In smectics, the signature of this spatially rapid oscillation is the appearance of "Bragg peaks" in the Fourier component of the density correlation function at $\mathbf{q} = (2m\pi/a)\hat{n}$ $(m = 1, 2, \ldots)$. These peaks can be observed experimentally via x-ray scattering [49].

Like our 2D system in the linear regime, three-dimensional smectics also exhibit logarithmically divergent phase fluctuations [49,50]. These manifest themselves in the shape of the x-ray scattering, which exhibit algebraic divergences as the peaks are approached [49,50].

Here we make a similar prediction for the temporal Fourier transform of the velocity correlation function near the "Bragg peaks" at frequencies $\omega = \pm b$ in frequency [51]. This Fourier transform is given by

$$I(\mathbf{R},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\mathbf{v}(\mathbf{r}+\mathbf{R},t+T)\cdot\mathbf{v}(\mathbf{r},t)\rangle\, e^{\mathrm{i}\omega T}\,dT. \tag{9.1}$$

Using the relation (3.2) between the velocity $\mathbf{v}(\mathbf{r},t)$ and the phase variable $\phi(\mathbf{r},t)$, the velocity correlations can be rewritten in terms of the phase correlations:

$$\begin{aligned} &\langle\mathbf{v}(\mathbf{r}+\mathbf{R},t+T)\cdot\mathbf{v}(\mathbf{r},t)\rangle \\ &= v_0^2\cos(bT)\langle\exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle. \end{aligned} \tag{9.2}$$

Inserting this into our expression (9.1) for the temporal Fourier transform of the velocity correlation function gives

$$\begin{aligned} I(\mathbf{R},\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} v_0^2\cos(bT)\langle\exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle e^{\mathrm{i}\omega T}\,dT \\ &= \frac{v_0^2}{4\pi}\int_{-\infty}^{\infty}\langle\exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}(e^{\mathrm{i}bT}+e^{-\mathrm{i}bT})e^{\mathrm{i}\omega T}\,dT \\ &= \frac{v_0^2}{4\pi}\int_{-\infty}^{\infty}\langle\exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle(e^{\mathrm{i}\delta\omega_+T}+e^{\mathrm{i}\delta\omega_-T})\,dT \\ &= G(\mathbf{R},\delta\omega_+)+G(\mathbf{R},\delta\omega_-), \end{aligned} \tag{9.3}$$

where we have defined

$$\delta\omega_{\pm} \equiv \omega \pm b \tag{9.4}$$

and

$$G(\mathbf{R}, \delta\omega) \equiv \frac{v_0^2}{4\pi}\int_{-\infty}^{\infty} \langle \exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R}, t+T) - \phi(\mathbf{r}, t)]\}\rangle e^{\mathrm{i}\delta\omega T}\,\mathrm{d}T. \tag{9.5}$$

Note that $\delta\omega_+$ measures the distance in frequency from the peak at $\omega = b$, while $\delta\omega_-$ similarly measures the distance in frequency from the peak at $\omega = -b$.

If there were no fluctuations in the phase ϕ, the average

$$\langle \exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R}, t+T) - \phi(\mathbf{r}, t)]\}\rangle = 1, \text{ no fluctuations.} \tag{9.6}$$

Inserting this into (9.3) reveals that the temporal Fourier transform of the velocity correlation function in the absence of fluctuations would consist of two delta-function Bragg peaks, at $\omega = \pm b$:

$$\begin{aligned} I(\mathbf{R}, \omega) &= \frac{v_0^2}{2}[\delta(\delta\omega_+) + \delta(\delta\omega_-)] \\ &= \frac{v_0^2}{2}[\delta(\omega+b) + \delta(\omega-b)], \text{ no fluctuations.} \end{aligned} \tag{9.7}$$

In the presence of fluctuations in ϕ, these peaks broaden. In light of the fact that the fluctuations of ϕ are slow (since ϕ is a hydrodynamic variable), $G(\delta\omega)$ will be appreciable only when $\delta\omega$ is small. Hence, in light of (9.3), $I(\mathbf{R}, \omega)$ will be large only when either $\delta\omega_+$ or $\delta\omega_-$ is small; that is, when ω is near either $+b$ or $-b$. When ω is near $+b$, $I(\mathbf{R}, \omega)$ will be dominated by the contribution from $G(\delta\omega_-)$, since only $\delta\omega_-$ is small in that range of ω; likewise, $I(\omega)$ will be dominated by the contribution from $G(\delta\omega_+)$ when ω is near $-b$.

Furthermore, the integral over the time difference T in (9.5) is dominated by $T \sim |\delta\omega|^{-1}$. Therefore, for $|\delta\omega|$ in the range $b^{-1} \gg |\delta\omega| \gg \tau_{\rm NL}^{-1}$, we can calculate the velocity correlation function in (9.5) from the linear theory. In that linear theory, fluctuations are Gaussian, so we can write

$$\begin{aligned} &\langle \exp\{\mathrm{i}[\phi(\mathbf{r}+\mathbf{R}, t+T) - \phi(\mathbf{r}, t)]\}\rangle \\ &\quad = \exp\left\{-\frac{1}{2}\langle[\phi(\mathbf{r}+\mathbf{R}, t+T) - \phi(\mathbf{r}, t)]^2\rangle\right\} \\ &\quad = \exp\left(-\frac{1}{2}C_\phi(\mathbf{R}, T)\right), \end{aligned} \tag{9.8}$$

where we have used the well-known identity for zero-mean Gaussian random variables $\langle e^{\mathrm{i}x}\rangle = e^{-\langle x^2\rangle/2}$.

Using this and (8.12), we can rewrite our expression (9.5) for $G(\mathbf{R}, \delta\omega)$ as

$$G(\mathbf{R}, \delta\omega) = \frac{v_0^2}{4\pi}\left(\frac{Re^{\gamma/2}}{2L_c}\right)^{-\alpha} h(\mathbf{R}, \delta\omega), \tag{9.9}$$

where we have defined

$$h(\mathbf{R}, \delta\omega) \equiv \int_{-\infty}^{\infty} \exp\left[\mathrm{i}\delta\omega T + \frac{\alpha}{2}\mathrm{Ei}\left(-\frac{R^2}{4\nu|T| + L_c^2}\right)\right]\mathrm{d}T. \tag{9.10}$$

The integral in (9.10) is dominated by the range $T \sim |\delta\omega|^{-1}$. Since $b^{-1} \gg |\delta\omega| \gg \tau_{\rm NL}^{-1}$ and $L_c^2 \sim \nu b$, we can neglect the L_c^2 in the argument of the Ei function. Therefore, we approximate (9.10) as

$$h(\mathbf{R}, \delta\omega) \equiv \int_{-\infty}^{\infty} \exp\left[\mathrm{i}\delta\omega T + \frac{\alpha}{2}\mathrm{Ei}\left(-\frac{R^2}{4\nu|T|}\right)\right]\mathrm{d}T. \tag{9.11}$$

Making the change of variables of integration from T to u with

$$u \equiv \frac{4\nu T}{R^2} \tag{9.12}$$

in (9.11), we can write $h(R, \delta\omega)$ in a scaling form:

$$h(\mathbf{R}, \delta\omega) = \frac{R^2}{4\nu}F_h\left(\frac{\delta\omega R^2}{4\nu}\right), \tag{9.13}$$

where the scaling function $F_h(S)$ is given by

$$F_h(S) = \int_{-\infty}^{\infty} \exp\left[\mathrm{i}Su + \frac{\alpha}{2}\mathrm{Ei}\left(-\frac{1}{|u|}\right)\right]\mathrm{d}u. \tag{9.14}$$

We will first derive the limiting behavior of this scaling function for small argument S, which will give us the behavior of $I(R, \delta\omega)$ for $\delta\omega \ll \frac{\nu}{R^2}$.

In this limit, the integral in (9.14) is dominated by $|u| \sim 1/|S| \gg 1$, which obviously implies, $1/|u| \ll 1$, so we can use the small argument limit of the exponential integral function $\mathrm{Ei}(-\frac{1}{|u|}) \approx -\ln|u| + \gamma$. Inserting this into (9.14), we obtain

$$\begin{aligned} F_h(S) &\approx \int_{-\infty}^{\infty} e^{\mathrm{i}Su}|u|^{-\alpha/2}e^{\alpha\gamma/2}\,\mathrm{d}u \\ &= \int_0^{\infty} e^{\mathrm{i}|S|u}u^{-\alpha/2}e^{\alpha\gamma/2}\,\mathrm{d}u + \text{c.c.} \end{aligned} \tag{9.15}$$

The integral in this expression is easily done by rotating the contour of integration over u in the complex plane to the imaginary axis. The result is

$$F_h(S) \approx C_h|S|^{\frac{\alpha}{2}-1} \tag{9.16}$$

with the constant $C_h(\alpha)$ (which is "constant" in the sense of being independent of S) given by

$$C_h(\alpha) = 2e^{\frac{\alpha\gamma}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)\sin\left(\frac{\alpha\pi}{4}\right). \tag{9.17}$$

Inserting the result (9.16) into (9.13) for $h(\mathbf{R}, \delta\omega)$, and inserting *that* result into our expression (9.9) for $G(\mathbf{R}, \delta\omega)$, we see that the separation R between the two spatial points being correlated cancels, leaving us with

$$G(\mathbf{R}, \delta\omega) = \frac{C_h v_0^2}{4\pi}\left(\frac{L_c^\alpha|\delta\omega|^{\alpha/2-1}}{(\nu e^\gamma)^{\alpha/2}}\right), \quad |\delta\omega| \ll \nu/R^2. \tag{9.18}$$

The fact that this result is independent of the spatial separation R in this limit is unsurprising. It is because the timescales being probed in this range of $\delta\omega$ are $\gg R^2/\nu$, which is simply the time it takes distortions of the phase field ϕ to diffuse a distance R. Thus, for longer times, we expect the correlations between these two points to be the same as if they were at the *same* point in space. Hence, R should drop out, as we just found it does.

Now recalling that the temporally Fourier transformed correlation function is dominated by the $G(\mathbf{R}, \delta\omega_-)$ term for ω near b, and by the $G(\mathbf{R}, \delta\omega_+)$ term for ω near $-b$, we see that

$$I(\mathbf{R},\omega) \approx \frac{C_h v_0^2}{4\pi}\left(\frac{|\delta\omega|^{\alpha/2-1}}{\left(\nu L_c^2 e^{\gamma}\right)^{\alpha/2}}\right), \quad |\delta\omega| \ll \nu/R^2 \tag{9.19}$$

with $\delta\omega$ the smaller of $\delta\omega_\pm$; i.e., the distance in frequency from the nearer of the Bragg peak positions.

In the opposite limit $|S| \gg 1$, which will determine the behavior of $I(\mathbf{R},\omega)$ for $|\delta\omega| \gg \frac{\nu}{R^2}$, the integral in (9.14) is dominated by $|u| \sim 1/|S| \ll 1$, which means $1/|u| \gg 1$. Hence, we can use the large argument approximation to the exponential integral function $\mathrm{Ei}(-\frac{1}{|u|}) \approx -|u|\exp[-\frac{1}{|u|}]$.

Since $|u|$, and, therefore, $e^{-1/|u|}$, is small, $\mathrm{Ei}(-\frac{1}{|u|})$ is also small, and so we can expand

$$\exp\left[-\frac{\alpha}{2}|u|\exp\left(-\frac{1}{|u|}\right)\right] \approx 1-\frac{\alpha}{2}|u|\exp\left(-\frac{1}{|u|}\right). \tag{9.20}$$

The Fourier transform of the first term is zero [strictly speaking, it's $\delta(S)$, but that of course is zero for $S \neq 0$, which is what we are considering here]. So $F_h(S)$ is dominated by the Fourier transform of the second term, which gives

$$\begin{aligned} F_h(S) &\approx -\int_{-\infty}^{\infty} e^{iSu}\frac{\alpha}{2}|u|\exp\left(-\frac{1}{|u|}\right)du \\ &= -\frac{\alpha}{2}\int_{-\infty}^{\infty} e^{iSu+\ln(|u|)-1/|u|}\,du. \end{aligned} \tag{9.21}$$

This equation can be rewritten

$$F_h(S) \approx -\frac{\alpha}{2}\left(\int_0^{\infty} e^{\Phi(u)}\,du + \text{c.c.}\right) \tag{9.22}$$

with

$$\Phi(u) = i|S|u + \ln(u) - 1/u. \tag{9.23}$$

The integral in (9.22) can be evaluated using the method of steepest descents. Details are given in Appendix D. The result is

$$\int_0^{\infty} e^{\Phi(u)}\,du \approx \frac{\sqrt{\pi}}{|S|^{5/4}}\exp\left[2e^{\frac{3i\pi}{4}}\sqrt{|S|}+\frac{i5\pi}{8}\right]. \tag{9.24}$$

Inserting this into (9.22), we obtain

$$F_h(S) \approx \frac{\alpha\sqrt{\pi}}{|S|^{5/4}}\sin\left(\sqrt{2|S|}+\frac{\pi}{8}\right)e^{-\sqrt{2S}}. \tag{9.25}$$

This decays rapidly with the scaling variable S; hence, the temporally Fourier transformed correlation function $I(\mathbf{R},\omega)$ decays rapidly for $|\delta\omega| \gg \nu/R^2$, where by $\delta\omega$ we mean, as before, the distance of the frequency ω from the nearest Bragg peak at $\pm b$.

Now we turn to the region extremely close to the peaks; that is, $|\delta\omega| \ll \tau_{\text{NL}}^{-1}$. On these timescales, the distribution of the field ϕ is no longer Gaussian. Therefore, even though we know the behavior of the mean squared phase difference $\langle[(\phi(\mathbf{r},t)-\phi(\mathbf{r}+\mathbf{R},t+T))^2]\rangle = C_\phi(\mathbf{R},T)$, we cannot use (9.8) to determine the behavior of $\langle\exp\{i[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle$, since (9.8) holds only for Gaussian distributions.

However, it seems reasonable to assume that the much bigger fluctuations in ϕ that occur in the nonlinear regime of the KPZ equation imply that the correlation function $\langle\exp\{i[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle$ decays more rapidly in the nonlinear regime than it does in *any* linear EW model. Since $\langle\exp\{i[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle$ decays like $|T|^{-\alpha/2}$ in the limit $|T| \to \infty$ in the EW model [see (8.14)], and in principle α can be made arbitrarily large, this would imply that the correlation function $\langle\exp\{i[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle$ decays more rapidly than any power law in the nonlinear regime.

If this is correct, then, even when $\delta\omega = 0$, $G(\mathbf{R}, \delta\omega = 0)$ is finite, since the integral over all time of $\langle\exp\{i[\phi(\mathbf{r}+\mathbf{R},t+T)-\phi(\mathbf{r},t)]\}\rangle$ will be finite.

Hence, the temporal Fourier transform of the velocity correlation function $I(\mathbf{R},\omega)$ remains finite even right *at* the Bragg peaks at $\omega = \pm b$. This implies that the $|\delta\omega|^{\alpha/2-1}$ divergence of $I(\mathbf{R},\omega)$ must round off for $|\delta\omega| \lesssim \tau_{\text{NL}}^{-1}$.

The resultant behavior of $I(\mathbf{R},\omega)$ is summarized in Fig. 3. Note that the oscillating and exponentially decaying tail implied by Eq. (9.25) is too small to be visible in this figure.

X. DENSITY CORRELATIONS IN THE LINEAR KPZ REGIME

We can now use Eq. (6.7) to obtain the density correlations in the linear regime. We will focus on the equal-time correlation functions for large spatial separations; specifically $|\mathbf{r}-\mathbf{r}'| \gg L_c$. In this range, the exponential integral in Eq. (8.12) is negligible (since its argument is large), and so we have

$$C_\phi(\mathbf{r}-\mathbf{r}', t=0) = \frac{D_\phi}{\pi\nu}\left[\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{2L_c}\right)+\gamma\right]. \tag{10.1}$$

This implies that the first term in (6.7) is short-ranged:

$$\begin{aligned} &e_3^2\nabla^2\nabla'^2\langle\phi(\mathbf{r},t)\phi(\mathbf{r}',t)\rangle \\ &\quad= -2e_3^2\nabla^2\nabla'^2\langle[\phi(\mathbf{r},t)-\phi(\mathbf{r}',t)]^2\rangle \\ &\quad= -2e_3^2\left(\frac{D_\phi}{\pi\nu}\right)\nabla^2\nabla'^2\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{2L_c}\right) \\ &\quad= -\left(\frac{4e_3^2 D_\phi}{\nu}\right)\nabla^2\delta(\mathbf{r}-\mathbf{r}'). \end{aligned} \tag{10.2}$$

The second term in (6.7) vanishes in the linear regime, since it is odd in ϕ, while the distribution of ϕ is even (indeed, a zero-mean Gaussian) in the linear regime.

However, the remaining term in the our expression (6.7) for $\delta\rho$ gives a long-ranged $|\mathbf{r}-\mathbf{r}'|^{-4}$ contribution to $\langle\delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\rangle$, as we will now show.

Fourier transforming, we have

$$\langle|\nabla\phi(\mathbf{r},t)|^2|\nabla\phi(\mathbf{r}',t)|^2\rangle = \frac{1}{V^2}\sum_{\mathbf{p}_{1,2,3,4}} e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot(\mathbf{r}-\mathbf{r}')}(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_3\cdot\mathbf{p}_4)\langle\phi(\mathbf{p}_1,t)\phi(\mathbf{p}_2,t)\phi(\mathbf{p}_3,t)\phi(\mathbf{p}_4,t)\rangle, \tag{10.3}$$

where V is the volume of the 2D system. Since the distribution of ϕ is Gaussian in the linear regime, we can Wick decompose the four-point average in (10.3), obtaining

$$\begin{aligned}&\langle\phi(\mathbf{p}_1,t)\phi(\mathbf{p}_2,t)\phi(\mathbf{p}_3,t)\phi(\mathbf{p}_4,t)\rangle\\ &= \langle\phi(\mathbf{p}_1,t)\phi(\mathbf{p}_2,t)\rangle\langle\phi(\mathbf{p}_3,t)\phi(\mathbf{p}_4,t)\rangle\\ &\quad+\langle\phi(\mathbf{p}_1,t)\phi(\mathbf{p}_3,t)\rangle\langle\phi(\mathbf{p}_2,t)\phi(\mathbf{p}_4,t)\rangle\\ &\quad+\langle\phi(\mathbf{p}_1,t)\phi(\mathbf{p}_4,t)\rangle\langle\phi(\mathbf{p}_2,t)\phi(\mathbf{p}_3,t)\rangle.\end{aligned} \tag{10.4}$$

We will now use our earlier result for the two-point correlations of ϕ in the linear theory, which can be written

$$\langle\phi(\mathbf{p}_1,t)\phi(\mathbf{p}_3,t)\rangle = \frac{D_\phi\delta^K_{\mathbf{p}_1+\mathbf{p}_3}}{\nu p_1^2}. \tag{10.5}$$

Using this in (10.4), and using the result in (10.3), we get, after doing two sums using the Kronecker deltas, and some relabeling,

$$\langle|\nabla\phi(\mathbf{r},t)|^2|\nabla\phi(\mathbf{r}',t)|^2\rangle = \frac{D_\phi^2}{\nu^2V^2}\sum_{\mathbf{p}_1,\mathbf{p}_2}\left(1+2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot(\mathbf{r}-\mathbf{r}')}}{p_1^2p_2^2}\right). \tag{10.6}$$

We see that the first term in (10.6) just gives a constant. This constant is just $\langle\delta\rho\rangle^2$, which is nonzero because, in the presence of fluctuations, the mean value of the density is changed by the fluctuations and is no longer the value at which $g(\rho,\ldots)$ vanishes. Because it is constant, this piece of the density-density correlation function is trivial, and, indeed, will drop out of the connected density-density correlation function. In other words, if we define $\delta\rho(\mathbf{r},t)$ to be the departure of $\rho(\mathbf{r},t)$ from its true mean value, rather than (as we have up to now) defining it as the departure of $\rho(\mathbf{r},t)$ from the value at which $g(\rho,\ldots)$ vanishes, then only the last two terms of (10.6) will contribute to $\langle\delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\rangle$.

The last two terms depend on $\mathbf{r}-\mathbf{r}'$ and survive in the connected correlation function. They are equal and lead to the following expression for the connected $|\nabla\phi|^2$ correlation function $\langle|\nabla\phi(\mathbf{r},t)|^2|\nabla\phi(\mathbf{r}',t)|^2\rangle_c$:

$$\begin{aligned}\langle|\nabla\phi(\mathbf{r},t)|^2|\nabla\phi(\mathbf{r}',t)|^2\rangle_c &= 2\left(\frac{D_\phi}{\nu}\right)^2\sum_{\mathbf{p}_1,\mathbf{p}_2}\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot(\mathbf{r}-\mathbf{r}')}}{V^2p_1^2p_2^2}\\ &= 2\left(\frac{D_\phi}{\nu}\right)^2\left(\sum_{\mathbf{p}_1}\frac{p_{1i}p_{1j}e^{\mathrm{i}[\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{r}')]}}{Vp_1^2}\right)\left(\sum_{\mathbf{p}_2}\frac{p_{2i}p_{2j}e^{-\mathrm{i}[\mathbf{p}_2\cdot(\mathbf{r}-\mathbf{r}')]}}{Vp_2^2}\right)\\ &\equiv 2\left(\frac{D_\phi}{\nu}\right)^2\sigma_{ij}(\mathbf{R})\sigma^*_{ij}(\mathbf{R}),\end{aligned} \tag{10.7}$$

where we have defined $\mathbf{R}\equiv\mathbf{r}-\mathbf{r}'$, and

$$\begin{aligned}\sigma_{ij}(\mathbf{R}) &\equiv \left(\sum_{\mathbf{p}}\frac{p_ip_je^{\mathrm{i}\mathbf{p}\cdot\mathbf{R}}}{Vp^2}\right)\\ &= \frac{\partial}{\partial R_i}\frac{\partial}{\partial R_j}\left(\sum_{\mathbf{p}}\frac{(1-e^{\mathrm{i}\mathbf{p}\cdot\mathbf{R}})}{Vp^2}\right)\\ &= \frac{\partial}{\partial R_i}\frac{\partial}{\partial R_j}\left(\int\frac{d^2p}{(2\pi)^2}\frac{(1-e^{\mathrm{i}\mathbf{p}\cdot\mathbf{R}})}{p^2}\right),\end{aligned} \tag{10.8}$$

where the 1 has been added to the second and third equalities simply to make the integrals over $\mathbf{p}$ converge as $\mathbf{p}\to\mathbf{0}$. (Its contribution to σ_{ij} clearly vanishes once the derivatives are taken). We have also introduced an ultraviolet cutoff $\Lambda_c\sim L_c^{-1}$ to avoid having to worry about any ultraviolet divergences.

Performing the integral over $\mathbf{p}$ in the last equality in polar coordinates, we obtain, after the angular integration,

$$\int\frac{d^2p}{(2\pi)^2}\frac{(1-e^{\mathrm{i}\mathbf{p}\cdot\mathbf{R}})}{p^2} = \frac{1}{2\pi}\int_0^{\Lambda_c}\frac{dp}{p}[1-J_0(pR)], \tag{10.9}$$

where $J_0(x)$ is the zeroth-order Bessel function. Hence

$$\begin{aligned}&\frac{\partial}{\partial R_j}\left(\int\frac{d^2p}{(2\pi)^2}\frac{(1-e^{\mathrm{i}\mathbf{p}\cdot\mathbf{R}})}{p^2}\right)\\ &= \frac{\partial}{\partial R_j}\frac{1}{2\pi}\int_0^{\Lambda_c}\frac{dp}{p}[1-J_0(pR)]\\ &= -\left(\frac{R_j}{R}\right)\left(\frac{1}{2\pi}\int_0^{\Lambda_c}dp\,J_0'(pR)\right)\\ &= \left(\frac{R_j}{2\pi R^2}\right)(J_0(0)-J_0(\Lambda_cR))\approx\frac{R_j}{2\pi R^2},\end{aligned} \tag{10.10}$$

where in the final approximate equality we have used the fact that $J_0(x \to \infty) \to 0$ to drop $J_0(\Lambda_c R)$ in the large R limit (specifically, the $R \gg \Lambda_c^{-1}$limit). We have also used the fact that $J_0(0) = 1$.

From (10.10), it follows that

$$\sigma_{ij}(\mathbf{R}) = \frac{R^2\delta_{ij} - 2R_iR_j}{2\pi R^4}, \tag{10.11}$$

which readily implies

$$\sigma_{ij}(\mathbf{R})\sigma_{ij}^*(\mathbf{R}) = \frac{1}{2\pi^2 R^4}. \tag{10.12}$$

Using this in (6.7) and (10.7), we have

$$\langle\delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\rangle = \frac{e_4^2 D_\phi^2}{\pi^2\nu^2|\mathbf{r}-\mathbf{r}'|^4}. \tag{10.13}$$

Performing a similar calculation for the equal-position case, we find

$$\langle\delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r},t')\rangle = \frac{e_4^2 D_\phi^2}{16\pi^2\nu^4|t-t'|^2}\left(1 - \mathrm{e}^{-\nu\Lambda_c^2|t-t'|}\right). \tag{10.14}$$

Since we are now in the regime $|t - t'| \gg \tau_c$, and since $\tau_c \gg \nu\Lambda_c^2$ since $\tau \sim b^{-1}$, $\nu \sim b^{-3/5}$, and $\Lambda_c \sim L_c^{-1} \sim b^{4/5}$, the exponential in (10.14) is much less than one and hence negligible. Then we have

$$\langle\delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r},t')\rangle = \frac{e_4^2 D_\phi^2}{16\pi^2\nu^4|t-t'|^2}. \tag{10.15}$$

XI. VORTICES

If we assume that the analogy between the linearized KPZ equation and the *XY* model holds for vortices as well as at the level of "spin-wave theory," then the recursion relation for the vortex fugacity y reads [40]

$$\frac{\mathrm{d}y}{\mathrm{d}\ell} = 2\left(1 - \frac{1}{4\alpha}\right)y, \tag{11.1}$$

where α is defined in (1.19). Since α vanishes like b^{η_v} with $\eta_v \approx 3/5$ for small chirality b, the second ($\frac{1}{4\alpha}$) term in (11.1) will dominate in that limit. Furthermore, α is essentially constant for $\ell < \ell^*$, where

$$\ell^* \equiv \ln\left(\frac{L_{\rm NL}}{L_c}\right) = C_{\rm NL} b^{-\eta_g} \tag{11.2}$$

is the RG "time" at which the λ_k nonlinearity becomes important and we have used (7.38) in the second equality. Hence, solving (11.1) gives

$$y(\ell^*) = y_0 \exp\left[2\ell^*\left(1 - \frac{1}{4\alpha}\right)\right]. \tag{11.3}$$

For small chirality b, the argument of this exponential is dominated by the $\frac{1}{4\alpha}$ term, which scales with b like $\frac{\ell^*}{\alpha} \propto b^{-\eta_y}$, where we have defined

$$\eta_y \equiv \eta_g + \eta_v \approx 22/5, \tag{11.4}$$

a result obtained by using our earlier results $\alpha = C_\alpha b^{\eta_v}$ and $\ell^* = C_{\rm NL} b^{-\eta_g}$.

Thus we see that

$$y(\ell^*) \propto \exp(-E b^{-\eta_y}), \tag{11.5}$$

with $E = \frac{C_{\rm NL}}{2C_\alpha}$ an $O(1)$ constant. This fugacity $y(\ell^*)$ will be extremely small for small chirality b.

With this result (11.5) in hand, we can now ask for the length scale L_v at which vortices actually become important. There are two equivalent ways to calculate this, which (reassuringly!) give the same result.

The first approach is to say that the "free" vortex density n_f at length scales longer that $L_{\rm NL}$ is proportional to $y(\ell^*)$. One then estimates L_v as the typical distance between vortices of density n_f per unit area, which is, of course, simply

$$L_v \approx n_f^{-1/2} \propto \exp(C_{vL} b^{-\eta_y}), \tag{11.6}$$

where

$$C_{vL} \equiv \frac{E}{2} = \frac{C_{\rm NL}}{4C_\alpha}. \tag{11.7}$$

The second approach is to continue to use the recursion relation (11.1) for RG times $\ell > \ell^*$, but assume that α has now been driven to infinity by the nonlinear term. Hence that recursion relation reduces to

$$\frac{\mathrm{d}y}{\mathrm{d}\ell} = 2y, \tag{11.8}$$

whose solution is trivially

$$y(\ell) = y(\ell^*) e^{2(\ell - \ell^*)}, \tag{11.9}$$

where we have matched this solution onto the solution (11.5) that we found earlier for $\ell < \ell^*$. We can now ask for the RG time ℓ_v at which this renormalized fugacity becomes $O(1)$. This is readily found to be

$$\ell_v = \ell^* - \frac{\ln[y(\ell^*)]}{2} \approx C_{vL} b^{-\eta_y}, \tag{11.10}$$

where in the last, approximate equality, we have used the fact that $(\frac{\ln[y(\ell^*)]}{2}) \approx C_{vL} b^{-\eta_y} \gg \ell^* \propto b^{-\eta_g}$, since $\eta_y - \eta_g = \eta_v > 0$.

The length scale L_v at which vortices become important is simply proportional to e^{ℓ_v}; this recovers (11.6).

Note that this value (11.6) of L_v is, for small chirality b, much greater than the nonlinear length scale $L_{\rm NL}$. Hence, there will be a large range of length scales over which the results we derive based on the KPZ equation in the linear and nonlinear regimes will hold, before the unbinding of vortices destroys all remnants of order.

XII. SUMMARY

We have shown that the hydrodynamic behavior of a generic planar chiral Malthusian flock in the rotating phase can be mapped onto the Kardar-Parisi-Zhang surface growth model in $(2+1)$ dimensions. Specifically, the fluctuating angular field in the chiral flocks corresponds to the height function in the KPZ model. Surprisingly, although *achiral* Malthusian flocks *can* order in two dimensions, *any* degree of chirality will destroy the long-ranged orientational order. We then focus on the small chirality limit and elucidate the distinct regimes that depend on the length and time scales of interest (Fig. 2). Our work thus highlights the intricate interplay between chirality and motility. Looking ahead, one interesting direction would be to relate our findings here to multicomponent systems with nonreciprocal interactions where a chiral phase can also be present [52].

Note added. Recently, we became aware of three related papers addressing chirality and KPZ-type scaling in various systems [37,53,54]. Reference [53] studies chiral active hexatics, whereas the present work concerns chiral active fluids (Malthusian flocks). Although both involve chirality in active matter, the two systems are governed by distinct hydrodynamic descriptions, and the emergence of KPZ scaling in chiral active fluids cannot be inferred from the hexatic case alone. Demonstrating this correspondence therefore requires a separate analysis, which is provided here. Reference [54] discusses this connection primarily at a qualitative level, while in the present work the mapping to the KPZ equation is established explicitly through a derivation from the hydrodynamic equations. Reference [37] reports KPZ scaling in time-crystalline matter, a different physical system with distinct dynamical equations. The principal results of our paper were obtained independently of those of Refs. [37,54].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DEMONSTRATION THAT THE MOST GENERAL MODEL STILL LEADS TO THE KPZ EQUATION

In this Appendix, we demonstrate that our fundamental result, which is that the dynamics of a chiral Malthusian flock is described by the KPZ equation, is not an artifact of the simplifications we made to our truncated dynamical EOM (2.4), but in fact holds even if all relevant terms in the EOM are included.

We will do this by considering all possible terms in the EOM involving one and two derivatives of the velocity fields, and showing that all such terms either average to zero when averaged over one cycle, or lead to terms of the same form as the two ϕ-dependent terms in the KPZ equation; that is, either $\nabla^2\phi$ or $|\nabla\phi|^2$. Terms involving more than two spatial derivatives are clearly irrelevant in the long-wavelength limit relative to these two terms, and so can be ignored.

1. One-derivative terms

a. Achiral one-derivative terms

The most general *achiral* one-derivative term in the EOM (that is, the expression for $\partial_t v_i$) can be written [55] as

$$\lambda_\alpha^{(n)} T_{ijkl\ldots mnp} v_j v_k . v_l \ldots \partial_n v_p, \tag{A1}$$

where the $2n$ index tensor $\mathbf{T}$ (with n an integer) is composed of the product of n Kronecker deltas, which contract all but one of the indices in the product $v_j v_k v_l \ldots \partial_n v_p$ of velocities and the derivative of the velocity, so as to leave a single free index i to match the free index in $\partial_t v_i$. The Cartesian index "n" (e.g., on the ∂_n) in this appendix should not be confused with the label "n" on the coefficient $\lambda_\alpha^{(n)}$. The distinction should always be clear from context.

Hence, the total number of indices in $v_j v_k v_l \ldots \partial_n v_p$ must be $2n+1$, and so the total number of velocities to the left of the derivative in (A1) must be $2n-1$. Most of these must be contracted with each other, rather than with the indices n and p of the derivative of the velocity. Indeed, the only possibilities are that there is one velocity on the left "left over," with all of the others contracted with others on the left, or three velocities "left over."

We will now consider these cases in turn.

One velocity "left over." There are three possibilities here:

(1) The remaining velocity component on the left is contracted with the n of the $\partial_n v_p$,

(2) The remaining velocity component on the left is contracted with the p of the $\partial_n v_p$,

(3) The remaining velocity component on the left is the "free" index i.

In case 1, the remaining index "p" must be i, and (A1) reduces to

$$\lambda_\alpha^{(n)} v_0^{2n-2} v_j \partial_j v_i = \lambda_\alpha^{(n)} v_0^{2n-2} (\mathbf{v}\cdot\nabla\mathbf{v})_i. \tag{A2}$$

Since v_0 is a constant, the sum of all such possible terms is simply

$$\lambda_1 (\mathbf{v}\cdot\nabla\mathbf{v})_i, \tag{A3}$$

with $\lambda_1 = \sum_{n=1}^{\infty} \lambda_\alpha^{(n)} v_0^{2n-2}$. This is exactly the λ_1 term we had in our EOM (2.4).

In case 2, we have

$$\lambda_\alpha^{(n)} v_0^{2n-2} v_j \partial_i v_j = \tfrac{1}{2} \lambda_\alpha^{(n)} v_0^{2n-2} \partial_i (v_j v_j) = 0. \tag{A4}$$

So all such terms vanish identically.

Finally, for case 3, we have

$$\lambda_\alpha^{(n)} v_0^{2n-2} v_i \partial_j v_j = \lambda_\alpha^{(n)} v_0^{2n-2} [\mathbf{v}(\nabla\cdot\mathbf{v})]_i. \tag{A5}$$

All such terms are along the velocity. Hence, when we project perpendicular to the velocity by acting on the EOM with $\epsilon_{ik} v_k$, they vanish.

We now turn to the terms in which three of the velocities to the left of the derivative are left to be contracted with the derivative of the velocity term. We first note that if any of the three indices to the left contract with the index of the velocity to the right, we will get a term proportional to

$$v_j \partial_w v_j = \partial_w (v_j v_j) = \partial_w \left(v_0^2\right) = 0, \tag{A6}$$

where w is some index, and it does not matter which. Hence all of these terms vanish as well.

The only other possibility is that one of the three velocities to the left of the derivative contracts with the derivative, and the other two contract with each other. It is easy to see that this simply leads to another contribution to the λ_1 term Eq. (2.4).

So all possible achiral first derivative terms can be absorbed into the λ_1 term that we already had in Eq. (2.4).

b. Chiral one-derivative terms

The chiral one-derivative terms can be reduced to terms involving only one $\boldsymbol{\epsilon}$ matrix since the product of two of them annihilate into Kronecker deltas:

$$\epsilon_{jk}\epsilon_{\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}. \tag{A7}$$

In addition, any such term must have an odd number of indices, so that after contraction one index remains to be i. So in general, the chiral one-derivative term is a combination of one matrix $\boldsymbol{\epsilon}$, one derivative, and even number of v components. Again, once we act on such a term with the "projection operator" $\left(\frac{\epsilon_{ik}v_k}{v_0}\right)$ the resultant piece vanishes upon averaging over short-time oscillation cycles of the rotating $\mathbf{v}$ since it involves an odd number of v components.

In conclusion, the only one-derivative term in the EOM for $\mathbf{v}$ is the $\lambda_1(\mathbf{v}\cdot\nabla\mathbf{v})$ term which we had in our original EOM (2.4). As we showed in our analysis in the main text (Sec. IV), this term makes no contribution to the EOM for the phase field ϕ. This is not a coincidence, but determined by the symmetry. Since the 2D chiral state itself is isotropic in the large-scale time limit, there is no way to create one-derivative linear terms in the hydrodynamic equation for ϕ without violating this isotropy.

2. Terms with two derivatives

a. Achiral two-derivative terms

There are two types of such terms:

$$\lambda_\alpha^{(2.1)} T^{(2.1)}_{ijkl\ldots mnps} v_n v_p v_s \ldots (\partial_j v_k)(\partial_l v_m) \tag{A8}$$

and

$$\lambda_\alpha^{(2.2)} T^{(2.2)}_{ijkl\ldots mnps} v_n v_p v_s \ldots (\partial_j \partial_l v_m), \tag{A9}$$

where, as before the tensors $\mathbf{T}^{(2.1)}$ and $\mathbf{T}^{(2.2)}$ are products of Kronecker deltas that contract the indices of the velocities and partial derivatives in pairs.

We will consider first the $\lambda_\alpha^{(2.1)}$ terms.

After contracting all the velocity components to the left of the derivatives that must be contracted with each other, we are left only with terms of the form

$$\gamma_\alpha^{(2.1)} M^{(2.1)}_{ijklmnps} v_n v_p v_s (\partial_j v_k)(\partial_l v_m), \tag{A10}$$

where the coefficient $\gamma_\alpha^{(2.1)}$ absorbs all of the powers of v_0^2 that arose from contracting velocity components with themselves, and the tensor $\mathbf{M}^{(2.1)}$ is again a product of Kronecker deltas which contract the free indices.

Now consider the possibilities for those contractions. If any of the indices n, p, or s of the velocity components outside the derivatives contracts with either of the velocity components k or m inside the derivatives, then this term vanishes, by Eq. (A6). Hence, the velocities outside the derivatives can contract either with only the derivatives or each other. This leaves two possibilities:

(1) Two of the outside velocities contract with the derivatives, and the remaining one is the free index i (remember that we are looking at a term in the expression for $\partial_t v_i$). In this case, when we "project" this term orthogonal to $\mathbf{v}$ by acting on it with $\epsilon_{is}v_s$, it will vanish.

(2) Two of the outside indices contract with each other, and the remaining index contracts with one of the derivatives. This leaves three possibilities:

$$v_j(\partial_j v_i)(\partial_l v_l), \quad (1.2.1), \tag{A11}$$

$$v_k(\partial_k v_j)(\partial_j v_i), \quad (1.2.2), \tag{A12}$$

and

$$v_k(\partial_k v_j)(\partial_i v_j), \quad (1.2.3). \tag{A13}$$

Consider first (1.2.1). Using

$$\partial_j v_i = \epsilon_{ik} v_k \partial_j \phi \tag{A14}$$

we can rewrite that term as

$$v_j(\partial_j v_i)(\partial_l v_l) = \epsilon_{ik}\epsilon_{lm} v_j v_k v_m (\partial_j\phi)(\partial_l\phi). \tag{A15}$$

Using the identity (A7) for the product of two ϵ matrices, this becomes

$$\begin{aligned} v_j(\partial_j v_i)(\partial_l v_l) &= (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) v_j v_k v_m (\partial_j\phi)(\partial_l\phi) \\ &= v_0^2 v_j (\partial_j\phi)(\partial_i\phi) - v_i v_j v_l (\partial_j\phi)(\partial_l\phi). \end{aligned} \tag{A16}$$

The second term is proportional to v_i, and so it will vanish when we act on it with $\epsilon_{is}v_s$. Acting on the first term with $\epsilon_{is}v_s$ gives

$$\epsilon_{is} v_s v_j (\partial_j v_i)(\partial_l v_l) = v_0^2 \epsilon_{is} v_j v_s (\partial_j\phi)(\partial_i\phi). \tag{A17}$$

Taking the average of this over one cycle using (4.5) gives

$$\begin{aligned} \langle \epsilon_{is} v_s v_j (\partial_j v_i)(\partial_l v_l)\rangle_c &= \frac{v_0^4}{2}\epsilon_{is}\delta_{js}(\partial_j\phi)(\partial_i\phi) \\ &= \frac{v_0^4}{2}\epsilon_{ij}(\partial_j\phi)(\partial_i\phi) = 0, \end{aligned} \tag{A18}$$

where the last equality holds because ϵ_{ij} is antisymmetric under interchange of i and j, while the product $(\partial_j\phi)(\partial_i\phi)$ is symmetric.

So (1.2.1) makes no contribution to the EOM for ϕ.

Now we turn to (1.2.2). Using (A14) again, this becomes

$$v_k(\partial_k v_j)(\partial_j v_i) = \epsilon_{im}\epsilon_{jl} v_k v_l v_m (\partial_j\phi)(\partial_k\phi), \tag{A19}$$

which is readily seen to be identical to (1.2.1) [see (A15)].

Finally, (1.2.3) gives, upon using (A14),

$$\begin{aligned} v_k(\partial_k v_j)(\partial_i v_j) &= \epsilon_{jl}\epsilon_{jm} v_k v_l v_m (\partial_i\phi)(\partial_k\phi) \\ &= \delta_{lm} v_k v_l v_m (\partial_i\phi)(\partial_k\phi) \\ &= v_0^2 v_k (\partial_i\phi)(\partial_k\phi), \end{aligned} \tag{A20}$$

where in the second equality we have used the identity (A7) for the $\boldsymbol{\epsilon}$ tensor.

Multiplying this term by $\epsilon_{is}v_s$ and avergaing over one cycle using (4.5) gives

$$\begin{aligned}\langle\epsilon_{is}v_s v_k(\partial_k v_j)(\partial_i v_j)\rangle_c &= \frac{v_0^4}{2}\epsilon_{is}\delta_{sk}(\partial_i\phi)(\partial_k\phi)\\ &= \frac{v_0^4}{2}\epsilon_{ik}(\partial_i\phi)(\partial_k\phi)=0,\end{aligned}\tag{A21}$$

where the final equality again follows because we are contracting a symmetric and an antisymmetric tensor.

Now we turn to the final achiral two derivative terms, (A9). Since we cannot have any of the velocities outside the derivatives carrying the index i, for the reasons discussed above, there are only four possibilities here:

(1) All velocities outside the derivatives contract with each other, and the two derivatives contract with each other, leaving the differentiated velocity to have index i. This gives us a term $v_0^{2n}\partial_j\partial_j v_i$, which is proportional to the $\mu_1\nabla^2 v_i$ term we had in our original EOM (2.4).

(2) All velocities outside the derivatives contract with each other, and one of the derivative indices contracts with the differentiated velocity, and the other is i. This gives us a term proportional to the $\mu_2\partial_i\nabla\cdot\mathbf{v}$ term we had in our original EOM (2.4).

(3) All undifferentiated velocities but two contract with each other, and the two remaining undifferentiated velocities contract with the derivatives, leaving the differentiated velocity index to be i. This gives us the $\mu_3(\mathbf{v}\cdot\nabla)^2 v_i$ term in our original EOM (2.4).

(4) All undifferentiated velocities but two contract with each other, and one of the two remaining undifferentiated velocities contracts with the differentiated velocity, and the other contracts with one of the derivatives, leaving the remaining derivative index to be i. This gives $v_j v_k\partial_i\partial_j v_k$, which is equivalent to (A13) since

$$\begin{aligned}v_j v_k\partial_i\partial_j v_k &= v_j\partial_i(v_k\partial_j v_k) - v_j(\partial_i v_k)(\partial_j v_k)\\ &= -v_j(\partial_i v_k)(\partial_j v_k),\end{aligned}\tag{A22}$$

where in the second equality we have used $v_j\partial_i(v_k\partial_j v_k) = \frac{1}{2}v_j(\partial_i\partial_j v_0^2) = 0$.

So our original EOM (2.4) contained all possible achiral two derivative terms.

b. Chiral two-derivative terms

As for the one-derivative terms, chiral two-derivative terms can be reduced to terms involving only one $\boldsymbol{\epsilon}$ matrix since the product of two of them annihilate into Kronecker deltas, as illustrated by (A7). These again fall into two classes:

$$\lambda_\alpha^{(2.3)}T^{(2.3)}_{ijkl\ldots mnptw}v_j v_p v_s\ldots\epsilon_{tw}(\partial_k v_l)(\partial_m v_n)\tag{A23}$$

and

$$\lambda_\alpha^{(2.4)}T^{(2.4)}_{ijkl\ldots mnptw}v_n v_p v_s\ldots\epsilon_{tw}(\partial_k\partial_l v_m).\tag{A24}$$

Class (A23) can be considerably simplified by recognizing that only one of the nondifferentiated v's can contract with the $\boldsymbol{\epsilon}$ tensor; if two contract, the term vanishes because that is an odd-even contraction. We also know from our earlier discussion that this term vanishes if any of the nondifferentiated v's contracts with one of the *differentiated* v's. Therefore, there are only three available indices in the $\epsilon_{tw}(\partial_k v_l)(\partial_m v_n)$ factor with which the nondifferentiated velocities can contract. Therefore, either all but three of them, or all but one of them, must contract with each other.

Consider first the terms in (A23) with three "outside" velocities that do not contract with other outside velocities. These are of the form

$$\lambda_\alpha^{(2.3)}M^{(2.3.a)}_{imns}v_j v_k v_l\epsilon_{jm}(\partial_k v_n)(\partial_l v_s),\tag{A25}$$

where $\mathbf{M}^{(2.3.a)}$ is a product of two Kronecker deltas, which contract two of the three indices m, n, and s, and turn the remaining one into the free index i. Using (A14), we can write

$$\begin{aligned}&v_j v_k v_l\epsilon_{jm}(\partial_k v_n)(\partial_l v_s)\\ &\quad= \epsilon_{jm}\epsilon_{nt}\epsilon_{sw}v_j v_k v_l v_t v_w(\partial_k\phi)(\partial_l\phi)\\ &\quad= (\delta_{ns}\delta_{tw}-\delta_{nw}\delta_{st})v_j v_k v_l v_t v_w\epsilon_{jm}(\partial_k\phi)(\partial_l\phi)\\ &\quad= \left(v_0^2\delta_{ns}-v_n v_s\right)v_j v_k v_l\epsilon_{jm}(\partial_k\phi)(\partial_l\phi),\end{aligned}\tag{A26}$$

where in the second equality we have used the identity (A7) for the product $\epsilon_{nt}\epsilon_{sw}$.

Now multiplying this term (A25) by $\epsilon_{i\alpha}v_\alpha$ to obtain the contribution to $\partial_t\phi$, we get

$$\begin{aligned}&\epsilon_{i\alpha}v_\alpha\epsilon_{jm}(\partial_k\phi)(\partial_l\phi)\left(v_0^2\delta_{ns}-v_n v_s\right)v_j v_k v_l\lambda_\alpha^{(2.3)}M^{(2.3.a)}_{imns}\\ &\quad= (\delta_{ij}\delta_{\alpha m}-\delta_{im}\delta_{\alpha j})(\partial_k\phi)(\partial_l\phi)\left(v_0^2\delta_{ns}-v_n v_s\right)\\ &\qquad\times v_j v_k v_l v_\alpha\lambda_\alpha^{(2.3)}M^{(2.3.a)}_{imns},\end{aligned}\tag{A27}$$

where in the second equality we have used the identity (A7) for the product $\epsilon_{i\alpha}\epsilon_{jm}$.

The Kronecker deltas explicitly diplayed in (A27), and those implicit in $M^{(2.3.a)}_{imns}$, can only contract the indices on the velocity components with themselves; the only other indices are k and l, which are already contracted. Thus this term (if not vanishing) is ultimately proportional to

$$v_k v_l(\partial_k\phi)(\partial_l\phi).\tag{A28}$$

Averaging this over one cycle using Eq. (4.5) gives

$$\frac{v_0^2}{2}\delta_{kl}(\partial_k\phi)(\partial_l\phi) = \frac{v_0^2}{2}|\nabla\phi|^2,\tag{A29}$$

which is a contribution to the KPZ nonlinearity λ_κ term in (4.6).

We now turn to terms with only one nondifferentiated velocity that is not contracted with other undifferentiated velocities. These terms are of the form

$$M_{ijkplmns}\epsilon_{jk}v_p(\partial_l v_m)(\partial_n v_s),\tag{A30}$$

where again the $\mathbf{M}$ tensor is composed of Kronecker deltas that contract the indices.

Using our old standby (A14), we now get

$$\epsilon_{jk} v_p(\partial_l v_m)(\partial_n v_s) = \epsilon_{jk}\epsilon_{mt}\epsilon_{sw} v_p v_t v_w(\partial_n\phi)(\partial_l\phi)$$
$$= \epsilon_{jk}(\delta_{ms}\delta_{tw} - \delta_{mw}\delta_{st}) v_p v_t v_w(\partial_n\phi)(\partial_l\phi), \tag{A31}$$

where in the second equality we have used the identity (A7) for the product $\epsilon_{mt}\epsilon_{sw}$.

Now multiplying this term (A25) by $\epsilon_{i\alpha} v_\alpha$ to obtain the contribution to $\partial_t\phi$, we get

$$\epsilon_{i\alpha} v_\alpha \epsilon_{jk} v_p(\partial_l v_m)(\partial_n v_s)$$
$$= \epsilon_{i\alpha}\epsilon_{jk}(\delta_{ms}\delta_{tw} - \delta_{mw}\delta_{st}) v_\alpha v_p v_t v_w(\partial_n\phi)(\partial_l\phi)$$
$$= (\delta_{ij}\delta_{\alpha k} - \delta_{ik}\delta_{\alpha j})(\delta_{ms}\delta_{tw} - \delta_{mw}\delta_{st}) v_\alpha v_p v_t v_w(\partial_n\phi)(\partial_l\phi), \tag{A32}$$

where in the second equality we have used the identity (A7) for the product $\epsilon_{i\alpha}\epsilon_{jk}$.

When we act on this with $M_{ijkplmns}$, if the result does not vanish, there are two possibilities. One possibility is that two of the velocities $v_\alpha v_p v_t v_w$ contract with the two ∂'s, and the other two contract with each other. Thus we are left with a term proportional to

$$v_k v_l(\partial_k\phi)(\partial_l\phi). \tag{A33}$$

We considered this term earlier [see (A28)], and showed it contributes only to the λ_κ term in (4.6). So that is the form of the contribution from (A30) as well. The other possibility is that all the velocities $v_\alpha v_p v_t v_w$ contract with each other, and the two ∂'s contract with each other. Then we get a term proportional to $|\nabla\phi|^2$, which again contributes only to the λ_κ term in (4.6).

Finally, we turn to terms of type (A24). Again only one of the nondifferentiated v's can contract with the $\boldsymbol{\epsilon}$ tensor. Furthermore, the total number of undifferentiated v's in (A24) must be even; if not, the piece will have odd number of v's after multiplying $\epsilon_{i\alpha} v_\alpha$, and its time average over one cycle will vanish. Therefore, either all but four of nondifferentiated v's, or all but two of them, or all of them, must contract with each other.

First, consider the case in which all of the nondifferentiated v's contract with each other. In this case (A24) is proportional to

$$M_{ijklmp}\epsilon_{jp}(\partial_k\partial_l v_m). \tag{A34}$$

Using (A14) twice, we have

$$\partial_k\partial_l v_m = \epsilon_{mn}\partial_k(v_n\partial_l\phi)$$
$$= \epsilon_{mn}[(\partial_k v_n)\partial_l\phi + v_n\partial_k\partial_l\phi]$$
$$= \epsilon_{mn}[\epsilon_{ns} v_s(\partial_k\phi)\partial_l\phi + v_n\partial_k\partial_l\phi]$$
$$= -\delta_{ms} v_s(\partial_k\phi)\partial_l\phi + \epsilon_{mn} v_n\partial_k\partial_l\phi$$
$$= -v_m(\partial_k\phi)\partial_l\phi + \epsilon_{mn} v_n\partial_k\partial_l\phi, \tag{A35}$$

where we have also used the identity $\epsilon_{mn}\epsilon_{ns} = -\delta_{ms}$.

Inserting (A35) into (A34) and multiplying the resultant piece by $\epsilon_{i\alpha} v_\alpha$ to obtain the contribution to $\partial_t\phi$, we get

$$M_{ijklmp}\epsilon_{jp}(-\epsilon_{i\alpha} v_\alpha v_m(\partial_k\phi)\partial_l\phi + \epsilon_{i\alpha}\epsilon_{mn} v_\alpha v_n\partial_k\partial_l\phi)$$
$$= M_{ijklmp}(-(\delta_{ij}\delta_{p\alpha} - \delta_{ip}\delta_{j\alpha}) v_\alpha v_m(\partial_k\phi)\partial_l\phi + \epsilon_{jp}(\delta_{im}\delta_{\alpha n} - \delta_{in}\delta_{\alpha m}) v_\alpha v_n\partial_k\partial_l\phi)$$
$$= M_{ijklmp}\big(-(\delta_{ij}\delta_{p\alpha} - \delta_{ip}\delta_{j\alpha}) v_\alpha v_m(\partial_k\phi)\partial_l\phi + \epsilon_{jp}\big(\delta_{im} v_0^2 - v_i v_m\big)\partial_k\partial_l\phi\big). \tag{A36}$$

Next we take the time average of this using (4.5):

$$\langle M_{ijklmp}\epsilon_{jp}(-\epsilon_{i\alpha} v_\alpha v_m(\partial_k\phi)\partial_l\phi + \epsilon_{i\alpha}\epsilon_{mn} v_\alpha v_n\partial_k\partial_l\phi)\rangle_c$$
$$= M_{ijklmp}\left[-\frac{1}{2}(\delta_{ij}\delta_{p\alpha} - \delta_{ip}\delta_{j\alpha}) v_0^2\delta_{\alpha m}(\partial_k\phi)\partial_l\phi + \epsilon_{jp}\left(\delta_{im} v_0^2 - \frac{v_0^2}{2}\delta_{im}\right)\partial_k\partial_l\phi\right]$$
$$= \left(\frac{v_0^2}{2}\right) M_{ijklmp}((-\delta_{ij}\delta_{pm} + \delta_{ip}\delta_{jm})(\partial_k\phi)\partial_l\phi + \epsilon_{jp}\delta_{im}\partial_k\partial_l\phi). \tag{A37}$$

For the first and the second term on the last line of (A37), the explicitly displayed Kronecker deltas and those implicitly in the tensor $\mathbf{M}$ must ultimately contract the indices k and l, giving a piece proportional to $|\nabla\phi|^2$, which is a contribution to the KPZ nonlinearity λ_κ in (4.6). For the third term, the Kronecker deltas contract the indices j, p, k, and l. There are two possibilities: one is that j contracts with p and k contracts with l, and the other is that j and p contract with k and j. Both of these vanish, the former because $\epsilon_{jj} = 0$, the second because $\epsilon_{kl}\partial_k\partial_l\phi = 0$.

Next consider the case in which all but two of the nondifferentiated v's in (A24) contract with each other. In this case (A24) is proportional to

$$M_{ijklmpqw}\epsilon_{jp} v_q v_w(\partial_k\partial_l v_m). \tag{A38}$$

Inserting (A35) into (A38) and multiplying the resultant piece by $\epsilon_{i\alpha} v_\alpha$ to obtain the contribution to $\partial_t\phi$, we get

$$M_{ijklmpqw}\epsilon_{jp}(-\epsilon_{i\alpha} v_\alpha v_q v_w v_m(\partial_k\phi)\partial_l\phi + \epsilon_{i\alpha}\epsilon_{mn} v_\alpha v_q v_w v_n\partial_k\partial_l\phi)$$
$$= M_{ijklmpqw}(-(\delta_{ij}\delta_{p\alpha} - \delta_{ip}\delta_{j\alpha}) v_\alpha v_q v_w v_m(\partial_k\phi)\partial_l\phi + \epsilon_{jp}(\delta_{im}\delta_{\alpha n} - \delta_{in}\delta_{\alpha m}) v_\alpha v_q v_w v_n\partial_k\partial_l\phi)$$
$$= M_{ijklmpqw}\big(-(\delta_{ij}\delta_{p\alpha} - \delta_{ip}\delta_{j\alpha}) v_\alpha v_q v_w v_m(\partial_k\phi)\partial_l\phi + \epsilon_{jp}\big(\delta_{im} v_0^2 - v_i v_m\big) v_q v_w\partial_k\partial_l\phi\big). \tag{A39}$$

Again taking the time average using (4.5), we obtain

$$\langle M_{ijklmpqw}\epsilon_{jp}(-\epsilon_{i\alpha}v_\alpha v_q v_w v_m(\partial_k\phi)\partial_l\phi+\epsilon_{i\alpha}\epsilon_{mn}v_\alpha v_q v_w v_n\partial_k\partial_l\phi)\rangle_c$$
$$=\left(\frac{v_0^4}{8}\right)M_{ijklmpqw}(-(\delta_{ij}\delta_{p\alpha}-\delta_{ip}\delta_{j\alpha})(\delta_{\alpha q}\delta_{wm}+\delta_{\alpha w}\delta_{qm}+\delta_{\alpha m}\delta_{qw})(\partial_k\phi)\partial_l\phi+\epsilon_{jp}(3\delta_{im}\delta_{qw}-\delta_{iq}\delta_{mw}-\delta_{iw}\delta_{mq})\partial_k\partial_l\phi). \tag{A40}$$

The argument used for (A37) also applies here. The conclusion is that if (A40) gives something nonvanishing, it can only be a contribution to the KPZ nonlinearity λ_κ term in (4.6)

Finally consider the case in which all but four of the nondifferentiated v's in (A24) contract with each other. Since only one of the nondifferentiated v's can contract with the $\boldsymbol{\epsilon}$ tensor, the other three must contract the two derivatives and the differentiated v, leaving one if index of $\boldsymbol{\epsilon}$ to be i. Therefore, in this case (A24) is proportional to

$$\epsilon_{ij}v_j v_k v_l v_m(\partial_k\partial_l v_m). \tag{A41}$$

Inserting (A35) into (A41) and multiplying the resultant piece by $\epsilon_{i\alpha}v_\alpha$ to obtain the contribution to $\partial_t\phi$, we get

$$\epsilon_{i\alpha}v_\alpha\epsilon_{ij}v_j v_k v_l v_m(-v_m(\partial_k\phi)\partial_l\phi+\epsilon_{mn}v_n\partial_k\partial_l\phi)$$
$$=-v_0^2\epsilon_{i\alpha}v_\alpha\epsilon_{ij}v_j v_k v_l(\partial_k\phi)\partial_l\phi=-v_0^4 v_k v_l(\partial_k\phi)\partial_l\phi. \tag{A42}$$

We considered this term earlier [see (A28)], and showed it contributes only to the λ_κ term in (4.6).

Thus, we have exhaustively shown that all possible symmetry allowed terms in the velocity EOM for a chiral flock lead to one of the two terms in the KPZ equation for ϕ. Furthermore, the nonlinear λ_κ term comes only from the chiral terms in the velocity equation, which implies that λ_κ is linear in the chirality for small chirality, a result we needed to obtain the scaling laws relating the crossover length scales L_c, $L_{\rm NL}$, and L_v, and timescales τ_c, $\tau_{\rm NL}$, and τ_v to the chirality b for small chirality.

APPENDIX B: NONRACEMIC SMALL "CHIRALITY" SYSTEMS

Our discussion of the small chirality case throughout this paper has imagined the chirality is tuned to zero in a racemic way (that is, by mixing chiral components that are mirror images of each other), which leads to all chiral parameters vanishing at the same time. However, as discussed in the introduction, when chirality is tuned by, e.g., making nonracemic mixtures (i.e., mixtures of different components which are not mirror images of each other), one would expect in general that the rotation rate b and the chiral nonlinear KPZ term λ_κ would vanish at different concentrations. Therefore, in such systems, two possibilities arise:

(1) $\lambda_\kappa\to 0$ while b remains finite, and

(2) λ_κ remains finite while $b\to 0$.

We will now discuss the behavior of each of these cases in turn.

1. When $\lambda_\kappa\to 0$ with b remaining finite

In this case, the large value of b makes the crossover to chiral behavior happen almost immediately; that is, at small length and time scales. Hence, there is no substantial renormalization of the diffusion constants in the achiral regime, because that regime has essentially disappeared. Thus, the KPZ diffusion constant ν has no strong dependence on, e.g., the departure $\delta c\equiv c-c_*$ of the concentration of components of the competing chirality from the value c_* at which λ_κ vanishes. One can, of course, replace "concentration" in the definition of this departure with whatever parameter one is tuning to drive λ_κ towards zero.

Hence, we expect the dimensionless KPZ coupling

$$g\equiv\frac{\lambda_\kappa^2 D_\phi}{2\pi\nu^3}, \tag{B1}$$

which provides a measure of the importance of nonlinear effects in the KPZ equation [25], to acquire all of its dependence on δc from λ_κ, which we would expect to vanish linearly with δc; i.e.,

$$\lambda_\kappa\propto\delta c. \tag{B2}$$

Using this in (B1), and taking ν and D_ϕ to be independent of δc, we predict that the bare value g_0 of g vanishes quadratically with δc :

$$g_0\propto(\delta c)^2. \tag{B3}$$

Using our result (7.37) relating g_0 to the length scale $L_{\rm NL}$, at which nonlinear effects become important, we thereby obtain

$$L_{\rm NL}=L_c\exp\left[\frac{C_n}{(\delta c)^2}\right]. \tag{B4}$$

The argument of the exponential in this expression should be contrasted with the $b^{-\eta_g}$, with $\eta_g\approx 19/5$, scaling of the corresponding argument in Eq. (7.38).

2. When $b\to 0$ with λ_κ remaining finite

In this case, we still get the fluctuation induced enhancement of the diffusivity $\nu\propto b^{-\eta_\nu}$ given by Eq. (7.28). But now, because the nonlinearity λ_κ remains finite as $b\to 0$, the scaling law relating the bare value g_0 of g to b now changes to

$$g_0\propto\nu^{-3}\propto b^{3\eta_\nu} \tag{B5}$$

with the exponent $3\eta_\nu\approx 9/5$, which in turn leads, after again using our result (7.37) relating the value of the RG "time" at which nonlinear effects become important to g_0, to

$$L_{\rm NL}=L_c\exp\left[\frac{C_n}{b^{3\eta_\nu}}\right]. \tag{B6}$$

APPENDIX C: EVALUATION OF $C_\phi(\mathbf{r}, t)$

The integral over ω in our expression (8.11) for $C_\phi(\mathbf{r}, t)$ can be done by elementary complex contour techniques. The result is

$$C_\phi(\mathbf{r}, t)=\frac{2D_\phi}{\nu}\int\frac{d^2q}{(2\pi)^2}\left(\frac{1-e^{-\nu q^2|t|}(e^{i\mathbf{q}\cdot\mathbf{r}}+e^{-i\mathbf{q}\cdot\mathbf{r}})/2}{q^2}\right)e^{-(qL_c)^2}. \tag{C1}$$

Differentiating this with respect to $|t|$ gives

$$\frac{\partial C_\phi(\mathbf{r}, t)}{\partial|t|}=D_\phi\int\frac{d^2q}{(2\pi)^2}\left(e^{i\mathbf{q}\cdot\mathbf{r}-\nu q^2|t|}+e^{-i\mathbf{q}\cdot\mathbf{r}-\nu q^2|t|}\right)e^{-(qL_c)^2}. \tag{C2}$$

Both integrals in this expression are now simple Gaussian integrals. Evaluating them gives

$$\frac{\partial C_\phi(\mathbf{r}, t)}{\partial|t|}=\frac{D_\phi}{2\pi\left(\nu|t|+L_c^2\right)}\exp\left[-\left(\frac{r^2}{4\left(\nu|t|+L_c^2\right)}\right)\right]. \tag{C3}$$

Integrating this expression gives

$$C_\phi(\mathbf{r}, t)=\int_0^{|t|}\frac{D_\phi}{2\pi\left(\nu t'+L_c^2\right)}\exp\left[-\left(\frac{r^2}{4\left(\nu t'+L_c^2\right)}\right)\right]dt' + C_\phi(\mathbf{r}, t=0). \tag{C4}$$

Changing variable of integration from t' to $u\equiv\frac{r^2}{4(\nu t'+L_c^2)}$ in the integral over t' gives

$$\begin{aligned}C_\phi(\mathbf{r}, t)&=\frac{D_\phi}{2\pi\nu}\int_{\frac{r^2}{4(\nu|t|+L_c^2)}}^{\frac{r^2}{4L_c^2}}\left(\frac{e^{-u}}{u}\right)du+C_\phi(\mathbf{r}, t=0)\\&=\frac{D_\phi}{2\pi\nu}\left\{\mathrm{Ei}\left(-\frac{r^2}{4L_c^2}\right)-\mathrm{Ei}\left[-\left(\frac{r^2}{4\left(\nu|t|+L_c^2\right)}\right)\right]\right\}\\&\quad+C_\phi(\mathbf{r}, t=0),\end{aligned} \tag{C5}$$

where $\mathrm{Ei}(x)$ is the Exponential Integral Function defined as

$$\mathrm{Ei}(x)\equiv-\int_{-x}^{\infty}\left(\frac{e^{-u}}{u}\right)du. \tag{C6}$$

We have thus reduced the problem to finding the equal-time correlation function. This can be obtained from (C1) by setting $t=0$, which gives

$$C_\phi(\mathbf{r}, t=0)=\frac{2D_\phi}{\nu}\int\frac{d^2q}{(2\pi)^2}\left[\frac{1-(e^{i\mathbf{q}\cdot\mathbf{r}}+e^{-i\mathbf{q}\cdot\mathbf{r}})/2}{q^2}\right]e^{-(L_cq)^2}. \tag{C7}$$

Defining a related function of two variables $\mathbf{r}$ and k via

$$F(\mathbf{r}, k)=\frac{2D_\phi}{\nu}\int\frac{d^2q}{(2\pi)^2}\left[\frac{1-(e^{i\mathbf{q}\cdot\mathbf{r}}+e^{-i\mathbf{q}\cdot\mathbf{r}})/2}{q^2}\right]e^{-kq^2}, \tag{C8}$$

we see that

$$C_\phi(\mathbf{r}, t=0)=F\left(\mathbf{r}, k=L_c^2\right). \tag{C9}$$

We can evaluate $F(\mathbf{r}, k)$ by first differentiating (C8) with respect to k, which gives

$$\frac{\partial F(\mathbf{r}, k)}{\partial k}=-\frac{D_\phi}{2\pi^2\nu}\int d^2q[1-(e^{i\mathbf{q}\cdot\mathbf{r}}+e^{-i\mathbf{q}\cdot\mathbf{r}})/2]e^{-kq^2}. \tag{C10}$$

Evaluating the Gaussian integrals in (C10) gives

$$\frac{\partial F(\mathbf{r}, k)}{\partial k}=\frac{D_\phi}{2\pi\nu}\left(\frac{e^{-\frac{r^2}{4k}}-1}{k}\right). \tag{C11}$$

Integrating this from $k=L_c^2$ to ∞, noting that $F(\mathbf{r}, k=\infty)=0$ for all $\mathbf{r}$, and using (C9) to relate the result to $C_\phi(\mathbf{r}, t)$, we get

$$C_\phi(\mathbf{r}, t=0)=\frac{D_\phi}{2\pi\nu}\lim_{W\to\infty}\int_{L_c^2}^{W^2}\left[\frac{1}{k}-\frac{\exp\left(-\frac{r^2}{4k}\right)}{k}\right]dk. \tag{C12}$$

The integral of the first term in parentheses is, of course, elementary, while the second can be evaluated with the change of variable of integration from k to $u\equiv\frac{r^2}{4k}$. The result is

$$\begin{aligned}&\int_{L_c^2}^{W^2}\left(\frac{1}{k}-\frac{\exp\left[-\frac{r^2}{4k}\right]}{k}\right)dk\\&=\ln\left(\frac{W^2}{L_c^2}\right)+\mathrm{Ei}\left(-\frac{r^2}{4W^2}\right)-\mathrm{Ei}\left(-\frac{r^2}{4L_c^2}\right).\end{aligned} \tag{C13}$$

Using the well-known [45] limiting behavior of the exponential integral function for small argument $-u$

$$-\mathrm{Ei}(-u)\approx\ln\left(\frac{1}{u}\right)-\gamma \tag{C14}$$

and taking the limit $W\to\infty$ in (C13), we see that the dependence on W cancels in this limit, and we get our final result for $C_\phi(\mathbf{r}, t=0)$:

$$C_\phi(\mathbf{r}, t=0)=\frac{D_\phi}{2\pi\nu}\left[2\ln\left(\frac{r}{2L_c}\right)-\mathrm{Ei}\left(-\frac{r^2}{4L_c^2}\right)+\gamma\right]. \tag{C15}$$

Inserting this result in our expression (C5) for the full-, position-, and time-dependent correlation function gives

$$C_\phi(\mathbf{r}, t)=\frac{D_\phi}{2\pi\nu}\left[2\ln\left(\frac{r}{2L_c}\right)-\mathrm{Ei}\left(-\frac{r^2}{4\left(\nu|t|+L_c^2\right)}\right)+\gamma\right], \tag{C16}$$

which is Eq. (1.17).

APPENDIX D: STEEPEST DESCENTS CALCULATION OF $I(R, \omega)$ FOR $\delta\omega\gg\nu/R^2$

The scaling function $F_h(S)$ in (9.14) is given by

$$F_h(S)=\int_0^{\infty}e^{g(u)}\,du+\text{c.c.} \tag{D1}$$

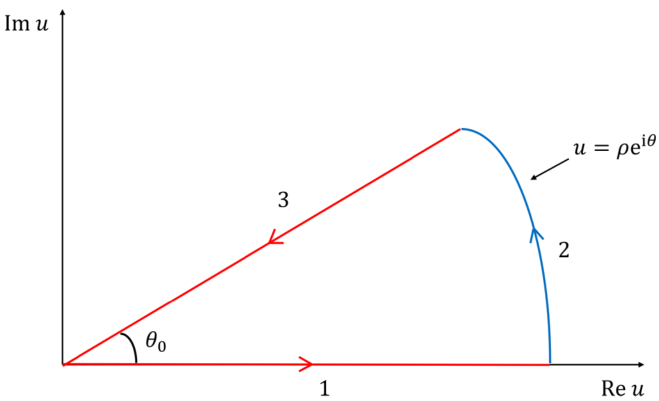

FIG. 4. The illustration of the closed contour of the contour integral (D3).

where

$$g(u) = i|S|u + \frac{\alpha}{2}\mathrm{Ei}\left(-\frac{1}{u}\right). \tag{D2}$$

Now consider the closed contour (see Fig. 4) and the integral

$$I = \oint e^{g(u)}\, du. \tag{D3}$$

The integral around the closed contour vanishes since $e^{g(u)}$ is analytic in the region enclosed by the contour. The integral over the circular section 2, which is defined by $u = Re^{i\theta}$, with R a positive real constant, which we will ultimately take to infinity, is given by

$$I_2 = i\int_0^{\theta_0} \exp\left[i|S|Re^{i\theta} + \frac{\alpha}{2}\mathrm{Ei}\left(-\frac{1}{Re^{i\theta}}\right)\right]Re^{i\theta}\, d\theta. \tag{D4}$$

For large R,

$$\mathrm{Ei}\left(-\frac{1}{Re^{i\theta}}\right) \approx -\ln(Re^{i\theta}) + \gamma. \tag{D5}$$

Hence,

$$I_2 = ie^{\frac{\alpha\gamma}{2}}R^{-\frac{\alpha}{2}}\int_0^{\theta_0} \exp\left[i|S|Re^{i\theta} - i\frac{\alpha\theta}{2}\right]Re^{i\theta}\, d\theta. \tag{D6}$$

For large R, this integral is dominated by small θ. Expanding the integrand for small θ and pushing the upper limit on θ from θ_0 to infinity (since the integral over θ converges rapidly as θ grows from zero). We get

$$\begin{aligned} I_2 &\approx ie^{\frac{\alpha\gamma}{2}+i|S|R}R^{1-\frac{\alpha}{2}}\int_0^{\infty} \exp\left[-\left(|S|R + i\frac{\alpha}{2} - i\right)\theta\right]d\theta \\ &= ie^{\frac{\alpha\gamma}{2}+i|S|R}R^{1-\frac{\alpha}{2}}\int_0^{\infty} \exp\left[-\left(|S|R + i\frac{\alpha}{2} - i\right)\theta\right]d\theta \\ &= \frac{ie^{\frac{\alpha\gamma}{2}+i|S|R}R^{1-\frac{\alpha}{2}}}{|S|R + i\frac{\alpha}{2} - i}, \end{aligned} \tag{D7}$$

which vanishes in the limit $R \to \infty$. Hence, since the integral (D3) vanishes, we have

$$\int_0^{\infty} e^{g(u)}\, du = \int_0^{\infty e^{i\theta_0}} e^{g(u)}\, du \tag{D8}$$

for any θ_0. That is, we can rotate the contour with complete impunity.

Inserting (D8) into (D1) gives

$$F_h(S) = \int_0^{\infty e^{i\theta_0}} e^{g(u)}\, du + \text{c.c.} \tag{D9}$$

For small S, we choose $\theta_0 = \frac{\pi}{2}$. That is, we rotate the contour of integration to the imaginary axis, as we have done in Sec. IX. This gives us the result in the small S limit.

For large S, we rotate the contour by $\theta_0 = \frac{\pi}{4}$ and change the variable by integration by writing u as

$$u = ve^{i\frac{\pi}{4}}. \tag{D10}$$

This gives

$$F_h(S) = e^{i\frac{\pi}{4}}\int_0^{\infty} e^{g(ve^{i\frac{\pi}{4}})}\, dv + \text{c.c.} \tag{D11}$$

Now we focus on the integral

$$I_3 = \int_0^{\infty} e^{g(ve^{i\frac{\pi}{4}})}\, dv. \tag{D12}$$

Let's separate this integral into small-v and large-v parts:

$$I_3 = \int_0^{\Delta_2} e^{g(ve^{i\frac{\pi}{4}})}\, dv + \int_{\Delta_2}^{\infty} e^{g(ve^{i\frac{\pi}{4}})}\, dv, \tag{D13}$$

where we choose

$$\Delta_2 = |S|^{-\sigma} \tag{D14}$$

with

$$0 < \sigma < \tfrac{1}{2}. \tag{D15}$$

In this range of σ, Δ_2 will be much less than 1 for large $|S|$, so that, in the first integral of (D13), namely,

$$I_{3<} = \int_0^{\Delta_2} e^{g(ve^{i\frac{\pi}{4}})}\, dv, \tag{D16}$$

the argument of the exponential integral function will be large. We can use the large argument expansion of $\mathrm{Ei}(-\frac{1}{u})$ in the first integral. In this regime $\mathrm{Ei}(-\frac{1}{u})$ will also be small, and we will use this to our advantage too.

For now, let's defer that discussion for later, and first show that the second integral in (D13), namely,

$$I_{3>} = \int_{\Delta_2}^{\infty} e^{g(ve^{i\frac{\pi}{4}})}\, dv, \tag{D17}$$

is negligible.

To do this, we will start by noting that the magnitude of $I_{3>}$ is bounded above:

$$|I_{3>}| \leqslant \int_{\Delta_2}^{\infty} \left|e^{g(ve^{i\frac{\pi}{4}})}\right| dv = \int_{\Delta_2}^{\infty} e^{\mathrm{Re}[g(ve^{i\frac{\pi}{4}})]}\, dv. \tag{D18}$$

Now one can show that

$$\mathrm{Re}\left[\mathrm{Ei}\left(-\frac{1}{ve^{i\frac{\pi}{4}}}\right)\right] < \mathrm{Re}\left[\mathrm{Ei}\left(-\frac{\pi e^{-i\frac{\pi}{4}}}{\sqrt{2}}\right)\right] = 0.0415139 \equiv C, \tag{D19}$$

for $v > 0$. To see this, we begin with the definition of the Ei function

$$\mathrm{Ei}\left(-\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\right) \equiv -\int_{\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}}^{\infty}\left(\frac{\mathrm{e}^{-u}}{u}\right)\mathrm{d}u. \tag{D20}$$

By changing variable of integration from u to $w \equiv u\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$, we can rewrite (D20) as [56]

$$\mathrm{Ei}\left(-\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\right) \equiv -\int_{\frac{1}{v}}^{\infty}\left[\frac{\mathrm{e}^{\frac{\sqrt{2}}{2}w(-1+\mathrm{i})}}{w}\right]\mathrm{d}w. \tag{D21}$$

The real part of this is given by

$$\mathrm{Re}\left[\mathrm{Ei}\left(-\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\right)\right] = -\int_{\frac{1}{v}}^{\infty}\left[\frac{\mathrm{e}^{-\frac{\sqrt{2}}{2}w}\cos\left(\frac{\sqrt{2}}{2}w\right)}{w}\right]\mathrm{d}w$$
$$\equiv f(v). \tag{D22}$$

It follows from this expression that the derivative

$$f'(v) = -\left[\frac{\mathrm{e}^{-\frac{1}{\sqrt{2}v}}\cos\left(\frac{1}{\sqrt{2}v}\right)}{v}\right]. \tag{D23}$$

Note first that this implies $f'(v) < 0$ for $v > \sqrt{2}/\pi$. Thus the first maximum of $f(v)$ that we will encounter when decreasing v from very large positive values will occur when

$$\frac{1}{\sqrt{2}v} = \frac{\pi}{2}. \tag{D24}$$

At this maximum, we have

$$f(v = \sqrt{2}/\pi) = \mathrm{Re}\left[\mathrm{Ei}\left(-\frac{\pi\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{\sqrt{2}}\right)\right] = 0.0415139, \tag{D25}$$

where the numerical value was obtained with *Mathematica*.

From (D23), it is clear that there are infinitely many other extrema of $f(v)$ at smaller values of v, whose positions satisfy

$$\frac{1}{\sqrt{2}v} = \left(m + \frac{1}{2}\right)\pi \tag{D26}$$

with m running over all positive integers. A moment's reflection reveals that the odd m's correspond to minima, while the even m's correspond to maxima. Hence, the positions of the *maxima* are given by

$$v_n = \frac{\sqrt{2}}{\pi(4n+1)}. \tag{D27}$$

All of these maxima, and, hence, all values of the entire function $f(v)$, are smaller than the first maximum (D24). To prove this, we first note that $f(v)$ satisfies a rigorous upper bound, which can be derived by noting that, for real v, $-\cos w \leqslant 1$ and, in the region of integration, $w > 1/v$. It follows that

$$f(v) = \int_{\frac{1}{v}}^{\infty}\left[\frac{\mathrm{e}^{-\frac{\sqrt{2}}{2}w}\left[-\cos\left(\frac{\sqrt{2}}{2}w\right)\right]}{w}\right]\mathrm{d}w$$
$$< \int_{\frac{1}{v}}^{\infty}\left[\frac{\mathrm{e}^{-\frac{\sqrt{2}}{2}w}}{\left(\frac{1}{v}\right)}\right]\mathrm{d}w = v\sqrt{2}\mathrm{e}^{-\frac{\sqrt{2}}{2v}}. \tag{D28}$$

Of all of the maxima at v_n with $n > 0$ as given by Eq. (D27), this upper bound is clearly largest for $n = 1$, at which point the bound is $f(v) < \frac{2}{5\pi}\mathrm{e}^{\frac{-5\pi}{2}} \approx 4.943 \times 10^{-5}$, which is less (*much less*) than the first ($n = 0$) maximum given by (D25).

Hence, that $n = 0$ maximum (D25) is the absolute maximum of $\mathrm{Re}[\mathrm{Ei}(-\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}})]$ over all positive v. This leads immediately to the bound (D19).

It follows that

$$\mathrm{Re}\left[g\left(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\right] = \mathrm{Re}\left(\mathrm{i}|S|v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right) + \frac{\alpha}{2}\mathrm{Re}\left[\mathrm{Ei}\left(-\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\right)\right]$$
$$< \mathrm{Re}\left(\mathrm{i}|S|v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right) + \frac{\alpha C}{2} < -\frac{|S|v}{\sqrt{2}} + \frac{\alpha C}{2}, \tag{D29}$$

where in the equality we have used the definition of $g(u)$ (D2). Hence, using (D29) in (D18) gives

$$|I_{3>}| < \mathrm{e}^{\frac{\alpha C}{2}}\int_{\Delta_2}^{\infty}\exp\left(-\frac{|S|v}{\sqrt{2}}\right)\mathrm{d}v$$
$$= \mathrm{e}^{\frac{\alpha C}{2}}\frac{\sqrt{2}}{|S|}\exp\left(-\frac{|S|\Delta_2}{\sqrt{2}}\right)$$
$$= \mathrm{e}^{\frac{\alpha C}{2}}\frac{\sqrt{2}}{|S|}\exp\left(-\frac{|S|^{1-\sigma}}{\sqrt{2}}\right), \tag{D30}$$

where in the last equality we have used (D14). Since $0 < \sigma < \frac{1}{2}$, (D30) implies

$$|I_{3>}| \ll \frac{1}{|S|^{\frac{5}{4}}}\exp(-\sqrt{2|S|}). \tag{D31}$$

This is because for large S and $0 < \sigma < \frac{1}{2}$, the dominating exponential $\exp(-\frac{|S|^{1-\sigma}}{\sqrt{2}})$ on the right-hand side of (D30) is less than $\exp(-\sqrt{2|S|})$ in (D31) (as σ is strictly less than $1/2$). Since the right-hand side of (D31) is the order of magnitude of the dominant contribution to I_3 coming from $I_{3<}$, as we will show below, $I_{3>}$ is negligible.

Now let's return to the first integral in (D13), namely, (D16). Since $\Delta_2 \ll 1$, it follows that, throughout this entire region of integration over v, the magnitude of $v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ is $\ll 1$. Therefore, the argument of the Ei function in $g(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})$:

$$g\left(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right) = \mathrm{i}|S|v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} + \frac{\alpha}{2}\mathrm{Ei}\left(-\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\right) \tag{D32}$$

is large, which means the exponential integral itself is small. Therefore, we can expand

$$\mathrm{e}^{g(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})} \approx \mathrm{e}^{\mathrm{i}|S|v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\left[1 + \frac{\alpha}{2}\mathrm{Ei}\left(-\frac{1}{v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\right)\right]. \tag{D33}$$

Inserting (D33) into (D16) we get

$$I_{3<} \approx \int_0^{\Delta_2}\mathrm{e}^{\mathrm{i}|S|v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\,\mathrm{d}v - \frac{\alpha}{2}\int_0^{\Delta_2}\mathrm{e}^{\Phi(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})}\,\mathrm{d}v, \tag{D34}$$

where

$$\Phi(u) = \mathrm{i}|S|u + \ln u - \frac{1}{u} \tag{D35}$$

as obtained by using the large argument expansion of the exponential integral. We note that the argument $\Phi(u)$ of the integral has a maximum in the complex plane at the complex values of u at which

$$\frac{\mathrm{d}\Phi}{\mathrm{d}u} = \mathrm{i}|S| + \frac{1}{u} + \frac{1}{u^2} = 0. \tag{D36}$$

For $|S| \gg 1$, the solution u_m of this equation is well approximated by

$$u_m = \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{\sqrt{|S|}}. \tag{D37}$$

The first integral in (D34) is

$$\int_0^{\Delta_2} \mathrm{e}^{\mathrm{i}|S|v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\,\mathrm{d}v = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{|S|}\big(1 - \mathrm{e}^{\mathrm{i}|S|\Delta_2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}\big). \tag{D38}$$

The first term in (D38) eventually leads to a contribution to $F_h(S)$ (D11):

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{|S|} + \mathrm{C.\,C.} = \frac{\mathrm{i} + (-\mathrm{i})}{|S|} = 0. \tag{D39}$$

The second term in (D38) can be written as

$$-\frac{\mathrm{e}^{-|S|\Delta_2 \sin\pi/4}\big(\mathrm{e}^{\mathrm{i}(|S|\Delta_2\cos\pi/4 + \pi/4)}\big)}{|S|}, \tag{D40}$$

and its contribution to $F_h(S)$ is thus

$$-\frac{\mathrm{e}^{-|S|^{1-\sigma}\sin\pi/4}\big(\mathrm{e}^{\mathrm{i}(|S|^{1-\sigma}\cos\pi/4 + \pi/2)}\big)}{|S|} + \mathrm{c.c.}, \tag{D41}$$

which, due to the fact that σ is strictly less than 1/2, is subdominant to other contributions to this contour integral, as we will soon see.

So the contribution to $F_h(S)$ is entirely dominated by the second term in (D34), the integral in which is given by

$$I_4 = \int_0^{\Delta_2} \mathrm{e}^{\Phi(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})}\,\mathrm{d}v. \tag{D42}$$

Now let's break this integral into three parts

$$I_{4.1} = \int_0^{\frac{1}{\sqrt{|S|}} - \Delta_3} \mathrm{e}^{\Phi(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})}\,\mathrm{d}v, \tag{D43}$$

$$I_{4.2} = \int_{\frac{1}{\sqrt{|S|}} - \Delta_3}^{\frac{1}{\sqrt{|S|}} + \Delta_3} \mathrm{e}^{\Phi(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})}\,\mathrm{d}v, \tag{D44}$$

$$I_{4.3} = \int_{\frac{1}{\sqrt{|S|}} + \Delta_3}^{\Delta_2} \mathrm{e}^{\Phi(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})}\,\mathrm{d}v, \tag{D45}$$

where Δ_3 is chosen to lie in the range

$$|S|^{-\frac{3}{4}} \ll \Delta_3 \ll |S|^{-\frac{2}{3}}. \tag{D46}$$

Our choice of the lower bound on Δ_3 is motivated by the fact that $\mathrm{e}^{\Phi(u)}$ is well approximated by a Gaussian near $u = u_m = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}/\sqrt{|S|}$; the width of this Gaussian is $O(|S|^{-\frac{3}{4}})$. We have chosen Δ_3 to be much bigger than this width, which proves to be convenient later.

Now we will prove that $I_{4.2}$, which is extremely well approximated by a Gaussian integral, is much larger than $I_{4.1}$ and $I_{4.3}$, which can therefore be neglected. Let's start with $I_{4.1}$. This can be written as

$$I_{4.1} = \mathrm{e}^{\Phi(u_m)} I_{4.1.a}, \tag{D47}$$

where we have defined

$$I_{4.1.a} \equiv \int_0^{\frac{1}{\sqrt{|S|}} - \Delta_3} \exp\big[\Phi\big(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\big) - \Phi(u_m)\big]\,\mathrm{d}v. \tag{D48}$$

Now we show that the magnitude of this is bounded. It is clear that

$$\begin{aligned} |I_{4.1.a}| &\leqslant \int_0^{\frac{1}{\sqrt{|S|}} - \Delta_3} \big|\exp\big[\Phi\big(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\big) - \Phi(u_m)\big]\big|\,\mathrm{d}v \\ &= \int_0^{\frac{1}{\sqrt{|S|}} - \Delta_3} \exp\big[\mathrm{Re}\big(\Phi\big(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\big) - \Phi(u_m)\big)\big]\,\mathrm{d}v. \end{aligned} \tag{D49}$$

Recall that $\Phi(u)$ is given by (D35). It follows that

$$\frac{\mathrm{d}}{\mathrm{d}v}\big[\mathrm{Re}\big(\Phi\big(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\big)\big)\big] = -\frac{|S|}{\sqrt{2}} + \frac{1}{v} + \frac{1}{\sqrt{2}v^2}. \tag{D50}$$

For all $v < \frac{1}{\sqrt{|S|}}$, it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}v}\big[\mathrm{Re}\big(\Phi\big(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\big)\big)\big] > 0. \tag{D51}$$

Hence,

$$\begin{aligned} &\mathrm{Re}\big(\Phi\big(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\big) - \Phi(u_m)\big) \\ &\quad < \mathrm{Re}\left\{\Phi\left[\left(\frac{1}{\sqrt{|S|}} - \Delta_3\right)\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right] - \Phi(u_m)\right\} \end{aligned} \tag{D52}$$

throughout the region of integration $0 < v < \frac{1}{\sqrt{|S|}} - \Delta_3$. Furthermore, we can approximate $\mathrm{Re}\{\Phi[(\frac{1}{\sqrt{|S|}} - \Delta_3)\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}]\}$ by expanding it around the maximum at u_m:

$$\begin{aligned} &\mathrm{Re}\left\{\Phi\left(\left[\frac{1}{\sqrt{|S|}} - \Delta_3\right)\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right] - \Phi(u_m)\right\} \\ &\approx \frac{1}{2}\frac{\mathrm{d}^2\{\mathrm{Re}[\Phi(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})]\}}{\mathrm{d}v^2}\bigg|_{v=\frac{1}{\sqrt{|S|}}} \Delta_3^2 = -\frac{1}{2}\sqrt{2|S|^3}\,\Delta_3^2. \end{aligned} \tag{D53}$$

Note that this expansion works provided the $O(\Delta_3^3)$ term in the expansion is $\ll 1$.

That term is

$$\frac{1}{6}\frac{\mathrm{d}^3\{\mathrm{Re}[\Phi(v\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})]\}}{\mathrm{d}v^3}\bigg|_{v=\frac{1}{\sqrt{|S|}}} \Delta_3^3 = \left(\frac{|S|^2}{\sqrt{2}}\right)\Delta_3^3. \tag{D54}$$

To ensure that this is $\ll 1$ as $S \to \infty$, we must have

$$\Delta_3 \ll |S|^{-\frac{2}{3}}, \tag{D55}$$

which is consistent with (D46). Now using (D53) in the bound (D52), and using that bound in the bound (D49), we have

$$|T_{4.1a|}| < \frac{1}{\sqrt{|S|}}\exp\left(-\sqrt{\frac{|S|^3}{2}}\Delta_3^2\right). \tag{D56}$$

Combining this with (D47) we get

$$\begin{aligned} |I_{4.1}| &< |\mathrm{e}^{\Phi(u_m)}|\frac{1}{\sqrt{|S|}}\exp\left(-\sqrt{\frac{|S|^3}{2}}\Delta_3^2\right) \\ &< |\mathrm{e}^{\Phi(u_m)}|\frac{1}{\sqrt{|S|}}\exp\left(-\frac{|S|^\alpha}{\sqrt{2}}\right), \end{aligned} \tag{D57}$$

where $\alpha > 0$. In the second "$<$" in (D57) we have used the range for Δ_3 given in (D46).

Now we turn to $I_{4.2}$. Since we have shown above that Δ_3 is small enough to expand $\Phi(u)$ to quadratic order in the regime

$$\frac{1}{\sqrt{|S|}} - \Delta_3 < v < \frac{1}{\sqrt{|S|}} + \Delta_3, \tag{D58}$$

the integral (D44) simply becomes a Gaussian integral:

$$\begin{aligned} I_{4.2} &= \mathrm{e}^{\Phi(u_m)} \int_{\frac{1}{\sqrt{|S|}}-\Delta_3}^{\frac{1}{\sqrt{|S|}}+\Delta_3} \exp\left[\frac{1}{2}\frac{\mathrm{d}^2\Phi(u)}{\mathrm{d}u^2}\bigg|_{u=u_m}(u-u_m)^2\right]\mathrm{d}v \\ &= \mathrm{e}^{\Phi(u_m)} \int_{\frac{1}{\sqrt{|S|}}-\Delta_3}^{\frac{1}{\sqrt{|S|}}+\Delta_3} \exp\left[-\frac{\sqrt{2}}{2}(1-\mathrm{i})|S|^{\frac{3}{2}}\left(v-\frac{1}{\sqrt{|S|}}\right)^2\right]\mathrm{d}v. \end{aligned} \tag{D59}$$

Since Δ_3 (see (D46)) is much larger than the width ($\sim|S|^{-3/4}$) of the Gaussian integral (D59), we can, with exponential accuracy, extend the range of this integral to $\pm\infty$. This gives

$$\begin{aligned} I_{4.2} &= \mathrm{e}^{\Phi(u_m)} \int_{-\infty}^{\infty} \exp\left[-\frac{\sqrt{2}}{2}(1-\mathrm{i})|S|^{\frac{3}{2}}\left(v-\frac{1}{\sqrt{|S|}}\right)^2\right]\mathrm{d}v \\ &= \frac{\sqrt{\pi}}{|S|^{\frac{3}{4}}}\exp\left[\Phi(u_m)+\mathrm{i}\frac{\pi}{8}\right] \\ &= \frac{\sqrt{\pi}}{|S|^{\frac{5}{4}}}\exp\left[-\sqrt{2|S|}+\mathrm{i}\left(\sqrt{2|S|}+\frac{3\pi}{8}\right)\right], \end{aligned} \tag{D60}$$

where in the last equality we have inserted $u_m = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}/\sqrt{|S|}$. Inserting (D60) into (D34) and then (D34) into (D11) recovers the result (9.25) of Sec. IX.

The argument that $I_{4.3}$ is negligible compared to $I_{4.2}$ closely parallels the argument given above for $I_{4.1}$, so we will leave it as an exercise for the reader.

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[56] The alert reader (the *preternaturally* alert reader) will notice that this change of variables makes the upper limit of the integral $e^{i\frac{\pi}{4}}\infty$, rather than real ∞. However, because the integrand here is analytic, we can deform the contour of integration into that illustrated in Fig. 4, with $\theta_0 = \pi/4$. The integral along the real axis then gives Eq. (D21), while the integral over the circular arc vanishes. We thereby recover Eq. (D21).

[57] L. Chen, P. Jentsch, C. F. Lee, A. Maitra, S. Ramaswamy, and J. Toner, The inconvenient truth about flocks, arXiv:2503.17064.