

Ergodicity and hydrodynamics: From quantum to classical spin systems

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We show that in classical spin systems, the precise nature of the late-time hydrodynamic tails of the autocorrelation functions of a generic observable is determined by (i) the dynamical critical exponent and (ii) the equilibrium thermodynamic properties of the corresponding observable. We provide numerical results for one- and two-dimensional systems and present theoretical considerations that only rely on the notion of ergodicity. Our result extends to the classical framework the *relaxation-overlap inequality*, first introduced in [Phys. Rev. X **15**, 011059 (2025)] for quantum many-body systems satisfying the eigenstate thermalization hypothesis.

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I. INTRODUCTION

Statistical mechanics is a cornerstone of our description of the physical world and is based on the crucial notion of *ergodicity*, the fact that after a sufficiently long time, a physical system has explored the entire set of configurations that are allowed by its conservation laws, so that time averages correspond to phase-space averages [1–3]. In macroscopic setups, ergodicity (see Refs. [4–6]) can be employed to describe the long-time averages of relevant observables using statistical Gibbs ensembles, such as the canonical and grand canonical ones.

Thermodynamic ensembles are stationary and do not evolve in time. On the other hand, one is often interested in setting them slightly out of equilibrium with a perturbation and studying how this effect evolves in time and space. Specifically, the *autocorrelation* functions of observables, and in particular their late-time properties, encode important information on such a dynamical behavior. For instance, an exponential decay in time typically reflects a fast erasure of memory of an initially applied perturbation [7,8], a phenomenology that is routinely observed in situations where no conservation laws are present.

In several common situations, the autocorrelation functions display a late-time algebraic decay to zero: this behavior marks the appearance of an emergent *hydrodynamic* behavior that can be linked to the presence of conservation laws, such as the energy or the number of particles. Hydrodynamics allows us to describe many-body systems in terms of local stationary

states, where the temperature and the chemical potential vary smoothly in space and time: such a coarse-grained description turns out to be predictive at large scales, both for classical [9–11] and quantum [12–23] systems, by replacing microscopic complicated details with a few transport equations.

Focusing only on the simplest setups, those featuring energy as a unique conservation law, it is considered that the hydrodynamic tails should be only determined by the *dynamical critical exponent* z , which is associated to the energy density spreading, and characterizes the ballistic universality class ($z = 1$), diffusive ($z = 2$), or else. This expectation is rooted in the fact that when energy is the only conserved quantity, its density spreads through the system, and the spreading of other generic local observables is typically determined by such a phenomenon; as such, the late-time dynamics of any observable is uniquely determined by the energy dynamics.

In a previous work [24] that we have co-authored (but see also Refs. [25–29]), it was shown that for quantum many-body systems, the precise nature of the hydrodynamic tails of the autocorrelation functions of observables is linked to both (i) the dynamical critical exponent z and (ii) the thermodynamic properties of the observable under analysis: such a relation was dubbed the *relaxation-overlap inequality*. That derivation relied on the eigenstate thermalization hypothesis (ETH) [30–32], which formalizes the notion of chaotic quantum systems, and on its relation to hydrodynamics. While we expect that the main results in Ref. [24] should be valid in a larger framework, including classical systems, many details that mostly rely on ETH need to be adapted to the classical framework.

A link between ETH and chaotic classical systems is well known in the literature, and it is provided by the well-established notion of *ergodicity* (see Ref. [33] and Refs. [34–36]): roughly speaking, it says that the dynamics

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spreads uniformly along the accessible phase-space, giving rise to memory erasing phenomena and the late-time appearance of the microcanonical ensemble. We revisit this problem, giving a fresh view on the subject, and we show, via the key assumption of ergodicity, how to extend the results of [24] to the classical framework.

In this article, we combine the aforementioned concepts and show with numerical and analytical tools that classical spin systems obey a relaxation-overlap inequality [24]. The cornerstones of our derivation are (i) the notion of ergodicity and (ii) its relation to the hydrodynamic spreading of energy. Compared to quantum systems, predictions for classical spin systems can be studied numerically on larger sizes, corroborating the predictions with high-precision data. We present those data, focusing on several classical spin models characterized by different energy-spreading properties. The results are fully compatible with a relaxation-overlap inequality, for which we propose an analytical derivation in a fully classical scenario.

The manuscript is organized as follows. In Sec. II, we outline the general mathematical framework that will be employed to study the classical spin systems, their thermodynamics, and their autocorrelation functions. In Sec. III, we present numerical simulations for several spin models, defined in one-dimensional and two-dimensional lattices and in setups with different transport properties. We show the existence of a link between the thermodynamical properties of certain observables and their autocorrelation functions. In Sec. IV, we present analytical considerations that prove such a connection. Since similar results have already been identified in many-body *quantum* spin systems, we present in Sec. V a critical comparison of the two settings, focusing on the classical analogues of ETH. Finally, in Sec. VI, we present our conclusions and a few perspectives on this work.

II. PRELIMINARIES

In this section, we provide some definitions useful for dealing with generic classical many-body systems, and specifically spin setups. The presentation follows a line that highlights similarities with the quantum setting.

A. Generic classical systems

We begin by describing classical systems in terms of observables and of their dynamics, and introduce a notion of pure states and statistical mixtures. Such a point of view, although slightly more abstract than elementary approaches, has the advantage of closely resembling the standard formulation of quantum systems, and therefore it allows us to bridge classical and quantum systems in a more direct way.

We start from a manifold $\mathcal{M}$ that we call *phase-space*, and we consider the *observables* as real-valued smooth functions supported on $\mathcal{M}$, denoted by $C^\infty(\mathcal{M}, \mathbb{R})$. In this context, a *state* is a probability distribution over $\mathcal{M}$ and it allows us to evaluate expectation values of observables. In particular, one can always associate *pure states* to the points of $\mathcal{M}$: given a point $p \in \mathcal{M}$, one defines the state $\langle \dots \rangle_p$ as follows:

$$\langle \mathcal{O} \rangle_p := \mathcal{O}(p), \quad \mathcal{O} \in C^\infty(\mathcal{M}, \mathbb{R}). \quad (1)$$

These classical pure states do not have correlations since $\langle \mathcal{O} \mathcal{O}' \rangle_{p,c} = \langle \mathcal{O} \mathcal{O}' \rangle_p - \langle \mathcal{O} \rangle_p \langle \mathcal{O}' \rangle_p = 0$ holds: this is a key difference with respect to quantum systems. On the other hand, any statistical mixture of pure states is associated with a probability measure dP_p on $\mathcal{M}$ which defines an expectation value

$$\langle \dots \rangle := \int_{\mathcal{M}} dP_p \langle \dots \rangle_p. \quad (2)$$

In particular, equilibrium states associated with the microcanonical, canonical, or grandcanonical ensembles are statistical mixtures; as it is well known, they show correlations whose origin is purely classical.

The *dynamics of a classical system* can be described in terms of the evolution of observables. For simplicity, here we only consider dynamics that arise from a Poisson-bracket structure and a Hamiltonian H which, for the sake of simplicity, does not depend on time. The evolution of an observable $\mathcal{O}$ is described by the equation

$$\frac{d}{dt} \mathcal{O} = \{H, \mathcal{O}\}, \quad (3)$$

which allows us to associate to any observable $\mathcal{O}$ its time-evolved counterpart $\mathcal{O}(t)$. As it is obvious, in a time-independent system, energy is conserved, namely $\frac{d}{dt} H = 0$ because the Poisson brackets are required to be antisymmetric in the two variables. Also, a *stationary state* $\langle \dots \rangle$ satisfies, by definition, $\langle \mathcal{O}(t) \rangle = \langle \mathcal{O} \rangle$, $\forall \mathcal{O}$, or, which is equivalent, $\frac{d}{dt} \langle \mathcal{O}(t) \rangle = 0$, $\forall \mathcal{O}$. This viewpoint is analogous to the *Heisenberg picture* in quantum mechanics.

B. Spin systems

In this work, we will focus on classical spin systems. Their interest comes from the fact that they correctly describe the large- S limit of quantum spin- S systems; in general, they are expected to capture the same relevant physics (for instance, diffusive hydrodynamics of energy) of quantum spin systems when S is finite (see, for instance, the intriguing results on the comparison of autocorrelation functions of classical and quantum spin setups in Ref. [37]); this is important because it motivates the attempt presented in this article to relate the findings of [24], tested on a spin-1 quantum Ising chain, to possible classical counterparts.

We first discuss the phase space of a single spin. We identify the phase space as $\mathcal{M} := \mathbb{R}^3$, where the three coordinates are the three directions of the spin, S^x , S^y and S^z . One introduces the Poisson brackets on $\mathcal{M}$ satisfying, for instance, $\{S^x, S^y\} = S^z$, and more generically,

$$\{S^\mu, S^\nu\} = \epsilon_{\mu\nu\lambda} S^\lambda, \quad (4)$$

where $\epsilon_{\mu\nu\lambda}$ is the fully antisymmetric Levi-Civita symbol. One can show that the squared modulus $(S^x)^2 + (S^y)^2 + (S^z)^2$, known as the *Casimir invariant*, is conserved under time evolution no matter the specific choice of the Hamiltonian: for this reason, the dynamics is restricted on submanifolds at fixed values of the Casimir invariant.

We can associate a rotational invariant measure (known as the *Haar measure*) to the spherical surface $(S^x)^2 + (S^y)^2 + (S^z)^2 = 1$, whose associated state plays the role of an infinite temperature state, i.e., $\beta = 0$: this is the only state with the

property $\langle \{S^a, \mathcal{O}\} \rangle_{\beta=0} = 0$ for $a = x, y, z$ and for any observable $\mathcal{O}$. Hence, the infinite-temperature state is stationary. With a bit of algebra, one can characterize the expectation values of observables systematically in the infinite temperature states and obtain, for example, that $\langle (S^a)^2 \rangle_{\beta=0} = 1/3$. Further details are given in Appendix A.

These building blocks allow us to construct in a natural way the phase space and infinite temperature state of a system with many classical spins. Important properties for the following are that the infinite temperature state is stationary and it does not correlate distinct spins, namely $\langle \mathcal{O}(\mathbf{r})\mathcal{O}'(\mathbf{r}') \rangle_{\beta=0,c} = 0$ for $\mathbf{r} \neq \mathbf{r}'$: the same properties hold for quantum mechanical systems as well.

C. Thermodynamics

In the previous paragraph, we have introduced the infinite temperature state $\langle \dots \rangle_{\beta=0}$, which can be considered as a universal reference stationary measure on the phase space. Similarly, for a Hamiltonian H , one can define a *thermal state* at finite inverse temperature β by

$$\langle \dots \rangle_{\beta} := \frac{\langle \dots e^{-\beta H} \rangle_{\text{Haar}}}{\langle e^{-\beta H} \rangle_{\text{Haar}}}, \quad (5)$$

where $\langle \dots \rangle_{\text{Haar}}$ denotes the average over the Haar measure and by definition, $\langle \dots \rangle_{\text{Haar}} = \langle \dots \rangle_{\beta=0}$. One can easily prove that the thermal state is stationary for the dynamics induced by H : this comes directly from the stationarity of the infinite temperature state and that of H ($H(t) = H$).

We can hence define the thermal expectation of an observable $\mathcal{O}$ as

$$\mathcal{O}(\beta) := \langle \mathcal{O} \rangle_{\beta}. \quad (6)$$

General mathematical results on spin systems [38] show that for infinite lattices with short-range interactions, the thermal states (at sufficiently high temperature, above any possible low-temperature symmetry breaking) satisfy the *clustering property*: the connected correlation functions $\langle \mathcal{O}(\mathbf{r})\mathcal{O}'(\mathbf{r}') \rangle_{\beta,c}$ decay exponentially fast in the distance $|\mathbf{r} - \mathbf{r}'|$. Also, as a consequence, the cumulants of extensive observables, namely, those that can be written as sums $\mathcal{Q} = \sum_{\mathbf{r}} q(\mathbf{r})$, with $q(\mathbf{r})$ the corresponding charge density, are extensive: for a system of volume V , then $\langle \mathcal{Q}^n \rangle_{\beta,c} \sim V$ is true and the central limit theorem holds (here, n is a positive integer and V is the number of spins).

Since for many-body systems the Hamiltonian is an extensive observable, it is convenient to define the energy density of a thermal state as

$$\varepsilon(\beta) := \frac{1}{V} \langle H \rangle_{\beta}. \quad (7)$$

We expect $\varepsilon(\beta)$ to be invertible: technically, this is guaranteed whenever the entropy is a strictly concave function of the energy and the heat capacity is positive. As a consequence, one can equivalently parametrize the thermal state either with β or with the corresponding energy density $\varepsilon(\beta)$. In particular, the expectation value of any observable $\mathcal{O}$ at a given energy density ε

$$\mathcal{O}(\varepsilon) := \langle \mathcal{O} \rangle_{\beta(\varepsilon)}, \quad (8)$$

will play an important role in our discussion.

Lastly, for completeness, we recall that, as long as expectation values of local observables are implied, the microcanonical and canonical ensembles become equivalent in the thermodynamic limit; specifically, $\mathcal{O}(\varepsilon)$ gives the microcanonical expectation value at a given energy density. Similarly, such an object enters the diagonal matrix elements of the ETH for quantum chaotic systems.

D. Autocorrelation functions

A common way of characterizing the dynamical properties of an equilibrium state is to compute multipoint correlation functions of local observables. In this work, we focus on the autocorrelation function of thermal states, defined as

$$C(t) = \langle \mathcal{O}(t)\mathcal{O} \rangle_{\beta,c} := \langle \mathcal{O}(t)\mathcal{O} \rangle_{\beta} - \langle \mathcal{O}(t) \rangle_{\beta} \langle \mathcal{O} \rangle_{\beta}. \quad (9)$$

The late-time behavior of such a quantity encodes relevant features of the underlying energy hydrodynamics, and it is associated with hydrodynamic tails of the form $\langle \mathcal{O}(t)\mathcal{O} \rangle_{\beta,c} \sim t^{-\nu}$; this scaling is expected to occur at large time in the infinite volume limit. In generic one-dimensional situations, it is rather natural to postulate that, after a short-time nonuniversal dynamics, the coarse-grained energy density $h(x, t)$ spreads diffusively according to the heat equation

$$\partial_t h(x, t) \simeq D \partial_x^2 h(x, t), \quad (10)$$

with D the diffusion constant. Based on this result, it is not difficult to predict that the late-time behavior of the autocorrelation function $\langle h(x=0, t)h(x=0, t=0) \rangle_c$ should decay as $\sim t^{-1/2}$. Hydrodynamic transport properties are thus intimately related to the appearance of the algebraic late-time tail in autocorrelation functions.

Even when energy is the only conserved quantity, and it diffuses through the setup, different observables may be characterized by other exponents ν , which can be regarded as an emergent feature of the dynamics of the operator $\mathcal{O}$. In the framework of this article, energy $H = \sum_{\mathbf{r}} h(\mathbf{r})$ is the only conserved charge: the autocorrelation function of the energy density $h(\mathbf{r})$ is characterized by $\nu = d/z$, where z is the *dynamical critical exponent* [11] of the energy transport and d the dimensionality of the setup. Common values for local Hamiltonians are $z = 1$, corresponding to situations where energy spreads ballistically, and $z = 2$, where it spreads diffusively. The former is typically associated with integrable models, and hence will not be encountered here, while the latter occurs in generic (chaotic) local systems. Furthermore, as we will see, when considering long-range interactions, other values of z can appear. The goal of this article is, in fact, to characterize the value of ν for observables that are not coinciding with the energy density and to discuss how the same underlying hydrodynamic energy transport can give rise to multiple different late-time tails.

III. NUMERICAL ANALYSIS

We begin our discussion by presenting some numerical results for one- and two-dimensional classical spin systems. Our goal is to highlight a connection between the hydrodynamic tail of the autocorrelation function of an observable $\mathcal{O}$ and its energy dependence as expressed by Eq. (8).

A. The models

The first model that we consider is a one-dimensional Ising chain with a tilted field,

$$H = \sum_j h_x S_j^x + h_z S_j^z + J S_j^z S_{j+1}^z, \quad (11)$$

where S_j^a denotes the components $a = x, y, z$ of a classical spin at site j . The chosen parameters are $h_x = 1.1$, $h_z = 0.9$ and $J = 1$.

The second model is a one-dimensional Ising chain with transverse field and long-range couplings,

$$H = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{J}{N_\alpha r_{ij}^\alpha} S_i^z S_j^z + \sum_i h_x S_i^x, \quad (12a)$$

where

$$r_{ij} = \min(|j - i|, L - |j - i|), \quad N_\alpha = \left(\sum_{i=2}^L \frac{1}{r_{i1}^{2\alpha}} \right)^{-\frac{1}{2}}. \quad (12b)$$

The energy transport is diffusive for $\alpha \geq 1.5$ and becomes anomalous at $1 < \alpha < 1.5$ with a dynamical exponent $z = 2\alpha - 1$ [39]. Here, we choose $h_x = 1.1$, $J = 2\sqrt{2}$ and two different values of α : $\alpha = 1.1$ and $\alpha = 1.5$.

Finally, as a third model, we consider a two-dimensional Ising model on a $\ell \times \ell$ square lattice,

$$H = \sum_i h_x S_i^x + \sum_{\langle ij \rangle} J S_i^z S_j^z, \quad (13)$$

where i and j label lattice sites, and the sum over $\langle ij \rangle$ runs over bonds connecting neighboring sites, and we choose $h_x = 1.1$ and $J = 1$. In all setups, we consider periodic boundary conditions. In all considered simulations, one-dimensional and two-dimensional, the total number of sites is L , and this parameter plays the role of the volume V in the thermodynamical discussions of Sec. II C.

B. Numerical technique

In the calculation of thermal expectation value $\langle \mathcal{O} \rangle_\beta$, we employ a Hamiltonian Monte Carlo method with a Metropolis-Hastings algorithm [40,41]. The thermal average is taken over 10^8 different configurations for each given temperature T .

In the calculation of autocorrelation function $C(t)$, we employ a Yoshida fourth-order symplectic integrator [42] with a time step of $\tau = 0.02$. The initial spin configurations are sampled using the Monte Carlo method. The results are averaged over around 10^8 independent initial configurations.

C. Numerical results at infinite temperature

We start by examining the thermodynamic properties of four observables, S_j^x , $S_j^x S_{j+1}^x$, $S_j^x S_{j+1}^x S_{j+2}^x$, and $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ for the Ising model with tilted field in Eq. (11). In Fig. 1, we show the energy dependence of the four observables, $\mathcal{O}(\varepsilon)$, computed according to the definition in Eq. (8). We consider a system of size up to $L = 200$, which is sufficient to highlight the infinite-size behavior and make $1/L$ corrections negligible on the scales of the plot. The observables have been chosen because the functions $\mathcal{O}(\varepsilon)$ display markedly different

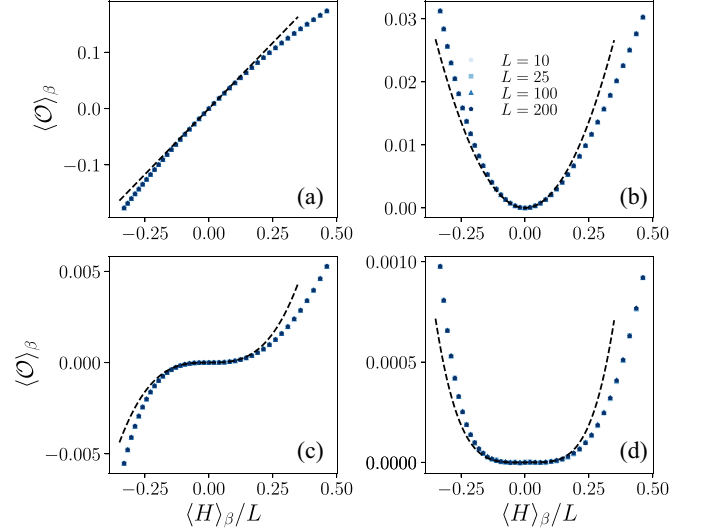


FIG. 1. $\langle \mathcal{O} \rangle_\beta$ versus $\langle H \rangle_\beta / L$ for the observables (a) S_j^x ; (b) $S_j^x S_{j+1}^x$; (c) $S_j^x S_{j+1}^x S_{j+2}^x$; and (d) $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ in the one-dimensional Ising model with tilted field. The dashed lines indicate the analytical prediction ε^m derived in Appendix B.

behaviours around $\varepsilon = 0$, and specifically, a small-energy behavior $\sim \varepsilon^m$ with $m = 1, 2, 3$, and 4 , respectively; the four scalings are proven analytically in Appendix B.

The behavior of $\mathcal{O}(\varepsilon)$ at energy density $\varepsilon = 0$ is particularly relevant because we will now numerically investigate the dynamical properties of the infinite-temperature state, which has energy density $\varepsilon = 0$ (this is true in all three models). Specifically, we compute the autocorrelation function $C(t)$ defined in Eq. (9) of the same four observables on the thermal state with $\beta = 0$. In the four panels of Fig. 2, we show the results of the calculations performed for several spin-chain

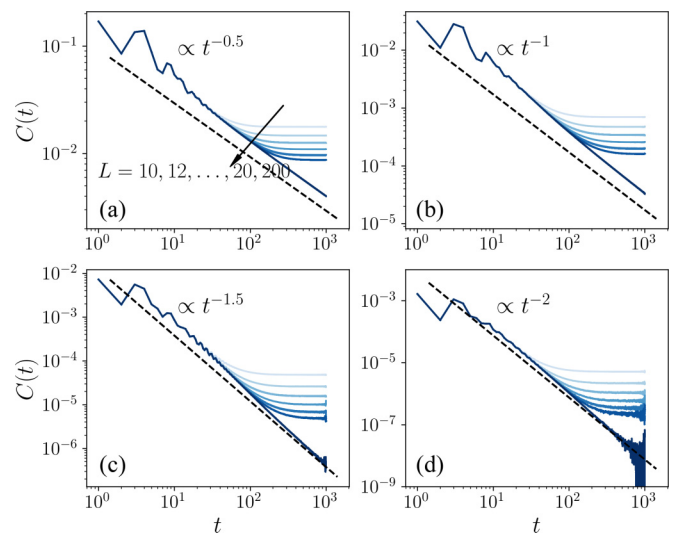


FIG. 2. $C(t)$ versus t for the infinite temperature $\beta = 0$ state for the four observables (a) S_j^x ; (b) $S_j^x S_{j+1}^x$; (c) $S_j^x S_{j+1}^x S_{j+2}^x$; and (d) $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ in the one-dimensional Ising model with tilted field. Dashed lines indicate $\propto t^{-dm/z}$ for $m = 1, 2, 3, 4$, respectively, with $z = 2$ and $d = 1$.

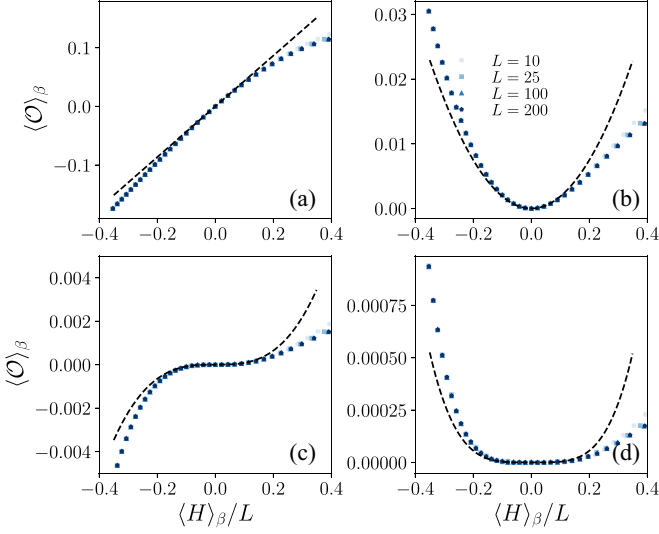


FIG. 3. $\langle \mathcal{O} \rangle_\beta$ versus $\langle H \rangle_\beta / L$ for observables (a) S_j^x ; (b) $S_j^x S_{j+1}^x$; (c) $S_j^x S_{j+1}^x S_{j+2}^x$; and (d) $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ in the one-dimensional long-range Ising model ($\alpha = 1.5$) with transverse field. The dashed lines indicate the analytical prediction ε^m derived in Appendix B.

lengths, from $L = 10$ to $L = 200$. In all cases, three time regimes can be identified. First, we observe an initial transient with oscillations, which extends up to timescales of order $\tau_1 \sim O(1)$. Afterward, we recognize the onset of a clear algebraic decay, interpreted as a hydrodynamic tail. Finally, the last time regime is characterized by a stationary behavior, and the autocorrelation function does not display a significant dependence on time. This latter regime is clearly a finite-size effect and indeed starts at a time τ_2 that scales with the chain length L , since the plateaus depart from the same universal power-law decay at times that increase the system size. As a consequence, the plateau values depend on L .

The exponent ν of the hydrodynamic tail depends on the observable. The values that we have identified are compatible with $1/2$, 1 , $3/2$, and 2 for the four observables, respectively, and are highlighted by a black dashed line in Fig. 2; the theory that we will present in Sec. IV will propose exactly these fractions. At this stage, it is possible to guess a link with the value m that characterizes the thermodynamic function $\mathcal{O}(\varepsilon)$, namely, $\nu = m/2$.

A similar analysis is presented in Figs. 3 and 4 for the one-dimensional long-range Ising model of Eq. (12) with long-range coupling set by the parameter $\alpha = 1.5$. By considering the same four observables, we show the behavior of $\mathcal{O}(\varepsilon) \sim \varepsilon^m$ with m varying from 1 to 4 as in the short-range model (see also Appendix B). The autocorrelation functions are more interesting and display two different algebraic decays (see also Ref. [39] for an in-depth study of the autocorrelation functions of this model). A first hydrodynamic tail $t^{-\nu^{(1)}}$ highlighted by a blue dotted line is compatible with the values $\nu^{(1)} = 0.75, 1.5, 2.25$, and 3 , and can be summarized by the formula $\nu^{(1)} = 3m/4$; this behavior is well highlighted by our numerics for setups with lengths up to $L = 20$, and it coincides exactly with the result that we co-presented in Ref. [24] for the corresponding quantum spin-1 setup. By exploiting the possibility offered by classical

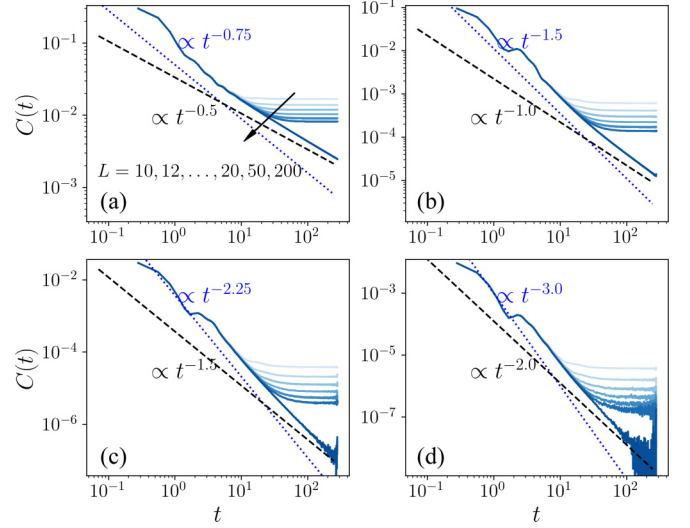


FIG. 4. $C(t)$ versus t for infinite temperature $\beta = 0$ for observables (a) S_j^x ; (b) $S_j^x S_{j+1}^x$; (c) $S_j^x S_{j+1}^x S_{j+2}^x$; and (d) $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ in one-dimensional long-range Ising model ($\alpha = 1.5$) with transverse field. The blue dotted lines indicate $\propto t^{-dm/z}$ for $m = 1, 2, 3, 4$, respectively with $z = 4/3$ (used in Ref. [24]) and $d = 1$. The black dashed lines indicate $\propto t^{-dm/z}$ with $z = 2$ and $d = 1$ for comparison (prediction of Ref. [39]).

systems of studying larger setups for longer times, we investigate a system of length $L = 200$ and discover that the power-law bends at late times toward a hydrodynamic tail $t^{-\nu^{(2)}}$ with $\nu^{(2)} = 0.5, 1, 1.5$, and 2 , highlighted in the figure by a black dashed line. This late-time result is predicted in Ref. [39].

By changing the range of the interactions setting $\alpha = 1.1$, the analysis presented in Figs. 5 and 6 shows that as long as the thermal expectation values of the four observables are considered, the behaviours are identical to those encoun-

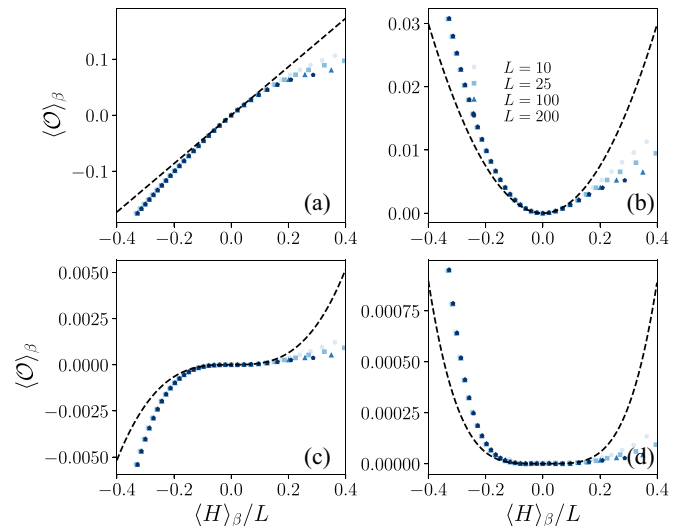


FIG. 5. $\langle \mathcal{O} \rangle_\beta$ versus $\langle H \rangle_\beta / L$ for observables (a) S_j^x ; (b) $S_j^x S_{j+1}^x$; (c) $S_j^x S_{j+1}^x S_{j+2}^x$; and (d) $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ in the one-dimensional long-range Ising model ($\alpha = 1.1$) with transverse field. The dashed lines indicate the analytical prediction ε^m derived in Appendix B.

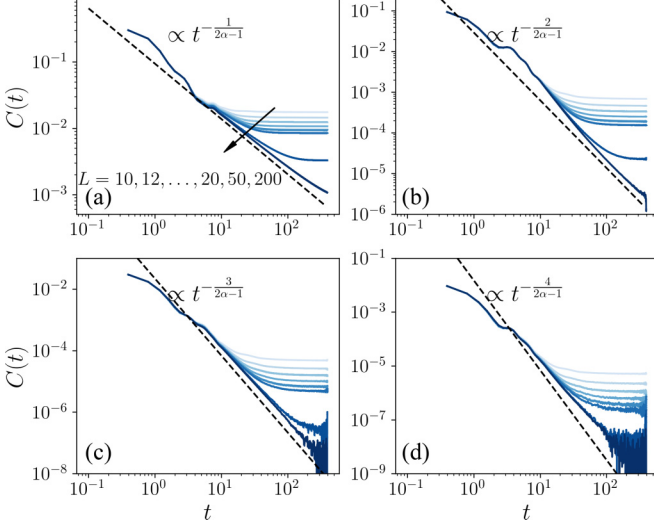


FIG. 6. $C(t)$ versus t for infinite temperature $\beta = 0$ for observables (a) S_j^x ; (b) $S_j^x S_{j+1}^x$; (c) $S_j^x S_{j+1}^x S_{j+2}^x$; and (d) $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ in one-dimensional long-range Ising model ($\alpha = 1.1$) with transverse field. The black dashed lines indicate $\propto t^{-dm/z}$ with $z = (2\alpha - 1)$ and $d = 1$ (prediction of Ref. [39]).

tered so far. The autocorrelation functions instead display one algebraic hydrodynamic tail that is compatible with the value $\nu = m/(2\alpha - 1)$ highlighted by the black dashed line.

Finally, in Figs. 7 and 8, we present the same analysis for the two-dimensional Ising model in Eq. (13). Whereas the thermodynamic properties of the model, represented by the function $\mathcal{O}(\varepsilon)$, are essentially coinciding with those of the two previous models, the autocorrelation functions display a marked hydrodynamic tail, $t^{-\nu}$. For this model, the values that we observe are compatible with $\nu = 1, 2, 3$, and 4 and can be summarized by the relation $\nu = m$.

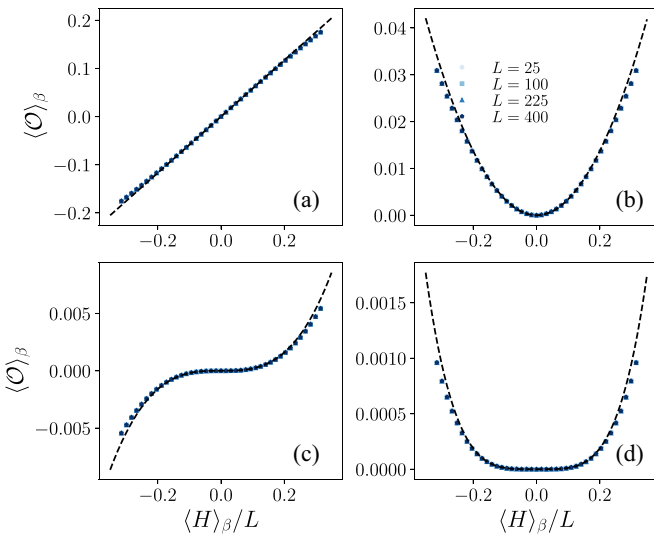


FIG. 7. $\langle \mathcal{O} \rangle_\beta$ versus $\langle H \rangle_\beta / L$ for observables (a) $S_{i,j}^x$; (b) $S_{i,j}^x S_{i+1,j}^x$; (c) $S_{i,j}^x S_{i+1,j}^x S_{i,j+1}^x$; and (d) $S_{i,j}^x S_{i+1,j}^x S_{i,j+1}^x S_{i+1,j+1}^x$ in the two-dimensional Ising model. The dashed lines indicate the analytical prediction ε^m derived in Appendix B.

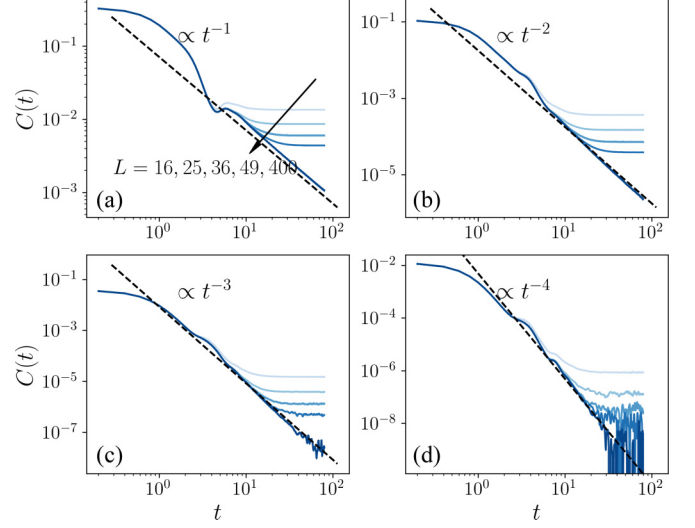


FIG. 8. $C(t)$ versus t for infinite temperature $\beta = 0$ for observables (a) $S_{i,j}^x$; (b) $S_{i,j}^x S_{i+1,j}^x$; (c) $S_{i,j}^x S_{i+1,j}^x S_{i,j+1}^x$; and (d) $S_{i,j}^x S_{i+1,j}^x S_{i,j+1}^x S_{i+1,j+1}^x$ in the two-dimensional transverse Ising model ($\ell \times \ell$ square lattice). Here, $L = \ell^2$, which indicates the total number of sites. Dashed lines indicate $\propto t^{-dm/z}$ for $m = 1, 2, 3, 4$, respectively, with $z = 2$ and $d = 2$.

D. Additional numerics at finite temperature

In addition to the infinite temperature case studied so far, we now consider finite and positive temperatures, $\beta > 0$. Different from the infinite-temperature state, one has $\langle S^x \rangle_\beta \neq 0$, $\langle S^y \rangle_\beta \neq 0$, and $\langle S^z \rangle_\beta = 0$. Similarly, one can obtain that $\langle (S^x)^{m_x} (S^y)^{m_y} (S^z)^{m_z} \rangle_\beta$ vanishes only when m_y is odd. As a result, following discussions similar to those in Appendix B, for all local observables $\mathcal{O} = S_1^x, S_1^x S_2^x, S_1^x S_2^x S_3^x, S_1^x S_2^x S_3^x S_4^x$ we considered $\langle H \mathcal{O} \rangle_{\beta > 0} \neq 0$. This implies that $\mathcal{O}(\varepsilon - \varepsilon(\beta)) \sim \varepsilon - \varepsilon(\beta)$ for $\varepsilon \rightarrow \varepsilon(\beta)$, and thus the four considered observables have the same $m = 1$; this is evident to the naked eye by looking at the curves in Figs. 1, 3, 5, and 7 at energy density different from zero.

We study autocorrelation functions of $\mathcal{O} = S_1^x, S_1^x S_2^x, S_1^x S_2^x S_3^x, S_1^x S_2^x S_3^x S_4^x$ at inverse temperature $\beta = 0.5$. Results are shown for the one-dimensional tilted field Ising chain (11) in Fig. 9 and for the two-dimensional Ising model (13) in Fig. 10. In both figures, it can be observed that $C(t)$ tends to converge toward the analytical prediction $C(t) \sim t^{-md/z}$ for $m = 1$, as the system size is increased. The result is particularly clear for the first two observables, in panels (a) and (b), while the presence of finite-size (or uncontrolled) effects is more noticeable for panels (c) and (d).

E. Summary

To conclude, two interesting quantitative properties of the autocorrelation functions emerge from this numerical analysis and must be stressed. First, the exponents ν that we have fitted are all compatible with the following formula:

$$\nu = \frac{dm}{z}, \quad (14)$$

where d is the dimensionality, m reflects the thermodynamic behavior $\mathcal{O}(\varepsilon) \sim \varepsilon^m$, and z will be identified in Sec. IV with

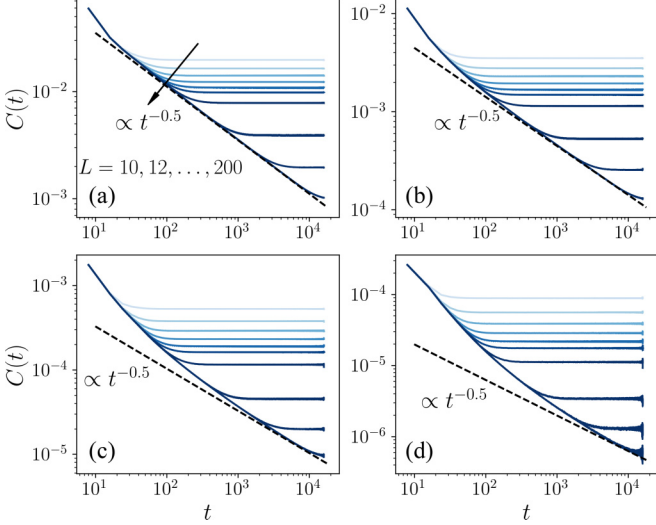


FIG. 9. $C(t)$ versus t for the finite temperature $\beta = 0.5$ state for the four observables (a) S_j^x ; (b) $S_j^x S_{j+1}^x$; (c) $S_j^x S_{j+1}^x S_{j+2}^x$; and (d) $S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x$ in the one-dimensional Ising model with tilted field, for system sizes $L = 10, 12, 14, 16, 18, 20, 25, 50, 100, 200$. Dashed lines indicate $\propto t^{-d/z}$ with $z = 2$ and $d = 1$.

the dynamical critical exponent. The results associated with the two short-ranged models are compatible, at late times, with $z = 2$, as the system is diffusive. The long-range one-dimensional Ising model shows an anomalous superdiffusive behavior with $z = 4/3$ when $\alpha = 1.5$, followed by a late-time diffusive behavior $z = 2$. When setting $\alpha = 1.1$, we instead get an anomalous value $z = 2\alpha - 1$ without late-time diffusion.

The second quantitative property is shown in Fig. 11 and shows that the infinite-time limit of the autocorrelation func-

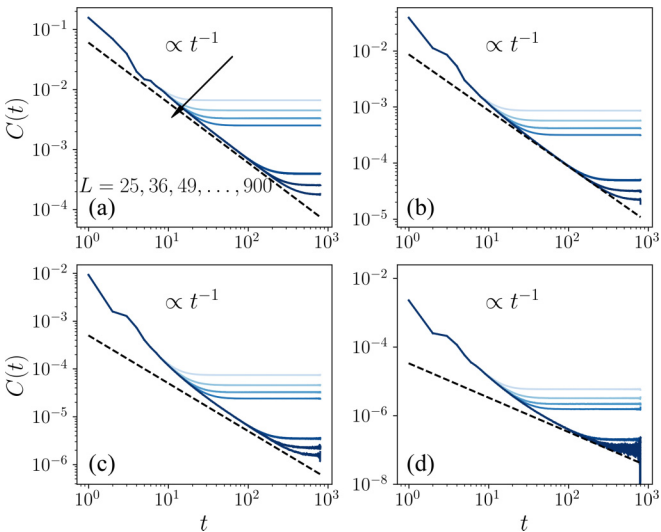


FIG. 10. $C(t)$ versus t for finite temperature $\beta = 0.5$ for observables (a) $S_{i,j}^x$; (b) $S_{i,j}^x S_{i+1,j}^x$; (c) $S_{i,j}^x S_{i+1,j}^x S_{i+1,j+1}^x$; and (d) $S_{i,j}^x S_{i+1,j}^x S_{i+1,j+1}^x S_{i+1,j+1}^x$ in the two-dimensional transverse Ising model ($\ell \times \ell$ square lattice), for $L = \ell^2 = 25, 36, 49, 64, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900$. Dashed lines indicate $\propto t^{-d/z}$ with $z = 2$ and $d = 2$.

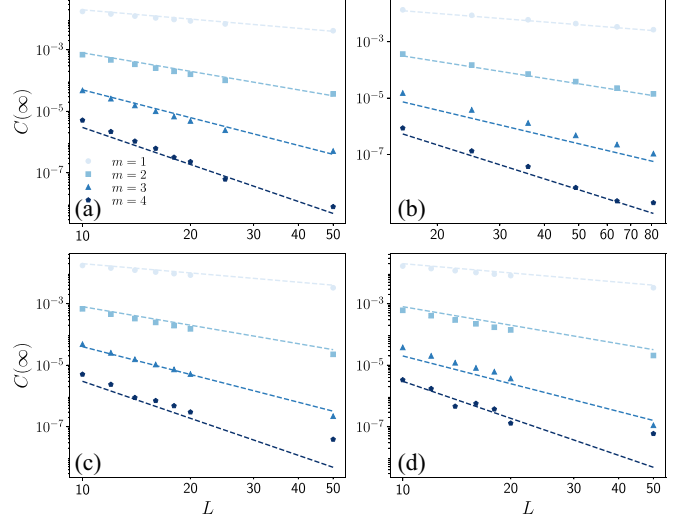


FIG. 11. Long-time average $C(\infty)$ versus L for (a) one-dimensional tilted field Ising model; (b) two-dimensional transverse Ising model; (c) one-dimensional long-range transverse Ising model $\alpha = 1.5$; and (d) one-dimensional long-range transverse Ising model $\alpha = 1.1$. The dashed lines indicate the analytical prediction $C(\infty) \propto L^{-m}$.

tion has a scaling of

$$\lim_{t \rightarrow \infty} \langle \mathcal{O}(t) \mathcal{O} \rangle_c \sim \frac{1}{L^m}, \quad (15)$$

and it is thus also determined by the thermodynamics of the model at zero energy density. The four different panels refer to the three models that we studied and highlight a universal scaling. The existence of a residual value in the late-time limit of autocorrelation functions that are expected to decay to zero in the thermodynamic limit has already been discussed in the context of chaotic quantum systems. The literature has highlighted the crucial difference between the algebraic decay to zero that we find also here in this classical framework and the exponential one, linking the two behaviors to the existence or absence of conservation laws [26,27,43].

The two relations in Eqs. (14) and (15) are a remarkable regularity connecting thermodynamic properties and autocorrelation functions, and the goal of the next section is to propose a theory that could explain them.

IV. THEORETICAL CONSIDERATIONS

In this section, we propose a theoretical interpretation of the numerical results presented in Sec. III, and summarized by Eqs. (14) and (15). The result that we will obtain will lead to a clear prediction for the hydrodynamic behavior of autocorrelation functions at finite temperature, that we will test numerically.

A. Ergodicity

We begin with some remarks on ergodicity for classical systems. We have seen that in closed classical systems the Hamiltonian H is always conserved and the thermal states are always stationary: we assume that no additional conserved quantities, independent from H , are present; to the best of our

knowledge, this is the case for the three models studied in Sec. III.

In order to formulate the notion of ergodicity, let us consider the observable 1_E that has support on the region of $\mathcal{M}$ with energy E , and that is defined by

$$\langle 1_E \rangle_p = \delta(E - H(p)), \quad p \in \mathcal{M}. \quad (16)$$

We say that the system is *ergodic* if any observable $\mathcal{O}$ spreads uniformly in time across the region of the phase space associated with the initial energy E . Mathematically, this means that the following relation holds at the observable level [44]:

$$\int_0^T \frac{dt}{T} \mathcal{O}(t) \xrightarrow{T \rightarrow \infty} \int dE \frac{\langle 1_E \mathcal{O} \rangle_{\text{Haar}}}{\langle 1_E \rangle_{\text{Haar}}} \times 1_E. \quad (17)$$

In particular, looking at Eq. (17), one identifies the microcanonical expectation value at energy E ,

$$\mathcal{O}(E) := \frac{\langle 1_E \mathcal{O} \rangle_{\text{Haar}}}{\langle 1_E \rangle_{\text{Haar}}}, \quad (18)$$

as the weight for the microcanonical distribution 1_E .

Equation (17) is crucial in our context, as it plays the role of the (diagonal part of the) ETH ansatz for quantum systems: physically, it means that the only parameter to describe stationary states in (closed) systems is the energy. Whereas in Ref. [24], we could prove the relaxation-overlap inequality in the quantum context assuming ETH, here we can prove it in the classical realm assuming ergodicity. The analogy between the two notions, discussed in depth in Ref. [33], but also in Refs. [34–36], is further analyzed in Sec. V.

An immediate consequence of ergodicity is the erasing of information after time evolution from a pure initial state,

$$\int_0^T \frac{dt}{T} \langle \mathcal{O}(t) \rangle_p \xrightarrow{T \rightarrow \infty} \mathcal{O}(E = H(p)). \quad (19)$$

In summary, if one averages over large time windows, ergodicity ensures that the value of an observable always converges to that associated to the energy fixed by the initial conditions. The expression in Eq. (19) can be regarded as a statement on the late-time dynamics of the observables, that converges to the microcanonical expectation value.

The writing employed in this section, where integrals over the phase space $\mathcal{M}$ are traded with integrals over energies, opens the windows to the introduction of a Boltzmann microcanonical entropy, defined as

$$e^{S(E)} = \langle 1_E \rangle_{\beta=0}, \quad (20)$$

where the expectation value is taken over the flat measure defined in Sec. II. Using this notation, algebra allows us to express the late-time limit in Eq. (19) on a thermal state as

$$\int_0^T \frac{dt}{T} \langle \mathcal{O}(t) \rangle_{\beta} \xrightarrow{T \rightarrow \infty} \frac{\int dE \mathcal{O}(E) e^{-\beta E + S(E)}}{\int dE e^{-\beta E + S(E)}}. \quad (21)$$

This relation demonstrates the convergence of any observable toward the stationary thermal expectation value.

The definitions and concepts above are general and do not require specific assumptions on the interactions of the

underlying system. When we consider a many-body system, say, locally interacting spins placed on a lattice of size V , the leading dependence on V enters through the energy density $\varepsilon = E/V$ for local observables. Specifically, the microcanonical average $\mathcal{O}(\varepsilon = E/V)$ has a definite value in the limit $V \rightarrow \infty$; it coincides with that of a thermal ensemble at inverse temperature β satisfying

$$\beta = \frac{dS}{dE}, \quad (22)$$

expressing the local indistinguishability between canonical and microcanonical ensembles for infinitely large many-body systems. We note that finite-size effects, arising as algebraically small discrepancies between the two aforementioned ensembles, are present and are part of the discussion of the next two subsections.

B. Finite-size scaling in Eq. (15)

In this and in the next subsections, we proceed with the proof of the relations in Eqs. (14) and (15). Remarkably, assuming ergodicity, it is possible to proceed with the proofs in the same way as we did in Ref. [24], where we assumed ETH and worked in the quantum framework. Hence, as anticipated, we will assume that our classical spin system is ergodic and that the energy is the only conserved quantity of the model.

We now focus on large, but finite, systems and study the autocorrelation function at finite temperature $\langle \mathcal{O}(t) \mathcal{O} \rangle_{\beta,c}$. We take the late-time average, and, assuming ergodicity, we compute

$$\int_0^T \frac{dt}{T} \langle \mathcal{O}(t) \mathcal{O} \rangle_{\beta,c} \xrightarrow{T \rightarrow \infty} \langle \bar{\mathcal{O}}^2 \rangle_{\beta,c}, \quad (23)$$

where $\bar{\mathcal{O}}$ is defined by

$$\bar{\mathcal{O}} := \lim_{T \rightarrow \infty} \int_0^T \frac{dt}{T} \mathcal{O}(t). \quad (24)$$

In order to obtain Eq. (23), we have only used the stationarity of the thermal state, together with the definition and the property $\mathcal{O}_1(t) \mathcal{O}_2(t) = (\mathcal{O}_1 \mathcal{O}_2)(t)$, which is a direct consequence of Eq. (3). Specifically, this allows writing the argument of the integral in the left-hand side of Eq. (23) as $\langle \mathcal{O}(t + t') \mathcal{O}(t') \rangle_{\beta,c}$ for an arbitrary value of t' , but also as an average over t and t' :

$$\int_0^T \frac{dt}{T} \langle \mathcal{O}(t) \mathcal{O} \rangle_{\beta,c} = \int_0^T \int_0^T \frac{dt}{T} \frac{dt'}{T} \langle \mathcal{O}(t + t') \mathcal{O}(t') \rangle_{\beta,c}. \quad (25)$$

Equation (23) follows after a simple change of variables.

Thanks to Eq. (23), we now need to estimate the variance of the observable $\bar{\mathcal{O}}$, which is, in general, nonvanishing since the energy fluctuates in the thermal state. We first assume ergodicity and, from Eqs. (17) and (21), we write explicitly

$$\begin{aligned} \langle \bar{\mathcal{O}}^2 \rangle_{\beta,c} &= \frac{\int dE e^{-\beta E + S(E)} \mathcal{O}(\varepsilon)^2}{\int dE e^{-\beta E + S(E)}} + \\ &\quad - \left(\frac{\int dE e^{-\beta E + S(E)} \mathcal{O}(\varepsilon)}{\int dE e^{-\beta E + S(E)}} \right)^2. \end{aligned} \quad (26)$$

In the strict thermodynamic limit $V \rightarrow \infty$, the integral localizes at the saddle $\varepsilon = \varepsilon(\beta)$, satisfying Eq. (22), and the

variance vanishes; to study the leading corrections around the saddle, we expand

$$\mathcal{O}(\varepsilon) \simeq \mathcal{O}(\varepsilon(\beta)) + \frac{1}{m!} \partial_\varepsilon^m \mathcal{O}(\varepsilon(\beta)) (\varepsilon - \varepsilon(\beta))^m, \quad (27)$$

where we have retained the first nonvanishing term in the Taylor expansion of $\mathcal{O}(\varepsilon)$ around $\varepsilon(\beta)$ (which is of order m).

Thanks to the central limit theorem for H , we estimate the variance of $(\varepsilon - \varepsilon(\beta))^m$ as

$$\langle (H/V - \varepsilon(\beta))^m \rangle_\beta - \langle (H/V - \varepsilon(\beta))^m \rangle_\beta^2 \sim \frac{1}{V^m}, \quad (28)$$

using the argument that we employed in Ref. [24]. Putting everything together, we find that

$$\lim_{T \rightarrow \infty} \int_0^T \frac{dt}{T} \langle \mathcal{O}(t) \mathcal{O} \rangle_{\beta,c} \sim \frac{1}{V^m}. \quad (29)$$

This concludes the first proof, determining the link between the thermodynamic behavior in Eq. (27) and the scaling of the late-time plateau of the autocorrelation function with the system size in Eq. (15).

In summary, the plateau of the autocorrelator at a given β depends on the profile of $\mathcal{O}(\varepsilon)$ around $\varepsilon(\beta)$ and, in particular, on the leading term of the Taylor expansion. For practical purposes, the index m can be extracted in a simpler way from the canonical expectation values as $\langle \mathcal{O} \rangle_{\beta'} - \langle \mathcal{O} \rangle_\beta \sim (\beta' - \beta)^m$. This is particularly useful when $\beta = 0$, as explained in Ref. [24], since one identifies m as the first integer such that $\langle H^m \mathcal{O} \rangle_{\beta=0} \neq 0$ (assuming $\langle \mathcal{O} \rangle_{\beta=0} = 0$) and such a quantity can be efficiently computed; we will discuss this in the context of the Ising model in Appendix A.

C. The relaxation-overlap inequality in Eq. (14) and hydrodynamics

In this section, we relate the finite-size plateau of the autocorrelator to the hydrodynamic tail. To do so, we report a physical argument that was first proposed in Ref. [24] for quantum many-body systems, since it applies straightforwardly for their corresponding classical counterpart.

First, the numerical studies of the autocorrelator $\langle \mathcal{O}(t) \mathcal{O} \rangle_c$ for local observables of many-body systems at finite-size V presented in Sec. III show that for sufficiently small t the thermodynamic limit is essentially achieved; it is the large- t behavior that suffers from finite-size effects. To understand the timescales where such effects become important, it is worth discussing the underlying hydrodynamics of the Hamiltonian, that is, in this context, the only conserved charge. For instance, given $h(\mathbf{x})$ the Hamiltonian density, this is expected to give rise [11] to the following scaling law:

$$\langle h(\mathbf{x}, t) h(\mathbf{0}, 0) \rangle_{\beta,c} \simeq \frac{1}{t^{d/z}} F\left(\frac{|\mathbf{x}|}{t^{1/z}}\right); \quad (30)$$

here, F is a universal function which depends on the transport properties, and z is the dynamical critical exponent. In particular, the exponents appearing in Eq. (30) are matched so that once the expression in (30) is summed over $\mathbf{x}$, one gets a constant independent of time: this is a very important property that follows directly from energy conservation. For example, for one-dimensional short-range systems where the

energy typically diffuses

$$\partial_t h(\mathbf{x}, t) \simeq D \partial_x^2 h(\mathbf{x}, t), \quad (31)$$

the dynamical exponent is $z = 2$ and F is a Gaussian. In general, from the hydrodynamics, one identifies a spatial region whose typical length scales as $\sim t^{1/z}$ where the energy density is spread at time t . As a consequence, before a timescale $t \lesssim V^{z/d}$, this region is smaller than the linear length of the setup, and finite-size effects are expected to be irrelevant.

While the discussion above is restricted to charge densities, the same idea can be extended to a generic local observable $\mathcal{O}$. The key mechanism, which goes under the name of *hydrodynamic projection*, is to project $\mathcal{O}$ over the charge densities (together with the powers and derivatives thereof, see Ref. [22,24] for details): this projection can be characterized systematically in terms of the correlation functions of $h(\mathbf{x}, t)$, while the rest, that is related to nonconserved fast decaying modes, is expected to give rise to subleading exponentially decaying contributions. Thus, the region where $\mathcal{O}$ spreads, relevant for the hydrodynamic tails of its autocorrelator, is approximately the same as the energy density. This implies that, for $t \lesssim V^{z/d}$ the autocorrelator decays as in the infinite volume limit, showing an asymptotic algebraic tail $t^{-\nu}$; at larger times it eventually saturates the plateau value, that has been estimated with $\sim V^{-m}$ in the previous sections.

Crucially, if one further assumes that the autocorrelator decays monotonically in time, which is a physically motivated hypothesis for the data observed in Sec. III, provided one averages out the little transient oscillations, one can compare different times to get useful information. For instance, choosing a time of order $\sim V^{z/d}$ where the crossover occurs and the finite-size effects appear, one expects from monotonicity that $V^{-z\nu/d} \gtrsim V^{-m}$. The latter inequality implies $\nu \leq md/z$: this is precisely the relaxation-overlap inequality in Eq. (14) that we presented in [24] for the quantum realm (see also Refs. [25–29] for related results).

Finally, it is worth commenting on the possible presence of additional conserved charges (such as magnetization). In that case, obtaining an accurate estimate of the finite-size plateau value would likely require knowledge of the equilibrium expectation values over a broader class of Gibbs states—e.g., grand-canonical ensembles—and not only the thermal ones. Moreover, although a connection between the algebraic decays and the hydrodynamic projection appears plausible in principle, the situation becomes conceptually more subtle when different charges exhibit different critical dynamical exponents. We plan to explore this scenario in future work.

V. DISCUSSION: ON THE RELATION BETWEEN ERGODICITY AND ETH

We now elaborate on the relation between ergodicity and the diagonal part of ETH (see also the discussion in Ref. [33], but also Refs. [34–36]). The point we want to stress here is that in generalizing our result from quantum to classical systems, it has been necessary to employ the standard notion of *ergodicity* whenever in the quantum setting we were employing the notion of ETH. The analogy between these two concepts, which has been highlighted in previous works, is found in this article in a different context, namely as a tool for extending

to the classical realm a result that was originally conceived in the quantum realm. This thus suggests that ETH is not really necessary in order to prove the relaxation-overlap inequality, as was done in Ref. [24], but only the notions of ergodicity and of smoothness of the observables as a function of energy. To prove that, in the following paragraphs, we show that a quantum system obeying ETH is ergodic, while smoothness is already assumed by ETH.

Let us begin by recalling the ETH: consider a quantum system with Hamiltonian $\hat{H}$ and an energy eigenstate $|E_j\rangle$, then the expectation value of a local observable $\hat{O}$ is given by a smooth function that only depends on the energy density $\varepsilon = E_j/V$, namely,

$$\langle E_j | \hat{O} | E_j \rangle = \mathcal{O}(\varepsilon). \quad (32)$$

In quantum many-body systems with a classical correspondence, it is reasonable to conjecture that the function $\mathcal{O}(\varepsilon)$ coincides with the classical one, given in Eq. (8); in the context of the spin systems considered here, it is reasonable to expect this to hold when comparing the classical model with a quantum model in the large- S limit.

Let us now briefly show that assuming ETH, one can prove that a quantum system is ergodic: as expressed by Eq. (17), the time average is equivalent to an ensemble average. Let us now consider the quantum version of the expression given in Eq. (17),

$$\int_0^T \frac{dt}{T} \hat{O}(t) = \sum_{j,k} \int_0^T e^{-i(E_j - E_k)t} \frac{dt}{T} \langle E_j | \hat{O} | E_k \rangle |E_j\rangle \langle E_k|. \quad (33)$$

In the limit $T \rightarrow \infty$, all the phases average out, and the expression becomes

$$\lim_{T \rightarrow \infty} \int_0^T \frac{dt}{T} \hat{O}(t) = \sum_j \langle E_j | \hat{O} | E_j \rangle |E_j\rangle \langle E_j|. \quad (34)$$

The analogy with the classical result in Eq. (8) is apparent, provided one identifies

$$\sum_{j, E \in [E_0, E_0 + \Delta E]} |E_j\rangle \langle E_j| \leftrightarrow \int_{E_0}^{E_0 + \Delta E} 1_E dE. \quad (35)$$

Specifically, this holds whenever the sum (integral) is restricted to a given (not too small) energy shell, sufficiently large to host a large number of eigenstates, making the classical/quantum correspondence meaningful. Using this fact, replacing the sum in (34) with partial sums over energy windows where $\langle E_j | \hat{O} | E_j \rangle$ is approximately constant, one gets the desired result, namely, an expression that is fully analogous to Eq. (17),

$$\int_0^T \frac{dt}{T} \hat{O}(t) \xrightarrow{T \rightarrow \infty} \sum_j \mathcal{O}(E/L) |E_j\rangle \langle E_j|. \quad (36)$$

VI. CONCLUSIONS AND PERSPECTIVES

In this work, we studied the classical counterpart of the theory developed in Ref. [24], where we linked the eigenstate

thermalization hypothesis with hydrodynamics. The key point of this article is to replace the ETH of quantum systems with the notion of classical ergodicity. Such a correspondence allows us to infer a relaxation-overlap inequality for classical many-body systems. The theoretical predictions describe in a very satisfactory way the numerical results for large systems, corroborating with high precision the proposed scenario, arising from the hydrodynamics of the energy, common to both quantum and classical systems.

Nonetheless, some questions regarding the link between ETH and ergodic many-body classical systems remain less clear. For example, it is not obvious what the classical counterpart of the off diagonal matrix elements with their random-matrix structure is. Such a direction would be particularly interesting for extending this work to integrable systems, where the off diagonal matrix elements are still under investigation [45–47].

Another intriguing direction would be to understand the fate of our theory when ergodicity is weakly broken and additional many-body stationary states, besides the Gibbs ensembles, emerge. This scenario is the classical counterpart of quantum many-body scars, which are accompanied by violations of ETH. For instance, recent observations of interesting large-scale phenomena in quantum spin chains that go beyond conventional hydrodynamics [48,49] present a challenge to incorporate into this work's framework.

Finally, we note that while this theory was developed to describe many-body systems at sufficiently high temperatures, distinct and relevant physics may emerge at lower temperatures. This is particularly pertinent for short-range spin chains in higher dimensions, where spontaneous symmetry breaking might occur, leading to the presence of dynamically disconnected phases. In such cases, one must reevaluate the definition of ergodicity and consider the potential for long-range correlations. We hope to revisit this problem in future work.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [50].

APPENDIX A: EXPECTATION VALUES IN THE INFINITE TEMPERATURE STATE

In this Appendix, we discuss the characterization of the expectation values of local observables in the infinite temperature state for classical spin systems. This is particularly relevant for the results of this work to compute the thermodynamical properties of local observables at small energy density ε . Specifically, we aim to compute the expectation values of monomials generated by the spin variables $\{(S_j^a)\}$. Since distinct sites are uncorrelated in the infinite temperature state, we focus on a given site and, from now on, we omit the site index j .

Before giving a quantitative characterization, it is worth providing a few symmetry considerations that ensure the vanishing of some quantities. For example, $\langle S^z \rangle = 0$: this comes from the invariance of the Haar measure under the reflection $z \rightarrow -z$ and that $S^z \rightarrow -S^z$ under this transformation. Similarly, using reflection properties across the three axes, one can ensure that $\langle (S^x)^{m_x} (S^y)^{m_y} (S^z)^{m_z} \rangle$ vanishes whenever at least one of the three integers m_a ($a = x, y, z$) is odd.

As a consequence, the only nonvanishing monomials can only be those generated by $(S^a)^2$. These are explicitly non-vanishing: the reason is that $(S^a)^2$ is a positive observable, meaning that, as a function of the unit sphere, it is positive semidefinite and it only vanishes on a measure-zero subset; similarly, the product of positive observables is positive, as well as their expectation values.

Finally, to give systematic predictions for the aforementioned expectation values, we can employ rotational invariance of the Haar measure. For example, using $(S^x)^2 + (S^y)^2 + (S^z)^2 = 1$ and $\langle (S^x)^2 \rangle = \langle (S^y)^2 \rangle = \langle (S^z)^2 \rangle$ (a consequence of the rotational symmetry), one easily gets $\langle (S^a)^2 \rangle = 1/3$. In general, it is convenient to define the generating function $\langle e^{\mu \cdot S} \rangle$, which gives the expectation values directly through its partial derivatives,

$$\langle (S^x)^{m_x} (S^y)^{m_y} (S^z)^{m_z} \rangle = \partial_{\mu_x}^{m_x} \partial_{\mu_y}^{m_y} \partial_{\mu_z}^{m_z} \langle e^{\mu \cdot S} \rangle|_{\mu=0}. \quad (\text{A1})$$

Using rotational invariance, we obtain $\langle e^{\mu \cdot S} \rangle = \langle e^{|\mu| S^z} \rangle$ and, from the expansion of the Haar measure in polar coordinates, we compute

$$\langle e^{\mu \cdot S} \rangle = \frac{\int_0^\pi d\theta \sin \theta e^{|\mu| \cos \theta}}{\int_0^\pi d\theta \sin \theta} = \frac{\sinh |\mu|}{|\mu|}. \quad (\text{A2})$$

From that, the expectation values of monomials can be computed efficiently and, for example, we have

$$\begin{aligned} \langle (S^x)^4 \rangle &= 1/5, & \langle (S^x)^6 \rangle &= 1/7, \\ \langle (S^x)^2 (S^y)^2 \rangle &= 1/15, & \langle (S^x)^2 (S^y)^4 \rangle &= 1/35. \end{aligned} \quad (\text{A3})$$

APPENDIX B: OVERLAPS FOR THE MODELS CONSIDERED IN THE ARTICLE

In this Appendix, we compute the overlaps between the local observables $\mathcal{O} = S_1^x, S_1^x S_2^x, S_1^x S_2^x S_3^x, S_1^x S_2^x S_3^x S_4^x$ and the Ising Hamiltonian in Eq. (11) in the infinite temperature state $\beta = 0$: specifically, we are interested in $\langle H^{m'} \mathcal{O} \rangle_{\beta=0}$ with a focus on the first values of m' for which it vanishes. To do that, we expand $H^{m'} \mathcal{O}$ as a polynomial in $\{S_j^a\}$ and use the results of Appendix A for the infinite temperature state.

First, we consider $\mathcal{O} = S_1^x$, whose expectation value vanishes $\langle \mathcal{O} \rangle_{\beta=0} = 0$. In the presence of a nonzero transverse field $h_x \neq 0$, a term $h_x (S_1^x)^2$ is generated in the expansion of $H\mathcal{O}$, which is the only one for which the expectation value does not vanish: as a consequence $\langle H\mathcal{O} \rangle_{\beta=0} \neq 0$, and the overlap order of S_1^x is $m = 1$.

More generally, for the observable $\mathcal{O} = S_1^x \dots S_m^x$, we can get a nonzero expectation value of $H^{m'} \mathcal{O}$ whenever $H^{m'}$ contains terms that are odd (under change of sign) in $S_1^x, \dots, S_m^x$. This is possible when $m' = m$, since H^m generates explicitly $h_x^m S_1^x \dots S_m^x$ in its polynomial expansion: on the other hand, one can easily check that it is not possible for $m' < m$. Therefore, $\langle H^m \mathcal{O} \rangle_{\beta=0} \neq 0$, and the overlap order of $\mathcal{O}$ is precisely m .

This calculation can be easily extended to the other models considered in the article, given in Eq. (12) and Eq. (13), obtaining the same result.

APPENDIX C: THE CASE OF A RELAXATION-OVERLAP INEQUALITY THAT IS NOT SATURATED

In this Appendix, we discuss cases where the relaxation-overlap inequality is not saturated.

A primary example is given by operators that are time derivatives of local observables,

$$\mathcal{O}' \equiv \frac{d}{dt} \mathcal{O} = \{\mathcal{O}, H\}. \quad (\text{C1})$$

A straightforward derivation yields

$$\langle \mathcal{O}'(t) \mathcal{O}' \rangle = \frac{d^2}{dt^2} \langle \mathcal{O}(t) \mathcal{O} \rangle \propto t^{-(v+2)}. \quad (\text{C2})$$

On the other hand, because $\mathcal{O}'$ is the time derivative of a local observable, its projections onto all powers of the Hamiltonian H vanish exactly. As a consequence, the overlap order m is effectively infinite, and the relaxation-overlap inequality is never saturated.

As an example of $\mathcal{O}'$, let us consider S_j^y in the tilted field Ising model (11), where one has

$$S_j^y \propto \{S_j^z, H\} = \frac{d}{dt} S_j^z(t), \quad (\text{C3})$$

and therefore,

$$\langle S_j^y(t) S_j^y \rangle \propto \frac{d^2}{dt^2} \langle S_j^z(t) S_j^z \rangle \propto t^{-5/2}. \quad (\text{C4})$$

In the derivation of Eq. (C4), we make use of the fact that S_j^z has overlap order $m = 1$ and

$$\langle S_j^z(t) S_j^z \rangle \sim t^{-1/2}. \quad (\text{C5})$$

In Fig. 12(a), a hydrodynamic tail $C(t) \sim t^{-1/2}$ is observed for the observable $\mathcal{O} = S_j^z$, in agreement with (C5). Results for autocorrelation functions of S_j^y are shown in Fig. 12(b). Strong oscillations around zero are observed, which obscure the scaling behavior $C(t) \sim t^{-5/2}$ expected from Eq. (C4). To better reveal this scaling, we show in panel (c) the Fourier

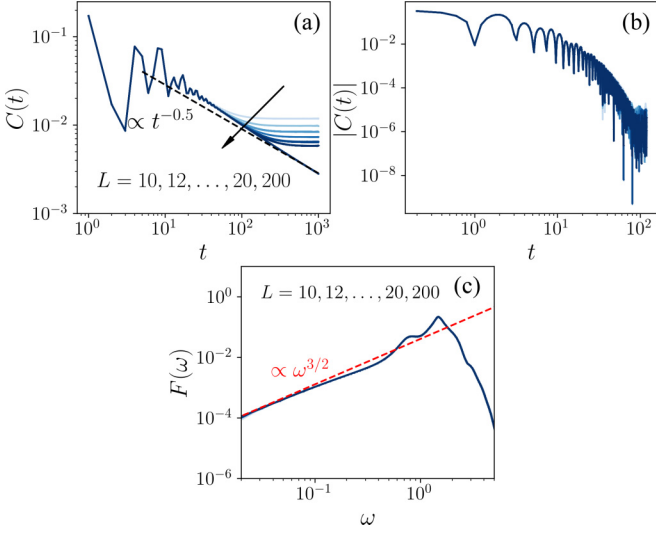


FIG. 12. $C(t)$ versus t for infinite temperature $\beta = 0$ for observables (a) S_j^x and (b) S_j^y . (c) Fourier transform of $C(t)$ for S_j^y . Dashed line in (a) indicates $\propto t^{-d/z}$ with $z = 2$ and $d = 1$. Dashed line in (c) indicates $\propto \omega^{3/2}$.

transform of $C(t)$,

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} C(t) e^{-i\omega t} dt, \quad (\text{C6})$$

where a scaling $F(\omega) \sim \omega^{3/2}$ is observed at $\omega \rightarrow 0$. This is consistent with the predicted real-time behavior $C(t) \sim t^{-5/2}$. In addition, an extra peak at finite ω is visible, which is related to the slowly decaying oscillations seen in Fig. 12(b).

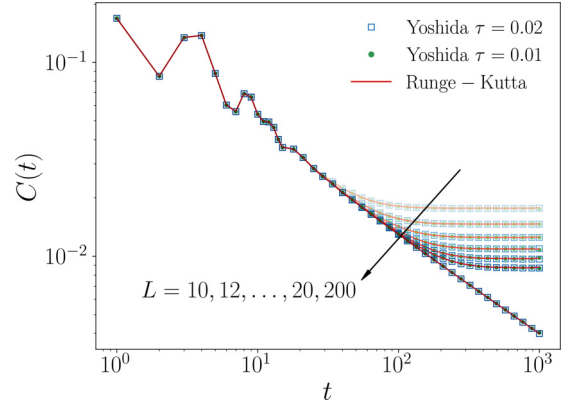


FIG. 13. $C(t)$ versus t for infinite temperature $\beta = 0$ for observable $\mathcal{O} = S_j^x$, calculated with Yoshida symplectic integrator $\tau = 0.02$ (open square), $\tau = 0.01$ (solid dot), and Tsitouras 5/4 Runge-Kutta integrator (solid line), for system size $L = 10, 12, \dots, 20, 200$.

APPENDIX D: ACCURACY CHECK OF THE NUMERICAL METHOD

In this Appendix, we check the accuracy of the numerical method employed in our study of the autocorrelation function, namely, the fourth-order Yoshida symplectic integrator with time step $\tau = 0.02$. To this end, we consider a different time step $\tau = 0.01$, as well as a different integrator—the Tsitouras 5/4 Runge-Kutta integrator, taking the tilted field Ising model (11) as an example. The results are shown in Fig. 13. We find almost perfect agreement among the results obtained using these different approaches, which demonstrates the high accuracy of our numerical results.

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