

Survival probability of particles inside the lemon billiard

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We study the escape of particles in the lemon billiard, a two-parameter family of billiard systems defined by the intersection of two identical circles. Using numerical simulations, we explore how the survival probability depends on the position and size of the hole, as well as on the billiard shape parameter. We find that the survival probability exhibits a two-stage decay pattern: an initial exponential regime followed by a long-time power-law tail, a signature of the stickiness effect. Our results show that the short-time exponential decay rate follows a power-law dependence on the hole size, with different scaling exponents for holes placed in chaotic regions versus mixed phase-space regions. For holes located in mixed phase-space regions, the decay exponent of the long-time power-law tail remains approximately constant, while the amplitude follows a power-law scaling with hole size. We also examine the dependence of short-time exponential decay rate on the billiard shape parameter and observe scaling behavior for small values of this parameter, which breaks down as the parameter increases.

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I. INTRODUCTION

Billiard systems are a type of dynamical system that originally emerged from statistical mechanics. In these systems, a particle moves freely along straight paths (or under the influence of potential forces) within a confined region on a plane [1]. When the particle hits the boundary of the billiard, it reflects elastically, changing its velocity according to the reflection law. Essentially, billiards serve as idealized models for situations where particles or waves are restricted to cavities or other uniform regions.

With advances in scientific computation, billiard systems have attracted considerable attention due to their intuitive nature and broad applicability. They provide useful frameworks for addressing complex problems across various scientific disciplines. Applications include biology [2,3], celestial mechanics [4,5], applied mathematics [6,7], plasma physics [8], waveguides [9], microwave billiards [10], dispersal of microorganisms in porous media [11], and superconductivity experiments [12].

The dynamics of billiard systems depend strongly on the billiard boundary's geometry, ranging from completely integrable (regular) to chaotic behaviors. A classic example is the circular billiard, whose dynamics are integrable because energy and angular momentum are conserved [1]. In this case, the phase space consists of periodic orbits (represented by straight lines). On the other hand, the Bunimovich stadium billiard is characterized by chaotic dynamics. Bunimovich originally studied a stadium billiard formed by two semi-circles of radius R connected by two straight segments of length $2a > 0$. When $a = 0$, the system reduces to the circular billiard [13]. More recently [14,15], systems with $a < 0$ have been studied, where straight segments disappear and the boundary forms two circular sectors resembling a lens or lemon shape. This configuration leads to a mixed phase space, characterized by Kolmogorov-Arnold-Moser (KAM) islands embedded within a chaotic sea.

In this paper, we investigate the survival probability of the lemon billiard with holes of varying sizes placed at different positions on the billiard's boundary. Our focus is on understanding whether this observable exhibits scaling invariance when varying key parameters such as the hole size h and the billiard shape parameter B . We analyze the behavior of both the exponential decay at short times and the power-law tails at long times, revealing distinct dynamics depending on the hole's location in chaotic or mixed regions of phase space.

This paper is organized as follows. In Sec. II, we introduce the lemon billiard model and discuss the equations governing the system's dynamics. Section III investigates the survival probability of particles escaping through a hole at two different positions, highlighting scaling behaviors under variations of the system parameters. Finally, Sec. IV presents our conclusions.

II. LEMON BILLIARD

Introduced by Heller and Tomsovic in 1993 [16], the family of lemon billiards has since been the subject of extensive research [17–21], including more recent studies [22–24]. These systems are defined by a boundary formed from the intersection of two identical circles of radius R , with their centers separated by a distance $2B$, where $2B < 2R$. The circles are positioned symmetrically along the horizontal axis: One is shifted left by B , the other right by B , so the centers of the circles lie at $x = \pm B$, with $B \in [0, 1)$. Figure 1 illustrates this configuration: the blue and red curves represent the left- and right-shifted circles, respectively, and the cyan region highlights their intersection, which defines the billiard domain.

The boundary of the billiard is described implicitly in Cartesian coordinates by the equations

$$(x + B)^2 + y^2 = R^2, \quad x > 0,$$

$$(x - B)^2 + y^2 = R^2, \quad x < 0.$$

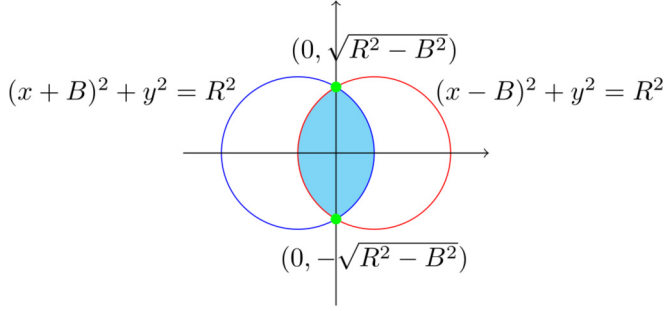


FIG. 1. Geometry of the lemon billiard. The boundary is formed by the intersection of two identical circles of radius R , whose centers are shifted symmetrically by a distance $2B$. The blue and red arcs represent the left- and right-shifted circles, respectively. The cyan area corresponds to the billiard domain. The green dots indicate the points of intersection between the two circles.

The two circles intersect at the points $(0, \pm\sqrt{R^2 - B^2})$, marked by green dots in Fig. 1. The total length of the billiard boundary is given by

$$\mathcal{L} = 2R \arcsin\left(\frac{\sqrt{R^2 - B^2}}{R}\right),$$

and a complete derivation is provided in the Appendix.

The dynamics of the system is characterized by the pair (s, α) , where s is the arclength along the boundary, measured counterclockwise from the point $(0, -\sqrt{R^2 - B^2})$, and $\alpha \in (0, \pi)$ is the angle formed between the trajectory of the particle and the tangent to the boundary at the point of collision, measured counterclockwise.

Figure 2 presents a schematic of the billiard motion and illustrates the definition of these angles at two successive collisions. The red angle represents the initial collision parameters, while the green angle shows those after the subsequent collision. Since no external forces or potentials act inside the billiard, the particle moves along straight-line segments at constant speed between collisions. These trajectories are shown as blue lines.

To describe the system's evolution, we begin with a particle initially located at arclength s_n with initial angle of injection

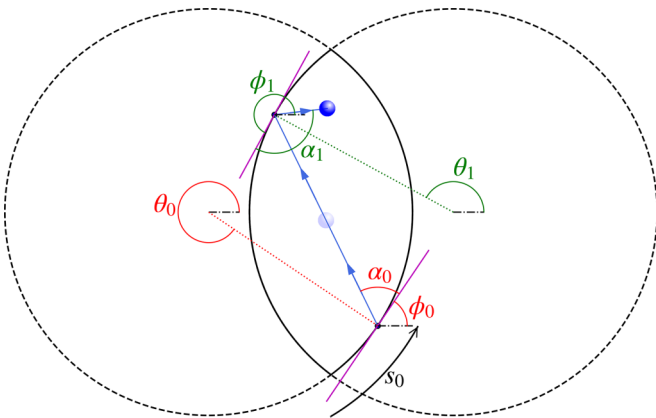


FIG. 2. An illustration of two consecutive collisions of a particle and the angles involved in the billiard.

α . The particle's Cartesian coordinates (x_n, y_n) can be obtained using the geometry of the billiard:

$$y_n = \begin{cases} R \sin\left[\frac{s_n}{R} + \arcsin\left(\frac{-\sqrt{1-B^2}}{R}\right)\right] & \text{if } s_n \leq \frac{L}{2} \\ -R \sin\left[\frac{s_n - \frac{L}{2}}{R} - \arcsin\left(\frac{\sqrt{1-B^2}}{R}\right)\right] & \text{if } s_n > \frac{L}{2}. \end{cases}$$

$$x_n = \begin{cases} -B + \sqrt{R^2 - y_n^2} & \text{if } s_n \leq \frac{L}{2} \\ B - \sqrt{R^2 - y_n^2} & \text{if } s_n > \frac{L}{2}. \end{cases} \quad (1)$$

To determine the global direction of the particle, in addition to the angle α_n , it is necessary to define the slope ϕ_n of the tangent line, measured counterclockwise from the horizontal axis. To do this, we introduce the polar angle θ_n , defined as the angle (in standard position, also measured counterclockwise from the horizontal axis) between the center of the arc on which the particle lies and its position (x_n, y_n) . Figure 2 illustrates these angles to provide a clearer visualization. The θ_n angle is given by

$$\theta_n = \begin{cases} \arctan\left(\frac{y_n}{x_n + B}\right) & \text{if } x_n > 0 \\ \arctan\left(\frac{y_n}{x_n - B}\right) & \text{if } x_n \leq 0. \end{cases}$$

From this, the Cartesian coordinates of the position (x_n, y_n) can be expressed in terms of θ_n as

$$\begin{aligned} x(\theta_n) &\equiv x_n = R \cos \theta_n, \\ y(\theta_n) &\equiv y_n = R \sin \theta_n, \end{aligned} \quad (2)$$

and then the angle ϕ_n is given by

$$\phi_n = \arctan\left[\frac{y'(\theta_n)}{x'(\theta_n)}\right] \mod 2\pi, \quad (3)$$

where the prime indicates the derivative with respect to θ_n .

Finally, the global direction of motion μ , measured counterclockwise from the horizontal axis, is then obtained by combining the local angle of incidence α_n with the tangent angle ϕ_n :

$$\mu_n = (\alpha_n + \phi_n) \mod 2\pi. \quad (4)$$

Since no forces act on the particle during free motion, its trajectory between collisions follows a straight line, parametrized by

$$\begin{aligned} x_{n+1} &= x_n + v_n \cos(\mu_n) \Delta t, \\ y_{n+1} &= y_n + v_n \sin(\mu_n) \Delta t, \end{aligned} \quad (5)$$

where Δt is the time interval between successive collisions. To determine Δt , we compute the times Δt_+ and Δt_- , which represent the time it takes for a particle initially at (x_n, y_n) , moving in the global direction μ_n , to reach the circles whose centers are shifted to the left and right by a distance B , respectively. Figure 3 illustrates the determination of Δt_+ and Δt_- . From the particle's current position (marked by a black dot), its trajectory is extended until it intersects the right-shifted circle (shown in pink) and the left-shifted circle (shown in brown). This line is illustrated in cyan. The intersection with the left-shifted circle is marked by a purple dot, and the corresponding time interval is denoted by Δt_+ . Similarly, the intersection with the right-shifted circle is marked by an orange dot, with the respective time interval denoted by Δt_- .

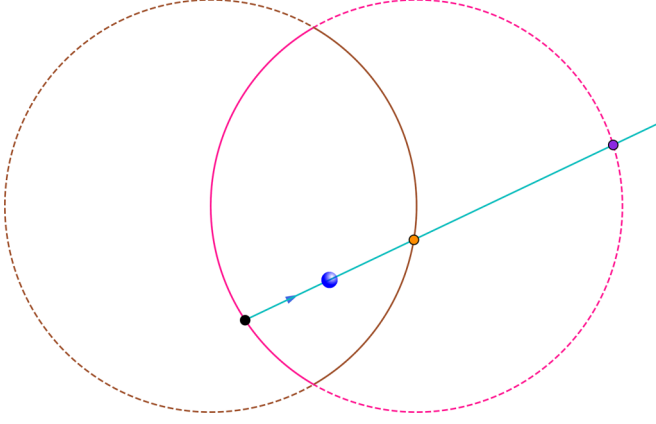


FIG. 3. Schematic representation of the calculation of the time to the next collision. From the particle's current position (black dot), an extended line is constructed along its trajectory (cyan line), intersecting both the left- and right-shifted circles. The intersection points are marked by orange and purple dots. The smaller of the two corresponding time intervals determines the effective time at which the next collision occurs.

These intersection points are obtained by solving

$$(x_{n+1} + B)^2 + y_{n+1}^2 = R^2$$

for Δt_+ , and

$$(x_{n+1} - B)^2 + y_{n+1}^2 = R^2$$

for Δt_- , where x_{n+1} and y_{n+1} are given by Eqs. (5). As a result, the time it takes for the particle to reach these points is

determined by solving the corresponding quadratic equations. For Δt_+ :

$$(\Delta t_+)^2 + 2v_n[(x_n + B)\cos\mu_n + y_n\sin\mu_n]\Delta t_+ + (x_n + B)^2 + y_n^2 - R^2 = 0, \quad (6)$$

and for Δt_- :

$$(\Delta t_-)^2 + 2v_n[(x_n - B)\cos\mu_n + y_n\sin\mu_n]\Delta t_- + (x_n - B)^2 + y_n^2 - R^2 = 0, \quad (7)$$

The solutions (considering only non-negative times) are given by

$$\Delta t_{\pm} = \frac{-b_{\pm} + \sqrt{b_{\pm}^2 - 4c_{\pm}}}{2}, \quad (8)$$

where

$$\begin{aligned} b_+ &= 2[(x_n + B)\cos\mu_n + y_n\sin\mu_n], \\ b_- &= 2[(x_n - B)\cos\mu_n + y_n\sin\mu_n], \\ c_+ &= (x_n + B)^2 + y_n^2 - R^2, \\ c_- &= (x_n - B)^2 + y_n^2 - R^2. \end{aligned} \quad (9)$$

Then, the time interval until the next collision, Δt , is given by the smaller of the two values:

$$\Delta t = \min(\Delta t_+, \Delta t_-).$$

Finally, the new arclength position s_{n+1} where the particle hits the boundary, is given by

$$s_{n+1} = \begin{cases} R[\arcsin(\frac{y_{n+1}}{R}) - \arcsin(\frac{-\sqrt{R^2 - B^2}}{R})] & \text{if } x_{n+1} \geq 0 \\ \frac{L}{2} + R[-\arcsin(\frac{y_{n+1}}{R}) + \arcsin(\frac{\sqrt{R^2 - B^2}}{R})] & \text{if } x_{n+1} < 0, \end{cases} \quad (10)$$

and the direction of the particle immediately after the collision is determined by

$$\alpha_{n+1} = (\phi_{n+1} - \mu_n) \mod \pi. \quad (11)$$

The structure of the phase space is shown in Fig. 4. We normalize the arclength s by dividing it by the total boundary length L to make the phase space dimensionless and comparable. Throughout this work, we consider $R = 1$ with-

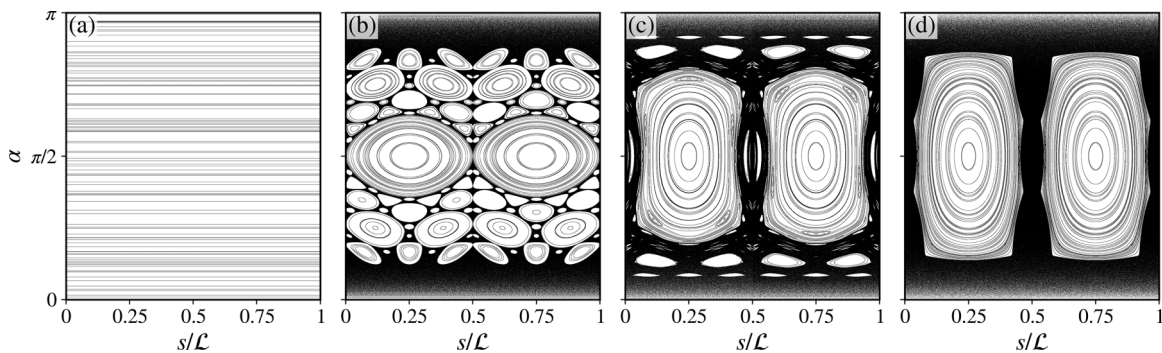


FIG. 4. Phase-space structure for different values of the parameter B . (a) For $B = 0$, the billiard is a perfect circle and the system is integrable; phase space is filled with invariant curves. (b) For $B = 0.1$, integrability breaks, and chaotic regions emerge alongside stability islands. (c) For $B = 0.7$, the central period-2 island becomes dominant. (d) For $B = 0.9$, the phase space is mostly chaotic, with a single large regular island.

out loss of generality. For $B = 0$, the boundary reduces to a perfect circle, and the system is integrable. As a result, the phase space is filled with straight lines corresponding to regular trajectories, as shown in Fig. 4(a). As B increases, integrability breaks down, and several islands emerge, as illustrated for $B = 0.1$ in Fig. 4(b). With a further increase of B , the large regular island around the period-2 orbit grows, while the smaller islands shrink and eventually disappear, leaving only the dominant period-2 island embedded in a fully chaotic sea. This evolution is shown for $B = 0.7$ and $B = 0.9$ in Figs. 4(c) and 4(d), respectively.

III. SURVIVAL PROBABILITY

A key quantity used to investigate the statistical behavior of transport phenomena is the survival probability, which reflects the chance that a particle remains in a certain region despite having an escape option. In this section, we explore the escape dynamics of particles through a hole of size h , measured in arclength units, positioned along the boundary of the billiard.

The survival probability quantifies the likelihood that a particle remains inside the system after n interactions. In other words, it corresponds to the fraction of particles that have not escaped by the n th iteration. It can be calculated using the following expression:

$$P(n) = \frac{N_{\text{surv}}(n)}{M}, \quad (12)$$

where M is the total number of initial conditions (particles), and N_{surv} is the number of particles that remain in the billiard up to the n th iteration.

To introduce escape dynamics into the billiard system, we consider two exits (activating only one at a time) placed along the boundary. Each hole has size h , measured in arclength units. These exits are located at $s_{\text{exit}} = 0.50\mathcal{L}$, situated in a fully chaotic region, and $s_{\text{exit}} = 0.75\mathcal{L}$, lying in a mixed region with both chaotic and regular dynamics.

The methodology involves initializing an ensemble of $M = 10^6$ particles randomly distributed across the entire phase space and tracking them for up to $N = 10^6$ boundary collisions, or until they escape through the active hole, which occurs when a particle hits the open exit. The statistical analysis is performed based on the number of collisions each particle undergoes before escaping. For each iteration n , the number of remaining particles $N_{\text{surv}}(n)$ inside the billiard is recorded. This data allows us to compute the survival probability for each iteration n , thereby enabling a statistical analysis of the escape dynamics.

We start our analysis considering $B = 0.1$ and varying the size of the exit h . Figure 5(a) shows the space phase for this configuration showing also the position of the two exits. The survival probability for the exits located in $s_{\text{exit}} = 0.50\mathcal{L}$ and $s_{\text{exit}} = 0.75\mathcal{L}$ are shown in Figs. 5(b) and 5(c), respectively, for five different exit sizes h . It is possible to see that we have similar behavior for either holes, which is the composition of two patterns; for short times, the survival probability decays as a stretched exponential in the form

$$P(n) \sim \exp(-(\kappa n)^\beta), \quad (13)$$

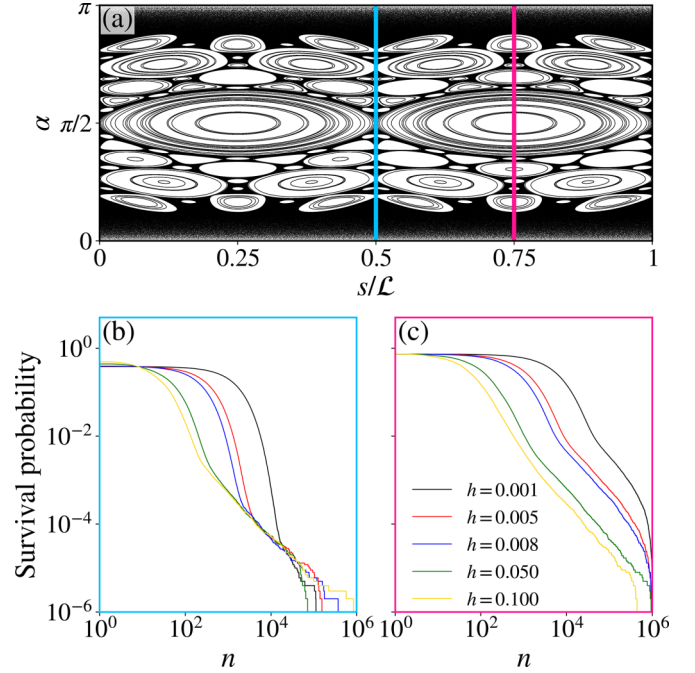


FIG. 5. (a) Phase space for $B = 0.1$, with hole positions indicated at $s_{\text{exit}} = 0.50\mathcal{L}$ (chaotic region) and $s_{\text{exit}} = 0.75\mathcal{L}$ (mixed region). (b) Survival probability curves for different hole sizes h at $s_{\text{exit}} = 0.50\mathcal{L}$. (c) Survival probability curves for different h at $s_{\text{exit}} = 0.75\mathcal{L}$.

where κ is the escape rate and β is the stretching exponent [25–27], while for longer times, a power-law tail emerges, described by

$$P(n) \sim An^{-\gamma}, \quad (14)$$

where A is also a non-negative constant and γ is the power-law decay rate, which is a characteristic feature of the stickiness effect [28,29]. Due to the stickiness effect, particles might be trapped near stability islands and resonance zones for a long, but finite, time leading to long escape times and causing the aforementioned deviations from the exponential decay.

The behavior of the exponential decay at short times is similar for both hole positions. We investigate how the decay rate κ and the shape parameter β depend on the hole size h , and the results are shown in Figs. 6(a) and 6(b), respectively. We find that the escape rate follows a power-law dependence over several orders of magnitude, $\kappa(h) \sim h^z$, where z is the scaling exponent. For the hole located at $s_{\text{exit}} = 0.75\mathcal{L}$, the exponent $z \approx 1.02$ is consistent with the linear scaling expected for uniformly mixed chaotic dynamics, where the escape probability is proportional to the hole size. In contrast, the hole at $s_{\text{exit}} = 0.50\mathcal{L}$ yields $z \approx 0.84$, indicating a sublinear scaling. This deviation from the linear prediction suggests that trajectories do not sample this portion of the boundary uniformly. The presence of sticky regions associated with KAM islands produces long-time correlations and modifies the density of collisions along the boundary. As a result, short-time dynamical correlations or local geometric effects reduce the effective flux of trajectories reaching the opening, leading to a slower increase of the escape rate as the hole size grows. Hence, these

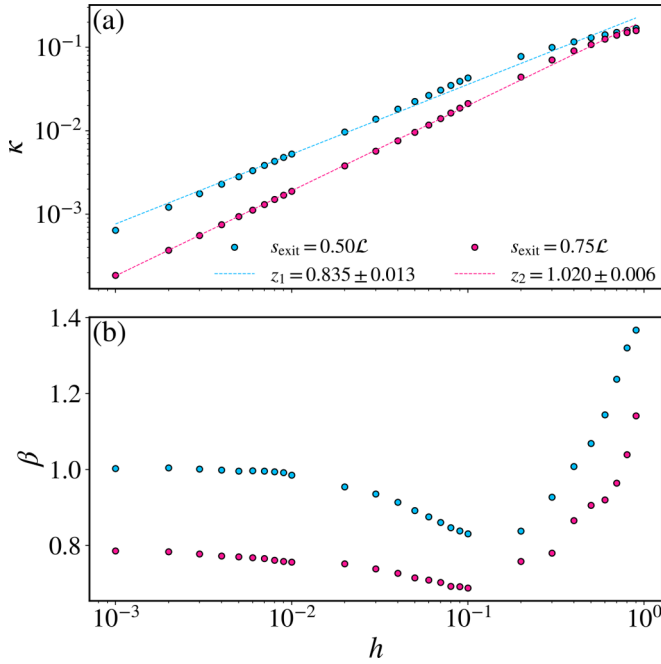


FIG. 6. (a) Log-log plot of the short-time escape rate κ and (b) semilogarithmic plot of the shape parameter β as a function of the hole size h . The observable κ shows a robust power-law scaling $\kappa(h) \sim h^z$ for both hole positions. Meanwhile, β initially decreases before experiencing a sudden increase at large h .

results show that the decay rate exhibits scaling invariance with respect to variations in h [30–32].

While the decay rate characterizes how quickly trajectories escape on average, the parameter β provides additional information about the distribution of escape times. The shape parameter β exhibits a nonmonotonic dependence on the hole size. For small holes, particles typically undergo many collisions before escaping, allowing trajectories to effectively explore the chaotic region. In this regime the survival probability is well described by a standard exponential decay ($\beta \approx 1$). As the hole size increases, β initially decreases, indicating a stretched exponential decay. This behavior suggests the presence of a broader distribution of short escape times, where trajectories starting closer to the hole escape rapidly while those located farther away require longer exploration times. For larger holes, β increases again and eventually exceeds unity. This trend indicates a narrowing of the escape-time distribution, consistent with the emergence of a characteristic escape timescale imposed by the geometry of the system. In this regime, trajectories typically reach the boundary and escape after a small number of collisions, producing a faster-than-exponential initial decay.

An interesting difference emerges when comparing the power-law tail behavior for holes located in chaotic regions versus those in mixed phase-space regions. For a hole placed in a chaotic region, the survival probability exhibits a nearly universal power-law tail, independent of the hole size h , converging to the same behavior. In contrast, when the hole is located in regions containing stability islands, the power-law decay is still present but shifted.

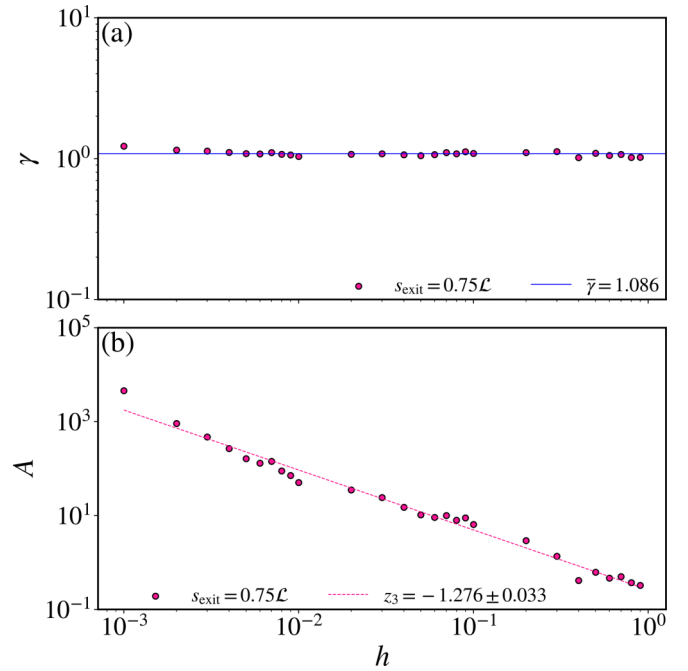


FIG. 7. Scaling analysis of the power-law tail when the hole is placed at $s_{\text{exit}} = 0.75L$. (a) Decay exponent γ as a function of h , indicating a constant behavior with average $\bar{\gamma} = 1.086$. (b) Coefficient A of the power-law decay versus h , exhibiting a power-law scaling $A(h) \sim h^{z_3}$.

This difference arises mainly because placing the hole partially or entirely over an island can eliminate all orbits near that island. In chaotic regions, the stickiness phenomenon remains statistically similar across different hole sizes. However, when the hole covers regions with islands, some sticky regions and resonance zones may be destroyed, affecting the power-law decay. Therefore, our results suggest that escape is faster when the hole is placed in chaotic regions without stability islands, consistent with findings in previous studies [33–35].

We further analyze the different patterns in the power-law tail for the survival probability in the case where the hole is located at $s_{\text{exit}} = 0.75L$, i.e., over regions with islands. By fitting this region with Eq. (14), we study how the parameters γ and A vary with h . The analysis of γ and A as functions of h [Figs. 7(a) and 7(b), respectively] reveals that the decay exponent γ is essentially constant, with a mean value $\bar{\gamma} = 1.086$, while A , similarly to κ , shows power-law scaling: $A(h) \sim h^{z_3}$, with $z_3 = -1.276 \pm 0.033$.

This weak dependence of γ on h can be attributed to the mechanism generating the power-law tail—the stickiness. Since this trapping mechanism is determined mainly by the local phase-space structure and not by the leak size (for sufficiently small holes), increasing h primarily changes the escape probability and therefore the prefactor A , while leaving the exponent γ nearly unchanged.

Given this result, a natural question arises: Does the system also exhibit scaling invariance when we vary the shape of the billiard, i.e., the parameter B ?

To address this question, Figs. 8(a) and 8(b) show the survival probability for fixed hole size $h = 0.1$ and several

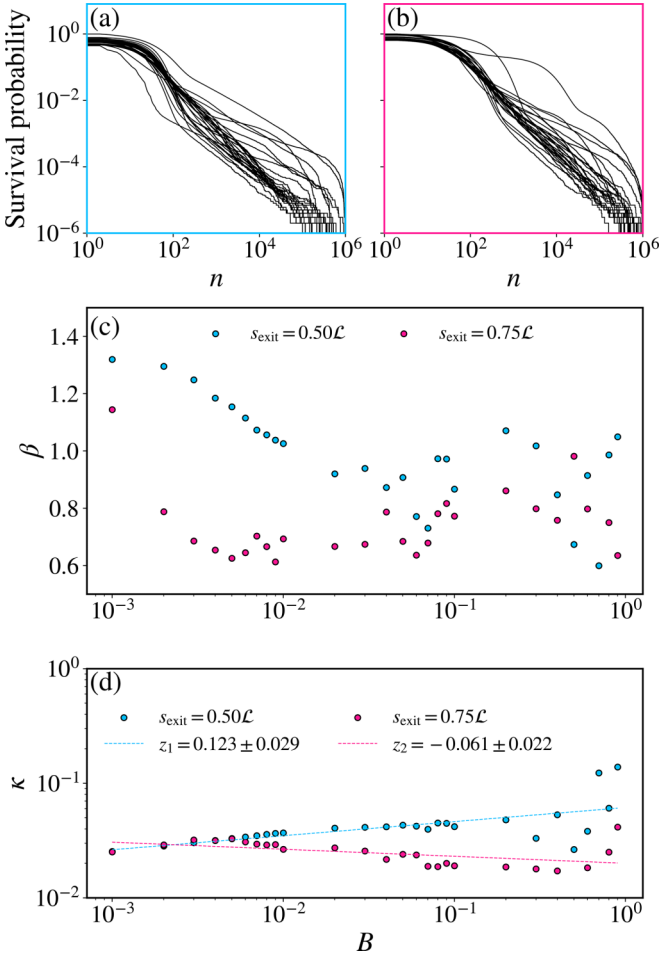


FIG. 8. Survival probability for fixed hole size $h = 0.1$ and different values of the shape parameter B . (a) Survival probability for holes at $s_{\text{exit}} = 0.50\mathcal{L}$. (b) Survival probability for holes at $s_{\text{exit}} = 0.75\mathcal{L}$. (c) semilogarithmic plot of the shape parameter β as a function of the hole size h . (d) Escape rate κ as a function of B for both hole positions. A power-law dependence is observed for small B , which deteriorates as B increases.

values of B , with the hole centered at $s_{\text{exit}} = 0.50\mathcal{L}$, and $s_{\text{exit}} = 0.75\mathcal{L}$, respectively. It is possible to observe that the curves exhibit the same general pattern as before: an initial stretched exponential decay followed by a power-law tail. Figure 8(c) confirms the presence of multiple stretched exponentials, each characterized by a different value of β . Figure 8(d) shows how the short-time exponential decay rate κ varies with B for holes at $s_{\text{exit}} = 0.50\mathcal{L}$ (blue) and $s_{\text{exit}} = 0.75\mathcal{L}$ (pink). For small values of B , a power-law behavior is evident, but this behavior deteriorates as B increases.

Scaling invariance is a property where a system's behavior remains unchanged under rescaling of its parameters, revealing underlying structures or universal patterns. Examples include fractional dynamics [36], social networks [37], and biological systems [3]. In the case of the lemon billiard, the system depends on two main parameters: the shape parameter B and the size of the hole h . We have already observed scaling behavior in the decay rate κ and in the coefficient A of the power-law tail when varying h .

Physical interpretation of the scaling laws

The escape dynamics in billiard systems with mixed phase space is strongly influenced by the nonuniform distribution of the invariant measure. In fully chaotic systems, a simple geometric argument predicts that the escape rate κ is proportional to the hole size h , since the probability that a trajectory reaches the opening is proportional to the fraction of the boundary occupied by the hole. However, in systems with mixed phase space, such as the lemon billiard, the invariant measure is not uniformly distributed along the boundary. Trajectories may spend long times in the vicinity of stability islands due to the stickiness mechanism, which produces strong spatial inhomogeneities in the distribution of collision points.

As a consequence, the effective flux of trajectories reaching the hole depends not only on its geometric size but also on the local structure of phase space. This explains the deviations from the purely geometric scaling $\kappa \propto h$ observed in our simulations. In particular, when the hole is located in regions where the invariant measure is depleted due to nearby sticky structures, the escape rate grows more slowly with h , leading to an exponent $z < 1$. Conversely, when the hole is located in regions that are more efficiently connected to the chaotic sea, the exponent approaches the geometric prediction.

The long-time power-law tail observed in the survival probability originates from trajectories temporarily trapped near the hierarchical structures surrounding KAM islands. These trajectories remain confined for long but finite times before returning to the chaotic sea and eventually escaping. Since the trapping mechanism is determined primarily by the local phase-space structure rather than the hole size, the exponent γ remains approximately constant when h varies, while the amplitude of the power-law tail changes.

A simple phenomenological interpretation of the escape dynamics can also be obtained using a stochastic transport model. If we assume that particles move chaotically inside the billiard and reach the boundary with an approximately constant average rate, the probability that a particle escapes at each collision can be written as

$$p \approx \frac{h}{L_{\text{eff}}}, \quad (15)$$

where L_{eff} represents an effective boundary length that takes into account the nonuniform invariant measure along the boundary.

Under this approximation, the survival probability satisfies

$$P(n+1) = (1-p)P(n), \quad (16)$$

which leads to the exponential decay

$$P(n) \sim e^{-\kappa n}, \quad (17)$$

with $\kappa \approx p$. In systems with mixed phase space, however, the presence of sticky regions modifies the effective transport toward the hole, producing deviations from the purely geometric prediction and leading to the scaling laws observed numerically in this work.

IV. CONCLUSION

In this work, we investigated the escape dynamics of the lemon billiard by analyzing the survival probability of

particles escaping through specific holes positioned along the boundary, focusing on whether this observable exhibits scaling invariance under variations of the system parameters, namely, the hole size h and the shape parameter B .

We started by providing a detailed geometric and dynamical description of the lemon billiard, which is formed by the intersection of two identical circles of radius R , separated by a distance $2B$. The dynamics were described using arclength and incidence angles, and we developed a framework to simulate the motion of particles as they collide with the billiard boundary.

To analyze the escape dynamics, we introduced a hole of size h at two distinct positions along the boundary: One located in a fully chaotic region ($s_{\text{exit}} = 0.50\mathcal{L}$) and another in a mixed region that includes stability islands ($s_{\text{exit}} = 0.75\mathcal{L}$). For each configuration, we computed the survival probability $P(n)$, defined as the fraction of particles that remain inside the billiard after n collisions. Our results show that the survival probability exhibits two distinct decay regimes: An initial exponential decay at short times, followed by a power-law decay at longer times. The exponential regime is associated with the chaotic dynamics, while the power-law tail results from the stickiness effect caused by particles temporarily trapped near regular islands.

We studied the dependence of the exponential decay rate κ on the hole size h , and found a power-law scaling $\kappa(h) \sim h^z$, where the exponent z depends on the hole's position. Additionally, in the mixed region, the amplitude A of the power-law tail also exhibits scaling behavior with respect to h , while the power-law exponent γ remains approximately constant.

Finally, we extended our analysis by fixing the hole size h and varying the shape parameter B . The survival probability continued to show the same general pattern of stretched exponential decay and power-law decay. We observed that for small values of B , the exponential decay rate κ again follows a power-law dependence, indicating scaling behavior. However, as B increases, this behavior becomes less clear.

In summary, our study reveals that the survival probability in the lemon billiard exhibits scaling invariance with respect to the hole size h , both in the exponential decay regime and in the power-law tail. Scaling invariance with respect to the shape parameter B is also observed, but only within a limited range of values.

The observed scaling laws suggest that escape dynamics in the lemon billiard is governed by the interplay between chaotic transport and sticky trajectories associated with stability islands. Developing a more detailed theoretical description of these mechanisms, possibly using stochastic transport models, remains an interesting direction for future work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: CALCULATION OF THE BILLIARD BOUNDARY LENGTH

To compute the total length of the billiard boundary, we start by calculating the arclength of a single circular segment. Due to the geometry of the lemon billiard, the full boundary consists of two symmetric arc segments of equal length. Therefore, it is sufficient to calculate just one arc and then double the result to obtain the total length.

The arc lies on a circle centered at $(-B, 0)$ with radius R , given implicitly by

$$(x + B)^2 + y^2 = R^2. \quad (\text{A1})$$

We apply the arclength formula for implicitly defined curves:

$$L = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy. \quad (\text{A2})$$

Differentiating equation (A1) with respect to y ,

$$\begin{aligned} \frac{d}{dy}((x + B)^2 + y^2) &= 0, \\ 2(x + B)\frac{dx}{dy} + 2y &= 0 \Rightarrow \frac{dx}{dy} = -\frac{y}{x + B}. \end{aligned} \quad (\text{A3})$$

Substituting Eq. (A3) into (A2),

$$L = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{y}{x + B}\right)^2} dy. \quad (\text{A4})$$

From Eq. (A1), we solve for $x + B$ in terms of y :

$$(x + B)^2 = R^2 - y^2. \quad (\text{A5})$$

Substituting Eq. (A5) into (A4),

$$\begin{aligned} L &= \int_{y_1}^{y_2} \sqrt{1 + \frac{y^2}{R^2 - y^2}} dy = \int_{y_1}^{y_2} \sqrt{\frac{R^2}{R^2 - y^2}} dy \\ &= \int_{y_1}^{y_2} \frac{R}{\sqrt{R^2 - y^2}} dy. \end{aligned} \quad (\text{A6})$$

The integral in Eq. (A6) evaluates to

$$L = R \left[\arcsin\left(\frac{y}{R}\right) \right]_{y_1}^{y_2}. \quad (\text{A7})$$

The circular arcs intersect at $y = \pm\sqrt{R^2 - B^2}$, so we set $y_1 = -\sqrt{R^2 - B^2}$ and $y_2 = \sqrt{R^2 - B^2}$. Then Eq. (A7) becomes

$$\begin{aligned} L &= R \left[\arcsin\left(\frac{\sqrt{R^2 - B^2}}{R}\right) - \arcsin\left(\frac{-\sqrt{R^2 - B^2}}{R}\right) \right] \\ &= 2R \arcsin\left(\frac{\sqrt{R^2 - B^2}}{R}\right). \end{aligned} \quad (\text{A8})$$

Since the billiard boundary consists of two identical arcs, the total boundary length is

$$\mathcal{L} = 2L = 4R \arcsin\left(\frac{\sqrt{R^2 - B^2}}{R}\right). \quad (\text{A9})$$

Comparison with the literature

In the work [14], the authors give the total boundary length as

$$\mathcal{L} = 4 \arctan(\sqrt{B^{-2} - 1}), \quad (\text{A10})$$

assuming $R = 1$.

To verify that Eqs. (A9) and (A10) are equivalent, define

$$\theta = \arcsin(\sqrt{1 - B^2}), \quad (\text{A11})$$

so that

$$\sin \theta = \sqrt{1 - B^2}, \quad (\text{A12})$$

$$\cos \theta = B. \quad (\text{A13})$$

Then the tangent of θ is

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{1 - B^2}}{B}. \quad (\text{A14})$$

Now define

$$\phi = \arctan(\sqrt{B^{-2} - 1}) = \arctan\left(\frac{\sqrt{1 - B^2}}{B}\right). \quad (\text{A15})$$

From Eqs. (A14) and (A15), it follows that $\theta = \phi$, and hence

$$\arcsin(\sqrt{1 - B^2}) = \arctan(\sqrt{B^{-2} - 1}). \quad (\text{A16})$$

Therefore, both expressions for $\mathcal{L}$, Eqs. (A9) and (A10), are equivalent when $R = 1$, and the identity confirms the consistency between our geometric derivation and the formula found in the literature.

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