

Using operator covariance to disentangle scaling dimensions in lattice models

Anders W. Sandvik ^{*}

Department of Physics, *Boston University*, 590 Commonwealth Avenue, Boston, Massachusetts 02215, USA
and School of Physical and Mathematical Sciences, *Nanyang Technological University*, Singapore



(Received 14 April 2025; revised 11 July 2026; accepted 22 July 2026; published 24 August 2026)

In critical lattice models, distance (r)-dependent correlation functions contain power laws $r^{-2\Delta}$ governed by scaling dimensions Δ of an underlying continuum field theory. In Monte Carlo simulations and other numerical approaches, the leading dimensions can be extracted by data fitting, which can be difficult when two or more powers contribute significantly. Here a method utilizing covariance between multiple lattice operators is developed where the r -dependent eigenvalues of the covariance matrix reflect scaling dimensions of individual field operators. This disentangling of scaling dimensions is demonstrated explicitly for conformal field theories. The computational scheme is first tested on the critical point of the two-dimensional Ising model, where the two primary scaling dimensions and their respective two lowest descendant dimensions are extracted. The three-dimensional Ising model is studied next, revealing the two relevant primaries and their lowest descendants to high precision. For a more challenging case, the tricritical Ising point in two dimensions is studied with the Blume-Capel (diluted Ising) model. Here the scaling dimensions of all three fully symmetric (under lattice point group and spin-inversion transformations) primary operators are successfully isolated along with the leading descendants. The eigenvectors in the space of the two relevant primary operators are also studied and give useful information on the boundary between the ordered and disordered phases in the neighborhood of the tricritical point. Finally, the crossover from regular to tricritical Ising scaling is investigated on several points on the phase boundary of the Blume-Capel model away from its tricritical point. The scaling of the eigenvalues corresponding to tricritical descendant operators are found to be remarkably stable even far from the tricritical point. The covariance method represents a simple extension of standard analysis of correlation functions and can significantly enhance the utility of Monte Carlo simulations and other computational methods in studies of classical and quantum criticality.

DOI: [10.1103/jq2v-wjkl](https://doi.org/10.1103/jq2v-wjkl)

I. INTRODUCTION

The most direct bridge between a critical point in a lattice model and its continuum field theoretical description is through the spectrum of scaling dimensions Δ_i . In numerical studies, e.g., Monte Carlo (MC) simulations, the dimensions are accessible in correlation functions, which contain contributions decaying with distance r as $r^{-2\Delta_i}$ for each field operator i compatible with the symmetries of the lattice operator used. For a system in d dimensions (space-time dimensions of a quantum system) the scaling dimensions of relevant operators ($\Delta_i < d$) in different symmetry sectors are related to the conventional critical exponents. While these relevant scaling dimensions are typically of primary interest, irrelevant operators ($\Delta_i > d$) can produce significant corrections that complicate the analysis of correlation functions. To more completely characterize a critical point, it is also useful in its own right to determine some of the irrelevant scaling dimensions. Modern conformal field theory (CFT) techniques, in particular the numerical bootstrap [1] and fuzzy sphere approaches [2], can access an extended range of scaling dimensions. It can be desirable to extract both relevant and

irrelevant dimensions also in lattice simulations, e.g., to determine whether or not the low-energy properties of a system are described by a CFT.

Indeed, some of the most challenging problems in contemporary condensed matter physics involve the problem of matching numerical simulation results to CFTs or other quantum field theories through scaling dimensions, often with the added complexity of multicritical behavior where correlation functions contain contributions from more than one relevant field. The case of the putative $d = 2 + 1$ deconfined quantum critical point [3] is perhaps the most prominent unresolved example. Here the precise matching of multiple power law decays of lattice correlators [4–9] to emerging CFT data [10–17] is crucial for building a consensus on the ultimate nature of the quantum phase transition. A more mundane but also very important class of problems with two relevant scaling dimensions is that of liquid-gas phase transitions. These are generically of Ising type [18] but, because the Z_2 symmetry is only emergent, require tuning of both the pressure and the temperature to reach the critical point (unlike the Ising model, in which the exact Z_2 symmetry implies that the critical point is located at zero field and only the temperature needs to be tuned). The presence of two relevant scaling fields makes it challenging to extract the critical point and the phase boundary in its neighborhood [19,20].

^{*}Contact author: sandvik@bu.edu

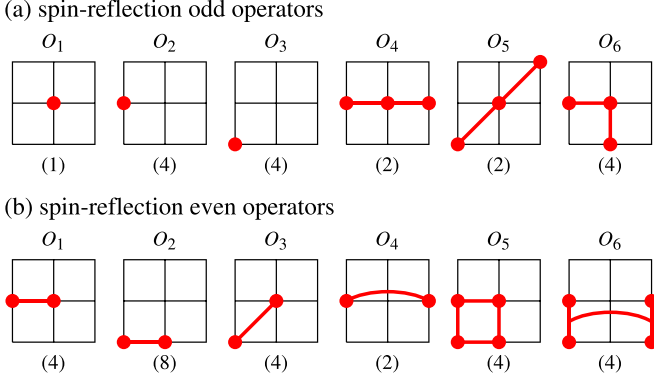


FIG. 1. Ising spin operators defined on cells of 3×3 spins, with (a) and (b) corresponding to Z_2 antisymmetry and symmetry, respectively. The red dots depict Ising operators σ_i , with lines connecting two or more operators implying products. Point group symmetrization is further carried out by adding all different operators resulting from rotations about the cell center and reflection about the central horizontal or vertical axis. The total number n_i of such terms is indicated beneath each of the cells.

In this article a simple method for disentangling of different power-law contributions to numerically computed correlation functions is proposed. The focus is on lattice models, but generalization to continuum systems will be straight-forward. Defining a set of a small number of lattice operators $O_i(\mathbf{x})$, $i = 1, \dots, N$ (with $N < 10$ typically) on cells centered on position $\mathbf{x}$, the full covariance matrix of these operators

$$C_{ij}(\mathbf{r}) = \langle O_i(\mathbf{x}_1) O_j(\mathbf{x}_2) \rangle - \langle O_i \rangle \langle O_j \rangle, \quad \mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1, \quad (1)$$

is evaluated versus the separation $\mathbf{r}$. In MC simulations, the expectation values include averaging over equivalent pairs of lattice sites $(\mathbf{x}_1, \mathbf{x}_2)$ and $\langle O_i \rangle$ is also a spatial average. By diagonalizing the symmetric covariance matrix, a set of eigenvalues $D_i(\mathbf{r})$ is obtained. It will be shown that, for sufficiently large separation r in a system described by a CFT (or other critical low-energy theory), these eigenvalues decay with different powers of r that correspond to the scaling dimensions of primary and descendant operators compatible with the symmetries of the operators used. Examples of lattice operators defined on cells of 3×3 Ising spins are shown in Fig. 1 and will be used to test the covariance method with the standard two-dimensional (2D) classical Ising model. These operators are all fully symmetric under the relevant point group transformations of the cells, thus contain only CFT operators with Lorentz spin $l = 0$. The operators in Figs. 1(a) and 1(b) are antisymmetric and symmetric, respectively, under spin inversion (Z_2 symmetry), as follows from the number of operators, even or odd, in each product. Using covariance matrices involving these operators, MC results for the 2D Ising model can reproduce the known scaling dimensions of the primary operators as well as their first two descendants. Likewise, for the 3D Ising model, using operators similar to those in Fig. 1, the two relevant primaries and their leading descendants are well resolved. The largest of the reproduced scaling dimensions corresponds to a power-law decay as fast

as r^{-10} , which is far beyond practical resolution with standard analysis of individual correlation functions.

The covariance method is particularly useful for analyzing multicritical points, where two or more primary operators are relevant. To test such a case, the 2D Blume-Capel (BC) model will be studied at its tricritical Ising point. The BC model, or diluted Ising model, has a three-state degree of freedom $\sigma_i \in \{-1, 0, +1\}$ on each site i and the Hamiltonian is

$$H = - \sum_{\langle ij \rangle} \sigma_i \sigma_j + \lambda \sum_i \sigma_i^2 \equiv E + \lambda F, \quad (2)$$

here on the square lattice. For positive $\lambda \lesssim 1.966$, the model undergoes a continuous Ising transition into a ferromagnetic state, with the critical temperature T_c decreasing as λ is increased, while for larger $\lambda \gtrsim 2$ the transition is first-order. The location of the tricritical point (λ_t, T_t) separating these behaviors has been estimated in several works [21–28], with the most precise reported parameters being $\lambda_t \approx 1.965815$, $T_t \approx 0.608578$ from transfer-matrix calculations [25]. This point belongs to the tricritical Ising universality class, having two relevant fully symmetric perturbations (operators) whose scaling dimensions are $\Delta_0 = 1/5$ and $\Delta_1 = 6/5$ [29,30]. Reproducing these values will be important but, as it turns out, a third irrelevant primary and some of the irrelevant dimensions can also be identified. Moreover, the method produces the linear combinations of the two operators E and F , defined in Eq. (2), corresponding to the relevant CFT operators. One of these eigenvectors gives the tangent of the phase boundary at the tricritical point, which here is determined to better precision than in previous works. This ability to reliably compute the eigenvectors in the space of two tuning parameters should be of great utility also in studies of liquid-gas phase transitions.

Operator covariance is often used in particle physics to extract masses [31–35]. This method is in part similar to the scheme developed here, but there are important differences. In particle physics, exponential decays in imaginary (Euclidean) time are used to extract the energy levels of a discrete spectrum, as opposed to the power law decays extracted here that correspond to a gapless spectrum in the limit when the system length $L \rightarrow \infty$. In other words, in particle physics covariance matrices are computed versus imaginary time separation τ in a system where the time dimension $L_\tau = 1/T$, T being the temperature, is much larger than the spatial length L . The ability of the eigenvalues to correctly separate the different exponential decays corresponding to different masses (discrete energy levels) was proven in that limit [31], and many aspects of the method are more involved than the scheme presented here; in particular the choice of suitable operators and the use of a reference time in a generalized eigenvalue problem.

In the present case, all dimensions in a classical system can be set equal, and in a quantum system L_τ can be of order L . In a critical quantum system with an underlying CFT description and $L_\tau \gg L$ imposed, the different power-law decays corresponding to CFT scaling dimensions would be applicable at time τ only up to $\tau \propto L$, not at $\tau \gg L$, where instead exponential decays corresponding to the finite-size discreteness of the spectrum would apply. Thus, the present method extends the application of covariance eigenvalues to the

different domain of classical and quantum criticality. Analytical arguments will be presented for the proper disentangling of the scaling dimensions. The numerical examples will cover only classical systems, but the application to quantum systems is straight-forward and will be considered elsewhere [36].

The outline of the rest of the article is as follows. In Sec. II, disentangling of scaling dimensions by the covariance method is first discussed in the framework of ideal lattice operators (in general, impossible to construct explicitly) realizing a single CFT operator. It is argued that unique linear combinations $\tilde{Q}_{pi}$ of these operators Q_{pi} belonging to a given symmetry sector p can always be found that are orthogonal in the correlation sense, i.e., $\langle \tilde{Q}_{pi}(\mathbf{r}) \tilde{Q}_{pj}(\mathbf{0}) \rangle = 0$ for $i \neq j$, and further that the decay with r of $i = j$ correlators is governed purely by Δ_i ; $\langle \tilde{Q}_{pi}(\mathbf{r}) \tilde{Q}_{pi}(\mathbf{0}) \rangle = r^{-2\Delta_i}$. Given that simple lattice operators (of the kind that can be easily constructed in practice) within a given symmetry sector can be written as linear combinations of the ideal operators $\tilde{Q}_{pi}$ (including all p compatible with the symmetries), it is then also true that diagonalization of an $N \times N$ covariance matrix, Eq. (5), produces eigenvalues whose asymptotic decays are controlled by the N smallest scaling dimensions. This assertion is exemplified by results for the 2D classical Ising model in Sec. III and the 3D Ising model in Sec. IV, where analogies with quantum systems in $2 + 1$ dimensions are also made. Tricriticality in the 2D BC model is studied in Sec. V. In Sec. VI the crossover from tricritical to conventional Ising scaling dimensions in the neighborhood of the tricritical point is investigated. The results and conclusions are further discussed in Sec. VII.

II. EFFECTIVE CORRELATION ORTHOGONALITY

The ability of the covariance method to deliver scaling dimensions for a low-energy field theory from correlation functions computed with a lattice Hamiltonian (or some other regularized model) will for definiteness here be framed within CFT terminology, though systems described by other field theories can be similarly treated. A CFT is characterized by a set of primary operators $Q_{p0}(\mathbf{r})$ and their descendants $Q_{pi}(\mathbf{r})$ obtained from the i th derivatives of $Q_{p0}(\mathbf{r})$. For a d -dimensional quantum system defined in the appropriate conformal geometry, for each conformal multiplet p the state-operator correspondence implies a tower of equally spaced eigenstates ϵ_i of a corresponding Hamiltonian H defined only on the system in the $d - 1$ spatial dimensions. The energy gaps $\epsilon_i - \epsilon_0$ are proportional to the spectrum of scaling dimensions Δ_i of the CFT, with the constant of proportionality being of order the inverse system size; $\epsilon_i - \epsilon_0 \propto \Delta_i/L$. The conformal geometry for $d = 1 + 1$ is an infinite cylinder [37], which for a quantum system can be realized with H defined on a ring of circumference L at temperature $T \rightarrow 0$. The imaginary-time evolution operator $e^{-\tau H}$ for $\tau \in [0, \beta]$, with $\beta = T^{-1}$, defines the second dimension, which is normally also taken to be periodic (as in quantum statistical mechanics).

The infinite cylinder (or torus with $\beta \gg L$) geometry is easy to implement in lattice calculations for systems (Hamiltonians) with potential CFT descriptions in one space dimension, and useful scaling dimensions can be extracted

from low-lying levels ϵ_i [38–40]. Though a critical point of a given microscopic model will only have a precise CFT description in the infinite-size limit, the flow of the energy levels ϵ_i (or the scaling dimensions Δ_i extracted from some other quantity) versus the system size is normally well behaved (being controlled by subleading corrections from irrelevant perturbations) and can be extrapolated. However, in $d = 2 + 1$ dimensions, the periodic boundary conditions commonly used in quantum MC simulations (alternatively, open or cylindrical boundaries that are more practical with some other methods [41]) are not compatible with the state-operator correspondence and, therefore, the level spectrum of the Hamiltonian defined on a torus does not correspond to the spectrum of CFT scaling dimensions in a known way [42]. The proper conformal spatial geometry is the surface of a sphere [43], which is very difficult to realize uniformly for lattice models [44–48]. Here the fuzzy sphere approach is making inroads [2,12,15,16,49,50], with the spherical geometry realized with fermions coupled to a magnetic monopole [51] and interactions designed to implement desired symmetries and quantum phase transitions when projected to the lowest Landau level. Though remarkable results have already been presented for some models, there are still significant limitations in the system size (number of fermions), and it is also often not obvious how to construct a model with a specific type of quantum phase transition.

The covariance method presented here does not rely on the specific conformal geometry, as will be explicitly demonstrated numerically in the later sections by classical MC results generated on standard periodic square and cubic lattices of volume L^2 and L^3 , respectively instead of the infinite cylinders in $d = 2$ or radial quantization in $d = 3$. From the CFT (or other field theory) perspective, it is from the outset not clear, however, whether diagonalizing a covariance matrix Eq. (5) will truly yield eigenvalues decaying asymptotically as $r^{-2\Delta_i}$ with (a) different scaling dimensions Δ_i and, related to this, (b) not including any off-diagonal $r^{-(\Delta_i + \Delta_j)}$ contributions. These complications might be expected on account of the fact that the CFT operators belonging to the same multiplet p are not orthogonal in the sense of correlation functions. It will be argued below that diagonalization of an $N \times N$ covariance matrix will nevertheless produce eigenvalues decaying asymptotically as $r^{-2\Delta_i}$, with the N smallest scaling dimensions Δ_i for a system described by a CFT. In practice, statistical errors in MC simulations will limit the number of such decay exponents that can be observed in practice, as will be seen in the later sections. However, even a rather small number of scaling dimensions (in each symmetry sector) beyond the normally accessible lowest primary can still provide very valuable information when probing a system for its CFT description.

A. Covariance matrices of ideal lattice operators

In principle, it is possible to construct lattice operators in a critical system that only contain a single primary or descendant CFT operator, though such operators would normally be extremely complicated and impossible to design in practice. Considering an infinite set $Q_i(\mathbf{r})$ of these ideal operators in a specific symmetry sector, including a fixed Lorentz spin l ,

arranged as a column vector $Q(\mathbf{r})$:

$$Q^T(\mathbf{r}) = [Q_0(\mathbf{r}), Q_1(\mathbf{r}), \dots]. \quad (3)$$

We will first assume that this group of operators only includes one primary Q_0 , with scaling dimension Δ_0 , along with all its descendants Q_n , $n = 1, 2, \dots$, within the same symmetry sector. Then,

$$\Delta_n = \Delta_0 + 2n, \quad (4)$$

since, for a primary operator with given l , the descendants with scaling dimensions $\Delta_0 + 2n - 1$ belong to different spin sectors; $l \pm 1$. The second descendant is therefore the lowest one with the same l as the primary.

With the restriction to a single conformal multiplet, none of the operators are orthogonal to each other in the sense of the correlation functions (only operators in different multiplets are), i.e.,

$$C_{ij}(\mathbf{r}) = \langle Q_i(\mathbf{r})Q_j(\mathbf{0}) \rangle = c_{ij}r^{-(\Delta_i+\Delta_j)}, \quad (5)$$

where the factors c_{ij} should be expected to be nonzero for all i, j . These correlation functions form the covariance matrix C , which can be written as

$$C(\mathbf{r}) = \langle Q(\mathbf{r})Q^T(\mathbf{0}) \rangle. \quad (6)$$

This matrix can be diagonalized by a unitary transformation U that depends on the separation $\mathbf{r}$,

$$\begin{aligned} D(\mathbf{r}) &= U^T(\mathbf{r})\langle Q(\mathbf{r})Q^T(\mathbf{0}) \rangle U(\mathbf{r}) \\ &= \langle U^T(\mathbf{r})Q(\mathbf{r})Q^T(\mathbf{0})U(\mathbf{r}) \rangle \\ &= \langle \tilde{Q}(\mathbf{r})\tilde{Q}^T(\mathbf{0}) \rangle, \end{aligned} \quad (7)$$

with $D_{ij}(\mathbf{r}) = 0$ for $i \neq j$ and the transformed operators $\tilde{Q}(\mathbf{r}) = U^T(\mathbf{r})Q(\mathbf{r})$ have been defined.

The question now is whether the eigenvalues of C decay according to the corresponding scaling dimensions of the original operators, i.e., whether it is true that

$$D_i(\mathbf{r}) \equiv D_{ii}(\mathbf{r}) = A_i r^{-2\Delta_i}, \quad (8)$$

where the prefactors A_i should be independent of r . These decay forms, purely governed by the individual exponents $2\Delta_i$, are not obvious even if D is diagonal, since the transformation might in principle mix the different power laws appearing in the original covariance matrix C . While a completely rigorous proof of Eq. (8) will not be presented here, a series of plausible arguments will be put forward in its support. Thus assuming that Eq. (8) holds, subsequently Sec. II B will proceed with a demonstration of the disentangling of scaling dimensions by diagonalization of a covariance matrix computed with a set of simple lattice operators that can be designed in practice.

Consider the elements u_{ij} of a matrix u that defines a transformation for a finite set of operators; the vector in Eq. (3) truncated at some $n = m$ that later can be taken to infinity:

$$\tilde{Q}_n = \sum_{i=0}^m u_{ni} Q_i, \quad n = 0, 1, \dots, m, \quad (9)$$

where initially u may be different from the diagonalizing transformation $U(\mathbf{r})$ but $u \equiv U(\mathbf{r})$ will be established along with the separation of eigenvalues according to Eq. (8). The

transformed covariance matrix is

$$D_{ij}(\mathbf{r}) = \sum_{kp} u_{ik} u_{jp} \langle Q_k Q_p \rangle = \sum_{kp} u_{ik} u_{jp} C_{kp}(\mathbf{r}), \quad (10)$$

which is not yet assumed to be diagonal. Using the covariance matrix in the form of Eq. (5) and the descendant dimensions in Eq. (4), we can also write $D(\mathbf{r})$ as

$$D_{ij}(\mathbf{r}) = r^{-(\Delta_i+\Delta_j)} \sum_{kp} u_{ik} u_{jp} c_{kp} r^{-2(k+p-i-j)}, \quad (11)$$

so that the original covariance matrix Eq. (5) is obviously recovered when $u_{ik} = \delta_{ik}$ and $u_{jp} = \delta_{jp}$.

If we require that the transformation leads to a diagonal $D(\mathbf{r})$, then the eigenvectors $u_i^T = [u_{i0}, \dots, u_{iN}]$ clearly must depend on r . However, we can make the plausible assumption that a collective solution for all $\mathbf{r}$ must exist, so that a single matrix diagonalization is sufficient for constructing a diagonal $D(\mathbf{r})$ for all $\mathbf{r}$. This collective solution can be found by assuming that the r dependence of u completely cancels $r^{-2(k+p-i-j)}$ in Eq. (11), i.e., defining a matrix v with

$$v_{ik} \equiv u_{ik} r^{-2(k-i)}, \quad (12)$$

and positing that this matrix can now be independent of r . Equation (11) then becomes

$$D_{ij}(\mathbf{r}) = r^{-(\Delta_i+\Delta_j)} \sum_{kp} v_{ik} v_{jp} c_{kp}, \quad (13)$$

which represents a standard orthogonal transformation of the symmetric matrix c . For any such transformation, the general original form Eq. (5) is maintained and the special unique transformation that diagonalizes c leads to eigenvalue separation according to Eq. (8), with the factors given by

$$A_i = \sum_{kp} v_{ik} v_{ip} c_{kp}. \quad (14)$$

It should be stressed again that the only assumption made here was that a collective solution for all $\mathbf{r}$ exists. The specific r -dependent prefactor $r^{-(\Delta_i+\Delta_j)}$ in front of the sum representing the collective transformation in Eqs. (11) and (13) is then required for the sum to contain the powers of r present originally and, eventually, to obtain the correct set of $m+1$ scaling dimensions. Finally we can also let $m \rightarrow \infty$ in Eq. (9).

B. Covariance of simple lattice operators

It is in general not possible in practice to uniquely define a lattice operator that contains a single field operator. Explicit translations from the lattice to the field operators are only known in special cases [30,52,53] and typical operators defined for generic lattice models contain all the symmetry compatible CFT operators. Using a finite set of N lattice operators, it should be possible, however, to find optimal linear combinations of these operators such that their correlation functions exhibit asymptotic decays according to the N lowest scaling dimensions. While these optimal operators of course depend on the original set of N chosen operators, in general one would expect some overlap between arbitrary lattice operators and the field operators.

For a given set of symmetries, many conformal multiplets with their full towers will in general contribute to a simple

lattice operator O_i , which therefore be written as a linear combination of the idealized operators Q_{pi} belonging to several multiplets p and $i = 0, 1, \dots, \infty$. Alternatively, we can expand in the correlation orthogonal operators $\tilde{Q}_{pi}$ discussed above in Sec. II A, which is what we will do here. For the sake of a simpler notation, we will just use a collective index $k = 1, 2, \dots, \infty$ to refer to all of these, since operators in different multiplets are automatically correlation orthogonal. Thus, our set of N lattice operators can in principle be written as

$$O_i = \sum_{k=1}^{\infty} f_{ik} \tilde{Q}_k, \quad i = 1, \dots, N, \quad (15)$$

where the exact form of the transformed field operators $\{\tilde{Q}_k\}$ defined by Eq. (7) depends on the distance between operators that eventually will be used in correlation functions, as detailed in Sec. II A. Without explicitly indicating this r dependence, the covariance matrix of the lattice operators is

$$C_{ij}(\mathbf{r}) = \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} f_{ik} f_{jp} \langle \tilde{Q}_k(\mathbf{r}) \tilde{Q}_p(\mathbf{0}) \rangle, \quad (16)$$

where the elements of f may also have some (weak) r dependence, as will be further discussed below. Since the $\tilde{Q}$ operators diagonalize the covariance matrix for given r , with eigenvalues given by Eq. (8), we have

$$C_{ij}(\mathbf{r}) = \sum_{k=1}^{\infty} f_{ik} f_{jk} A_k r^{-2\Delta_k}. \quad (17)$$

Diagonalizing C corresponds to finding specific linear combinations of the N lattice operators,

$$\tilde{O}_i = \sum_{a=1}^N g_{ia} O_a, \quad i = 1, \dots, N, \quad (18)$$

and we again denote by $D(\mathbf{r})$ the result of the orthogonal transformation of $C(\mathbf{r})$ that will eventually be diagonal:

$$D_{ij}(\mathbf{r}) = \sum_{a=1}^N \sum_{b=1}^N g_{ia} g_{jb} C_{ab}(\mathbf{r}), \quad (19)$$

which with Eq. (17) becomes

$$D_{ij}(\mathbf{r}) = \sum_{a=1}^N \sum_{b=1}^N \sum_{k=1}^{\infty} g_{ia} g_{jb} f_{ak} f_{bk} A_k r^{-2\Delta_k}. \quad (20)$$

Here the sums over a and b just represent transformed coefficients defining a matrix w ; thus,

$$D_{ij}(\mathbf{r}) = \sum_{k=1}^{\infty} w_{ik} w_{jk} A_k r^{-2\Delta_k}. \quad (21)$$

If the sum over k would extend only to $k = N$, then we could now simply conclude that a diagonalizing transformation corresponds to $w_{ik} = \delta_{ik}$ and a complete separation of the different scaling dimensions, with the eigenvalues D_i then decaying as in Eq. (8). The fact that the sum includes an infinite number of power laws implies that the decoupling is not perfect. However, with the rapidly increasing exponents $2\Delta_k$ with k , the remaining mixing should be completely negligible

even when N is not very large, as we will indeed see in the numerical examples in the following sections.

Note that, while it is only the elements of g in Eq. (19) that are determined by diagonalization of a numerically computed covariance matrix C , in principle the f coefficients in the expansion of the lattice operator into the correlation orthogonal field operators can also be determined from g , the trivial (unit) matrix w (under the assumption of no contributions from scaling dimensions Δ_k with $k > N$), and the prefactors A_k of the power law decays in Eq. (21). The expansion in the conventional operators Q requires additional CFT information in the form of the coefficients c_{ij} in Eq. (5). We will here not be concerned with these potential further useful aspects of the covariance method beyond the eigenvalues.

Unlike the ideal operators (with a single scaling dimension) considered in Sec. II A, the complete disentanglement in the exact form of Eq. (8) should only apply asymptotically for large r . Moreover, a critical point of a lattice model does not correspond exactly to the CFT fixed point but only flows to this point when the lattice size L and the distance r (in correlation functions) are taken to infinity. For finite length scales, irrelevant perturbations are present and lead to multiplicative corrections of the form $(1 + \sum_i \lambda_i L^{-\omega_i})$ or $(1 + \sum_i \lambda_i r^{-\omega_i})$, where $\omega_i = \Delta_i - d$ for all symmetry allowed operators with $\Delta_i > d$ (and the relevant perturbations with $\Delta_i < d$ have been tuned away to reach the critical or multicritical point). These corrections are well known for critical scaling of observables in finite lattices [54–56] and should also affect the eigenvalues of the covariance matrix. The corrections should stem from the expansion of the lattice operators in the field operators in Eq. (16), where the coefficients f would exhibit some L and r dependence. However, asymptotically the unknown matrix f and the diagonalizing transformation g in Eq. (19) should be independent of L and r .

The finite number N of operators is also of course a limitation, as already discussed above, but still one would expect to resolve N scaling dimensions in principle. However, numerical errors in the original covariance matrices $C(\mathbf{r})$, in particular statistical MC errors, will in practice make it impossible to determine more than a small number of scaling dimensions, considering the rapid increase in the decay power (in steps of 4) with the descendant index n , Eq. (4), for operators with fixed Lorentz spin l . The results presented below will show that one or two descendant dimensions can typically be extracted along with the primary, which is still very useful and exceeds what can be achieved by fitting conventional correlation functions.

III. TWO-DIMENSIONAL ISING MODEL

As a first test, we here study the conventional classical Ising model with Hamiltonian

$$H = - \sum_{\langle ij \rangle} \sigma_i \sigma_j, \quad \sigma_i \in \{-1, +1\}, \quad (22)$$

with the spins residing on an $L \times L$ square lattice with standard periodic boundary conditions. With the coupling constant set to unity above, the critical temperature is $T_c = 2 / \ln[1 + \sqrt{2}]$ and we study the model with MC simulations at this point, using a combination of Swendsen-Wang

[57] and Wolff [58] cluster updates, the latter with the number of clusters constructed in each MC step adapted so that roughly N spins are visited. The covariance matrices of all the operators illustrated in Fig. 1 are computed for $\mathbf{r} = (r, 0)$ and $(0, r)$ (and these are averaged) after every few MC steps and binned, with a data bin representing averages over 10^6 or more MC steps. Diagonalization is carried out post simulation using a large number (of the order 10^4) of stored data bins, using bootstrap analysis for statistical errors.

A full symmetrized operator O_i corresponds to starting from a single-operator product O_{ia} , $a = 1$, inside the 3×3 cell as in Fig. 1, which can always be written as

$$O_{ia}(\mathbf{r}) = \prod_{\{c(\mathbf{r})\}} \rho_c, \quad \rho_c \in \{\sigma_c, I\}, \quad (23)$$

where $\{c(\mathbf{r})\}$ is the set of nine sites belonging to the cell centered at $\mathbf{r}$ and ρ_c is either an Ising spin operator (for the sites with a red dot in Fig. 1) or the identity operator I (sites with no dot). Symmetrization corresponds to summing over all unique symmetry equivalent locations of the σ_c products, generated by rotating about the cell center and reflecting about the central horizontal or vertical axis. Thus, the symmetrized operators can be expressed as

$$O_i = \frac{1}{n_i} \sum_{a=1}^{n_i} O_{ia}(\mathbf{r}), \quad (24)$$

where a refers to the different transformed operator patterns and n_i is the number of such unique patterns, which is indicated for each operator in Fig. 1. The distance r between operators in a correlation function Eq. (5) is defined as the center-center distance between the 3×3 spin cells. We will first study the r -dependent eigenvalues for a fixed system size, mainly $L = 256$ but with some results also for $L = 128$ to check for finite-size effects. After systematically investigating the role of the number N of operators, we will also consider smaller systems and study the covariance matrices for $r = L/2$ versus L .

The goal here is to resolve some of the known scaling dimensions of the 2D Ising critical point. For 2D CFTs with central charge $c \leq 1$, the possible values of c are $c = 1 - 6/[m(m+1)]$ for integer $m \geq 3$. For given m , the primary scaling dimensions take the form [30]

$$\Delta_{jk} = \frac{[k + m(k-j)]^2 - 1}{2m(m+1)}, \quad 1 \leq j \leq k < m. \quad (25)$$

The case $m = 3$ ($c = 1/2$) corresponds to the conventional Ising critical point, while $m = 4$ ($c = 7/10$) is the tricritical Ising class that will be studied with the BC model in Sec. V. The two $l = 0$ primaries for $m = 3$ (corresponding to the spatial symmetrized operators described above) are $\Delta_{22} = 1/8$ (Z_2 odd) and $\Delta_{21} = 1$ (Z_2 even), which for simplicity we will just refer to as Δ_0 in their respective Z_2 sectors.

A. Single-operator correlations

The correlation functions of the six individual operators in both symmetry sectors, i.e., the diagonal elements of the covariance matrix $C_{ij}(r)$, are graphed versus r in log-log plots in Fig. 2. Given that all the lattice operators should contain the

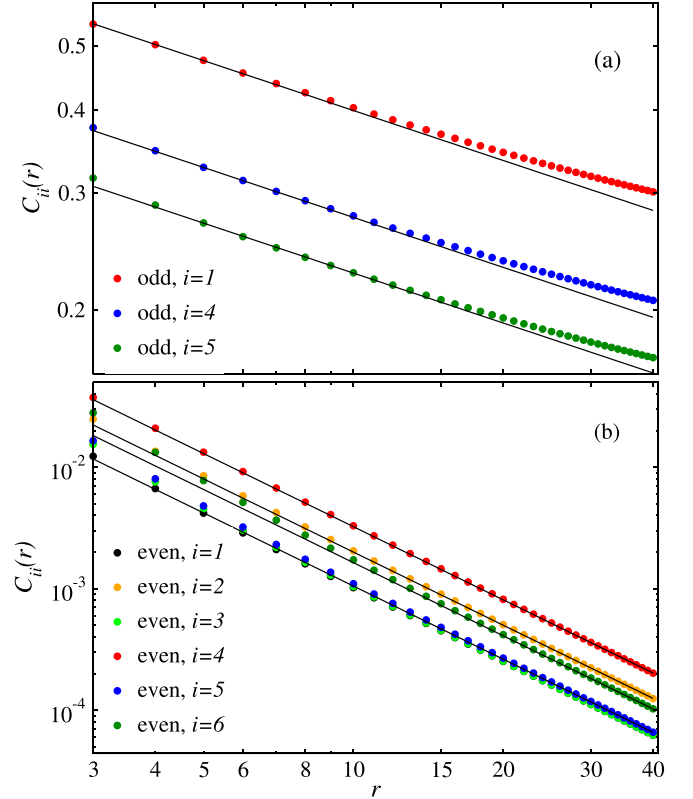


FIG. 2. Diagonal elements of covariance matrices for Z_2 odd (a) and even (b) operators, shown in log-log plots for lattice size $L = 256$ vs $r \ll L$. The index i corresponds to Fig. 1. In (a) the $i = 2$ and $i = 3$ correlations coincide very closely with those for $i = 1$ and are not shown. The $i = 6$ correlations are very close to those for $i = 4$ and are also not shown. The lines in (a) show the behavior $\propto r^{-1/4}$ expected for large r and L with $r \ll L$, and the lines in (b) similarly show the expected form $\propto r^{-2}$.

primary CFT operator in the respective Z_2 sectors, asymptotic decays of the form $r^{-2\Delta_0}$ are expected, with $\Delta_0 = 1/8$ in the odd sector and $\Delta_0 = 1$ in the even sector. The small value of the odd scaling dimension implies larger effects of the periodic boundaries, i.e., the $1 \ll r \ll L$ limiting behavior is harder to realize, even on lattices much larger than $L = 256$ used here. Indeed, clear deviations from the expected behavior are seen at the larger values of r in Fig. 2(a), while in the small range $3 < r < 10$ the expected behavior is reasonably well reproduced. In Fig. 2(b), the decay exponent is much larger and no boundary enhancements of the correlations are detectable within statistical errors.

In principle, it should be possible to gain some information also on the descendant scaling dimensions from these individual correlation functions, at least those in Fig. 2(b). However, as already discussed in Sec. II B, the short distance correlations not only mix contributions with different decays $r^{-2\Delta_n}$ but also contain other (related) power laws originating from various scaling corrections. It is therefore not possible in practice to reliably extract even the first visible descendant dimension $\Delta_1 = \Delta_0 + 4$, in particular in the odd Z_2 sector, Fig. 2(a), where also the boundary effects are very significant.

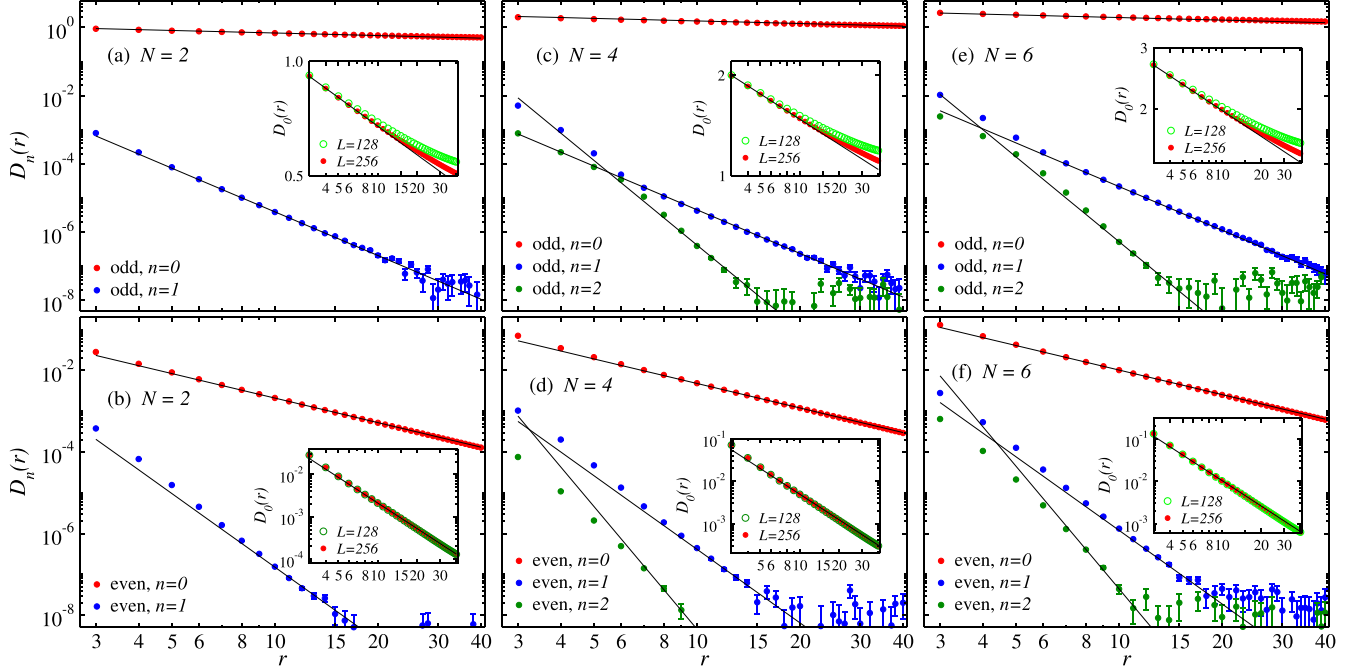


FIG. 3. Distance-dependent eigenvalues of covariance matrices involving $N = 2$ Ising operators in panels (a), (b), $N = 4$ in panels (c), (d), and $N = 6$ in panels (e), (f). In the notation of Fig. 1, the set of antisymmetric operators is $\{O_1, O_4\}$ in panel (a), $\{O_1, \dots, O_4\}$ in panel (c), and $\{O_1, \dots, O_6\}$ in panel (e). The symmetric operators are $\{O_1, O_5\}$ in panel (b), $\{O_1, O_4, O_5, O_6\}$ in panel (d), and $\{O_1, \dots, O_6\}$ in panel (f). In the main graphs the lattice size is $L = 256$ and the insets show $L = 128$ and $L = 256$ results for the largest eigenvalue in each case. The lines in these log-log plots show the expected power-law decay $\propto r^{-2\Delta_0}$, corresponding to the CFT primary operator with $\Delta_0 = 1/8$ in the top row and $\Delta_0 = 1$ in the bottom row, as well as the faster decaying power-laws originating from the leading descendants, $\propto r^{-2(\Delta_0+2)}$ and, in the case of panel (c)–(f), also $\propto r^{-2(\Delta_0+4)}$.

B. Covariance eigenvalues for large L and $r \ll L$

In the MC simulations all correlations were computed between $N = 6$ operators within the even and odd sets illustrated in Fig. 1, i.e., in each case 36 different correlation functions. Any subset of these correlations forming a smaller matrix can be selected post simulation for the diagonalization process. When selecting such subsets of operators (and of course also when constructing the full set), there are opportunities for optimization, and some results showing variations in results for different subsets will be discussed below. First, Fig. 3 shows $L = 256$ results for eigenvalues of covariance matrices computed with $N = 2, 4$ and 6 operators, with the operators included for $N = 2$ and $N = 4$ not necessarily the optimal ones but among the best ones from the full set of size operators. Here we also refer to the eigenvalues $D_n(r)$ for given r using the index $n = 0, \dots, N - 1$ from the smallest to the largest. Then the large- r decay form should match the scaling dimension with the same label; $D_n \sim r^{-2\Delta_n}$. However, in some cases the order of the eigenvalues is different for small r .

For $N = 2$, Figs. 3(a) and 3(b), the expected power law decays corresponding to the two lowest scaling dimensions can be resolved, as demonstrated with the straight lines showing the expected decay forms. There are some deviations from these ideal forms at short distances, on account of the corrections discussed in Sec. II B. In the case of the largest eigenvalues in the odd sector, Fig. 3(a), there are also large deviations at the longer distances, as shown in more detail in the insets for both $L = 128$ and $L = 256$. The upward

deviations from the power law decay, with larger deviations for the smaller system size, are clearly again due to effects of the periodic boundary conditions, as in Fig. 2(a). It is noteworthy, however, that the enhancement of the correlations for $L = 256$ are actually a bit smaller in the eigenvalue than in the diagonal elements C_{ii} , for reasons that are not obvious.

The reason for the overall significant boundary effects in Figs. 2(a) and 3(a), i.e., the difficulty in realizing the $1 \ll r \ll L$ limit, stems from the small value of the scaling dimension of the odd primary, the decay exponent being $2\Delta_0 = 1/4$. There are no apparent boundary effects in the fast decaying second eigenvalue in Figs. 3(a), though the error bars are large for $r \gtrsim 20$ and might in principle hide a small upward deviation. Significant boundary effects are unlikely, however, because the expected (and numerically confirmed) decay exponent is large, $2\Delta_1 = 4.25$, and the relative size of the boundary correction should be of order $[r/(L - r)]^{-2\Delta_1}$. In the even sector, Fig. 3(b), the exponent of the largest eigenvalue is $2\Delta_0 = 2$, and also here there are no discernible boundary effects or differences between $L = 128$ and 256 for the distances $r \leq 40$ graphed, and no upward deviations are seen at the largest r . In this case, the second eigenvalue is affected by larger relative corrections at short distances, and the expected exponent $2\Delta_1 = 6$ can be observed only rather close to the values of r at which the error bars begin to render the data useless for fitting.

Moving to the $N = 4$ results in Figs. 3(c) and 3(d), here also the second descendants can be resolved, though marginally in the case of the even sector, where only the points

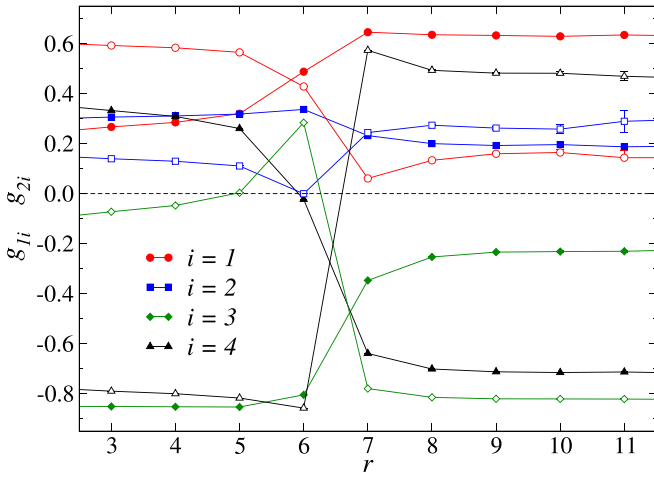


FIG. 4. Distance-dependent elements of the eigenvectors corresponding to the $n = 1$ and $n = 2$ eigenvalues in Fig. 3(c). The filled symbols of different colors show the $n = 1$ coefficients g_{1i} and the corresponding $n = 2$ values g_{2i} are shown with the open symbols of matching shapes and colors.

$r = 7, 8, 9$ fall close to the expected form before the error bars become too large. The fourth eigenvalues are completely noise dominated overall and are therefore not shown. In Fig. 3(c) the $n = 1$ and $n = 2$ eigenvalues cross each other close to $r = 6$. The color coding of the data points only refer to the ordering of the eigenvalues for each r , but it is clear from the crossing behavior that the association of the data points with scaling dimensions swap. This behavior is confirmed by examining the eigenvectors, the elements of which are graphed versus r in Fig. 4. All the $n = 1$ and $n = 2$ elements swap between $r = 6$ and $r = 7$, and then become essentially r independent for $r > 8$. Note that all four operators contribute to the diagonalizing linear combinations.

Increasing the number of operators to $N = 6$, Figs. 3(e) and 3(f) still only include three eigenvalues because the higher ones are again completely noise dominated. The results qualitatively do not look very different from those obtained with $N = 4$, but it should be noted that the overall amplitudes of the eigenvalues increase somewhat with N . While the impact of the statistical noise also increases slightly, the relative errors overall diminish with increasing N —this trend is clear at least for the $n = 1$ eigenvalues. Thus, using larger N may be helpful even if the number of resolved scaling dimensions is less than N . The favorable effect of the overall scale of the eigenvalues increasing with N likely reflects improving flexibility in forming linear combination of the lattice operators that maximize the overlaps with the field operators. However, it should be kept in mind that the computational effort of accumulating the covariance matrix scales as N^2 and can easily dominate the total effort of simulations. Realistically, N would therefore typically be no larger than five to ten.¹

¹In principle, two-point and multipoint correlation functions could be computed at arbitrary separation and an essentially unlimited range of covariance matrices could be constructed with them post simulation. In cases such as the simple Ising models, these

C. Statistical errors

It should be noted here that the noise level can be much smaller than what would be expected from the individual correlation functions $C_{ij}(r)$, such as the diagonal ones in Fig. 2, where the statistical errors of the Z_2 -odd operators are of the order 10^{-6} or above, while in Fig. 3 the eigenvalues are resolved down to values as small as 10^{-8} . The reason why the diagonalization procedure can resolve such small eigenvalues must be that the MC statistical fluctuations of C_{ij} for all i, j are correlated, i.e., the individual error bars do not give a complete picture of the information contained in the entire data set. This effect of noise correlations is less pronounced for the Z_2 -even operators, where the individual correlation functions also have statistical errors of order 10^{-8} , similar to the noise cut-offs in Fig. 3.

At face value, it may appear as if many eigenvalues in Fig. 3 for large r are stable (with relatively small error bars) around values of order 10^{-8} , above the expected much smaller values. However, this apparent flaw is mainly a quirk of the logarithmic scale when graphing errors defined as one standard deviation of the mean. As an example, the $n = 1$ eigenvalue at $r = 29$ in Fig. 3 is less than 2.5 times its own error bar, thus reasonably consistent with an extrapolated value $\approx 2 \times 10^{-9}$ if the power law decay for the smaller distances continues to apply as expected. There may still be a small positive bias on the overall errors. Statistically, a roughly equal number of positive and negative eigenvalues should be expected when the actual eigenvalues are extremely small. However, there are no negative $n = 1$ eigenvalues in Fig. 3(f) [while there are some in the $n = 2$ set and also other graphs, except for in Fig. 3(a)].

A small positive bias in the eigenvalues could possibly reflect the fact that matrix diagonalization is a nonlinear procedure, which implies that the statistical errors of the eigenvalues do not have zero expectation value for a finite data set. As a trivial example of such noise bias, consider an MC estimate $\bar{A}$ of some expectation value $\langle A \rangle$, which can be written as $\bar{A} = \langle A \rangle + \delta$, where the unknown error δ is of order $M^{-1/2}$, where M is the total number of MC measurements performed. Denoting an expectation value over different MC simulations by $[\]$, we have $[\bar{A}] = \langle A \rangle$ because $[\delta] = 0$. However, for the square of the MC estimate we have $[\bar{A}^2] = \langle A \rangle^2 + [\delta^2]$, i.e., it is biased toward higher values. The bias vanishes at the rate M^{-1} but can be statistically significant even for large M if $\langle A \rangle$ is very small or if the prefactor is large. In the case of matrix diagonalization, a quadratic error leading to an M^{-1} bias can similarly be expected in the eigenvalues (which can easily be confirmed for a 2×2 matrix), and it can also be expected that the prefactor of the M^{-1} scaling form increases with N .

correlation functions can also be computed using improved cluster estimators to reduce statistical errors and improve computational efficiency. Here the specific lattice operators were only implemented with standard estimators by simply evaluating the corresponding cell values for each spin configuration and accumulating the relevant products of those numbers.

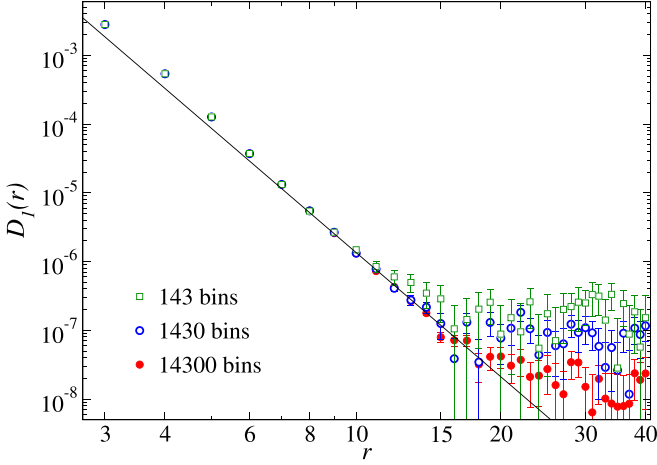


FIG. 5. $n = 1$ eigenvalues from Fig. 3(f) (red solid circles) graphed along with results obtained when using only 10% (blue open circles) and 1% (green open squares) of the available 14 300 data bins. Each bin corresponds to 10^6 MC steps. The line shows the decay form r^{-6} expected with the $n = 1$ scaling dimension $\Delta_1 = 3$.

To test the ability to resolve the eigenvalues with data noisier than in Fig. 3, in Fig. 5 the $n = 1$ eigenvalues from Fig. 3(f) are graphed along with results obtained when only a small fraction, 10% and 1%, of the original data set is used. When the size of the data set is reduced, the positive values of the noise dominated large- r eigenvalues increase overall roughly in the way statistically expected, i.e., with 1% of the data used, the noise level (taken as the standard deviation or typical magnitude of the positive eigenvalues for large r) increases by an order of magnitude. Again, the logarithmic scale can be misleading for the error bars, as the largest deviation from 0 in the 1% data set is about 2.5 error bars. There are no negative mean values in the data sets in Fig. 5, suggesting a positive bias. The bias should grow by a factor of 100 when reducing the data set to 1%, which is not seen in Fig. 5, though the increase may be marginally above the factor 10 expected for unbiased statistical fluctuations. The bias effect, quantified by the unknown prefactor of the M^{-1} scaling of the quadratic error discussed above, therefore must be very small, though it may still be responsible for the lack of negative eigenvalues for large r , in combination with the fact that the true eigenvalues are positive though very small. The positive bias may be of significance in the way the eigenvalues initially begin to deviate from the power law decay, i.e., above $r \approx 10$ in the 1% data but only above $r \approx 20$ when the full data set is used. The deviations are mainly upward, a behavior typically (not always) seen before the data become completely noise dominated with the apparent flattening-out behavior discussed above.

As an example of the computational effort spent on these calculations, in the case of the $L = 256$ lattice (all the data in Figs. 3–5) each data bin consisted of 10^6 MC steps and the covariance matrices were accumulated after every 10 steps. Each bin required about 10 CPU (core) hours of compute time and the total of more than 10^4 bins were generated by running independent simulations simultaneously on about 100 cores.

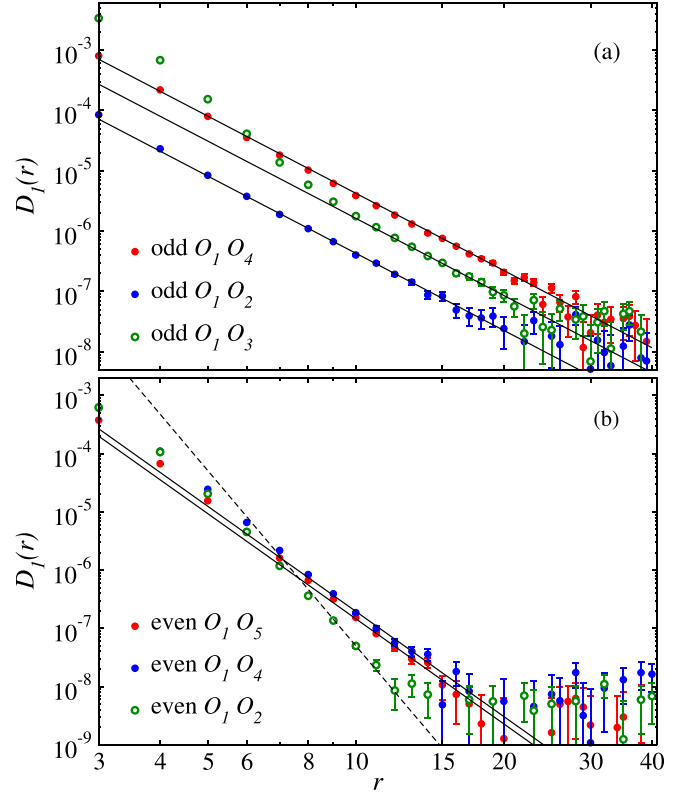


FIG. 6. Dependence on the choice of operators, illustrated by the second eigenvalues D_1 obtained when diagonalizing 2×2 covariance matrices in the odd (a) and even (b) Z_2 sectors. In each case, the operators used are indicated in the legends in the notation of Fig. 1. The solid lines show the $\propto r^{-2\Delta_1}$ behaviors expected asymptotically for the first descendant operators, with $\Delta_1 = 17/8$ in (a) and $\Delta_1 = 3$ in (b). The dashed line in (b) shows the behavior expected for the second even descendant, which has $\Delta_2 = 5$.

D. Dependence on the operator set

The choice of operators for given N can also play a crucial role in optimally resolving the scaling dimensions. The six operators used here in both Z_2 sectors of course do not exhaust all possibilities on the 3×3 cells but represent intuitive choices based on small numbers of σ_i operators. When using N smaller than the full number of operators implemented in the simulations, different combinations of operators can be investigated post simulation. Figure 6 shows examples using $N = 2$ with three different even or odd operator pairs from those in Fig. 1. Only the second eigenvalue is graphed in each case, corresponding to the first descendant dimension Δ_1 . In the case of the odd sector in Fig. 6(a) the correct decay exponent is well reproduced in all cases, but with the combination (O_1, O_3) the corrections for small r are much more prominent. The best choice among the three pairs is (O_1, O_4) , which produces the largest overall magnitude of the eigenvalues and, therefore, the smallest relative statistical errors. The scaling corrections for small r are also small.

In the case of the even operators in Fig. 6(b), the pair (O_1, O_2) does not produce the expected decay governed by Δ_1 , but for the larger r values, before the statistical cut off, the decay exponent is instead close to that expected with Δ_2 . It is

likely that the overlap with the first CFT descendant is very small with these two operators, while being relatively large with the second descendant. The cross-over to the asymptotic form governed by Δ_1 will then take place only for larger distances. This potential problem should not be serious for larger N , where better overlaps with the most important operators should be expected. The variations of the results with N can also be regarded as an opportunity to optimize the performance of the method.

These examples show that the set of operators included in the covariance matrix can have large effects on the quality of the results. Since the size N of the full set used in a simulation has to be rather small, because of the N^2 scaling, the operators included should be considered carefully by testing, before large scale simulations are carried out.

E. Small system sizes L and $r = L/2$

In $d = 2$ classical systems it should in general not be difficult to realize the range of distances $1 \ll r \ll L$ for which pure power law decays can be observed. However, in higher dimensions, especially in the context of electronic models in $d = 2 + 1$ dimensions simulated with auxiliary field quantum MC methods [59–62], only relatively small systems can be accessed in practice. In conventional studies of correlation functions it is then useful to set r to a fraction of the system length L , e.g., to study correlations versus $r = L/2$. Though the correlations there are maximally enhanced by the periodic boundary effects, the decay power is still the same as for $r \ll L$ in a large system. The utility of the covariance method will of course be much broader if it can also be used without modifications for scaling with $r \propto L$. Since the $L \times L$ square lattice does not have the ideal conformal geometry, the CFT arguments for disentangling of scaling dimensions in Sec. II A may not a priori hold unless $r \ll L$. However, one would still expect the same scaling dimensions to be at play, for the same reason that the standard correlation functions decay with the correct critical exponents $2\Delta_0$ at $r \propto L$. There is also nothing specific about the arguments in Sec. II A that would point to them not remaining valid for all covariance eigenvalues for $r \propto L$, as long as L is sufficiently large and the decay of all correlations of CFT operators still are of the form Eq. (5). We thus proceed to study the covariance eigenvalues of small Ising lattices at $r = L/2$.

Figure 7 shows results for system sizes up to $L = 36$ versus $r = L/2$, using covariance matrices with all the operators in Fig. 1. The results do look similar to those in Fig. 3, but for these small systems also the fourth ($n = 3$) eigenvalue can be well resolved beyond statistical errors because the overall larger amplitudes of the boundary enhanced eigenvalues. The decay exponent is not the expected $2\Delta_3$, however, but instead is very close to $2\Delta_2$ for the largest several system sizes in both Z_2 sectors. This failure to capture Δ_3 likely just reflects transient behavior, and crossovers to the expected forms should take place for larger systems. The eigenvectors for sufficiently large r also are the same in systems with $r \ll L$ and $r = L/2$, as shown in Fig. 8 for the case of $n = 0, 1$, and 2 in the Z_2 -odd sector. All these results indicate that the disentangling of the scaling dimensions works independently of the system geometry.

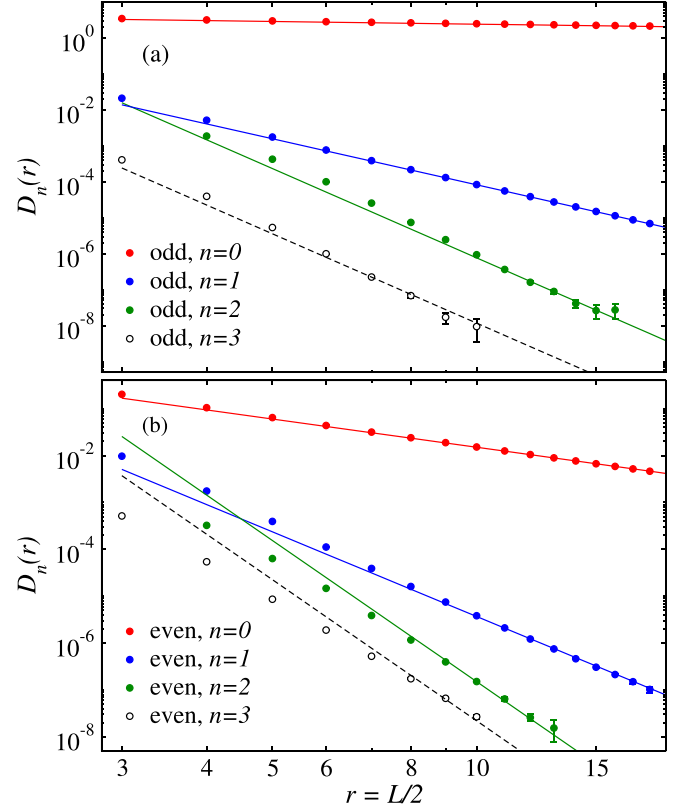


FIG. 7. Eigenvalues of the Z_2 odd (a) and even (b) covariance matrices computed with all the operators in Fig. 1 at distance $r = L/2$ for system sizes from $L = 6$ to $L = 36$. The red, blue, and green lines drawn adjacent to the $n = 0, 1, 2$ data have the slopes expected with the scaling dimensions of the respective Ising primaries and their first two descendants, as in Fig. 3. The dashed black lines adjacent to the $n = 3$ data also have slopes corresponding to Δ_2

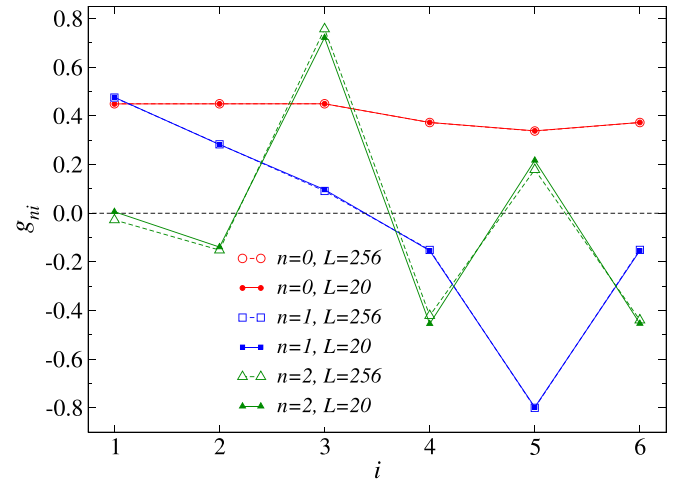
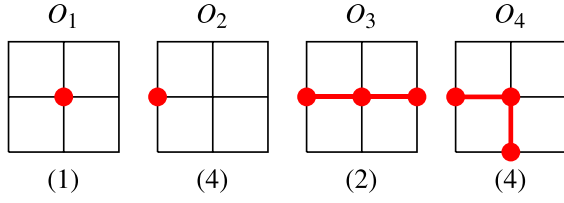


FIG. 8. Eigenvectors in the Z_2 -odd sector corresponding to the scaling dimensions Δ_0 , Δ_1 , and Δ_2 , obtained from $r = 10$ covariance matrices with $N = 6$ in a large system, $L = 256 \gg r$ (open symbols), and in a small system of size $L = 20$, where $r = L/2$. The corresponding eigenvalues for $L = 256$ and $L = 20$ are shown in Figs. 3(e) and 7(a), respectively. The two data sets for $n = 0$ and $n = 1$ overlap to a large extent while the minor differences between the $n = 2$ sets reflect the statistical errors for $L = 256$.

(a) spin-reflection odd operators



(b) spin-reflection even operators

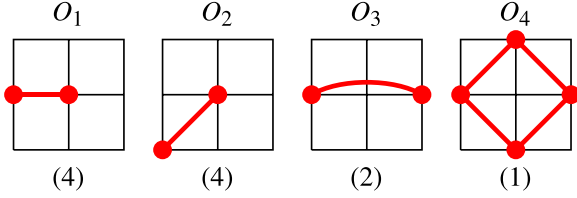


FIG. 9. Z_2 antisymmetric (a) and symmetric (b) operators on 3×3 plaquettes used for the covariance matrices of the 3D Ising model. The red circles and lines correspond, respectively, to spin operators and products as in Fig. 1, and the number of cell rotations used to construct point-group symmetric operators is indicated beneath each cell.

IV. THREE-DIMENSIONAL ISING MODEL

The scaling dimensions of the 3D Ising model are not known rigorously, but recent works with the numerical conformal bootstrap method have produced high-precision results that are believed to be exact within small windows of uncertainties originating from the “islands” of allowed values [1]. Here we use the same MC methods as in Sec. III to study the Ising model on the simple cubic lattice with L^3 spins at its critical point, testing scaling dimensions in both even and odd Z_2 sectors using the operators depicted in Fig. 9. These purely in-plane operators are similar to the ones studied above in the case of the 2D Ising model, some of them being identical to those in Fig. 1. While we could also study fully symmetric 3D operators, we here take the opportunity to connect to quantum models in $d = 2 + 1$ dimensions, where it is normally not practical to define operators with a finite extent in imaginary time. Even if such definitions could be implemented, there is still not complete space-time symmetry of the model—only emergent Lorentz symmetry with a model-dependent velocity relating space and time distances. With the in-plane operators of the classical 3D Ising model, we study separations that are, in the classical-quantum mapping, either purely spacelike, as illustrated in Fig. 10(a), or purely timelike, as in Fig. 10(b), when considering the z direction as discretized time in a quantum model.

The lack of full 3D symmetry of the operators implies that they do not correspond to a fixed Lorentz spin l but mix $l = 0$ and $l = 2$ contributions. The relevant operators are in the $l = 0$ sector, with the symmetric (ϵ) and antisymmetric (σ) ones having scaling dimension $\Delta_\epsilon = 1.412625(10)$ and $\Delta_\sigma = 0.5181489(10)$, respectively, where the numbers within () indicate the uncertainty in the preceding digit [1]. The following primaries in these sectors have dimensions $\Delta_{\epsilon'} = 3.82968(23)$ and $\Delta_{\sigma'} = 5.2906(11)$. Moreover, in the

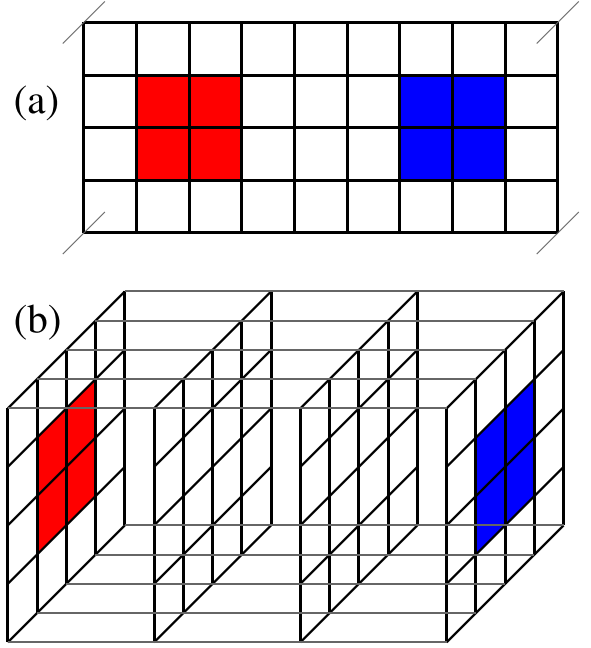


FIG. 10. Schematic illustration of correlation functions of the 3D Ising model defined with the 2D 3×3 site operators in Fig. 10. In panel (a) both operators reside in the same xy plane (corresponding to spacelike separations in a corresponding quantum system) at separation $\mathbf{r} = (5, 0, 0)$, while in panel (b) the operators are in different planes z (timelike separation) but with the same in-plane xy coordinates, with $\mathbf{r} = (0, 0, 3)$.

$l = 2$ sector the energy-momentum tensor is marginal, i.e., with scaling dimension $\Delta_T = 3$, and in the same sector the dimension of following operator is $\Delta_{T'} = 5.50915(44)$. Since descendants of all these operators exist in both the $l = 0$ and $l = 2$ sectors, pairs of eigenvalues with the same scaling dimension $\Delta_X + 2n$ should be expected for all the primaries X . With only four operators in the covariance matrices, we can at most detect the four lowest scaling dimensions in each of the two Z_2 sectors. We only study relatively small systems at $r = L/2$ in this case.

The critical temperature of the 3D Ising model is known to high precision from large-scale MC simulations; $T_c = 4.51152325(10)$ [63]. Figures 11(a) and 11(b) show spacelike eigenvalues at criticality, while Figs. 11(c) and 11(d) show the corresponding timelike eigenvalues. The results overall are very similar and the primaries and doubled first descendants can be convincingly detected in the decays for the largest distances $r = L/2$. However, the timelike separation appears to have an advantage in smaller scaling corrections, which intuitively may be related to the fact that the timelike operator pairs are more symmetric, maintaining the symmetries of the individual operators. Moreover, all the site-site distances between the two operators are closer to the center-center distance r on the case of timelike separations. For these reasons, most likely, it will also be better to study purely timelike correlations in quantum systems.

In the case of the 2D Ising model, though the $n = 2$ and $n = 3$ data sets in Fig. 7 also appear to be governed by the same scaling dimension, this is most likely a nonasymptotic

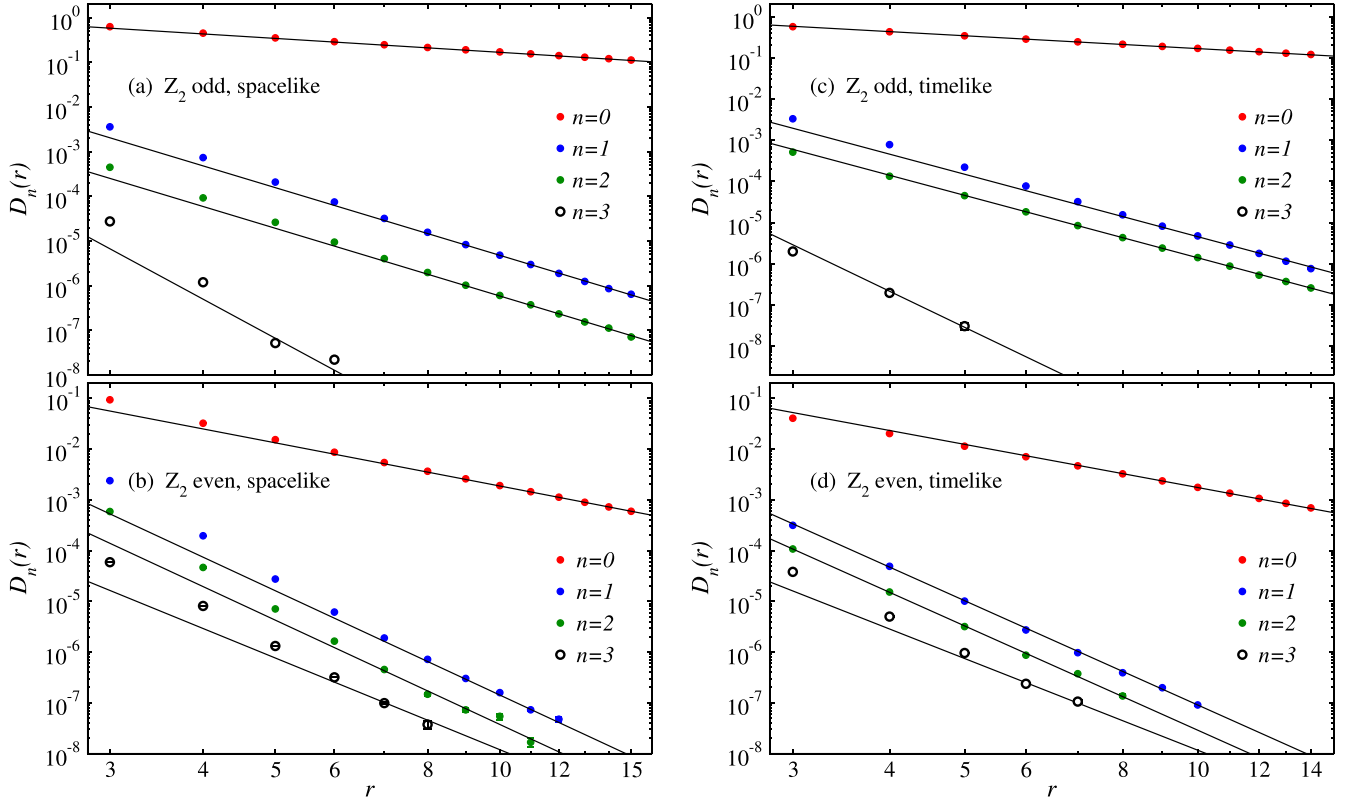


FIG. 11. Covariance eigenvalues of the 3D Ising model vs $r = L/2$ and $L \leq 30$. The operators included are defined in Fig. 9. Results from space- and timelike correlations are shown in the graphs to the left (a), (b) and right (c), (d), respectively, corresponding to relative orientation of the operators according to the illustrations in Fig. 10. The lines fitted to the $n = 0$ data sets in panels (a), (c) show the expected scaling with the known value of $\Delta_\sigma \approx 0.518149$, while those for the $n = 1, 2$ sets correspond to the $l = 0$ and $l = 2$ descendants, both with scaling dimension $\Delta_\sigma + 4$. Similarly, the solid lines in panels (b), (d) show the expected scaling with the known value of $\Delta_\epsilon \approx 1.41262$ and the corresponding descendant dimensions $\Delta_\epsilon + 4$. The solid lines drawn close to the $n = 3$ data sets in (a)–(d) have slopes corresponding to scaling dimensions that likely describe these eigenvalues asymptotically, as discussed in the text.

feature of the $n = 3$ set, which is rather noisy, and the $n = 4$ set (not shown) is completely noise dominated. In contrast, in the case of the 3D Ising model, in Fig. 11 the $n = 1, 2$ data sets develop either the same decay after the visible corrections have decayed away or display no significant corrections. Moreover, the $n = 3$ sets in Figs. 7(a) and 7(c) clearly decay much faster, close to what is expected with Δ_3 . There is, thus, little doubt that the $n = 1, 2$ pairs really reflect the first $l = 0$ and $l = 2$ descendants with common scaling dimensions. In the Z_2 symmetric sector, the scaling dimension Δ_T of the energy-momentum tensor is smaller than the first descendant dimensions Δ'_ϵ , and in Fig. 11(d) there is at least some indication of the fourth eigenvalue decaying in the way expected with Δ_T . In Fig. 11(c), the decay of the fourth eigenvalue is much faster, close to what is expected with Δ''_σ . In the case of the spacelike even eigenvalues, for $n = 3$ in Fig. 11(b) the decay in the available range of r is somewhat faster than the expected $r^{-2\Delta_T}$ form, being close to the same $r^{-2\Delta'_\epsilon}$ as the $n = 1, 2$ pair. This behavior most likely again just reflects slower convergence to the asymptotic form than the corresponding timelike eigenvalue. The fourth Z_2 -odd eigenvalue on Fig. 11(b) decays rapidly but the behavior expected with Δ_T cannot be fitted to the small number of points available.

Despite the large uncertainty in the fourth eigenvalues, the $n = 0, 1, 2$ data demonstrate that the covariance method also

works with $r = L/2$ in 3D lattices. The better results from timelike correlations point to likely successful applications also to quantum criticality in $d = 2 + 1$ dimensions.

V. BLUME-CAPEL MODEL

At a multicritical point, two or more model parameters have to be tuned for all the relevant perturbations (scaling fields) to vanish. Lattice operators will in general contain all the relevant scaling fields. Separating them reliably by fitting correlation functions to multiple power laws is difficult and unreliable, considering also the presence of scaling corrections. While corrections are also present in the covariance matrix, as we saw in the previous sections, the asymptotic behaviors of each eigenvalue reflects a different scaling dimension. This way of disentangling scaling dimensions may help immensely in studies of multicriticality, where two or more operators with the same symmetries as the Hamiltonian are relevant. Here this capability of the covariance method will be exemplified by a study of the tricritical Ising universality class with the 2D BC model defined in Eq. (2). The Ising coupling has been set at unity and the two tuning parameters are the temperature T and the chemical potential λ .

The model parameters $(\lambda_t, T_t) = (1.965815, 0.608578)$ from transfer-matrix calculations [25] was used for the tri-

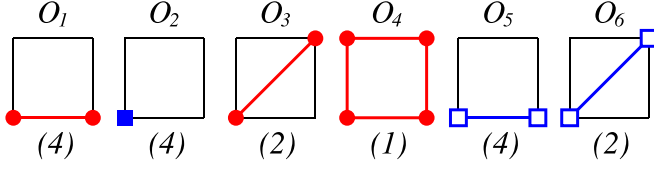


FIG. 12. Operators on 2×2 plaquettes used for the covariance matrices of the BC model. Red circles represent spin operators taking the values $\sigma_i \in \{-1, 0, +1\}$, blue solid squares $\sigma_i^2 \in \{0, 1\}$, open blue squares $1 - \sigma_i^2$. The colored lines imply products of the operators. The number of operations (cell rotations producing different patterns) used for symmetrizing is indicated beneath each plaquette.

criticality results presented here, but nearby points were also studied, including the estimate (1.9660, 0.6080) of the tricritical point obtained from Wang-Landau MC simulations [28]. While there are some minor differences in the results, the decay exponents and eigenvectors are stable for the system sizes considered here—further results supporting this stability of the scaling dimensions will be presented in Sec. VI in the context of the crossover from tricritical to regular Ising criticality on the continuous part of the phase boundary away from (λ_t, T_t) .

Standard MC simulations on $L \times L$ periodic lattices were again carried out, in this case combining Swendsen-Wang cluster updates [57] for the $|\sigma_i| = 1$ spins with Metropolis updates to change the number N of spins $|\sigma_i| = 1$ and to interchange $\sigma_i = 0$ vacancies and $|\sigma_j| = 1$ spins at different locations i, j . An MC step entails the construction of all Swendsen-Wang clusters and flipping each of them with probability $1/2$, followed by N and $2N$ attempts, respectively, of single-site and two-site Metropolis updates. Six operators are again used for the covariance matrices, but here we only consider the Z_2 -even sector and use plaquettes of 2×2 sites. The operators are illustrated in Fig. 12.

Given that the autocorrelation time is quite long in this model (because the vacancies are affected only by the local Metropolis updates) and the N^2 scaling of the covariance calculations, the matrices were accumulated only after every 100 MC steps in the case of the $L = 256$ system of main focus (and more often for smaller systems). The covariance matrices were again averaged over $\mathbf{r} = (r, 0)$ and $(0, r)$. A bin average of the covariance matrix involves 5×10^6 MC steps in this case and about 10^4 bins were generated.

The tricritical Ising point corresponds to $m = 4$ in the CFT hierarchy in Eq. (25). The values of the primary scaling dimensions corresponding to the fully symmetric operators studied here are, in the notation of Eq. (25), $\Delta_{33} = 1/5$, $\Delta_{32} = 6/5$, and $\Delta_{31} = 3$ (or $2h_\epsilon$, $2h_{\epsilon'}$, and $2h_{\epsilon''}$ with the conventions of Ref. [30]). Out of these, Δ_{33} and Δ_{32} are relevant and tuning them both to zero in a microscopic model will in general require two adjustable parameters; λ and T in the BC model. Since Δ_{33} and Δ_{32} differ exactly by 1, one should expect contributions to correlation functions that decay as $\propto r^{-2\Delta}$ with Δ taking all the values $1/5 + a$ for $a = 0, 1, 2, \dots$, as well as $3 + b$ with $b = 0, 2, \dots$. For simplicity of the notation, we will here just use Δ_i , $i = 1, 2, \dots$ to refer to all the scaling dimensions in order of increasing values (instead of $n = 0, 1, \dots$ used above for the Ising models, which has

only one primary, $n = 0$, in each Z_2 sector), and those that we in principle should be able extract using 6×6 covariance matrices are

$$\Delta_1 \equiv \Delta_{33} = 0.2, \quad (26a)$$

$$\Delta_2 \equiv \Delta_{32} = 1.2, \quad (26b)$$

$$\Delta_3 \equiv \Delta'_{33} = 2.2, \quad (26c)$$

$$\Delta_4 \equiv \Delta_{31} = 3.0, \quad (26d)$$

$$\Delta_5 \equiv \Delta'_{32} = 3.2, \quad (26e)$$

$$\Delta_6 \equiv \Delta''_{33} = 4.2, \quad (26f)$$

where primes refer to descendants. Similarly, we label the eigenvalues for given r by $D_i(r)$ with $i = 1, \dots, N$ in increasing order. For large r , the indices for the scaling dimensions and the eigenvalues should then match; $D_i \sim r^{-2\Delta_i}$. For small r , the ordering of the eigenvalues may again be different in some cases, as in Fig. 3. We will see some evidence of such crossing behavior of two of the eigenvalues for r beyond the largest accessible distance.

A. $N = 2$ eigenvalues and the tangent vector of the phase boundary

We first consider the 2×2 covariance matrix of O_1 and O_2 in Fig. 12, which make up the E and F terms, respectively, in the Hamiltonian Eq. (2). The diagonal elements (the conventional correlation functions) are graphed versus distance in Fig. 13(a). To test convergence with the system size, results for $L = 128$ and 256 are compared. The periodic boundary conditions again cause enhancements of the correlations that grow with r , but these boundary effects can largely be avoided at the shorter distances. There the two correlation functions are, as expected, completely dominated by the contribution from the smallest scaling dimension Δ_1 , but some deviations from the leading power law are visible at short distances. While there are finite-size effects left even in the limited range of r in Fig. 13(a), for $L = 256$ there is good agreement with the expected leading asymptotic exponent for $r \in [6, 20]$.

Figure 13(b) shows the two eigenvalues of the covariance matrix. The decay forms are indeed consistent with the two leading scaling dimensions Δ_1 and Δ_2 , though there are, as expected, corrections at the shortest distances. The eigenvectors $g_i = (g_{i1}, g_{i2})$ defining $Q_i = g_{i1}O_1 + g_{i2}O_2$ are $g_1 = (g_{11}, g_{12}) = (0.731, 0.682)$ and $g_2 = (g_{21}, g_{22}) = (0.682, -0.731)$ in the range $r \in [5, 20]$, with very small changes for larger distances.

The eigenvectors mixing the two energy terms in the Hamiltonian should give the special directions in the phase diagram along which critical scaling is governed only by the corresponding correlation length exponents $\nu_1 = (2 - \Delta_1)^{-1}$ and $\nu_2 = (2 - \Delta_2)^{-1}$. The above eigenvalues were, however, computed with relative weights different from those of E and F defined in Eq. (2). The proper weighting of the two operators is obtained with $2O_1$ and O_2 (corresponding to $2L^2$ Ising bond operators and L^2 single-site operators on the lattice), which gives the eigenvectors $g_1 = (0.906, 0.423)$

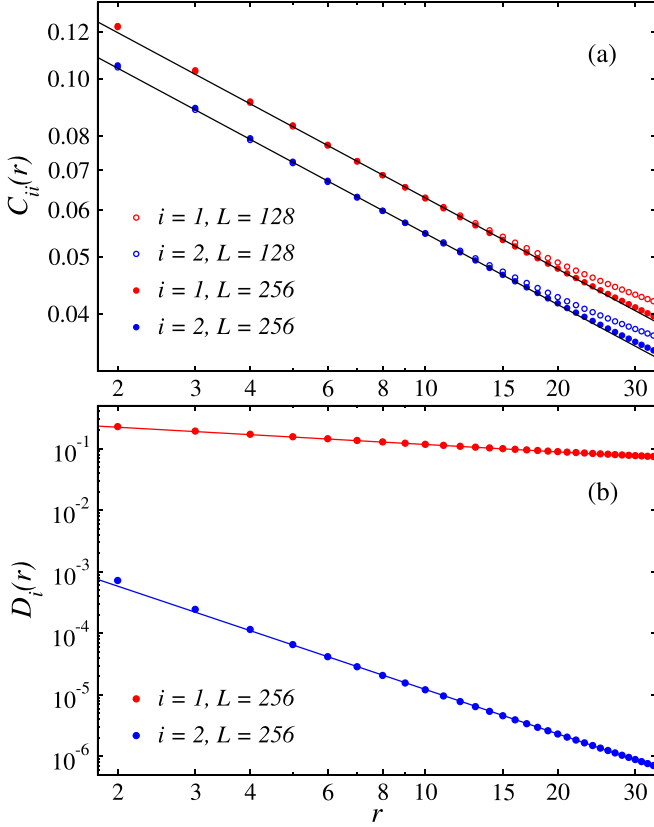


FIG. 13. (a) Distance-dependent correlation functions of the individual operators O_1 and O_2 in Fig. 12 computed on $L = 128$ and 256 lattices at the estimated tricritical point of the BC model. The lines show the power-law decays expected with the scaling dimensions of the smallest primary operator $\Delta_1 = 1/5$. (b) Eigenvalues of the distance-dependent covariance matrix of O_1 and O_2 for $L = 256$. The lines show the expected power laws with the scaling dimensions $\Delta_1 = 1/5$ and $\Delta_2 = 6/5$.

and $g_2 = (0.423, -0.906)$ (i.e., the ratio of the coefficients trivially change by a factor of 2). With the Ising interaction strength set to unity in Eq. (2), one of these vectors should define the tangential direction at the tricritical point of the phase boundary expressed as $\mu_c = -(\beta\lambda)_c$ versus β , where $\beta \equiv T^{-1}$. Examining the phase boundary λ_c versus T in Fig. 1 of Ref. [28] (which uses slightly different notation), a derivative $d\lambda_c/dT \approx -0.28$ at the tricritical point can be estimated. This translates to $d\mu_c/d\beta \approx -2.17$, which should equal the ratio g_{i2}/g_{i1} for one of the eigenvectors of the covariance matrix. Indeed, the derivative closely matches g_2 (corresponding to $\Delta_2 = 6/5$), with $g_{22}/g_{21} \approx -0.906/0.423 \approx -2.14$. This agreement within less than 2% is as good as could possibly be expected, given the uncertainty of the tangent estimated from the phase diagram in Ref. [28]. The present result for the tangent should be the best available so far, as it avoids the difficulties of constructing a challenging portion of the phase boundary to high precision.

The fact that Δ_2 , not Δ_1 , is the relevant scaling dimension here is also expected on account of the fact that the larger of the correlation length exponents, here $\nu_2 = (2 - \Delta_2)^{-1}$, in general governs the divergence of the correlation length on the line tangential to the phase boundary at a tricritical point.

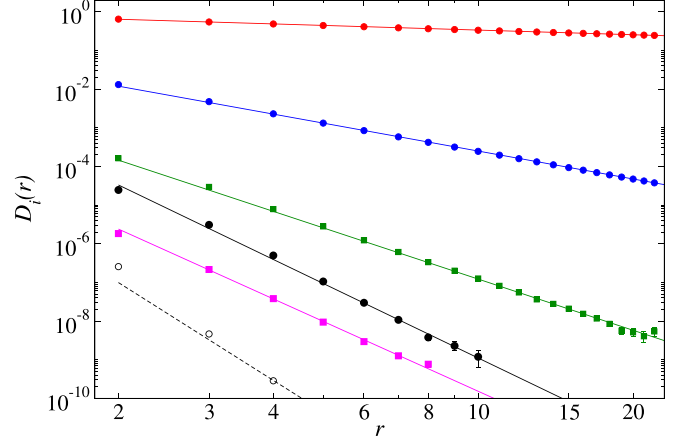


FIG. 14. (a) Eigenvalues vs distance of the BC covariance matrix of all the operators depicted in Fig. 12. The fitted lines show power laws with the exponents $2\Delta_i$ expected with the BC scaling dimensions listed in Sec. V. The lines correspond to $i = 1, 2, 3, 5, 4, 6$ from top to bottom.

B. $N = 6$ eigenvalues for $r \ll L$

Moving on to the full 6×6 covariance matrix, Fig. 14 shows $L = 256$ results for $D_i(r)$ up to $r = 22$, where the boundary effects are small. Remarkably, the six lowest exponents ($2\Delta_i = 0.4, 2.4, 4.4, 6, 6.4$, and 8.4) are all well reflected in the observed power-law decays, though in the case of $i = 6$ the decay is too fast to capture the asymptotic form, with large corrections likely affecting the three available data points. The $i = 4$ and $i = 5$ data sets appear to correspond to the scaling dimensions Δ_5 and Δ_4 , i.e., in the reverse order, as we saw examples of in the case of the Ising model in Sec. III. However, in Fig. 14 an eventual crossing of the eigenvalues for much larger r is less clear, given the closeness of the scaling dimensions $\Delta_4 = 3$ and $\Delta_5 = 3.2$, and it is also possible that the convergence to the ultimate power laws is just slow and there is no crossing of the eigenvalues.

The $i = 1, \dots, 5$ data sets can be fitted to power laws, which we do here in addition to showing the power laws with the known exponents $2\Delta_i$ in Fig. 14. We fit without corrections (additional power-law terms), including a range of r values for which the corrections appear to be relatively small and high quality fits can be obtained. A more careful procedure would consider corrections and also systematic investigations of small remaining finite-size effects (which are important only for the slowest decaying modes). The fitting results here should just be taken as examples of reasonable fits, not an exhaustive analysis.

Since the data for different r values are correlated, a statistically sound fitting procedure must include the full covariance matrix of the MC fluctuations, instead of just the conventional standard deviations used for uncorrelated data. Note that such an “error covariance matrix” $C_{\text{err}}(r, r')$ involves a range of distances included in the fits, $r, r' \in [r_{\min}, r_{\max}]$, for the same eigenvalue and is very different from the operator covariance matrix $C_{ij}(r)$. Some of the eigenvalues of $C_{\text{err}}(r_i, r_j)$ are typically very small, i.e., there are modes with very small fluctuations. Including all of them in the definition of the goodness-of-fit χ^2 is not possible, because very small

deviations of the true functional form from a single power law (and such deviations cannot be eliminated when fitting without any corrections to scaling) can be reflected in very large χ^2 values. A meaningful fit to a single power law then must include only a few (some times only two) $C_{\text{err}}(r_i, r_j)$ eigenmodes. The final error bars on the exponents can be estimated by repeating the fits multiple times with Gaussian noise added to the eigenvalues of $C_{\text{err}}(r_i, r_j)$.

The fitting windows $r \in [r_{\min}, r_{\max}]$ are here chosen with $r_{\min}$ sufficiently large to eliminate most of the scaling corrections and with $r_{\max}$ small enough to avoid large boundary effects and to exclude data with error bars too large to be useful. For some consistency among the different data sets, the fits reported below are for $r \in [8, 16]$ for data sets $i = 1, 2, 3$, while for sets $i = 4, 5$, where only a small number of data points are available, $r \in [6, 10]$ for $i = 4$ and $r \in [5, 9]$ for $i = 5$; for the latter two sets $r_{\max}$ is the largest r shown Fig. 14. The results for the scaling dimensions are then, with statistical errors representing one standard deviation: $\Delta_1 = 0.1993 \pm 0.0002$, $\Delta_2 = 1.1973 \pm 0.0002$, $\Delta_3 = 2.22 \pm 0.02$, $\Delta_4 = 3.24 \pm 0.15$, and $\Delta_5 = 2.98 \pm 0.12$.

Comparing with the expected values in Sec. V, the two leading scaling dimensions are very close to their exact values, with deviations of only about 0.4% and 0.3%, respectively, for Δ_1 and Δ_2 . In the case of Δ_2 , the deviation represents more than 10 standard deviations, however, reflecting some remaining systematic errors from scaling corrections, as discussed in Sec. IIB. Changing the fitting window slightly does not affect the result appreciably. The result for Δ_3 is within 1% of the correct value, in both absolute and statistical terms. The $i = 4, 5$ results have large error bars but do not otherwise deviate from their expected values, apart from the fact that the order of these eigenvalues is switched in the range of r for which reasonably good data are available. It is again not possible to conclude with certainty that the order of these eigenvalues really is reversed.

It would be impossible to extract all of these scaling dimensions, to the level of statistical precision achieved here, from a conventional correlation function. At most the two smallest dimensions can be extracted by fitting power laws, at worse fidelity than achieved here. While there is some uncertainty in the data corresponding to Δ_4 and Δ_5 , and the $i = 6$ data set only shows rough consistency with Δ_6 , the good results for Δ_3 is particularly noteworthy.

The eigenvectors corresponding to the three smallest scaling dimensions are

$$g_1 \approx \begin{pmatrix} 0.436 \\ 0.407 \\ 0.444 \\ 0.403 \\ -0.379 \\ -0.375 \end{pmatrix} \quad g_2 \approx \begin{pmatrix} 0.179 \\ -0.138 \\ 0.328 \\ 0.549 \\ 0.448 \\ 0.583 \end{pmatrix} \quad g_3 \approx \begin{pmatrix} 0.262 \\ 0.017 \\ 0.616 \\ -0.715 \\ 0.139 \\ 0.144 \end{pmatrix},$$

where the elements from top to bottom are the coefficients of $O_1, \dots, O_6$ in Fig. 12(b). Here g_1 and g_2 change very little with r , while g_3 is statistically noisy for $r > 10$ but does not change appreciably between $r = 4$ and $r = 10$. The above results are all for $r = 10$ and have statistical errors beyond the digits shown in the case of g_1 and g_2 , while the g_3 ele-

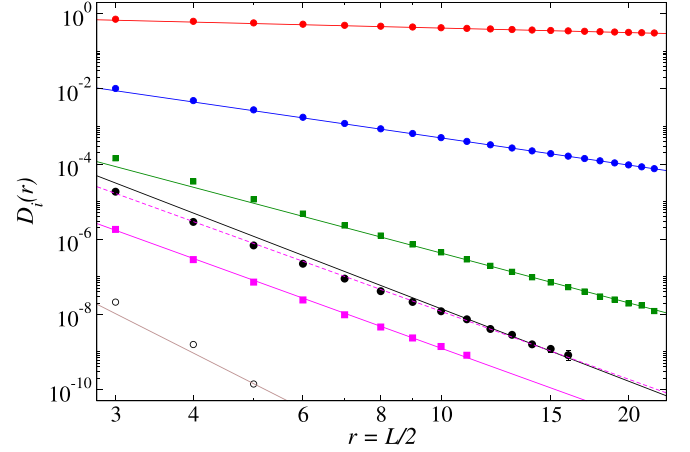


FIG. 15. Size-dependent eigenvalues of the BC model vs $r = L/2$ for even lengths $L \leq 44$. The solid lines show power laws with the exponents $2\Delta_i$ expected with the BC scaling dimensions listed in Eqs. (26), with the index i in the order $i = 1, 2, 3, 5, 4, 6$ from top to bottom. The dashed magenta line shows a fit assuming that the $i = 4$ and $i = 5$ eigenvalues are not in the reverse order, i.e., with Δ_4 governing the $i = 4$ set. If that is the case, then the $i = 5$ set must not yet be in its asymptotic regime governed by Δ_5 (for which no line is shown with the $i = 5$ set).

ments have statistical errors of roughly 0.003 (i.e., affecting the last digit displayed). Unlike the $N = 2$ case discussed earlier, for $N = 6$ the relationship between the eigenvectors and the special directions in the two-dimensional phase diagram is nontrivial, given that the original operators O_i are not orthogonal. It is interesting to note, however, that the eigenvector g_1 has almost exactly the same ratio of coefficients, $g_{11}/g_{12} \approx 1.07$ as in the 2×2 case, which is not true for the second eigenvector, where now $g_{22}/g_{21} \approx -0.78$ (instead of -1.07 in the 2×2 case).

C. $N = 6$ eigenvalues for $r = L/2$

Eigenvalues for system sizes L up to 44 are shown in Fig. 15 versus $r = L/2$. The behaviors are very similar to those for $r \ll L$ (Fig. 14), as also found in the case of the 2D Ising model in Sec. III. Here the $i = 4$ and $i = 5$ eigenvalues are not as clearly associated with the scaling dimensions Δ_5 and Δ_4 in the reverse order, with the $i = 4$ set also being well described by Δ_4 . In Fig. 14 the reversal of the eigenvalues is more convincing (though not completely clear) and suggests a crossing of the two power laws for r much larger than the maximum accessible distance. The $i = 4$ and $i = 5$ eigenvectors from the $r = L/2$ and $r \ll L$ calculations are very similar, suggesting that the eventual crossing of these eigenvalues should either happen in both cases or in neither case. Given that the $i = 5$ data set in Fig. 15 matches Δ_4 (which is the dimension of the third primary) quite well and the corrections to scaling even for the smallest r value are small, the most likely scenario is a slow convergence of the $i = 4$ data set to the behavior governed by Δ_5 , which is the dimension of the first descendant of the Δ_2 operator. It is not clear, however, why the convergence of this eigenvalue should be slower in

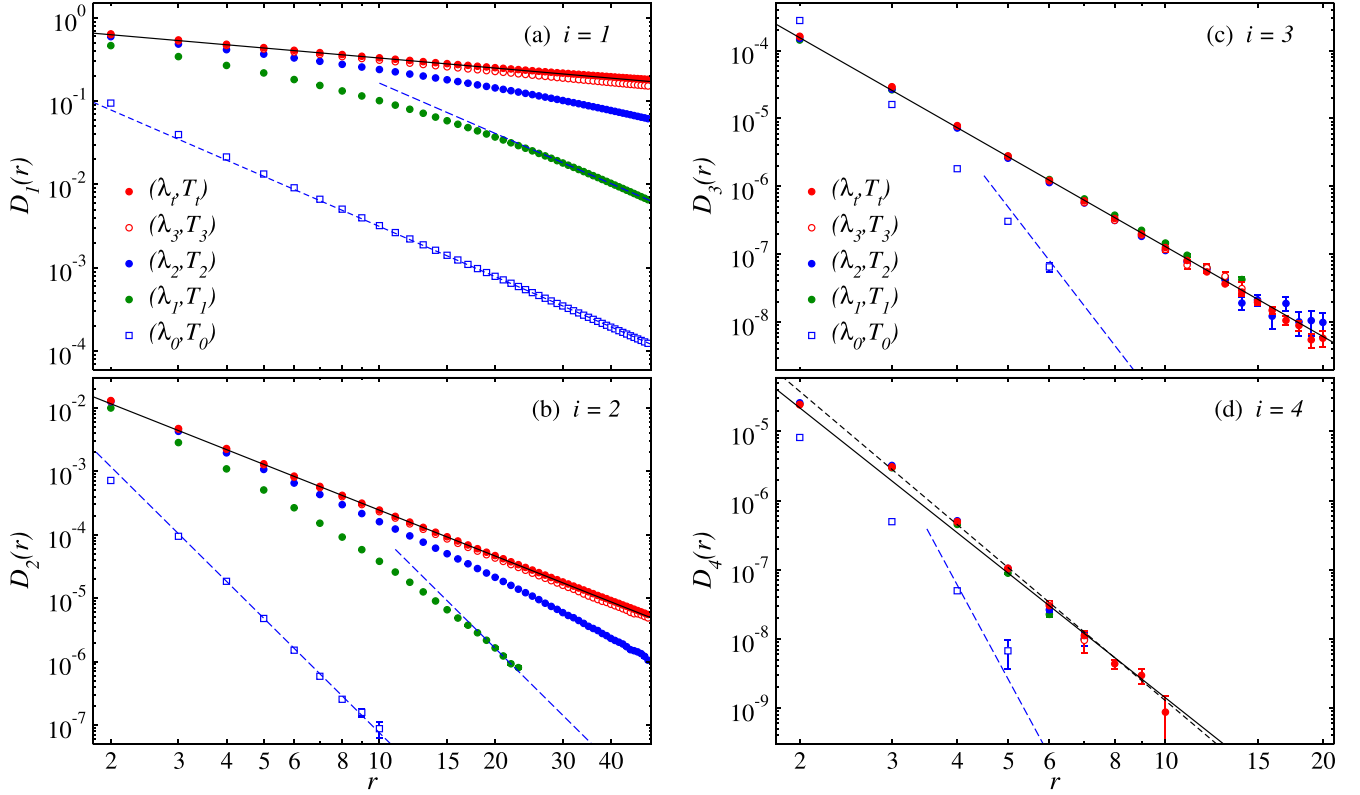


FIG. 16. Four largest eigenvalues of the 6×6 covariance matrix of all the operators in Fig. 12 graphed vs the distance r at five points (λ_k, T_k) on (within the precision of the calculations) the part of the phase boundary of the BC model where the transitions are continuous. In all the graphs, these points are $(\lambda_0, T_0) = (0, 1.69356)$, $(\lambda_1, T_1) = (1.95, 0.65778)$, $(\lambda_2, T_2) = (1.96253, 0.62)$, $(\lambda_3, T_3) = (1.96541, 0.61)$, and $(\lambda_t, T_t) = (1.965815, 0.608578)$ is the estimated tricritical point. The transition is of regular Ising type for $k = 0, \dots, 3$. The black lines have slopes corresponding to the expected tricritical exponents $2\Delta_i$ with $i = 1, \dots, 4$ in panels (a)–(d), i.e., $2\Delta_i = 0.4, 2.4, 4.4$, and 6 , respectively. In panel (d), the black dashed line shows the slope if the data set instead corresponds to $2\Delta_5 = 6.4$ (as the black line with the $i = 4$ data set in Fig. 14). The blue dashed lines show the slopes corresponding to the four smallest Z_2 -even Ising scaling dimensions, i.e., the exponents $2, 6, 10, 14$ in panels (a)–(d), respectively. In panels (c), (d), many of the plotting symbols are not visible because of the closeness of the data for all cases but (λ_0, T_0) .

the $r = L/2$ case and also slower than that of the third primary dimension.

VI. CRITICAL CROSSOVER

The tricritical Ising point (λ_t, T_t) of the BC model studied in Sec. V is the end point of a curve (λ_c, T_c) on which the transition is of regular Ising type for $\lambda_c < \lambda_t$ and $T_c > T_t$. For $\lambda_c > \lambda_t$ and $T_c < T_t$ the transition is of first order. On the critical part of the phase boundary, for (λ_c, T_c) close to (λ_t, T_t) the physical quantities should exhibit crossover behaviors, with tricritical behavior applying up to a length scale $\Lambda \propto |\delta|^{-\nu_{\parallel}}$, where δ is the distance to (λ_t, T_t) and $\nu_{\parallel}$ is the exponent governing the correlation length when approaching the tricritical point in the tangent direction of the phase boundary. As discussed in Sec. V A, this exponent is related to the larger of the two correlation-length exponents of the tricritical point, $\nu_{\parallel} = (2 - \Delta_2)^{-1} = 5/4$. On length scales larger than Λ , the regular Ising criticality should apply, and, therefore, a crossover regime in r also must exist where none of the critical behaviors is cleanly exhibited.

It is interesting to investigate how the crossover behavior is manifested in the eigenvalues of the covariance matrix. For

this purpose, several values of (λ_c, T_c) on the critical phase boundary were determined, using the standard Binder cumulant method. Among these points, the one furthest away from the tricritical point has zero chemical potential, $(\lambda_0, T_0) = (0, 1.69356)$, where the transition temperature T_0 agrees well with a recent result [64]. The other points on the critical line are labeled (λ_k, T_k) , $k = 1, 2, 3$, and are located much closer to the tricritical point. The values of λ_k and T_k are listed in the caption of Fig. 16. This figure shows the four smallest covariance eigenvalues (using the same six operators as in Sec. V) versus the separation r . Figures 16(a)–16(d) correspond to data sets $i = 1, \dots, 4$ and the eigenvalues for all five critical points are graphed in the same panel, with lines showing the power laws corresponding to both regular Ising and tricritical Ising behaviors.

In Figs. 16(a) and 16(b), regular critical behavior is well manifested at the point (λ_0, T_0) , which is very far from the tricritical point. The crossover to regular Ising behavior is also seen for (λ_1, T_1) when r exceeds $15 \sim 20$. However, no clear-cut tricritical behavior is observed at smaller r ; instead the flow to the ultimate regular Ising fixed point has already commenced at the shortest distances. In contrast, at (λ_3, T_3) the eigenvalues follow the tricritical behavior up

to $r = 10 \sim 15$ and only deviate slightly (though clearly visibly) from it up to the largest distances considered. At (λ_2, T_2) the tricritical behavior is less pronounced and the regular Ising criticality also has not set in for r up to 40. These behaviors of the first and second eigenvalues are qualitatively expected.

The third and fourth eigenvalues in Figs. 16(c) and 16(d) are more interesting. Surprisingly, here the tricritical behavior is observed for all the points except for (λ_0, T_0) , and at the latter the results are noisy and the asymptotic regular Ising behavior is not observed clearly. In Fig. 16(c) the last points are nevertheless consistent with the expected decay. In 16(d) the decay with r is very fast but not yet as fast as the expected r^{-14} . The main new insight here is that the tricritical behavior of the third and fourth eigenvalues is very stable for all the points (λ_k, T_k) , $k = 1, 2, 3$, even though the regular Ising model should eventually set in, as it does already for small r in the case of (λ_0, T_0) (which is not a special point even though $\lambda = 0$). The stability of these eigenvalues is likely a consequence of the fact that the corresponding regular Ising scaling dimensions are so large, thus the perturbations of the corresponding levels in the tricritical CFT are very weak.

VII. CONCLUSIONS AND DISCUSSION

The results obtained here demonstrate that the covariance method is capable of resolving multiple scaling dimensions in remarkable detail, far beyond what is possible by fitting conventional correlation functions to a sum of power laws. In the conventional way of fitting individual correlation functions, or even combining several correlation functions, the analysis is complicated by the fact that there are corrections of the form $r^{-(\Delta_j-d)}$ from all irrelevant perturbations ($\Delta_j > d$), in addition to the contributions to the correlation functions from several power laws $r^{-2\Delta_i}$ and $r^{-(\Delta_i+\Delta_j)}$. In practice, it is not possible to obtain reliable results for the fast decaying contributions from descendants or higher primaries under these conditions. The covariance method solves this problem by disentangling the different $r^{-2\Delta_i}$ decays. Thus, while the corrections from irrelevant perturbations are still present in the eigenvalues (and it may be useful to include corrections when fitting), their asymptotic forms governed by different power laws makes it possible to extract multiple scaling dimensions.

A well known method for reducing scaling corrections is to introduce other interactions in a model so that the field corresponding to the leading correction can be tuned away, or at least made very small [65,66]. This method has helped to produce some of the best MC results for critical exponents [67,68], but its original intent was not to extract irrelevant scaling dimensions or multiple relevant dimensions. In cases where the descendants are of interest, tuning away a scaling correction may also be helpful, though fitting of the eigenvalues can also be done with corrections. It should also be kept in mind that identifying the optimal interactions to minimize the scaling corrections can be a very time consuming task.

The exact number of scaling dimensions that can be obtained reliably from the eigenvalues of course depends on the system under study. Here it was possible to observe a scaling dimension as large $\Delta = 5$ in the 2D Ising model and $\Delta \approx 3.5$ in the 3D Ising model. In the latter case, the following not yet resolved scaling dimension should be slightly larger than

5 and may also be accessible with more (and more optimally selected) operators included in the covariance matrix. At a multicritical point, here in the 2D BC model, the larger number of relevant primary operators, with their descendants, implies a larger number of resolvable scaling dimensions with reasonably small values. To resolve many operators, a correspondingly sized covariance matrices of course have to be used, and longer MC simulations may also be required to properly disentangling the many power laws with rather close decay exponents.

The ability to extract multiple scaling dimensions and eigenfunctions corresponding to relevant orthogonal operators at a tricritical point (and likely also at higher-order multicritical points) is also an appealing feature. There are many potential applications to multicriticality, e.g., further numerical studies of deconfined quantum-critical points [8,9,14,15] and liquid-gas transitions [19,20]. While only the 2D BC model was used here to study the tricritical Ising point and associated crossover behavior, this model is highly nontrivial and the good results obtained for several scaling dimensions is highly suggestive of excellent performance more broadly.

An important feature of the covariance method is that it works not only for $r \ll L$ on large lattices, but also for $r \propto L$ in relatively small systems. This aspect should be particularly useful in quantum MC studies of fermion systems, where the accessible system lengths are often only of the order 20 [59–62]. In 3D classical models and 2+1-dimensional quantum spin models (those for which the negative sign problem can be avoided [69]), larger system sizes can be studied. However, with small scaling dimensions the boundary effects can still be problematic [9] and studies for $r = L/2$ will be useful also in those cases. The method is easy to implement with any MC or quantum MC simulation method and should be applicable also with other numerical approaches. While only classical models were studied here, the 2D operators (3×3 site plaquettes) used with the 3D Ising model represent a close analogy of operators typically implemented in a quantum system—operating on states at fixed imaginary time. Here it was found that timelike correlators have an advantage over spacelike ones, likely because the finite spatial extent of the operators cause less corrections when separated in imaginary time, i.e., all the inter-plaquette operator distances are closer to the plaquette center-center distance r .

An important aspect of CFTs is the fact that the conformal multiplets correspond to equally spaced scaling dimensions, with the step size equal to 2 when a sector of fixed Lorentz spin l is considered. This fact can be very useful for testing whether there is an underlying CFT description of a critical point observed numerically. As we have seen in the demonstrations here, the covariance method is well capable of reproducing relevant $l = 0$ primaries and their first $l = 0$ descendants, and in some cases a second $l = 0$ descendant can also be reasonably well reproduced. Tests with higher l incorporated in the square-shaped lattice operators (allowing for $l = 1$ and $l = 2$) confirm that the expected first two or three scaling dimensions can also be reproduced there. It was also pointed out here that lattice operators that do not incorporate the full symmetry of the lattice will in general include both $l = 0$ and $l = 2$ field operators, and this was tested explicitly in the 3D Ising model with 2D plaquette

operators in the form of two equal dimensions from $l = 0$ and $l = 2$ descendants. This fact should also be kept in mind in applications to quantum systems.

To test for a CFT corresponding to a regular critical point with a single relevant primary, the correct spacing between their first two $l = 0$ scaling dimensions already provides strong support. In light of recent progress and remaining controversies in numerical studies of deconfined quantum-critical points, it would be very interesting and useful to implement the covariance method and try to extract multiple scaling dimensions in quantum MC simulations of spin and fermion models hosting such transitions [8,9,15,16,62].

It would be interesting to apply the covariance method also to critical systems not described by CFTs, e.g., generic Mott-superfluid transitions [70] or systems with quenched disorder, including classical and quantum spin glasses. The concept of conformal multiplets with their towers of equally spaced levels does not apply there, and much less is known about the spectrum of scaling dimensions. At least on an intuitive level, correlations with different power-law decays should correspond to different fluctuation modes and their associated scaling dimensions should therefore be resolved in the covariance eigenvalues. Tests on various models are planned for future work.

In quantum systems, it will also be useful to study the covariance matrix as a function of imaginary time τ . A method in spirit similar to the one used here to separate different power laws was years ago employed to study exponentially decaying correlations in a system with discrete energy levels [31]. This method is now often used to extract particle masses [32–35]. A CFT corresponds to a gapless point, but for a finite quantum system at $T = 0$, the discreteness of the level spectrum $\{\epsilon_i\}$ implies correlation functions $C(\tau)$ mixing exponential decays $\propto e^{-\tau\epsilon_i}$. The covariance method should resolve these levels in the same way as in particle physics (where the method is often formulated as a generalized eigenvalue problem [31]), but for a CFT the levels become more closely spaced with increasing system size, $\epsilon_i \propto \Delta_i/L$. For a sufficiently large critical system described by a CFT, the τ dependence for τ up to order L should be a sum of power laws $\tau^{-2\Delta_i}$ governed by scaling dimensions, just like the real-space correlations analyzed here (and more generally space and time correlations are related by a dynamic exponent [70]). The crossover from power-law $\tau^{-2\Delta_i}$ to exponential $\propto e^{-a\tau\Delta_j/L}$ decays (where a is a constant and $\Delta_i = \Delta_j$ does not always hold) will be studied in $d = 1 + 1$ in a forthcoming publication [36]. It would also be interesting to study this

crossover in the eigenvalues and -vectors of the covariance matrix in quantum MC simulations within the fuzzy sphere approach [49], provided that large enough system sizes can be reached to detect the power-law form (while the exponential decay should always hold asymptotically, though with corrections to the scaling dimensions). An important question is whether the covariance method can also access CFT data beyond scaling dimensions (central charge, coefficients of the operator-product expansion, etc.).

The closest MC based competitors to the covariance scheme would likely be MC renormalization-group (RG) methods, which have a long history [71–73], including in the context of the same BC model studied here [21,22]. This approach involves eigenvalues of an RG transformation matrix, which also produces scaling dimensions. However, the method is less general than the covariance matrix approach, for a given model requiring a decision on what type of RG transformation (coarse graining) to carry out. Depending on the model this procedure may be far from obvious, especially in the case quantum systems. In contrast, the method proposed here is very general and only requires rather trivial design of a set of operators.

Recently the MC RG approach has been further developed by adapting neural networks (machine learning) to construct lattice realizations of continuum field operators [74]. Training a neural network using MC samples, some of these methods make use of covariance and principal value analysis to optimize a set of orthogonal operators [75–78]. The general setup is, however, very different from, and much more complicated than, the covariance method demonstrated here, where the initial operators can be essentially arbitrary and their linear combinations found by diagonalization are the optimized operators. It would certainly be interesting to compare the performance of the methods for some challenging model.

ACKNOWLEDGMENTS

The author thanks Jonathan D’Emidio, Emilie Huffman, Ami Katz, Dong-Hee Kim, Masaki Oshikawa, and Jun Taka-hashi for valuable discussions and comments. This research was supported by the Simons Foundation under Grant No. 511064, by a Fellowship from the John Simon Guggenheim Memorial Foundation, and by Grant No. NSF PHY-2309135 to the Kavli Institute for Theoretical Physics (KITP). The numerical calculations were carried out on the Shared Computing Cluster managed by Boston University’s Research Computing Services.

-
- [1] D. Poland, S. Rychkov, and A. Vichi, The conformal bootstrap: Theory, numerical techniques, and applications, *Rev. Mod. Phys.* **91**, 015002 (2019).
 - [2] W. Zhu, C. Han, E. Huffman, J. S. Hofmann, and Y.-C. He, Uncovering conformal symmetry in the 3D Ising transition: State-operator correspondence from a quantum fuzzy sphere regularization, *Phys. Rev. X* **13**, 021009 (2023).
 - [3] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Deconfined quantum critical points, *Science* **303**, 1490 (2004).
 - [4] R. K. Kaul and A. W. Sandvik, Lattice model for the SU(N) Néel to valence-bond solid quantum phase transition at large N, *Phys. Rev. Lett.* **108**, 137201 (2012).
 - [5] M. S. Block, R. G. Melko, and R. K. Kaul, Fate of CP^{N-1} fixed points with q monopoles, *Phys. Rev. Lett.* **111**, 137202 (2013).
 - [6] A. Nahum, J. T. Chalker, P. Serna, M. Ortuño, and A. M. Somoza, Deconfined quantum criticality, scaling violations, and classical loop models, *Phys. Rev. X* **5**, 041048 (2015).
 - [7] G. J. Sreejith, S. Powell, and A. Nahum, Emergent SO(5) symmetry at the columnar ordering transition in the

- classical cubic dimer model, *Phys. Rev. Lett.* **122**, 080601 (2019).
- [8] B. Zhao, J. Takahashi, and A. W. Sandvik, Multicritical deconfined quantum criticality and Lifshitz point of a helical valence-bond phase, *Phys. Rev. Lett.* **125**, 257204 (2020).
- [9] J. Takahashi, H. Shao, B. Zhao, W. Guo, and A. W. Sandvik, SO(5) multicriticality in two-dimensional quantum magnets, [arXiv:2405.06607](#).
- [10] Y. Nakayama and T. Ohtsuki, Necessary condition for emergent symmetry from the conformal bootstrap, *Phys. Rev. Lett.* **117**, 131601 (2016).
- [11] Z. Li, Bootstrapping conformal QED₃ and deconfined quantum critical point, [arXiv:1812.09281](#).
- [12] Z. Zhou, L. Hu, W. Zhu, and Y.-C. He, The SO(5) deconfined phase transition under the fuzzy sphere microscope: Approximate conformal symmetry, pseudo-criticality, and operator spectrum, *Phys. Rev. X* **14**, 021044 (2024).
- [13] X. Zou, S. Yin, Z.-X. Li, and H. Yao, Unraveling deconfined quantum criticality in non-Hermitian easy-plane J - Q model, [arXiv:2511.03456](#).
- [14] S. M. Chester and N. Su, Bootstrapping deconfined quantum tricriticality, *Phys. Rev. Lett.* **132**, 111601 (2024).
- [15] B.-B. Chen, X. Zhang, Y. Wang, K. Sun, and Z. Y. Meng, Phases of (2+1)D SO(5) non-linear sigma model with a topological term on a sphere: Multicritical point and disorder phase, *Phys. Rev. Lett.* **132**, 246503 (2024).
- [16] B.-B. Chen, X. Zhang, and Z. Y. Meng, Emergent conformal symmetry at the multicritical point of (2+1)D SO(5) model with Wess-Zumino-Witten term on sphere, *Phys. Rev. B* **110**, 125153 (2024).
- [17] Z. Li and T. Shen, Bootstrap cone and the multicritical deconfined quantum critical point, [arXiv:2606.09985](#).
- [18] L. P. Kadanoff, W. Götze, D. Hamblen, R. Hecht, E. A. S. Lewis, V. V. P. Ciauskas, M. Rayl, and J. Swift, Static phenomena near critical points: Theory and experiment, *Rev. Mod. Phys.* **39**, 395 (1967).
- [19] M. Yarmolinsky and A. Kuklov, Revisiting universality of the liquid-gas critical point in two dimensions, *Phys. Rev. E* **96**, 062124 (2017).
- [20] M. Yarmolinsky and A. Kuklov, Strong and weak field criticality of 2D liquid gas transition, [arXiv:1806.10590](#).
- [21] D. P. Landau and R. H. Swendsen, Tricritical universality in two dimensions, *Phys. Rev. Lett.* **46**, 1437 (1981).
- [22] D. P. Landau and R. H. Swendsen, Monte Carlo renormalization-group study of tricritical behavior in two dimensions, *Phys. Rev. B* **33**, 7700 (1986).
- [23] P. D. Beale, Finite-size scaling study of the two-dimensional Blume-Capel model, *Phys. Rev. B* **33**, 1717 (1986).
- [24] N. B. Wilding and P. Nielaba, Tricritical universality in a two-dimensional spin fluid, *Phys. Rev. E* **53**, 926 (1996).
- [25] Y. Deng and H. Blöte, Edge phase transitions of the tricritical Potts model in two dimensions, *Phys. Rev. E* **71**, 026109 (2005).
- [26] C. J. Silva, A. A. Caparica, and J. A. Plascak, Wang-Landau Monte Carlo simulation of the Blume-Capel model, *Phys. Rev. E* **73**, 036702 (2006).
- [27] J. A. Plascak and P. H. L. Martins, Probability distribution function of the order parameter: Mixing fields and universality, *Comput. Phys. Commun.* **184**, 259 (2013).
- [28] W. Kwak, J. Jeong, J. Lee, and D.-H. Kim, First-order phase transition and tricritical scaling behavior of the Blume-Capel model: A Wang-Landau sampling approach, *Phys. Rev. E* **92**, 022134 (2015).
- [29] B. Nienhuis, Analytical calculation of two leading exponents of the dilute Potts model, *J. Phys. A: Math. Gen.* **15**, 199 (1982).
- [30] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, NY, 1997), Vol. 1.
- [31] M. Lüscher and U. Wolff, How to calculate the elastic scattering matrix in two-dimensional quantum field theories by numerical simulation, *Nucl. Phys. B* **339**, 222 (1990).
- [32] C. J. Morningstar and M. Peardon, Efficient glueball simulations on anisotropic lattices, *Phys. Rev. D* **56**, 4043 (1997).
- [33] C. J. Morningstar and M. Peardon, The glueball spectrum from an anisotropic lattice study, *Phys. Rev. D* **60**, 034509 (1999).
- [34] B. Blossier *et al.* (ALPHA collaboration), On the generalized eigenvalue method for energies and matrix elements in lattice field theory, *J. High Energy Phys.* **04** (2009) 094.
- [35] L. Liu *et al.* (For the Hadron Spectrum Collaboration), Excited and exotic charmonium spectroscopy from lattice QCD, *J. High Energy Phys.* **07** (2012) 126.
- [36] Z. Lin, G. Schumm, and A. W. Sandvik (unpublished).
- [37] J. L. Cardy, Conformal invariance and universality in finite-size scaling, *J. Phys. A: Math. Gen.* **17**, L385 (1984).
- [38] A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, and M. H. Freedman, Interacting anyons in topological quantum liquids: The golden chain, *Phys. Rev. Lett.* **98**, 160409 (2007).
- [39] H. Suwa and S. Todo, Generalized moment method for gap estimation and quantum Monte Carlo level spectroscopy, *Phys. Rev. Lett.* **115**, 080601 (2015).
- [40] R. Wang, Y. Zou, and G. Vidal, Emergence of Kac-Moody symmetry in critical quantum spin chains, *Phys. Rev. B* **106**, 115111 (2022).
- [41] E. M. Stoudenmire and S. R. White, Studying two dimensional systems with the density matrix renormalization group, *Annu. Rev. Condens. Matter Phys.* **3**, 111 (2012).
- [42] M. Schuler, S. Whitsitt, L.-P. Henry, S. Sachdev, and A. M. Läuchli, Universal signatures of quantum critical points from finite-size torus spectra: A window into the operator content of higher-dimensional conformal field theories, *Phys. Rev. Lett.* **117**, 210401 (2016).
- [43] J. L. Cardy, Universal amplitudes in finite-size scaling: Generalisation to arbitrary dimensionality, *J. Phys. A: Math. Gen.* **18**, L757 (1985).
- [44] R. C. Brower, M. Cheng, G. T. Fleming, A. D. Gasbarro, T. G. Raben, C.-I. Tan, and E. S. Weinberg, Lattice ϕ^4 field theory on Riemann manifolds: Numerical tests for the 2D Ising CFT on S^2 , *Phys. Rev. D* **98**, 014502 (2018).
- [45] R. C. Brower, G. T. Fleming, A. D. Gasbarro, D. Howarth, T. G. Raben, C.-I. Tan, and E. S. Weinberg, Radial lattice quantization of 3D ϕ^4 field theory, *Phys. Rev. D* **104**, 094502 (2021).
- [46] B.-X. Lao and S. Rychkov, 3D Ising CFT and exact diagonalization on icosahedron: The power of conformal perturbation theory, *SciPost Phys.* **15**, 243 (2023).
- [47] V. Ayyar, R. C. Brower, G. T. Fleming, A.-M. E. Glück, E. K. Owen, T. G. Raben, and C.-I. Tan, The operator product expansion for radial lattice quantization of 3D ϕ^4 theory, *Phys. Rev. D* **109**, 114518 (2024).
- [48] R. C. Brower and E. K. Owen, The Ising model on S^2 , [arXiv:2407.00459](#).

- [49] J. S. Hofmann, F. Goth, W. Zhu, Y.-C. He, and E. Huffman, Quantum Monte Carlo simulation of the 3D Ising transition on the fuzzy sphere, *SciPost Phys. Core* **7**, 028 (2024).
- [50] G. Fardelli, A. L. Fitzpatrick, and E. Katz, Constructing the infrared conformal generators on the fuzzy sphere, *SciPost Phys.* **18**, 086 (2025).
- [51] F. D. M. Haldane, Fractional quantization of the Hall effect: A hierarchy of incompressible quantum fluid states, *Phys. Rev. Lett.* **51**, 605 (1983).
- [52] I. Affleck and F. D. M. Haldane, Critical theory of quantum spin chains, *Phys. Rev. B* **36**, 5291 (1987).
- [53] P. Patil, E. Katz, and A. W. Sandvik, Numerical investigations of SO(4) emergent extended symmetry in spin-1/2 Heisenberg antiferromagnetic chains, *Phys. Rev. B* **98**, 014414 (2018).
- [54] M. E. Fisher and M. B. Barber, Scaling theory for finite-size effects in the critical region, *Phys. Rev. Lett.* **28**, 1516 (1972).
- [55] K. Binder, Critical properties from Monte Carlo coarse graining and renormalization, *Phys. Rev. Lett.* **47**, 693 (1981).
- [56] M. N. Barber, *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz, Vol. 8 (Academic, London, 1983).
- [57] R. H. Swendsen and J.-S. Wang, Nonuniversal critical dynamics in Monte Carlo simulations, *Phys. Rev. Lett.* **58**, 86 (1987).
- [58] U. Wolff, Collective Monte Carlo updating for spin systems, *Phys. Rev. Lett.* **62**, 361 (1989).
- [59] F. F. Assaad, M. Bercx, F. Goth, A. Götz, J. S. Hofmann, E. Huffman, Z. Liu, F. Parisen Toldin, J. S. E. Portela, and J. Schwab, The ALF (Algorithms for lattice fermions) project release 2.0. Documentation for the auxiliary-field quantum Monte Carlo code, *SciPost Phys. Codebases* **1** (2022).
- [60] Y. Liu, Z. Wang, T. Sato, W. Guo, and F. F. Assaad, Gross-Neveu Heisenberg criticality: Dynamical generation of quantum spin Hall masses, *Phys. Rev. B* **104**, 035107 (2021).
- [61] Z. H. Liu, M. Vojta, F. F. Assaad, and L. Janssen, Critical properties of metallic and deconfined quantum phase transitions in Dirac systems, *Phys. Rev. B* **110**, 125123 (2024).
- [62] J. D’Emidio, R. Orús, N. Laflorencie, and F. de Juan, Universal features of entanglement entropy in the honeycomb Hubbard model, *Phys. Rev. Lett.* **132**, 076502 (2024).
- [63] A. M. Ferrenberg, J. Xu, and D. P. Landau, Pushing the limits of Monte Carlo simulations for the three-dimensional Ising model, *Phys. Rev. E* **97**, 043301 (2018).
- [64] L. Moueddene, N. G. Fytas, Y. Holovatch, R. Kenna, and B. Berche, Critical and tricritical singularities from small-scale Monte Carlo simulations: The Blume-Capel model in two dimensions, *J. Stat. Mech.* (2024) 023206.
- [65] H. G. Ballesteros, L. A. Fernández, V. Martín-Mayor, and A. Muñoz Sudupe, Finite size scaling and “perfect” actions: The three dimensional Ising model, *Phys. Lett. B* **441**, 330 (1998).
- [66] A. Pelissetto and E. Vicari, Critical phenomena and renormalization-group theory, *Phys. Rep.* **368**, 549 (2002).
- [67] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Critical exponents and equation of state of the three-dimensional Heisenberg universality class, *Phys. Rev. B* **65**, 144520 (2002).
- [68] M. Hasenbusch, Finite size scaling study of lattice models in the three-dimensional Ising universality class, *Phys. Rev. B* **82**, 174433 (2010).
- [69] P. Henelius and A. W. Sandvik, Sign problem in Monte Carlo simulations of frustrated quantum spin systems, *Phys. Rev. B* **62**, 1102 (2000).
- [70] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Boson localization and the superfluid-insulator transition, *Phys. Rev. B* **40**, 546 (1989).
- [71] S.-k. Ma, Renormalization group by Monte Carlo methods, *Phys. Rev. Lett.* **37**, 461 (1976).
- [72] R. H. Swendsen, Monte Carlo renormalization group, *Phys. Rev. Lett.* **42**, 859 (1979).
- [73] R. H. Swendsen, Monte Carlo Renormalization, in *Real Space Renormalization*, edited by T. Burkhardt and J. M. J. van Leeuwen (Springer-Verlag, Berlin, 1982), Vol. 30.
- [74] M. Koch-Janusz and Z. Ringel, Mutual information, neural networks and the renormalization group, *Nat. Phys.* **14**, 578 (2018).
- [75] A. Gordon, A. Banerjee, M. Koch-Janusz, and Z. Ringel, Relevance in the renormalization group and in information theory, *Phys. Rev. Lett.* **126**, 240601 (2021).
- [76] D. E. Gökmen, Z. Ringel, S. D. Huber, and M. Koch-Janusz, Statistical physics through the lens of real-space mutual information, *Phys. Rev. Lett.* **127**, 240603 (2021).
- [77] D. E. Gökmen, Z. Ringel, S. D. Huber, and M. Koch-Janusz, Symmetries and phase diagrams with real-space mutual information neural estimation, *Phys. Rev. E* **104**, 064106 (2021).
- [78] L. Oppenheim, M. Koch-Janusz, S. Gazit, and Z. Ringel, Machine learning the operator content of the critical self-dual Ising-Higgs gauge model, *Phys. Rev. Res.* **6**, 043322 (2024).