Critical Hawkes processes with random fertilities: Stationarity in law beyond infinite mean activity

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Genuinely critical dynamics have been proposed to organize many natural and social systems, yet exact criticality is usually thought to preclude stationarity because the mean activity diverges. I show that this conclusion is not generally valid for self-exciting Hawkes point processes. At criticality, stationarity in law is controlled not by the mean intensity but by local finiteness of the infinite-past Poisson-cluster construction. The relevant quantity is the fixed-window hitting probability $H_T(u)$, the probability that a cluster born at time $-u$ contributes at least one event to a window of length T. For offspring waiting-time tails $\mathbb{P}(S > t) \sim t^{-\theta}$ and fertility tails $\mathbb{P}(\kappa > x) \sim x^{-\gamma}$, a finite-mean-lifetime argument rigorously establishes stationarity for $1 < \gamma < 2$ and $\theta > \gamma$, while the finite-memory, finite-variance critical regime is nonstationary. A direct asymptotic analysis of the exact fixed-window nonlinear renewal equation identifies the precise local-finiteness boundary $\theta = \gamma - 1$ in the infinite-fertility-variance regime. Below this boundary, H_T is nonintegrable. At the marginal point $\theta = \gamma - 1$, a refined Laplace-Tauberian analysis yields $H_T(u) \sim \frac{[\ln u]^{1/(\gamma-1)}}{u}$, which is also nonintegrable, demonstrating that the critical Hawkes process remains nonstationary at the boundary $\theta = \gamma - 1$. Direct numerical solutions of the exact renewal equation confirm the predicted power-law and logarithmic regimes across the transition. Thus broad fertility fluctuations can sustain a locally finite stationary critical Hawkes process with positive immigration even though its stationary mean intensity is infinite.

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I. INTRODUCTION

Self-exciting point processes introduced by Hawkes [1,2] and the epidemic-type aftershock sequence (ETAS) models used in seismology [3,4] provide a minimal representation of cascades, aftershocks, contagion, and social or financial excitation. Their applications range from collective social dynamics and online information diffusion [5–9] to high-frequency finance and financial contagion [10–13], as well as crime, epidemics, terrorism, and cyber risk [14–17].

In the scalar case, the central control parameter is the branching ratio n, the expected number of first-generation offspring of one event. The order parameter is the stationary mean intensity, which reads $\bar{\lambda} = \nu/(1-n)$, where ν is the immigrant or exogenous rate. For $n \uparrow 1$, it diverges. This observation is often taken to mean that critical Hawkes processes are nonstationary or ill defined. But such a conclusion relies only on stationarity in expectation. It determines neither whether the point process itself can remain locally finite nor whether finite-window counts or intensities can possess time-invariant distributions with an infinite first moment. The critical problem is therefore to determine what really fails at $n = 1$: the existence of the process, its local finiteness, strict stationarity in law, or only the finiteness of its mean intensity.

This is the question addressed in this paper. Can there be a stationary critical Hawkes process in the sense of invariant finite-window count distributions, even though stationarity in expectation fails? More precisely, when the process is constructed on the whole line by starting the immigration mechanism at time $-\infty$, is the resulting Poisson-cluster superposition locally finite and time-shift invariant or does the infinite past contribute infinitely many ancestral clusters to any bounded observation window?

The importance of this question comes from the empirical and theoretical status of criticality itself. Across several domains, calibrated branching or Hawkes- or ETAS-type models have reported branching ratios close to the critical value, notably in seismicity [4,18], high-frequency financial markets [19,20], and neuronal avalanches [21,22]. These estimates have often been interpreted as evidence that the underlying system is poised at, or self-organized near, a critical state. This interpretation may of course be only asymptotic: Real systems may be slightly subcritical, while finite samples, incomplete observations, nonstationary backgrounds, or model misspecification push estimates toward $n \simeq 1$. But precisely for this reason the critical boundary cannot be treated as a mere pathological end-point. It is the reference point against which near-critical calibrations, scaling laws, and claims of endogeneity are interpreted.

The issue is therefore to understand what the limit $n \uparrow 1$ is a limit toward. If $n = 1$ necessarily destroyed the point-process construction, then near-criticality would have to be understood only as a subcritical approximation. If, on the contrary, a critical Hawkes process can remain locally finite and stationary in law while losing only finite-mean stationarity, then empirical observations of superlinear fluctuations, divergent sample averages, or large endogenous cascades need not signal nonstationarity of the underlying law. They may instead be the observable signatures of a stationary heavy-tailed critical regime. The classification of critical Hawkes processes

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is thus essential for deciding what is meant, mathematically and physically, when a self-exciting system is said to operate at or near criticality.

Critical stationary Hawkes processes have previously been considered by Brémaud and Massoulié [23], who established the existence of Hawkes branching point processes without ancestors under a rather specific heavy-tail condition on the excitation kernel. In particular, their existence theorem assumes criticality, $\int_0^\infty \phi(t)\,\mathrm{d}t = 1$, together with an asymptotic power-law kernel of the form $\phi(t) \sim t^{-1-\theta}, 0 < \theta < \frac{1}{2}$, supplemented by uniform regularity bounds on $t^{1+\theta}\phi(t)$ and additional technical assumptions. Their stationary critical process relies on an exceptionally strong form of long memory, corresponding to the heavy-tailed regime $\theta < 1/2$. This requirement is substantially more restrictive than merely requiring an infinite mean offspring waiting time, which already occurs for $\theta < 1$. Their construction therefore lies well inside the infinite-mean waiting-time regime and is obtained as a critical limit in which the immigrant intensity vanishes as the branching ratio approaches one. At criticality the resulting process consequently has no ancestors and retains a finite average intensity.

This construction is conceptually and mathematically distinct from the problem studied here. We keep a strictly positive homogeneous immigration rate $\nu > 0$ on the whole real line and ask whether the superposition of the critical clusters generated by immigrants from the infinite past remains locally finite, even though its stationary mean activity is infinite. Moreover, the present framework allows random fertilities, including the infinite-variance regime $1 < \gamma < 2$, and classifies local finiteness through the interplay between the fertility-tail exponent γ and the memory exponent θ. Thus the novelty here is not the generic possibility of critical stationarity but a different and considerably broader stationarity mechanism: local finiteness of a positive-immigration, infinite-past critical cluster construction with random fertilities and potentially infinite mean intensity.

II. HAWKES PROCESS, BRANCHING REPRESENTATION, AND CRITICALITY

A linear Hawkes process is exactly equivalent to an age-dependent branching, or immigration-birth, process [2]. Immigrants arrive according to a homogeneous Poisson process with rate ν. Each event i with fertility κ_i gives birth, independently of all other events, to first-generation offspring according to an inhomogeneous Poisson process on $\mathbb{R}_+$ with intensity measure $\kappa_i\phi(u)\,\mathrm{d}u$, where ϕ is the normalized memory kernel. These offspring then reproduce independently according to the same rule, with their own fertilities. Thus each immigrant generates a random genealogical tree, or cluster, whose event times are measured relative to the immigrant.

The conditional intensity of the Hawkes process reads

$$\lambda(t) = \nu + \sum_{t_i<t} \kappa_i\,\phi(t-t_i), \qquad \int_0^\infty \phi(s)\,\mathrm{d}s = 1, \tag{1}$$

where $\nu > 0$ is the immigrant or background rate, ϕ is the normalized memory kernel, and $\kappa_i \geqslant 0$ is the fertility of event i. The fertility mark κ_i is sampled independently when event i is born and is revealed at its event time. Equation (1) is therefore the predictable conditional intensity with respect to the completed, right-continuous marked filtration $(\mathcal{F}_t)$, where $\mathcal{F}_t$ is generated by event times and their fertility marks observed up to time t (and augmented by immigrant labels in the cluster construction). The intensity at t is conditioned on $\mathcal{F}_{t-}$, the history strictly before t. In Sec. III, we return to the pathwise local-integrability of this intensity in the infinite-mean regime.

We will consider in particular the Omori-type memory kernel,

$$\phi(t) \sim \frac{A}{t^{1+\theta}}, \quad t \to \infty, \tag{2}$$

with $\theta > 0$. In ETAS and other heterogeneous systems, fertilities are independent and identically distributed random variables that are broadly distributed; we consider in particular the case

$$\mathbb{P}(\kappa > x) \sim C_\kappa x^{-\gamma}, \quad \gamma > 1. \tag{3}$$

In the branching representation, the Hawkes branching ratio is the mean fertility,

$$n = \mathbb{E}[\kappa]. \tag{4}$$

The standard constant-fertility Hawkes process corresponds to the degenerate case $\kappa_i \equiv n$. The condition $\gamma > 1$ guarantees the existence of the mean fertility and allows one to tune the system to criticality, $n = 1$. By contrast, for $0 < \gamma < 1$, $\mathbb{E}[\kappa]$ diverges; the process then exhibits stochastic finite-time singularities [24].

For $n < 1$, taking the statistical average of (1) gives the familiar mean-rate identity

$$\bar\lambda = \nu + n\bar\lambda, \quad \Rightarrow\ \bar\lambda = \frac{\nu}{1-n}. \tag{5}$$

This identity is the basis of the usual finite-mean stability criterion and of the conclusion that no finite stationary occurrence rate exists at $n = 1$ [18,25]. The inference from Eq. (5) to nonexistence of the process is, however, logically stronger than what the equation proves. It proves the absence of a finite first moment at $n = 1$. It does not, by itself, rule out a stationary distribution with infinite mean.

The distinction is elementary but important. Let $(X_j)_{j\in\mathbb{Z}}$ be independent and identically distributed random variables with $\mathbb{P}(X > x) \sim Cx^{-\mu}$ and $0 < \mu < 1$. The sequence is strictly stationary, but $\mathbb{E}[X_j] = \infty$. Moreover, sums over m observations grow typically as $m^{1/\mu}$ because they are controlled by extremes, as in the generalized central-limit–stable-law mechanism for broad distributions [26,27]. Superlinear scaling of aggregates is therefore not a diagnostic of nonstationarity. It can be the signature of strict stationarity with infinite first moment.

Saichev and Sornette [28] made a first step in characterizing the statistical properties of the realizations of the Hawkes process at criticality. For a finite but large observation window of duration τ, containing on average $\nu\tau$ immigrant sources, and assuming that no sources occurred before the beginning of the window, Saichev and Sornette showed that the typical total number of descendants, summed over all generations, scales

at criticality $n = 1$ as

$$r_{\mathrm{typ}}(\nu\tau) \sim \begin{cases} (\nu\tau)^{\gamma}, & 1 < \gamma < 2, \\ (\nu\tau)^{2}, & \gamma > 2 \text{ or finite fertility variance}, \end{cases} \tag{6}$$

while the corresponding mean is dominated by extremely rare realizations and diverges. This proves that finite-source, finite-window critical clusters are meaningful random objects with proper distributions even when first moments fail. However, this result (6) [28] does not automatically ensure the existence of an infinite-past stationary Hawkes process at $n = 1$, which is the focus of the present study.

III. WHOLE-LINE CONSTRUCTION AND STATIONARITY CRITERION

Throughout, a candidate stationary Hawkes process is understood as a whole-line process, constructed from an immigrant process defined on the whole time axis $\mathbb{R}$, equivalently from immigration started at time $-\infty$. Since each cluster evolves only as a function of its age relative to its immigrant, the formal construction is automatically invariant under time translations. The only nontrivial question is whether this formal superposition defines a genuine point process, namely whether every finite observation window contains only finitely many events almost surely.

Let $(T_j)_{j\in\mathbb{Z}}$ be the immigrant times and let $\mathcal{C}_j$ be independent copies of the critical Hawkes cluster rooted at the origin. Writing $u \in \mathcal{C}_j$ for the age of an event in the jth cluster, the absolute event time is $T_j + u$ and the candidate random measure is $N(\mathrm{d}s) = \sum_{j\in\mathbb{Z}} \sum_{u\in\mathcal{C}_j} \delta_{T_j+u}(\mathrm{d}s)$. Hence, for any bounded observation window $B \subset \mathbb{R}$, the window count

$$N(B) = \int_B N(\mathrm{d}s) = \sum_{j\in\mathbb{Z}} \sum_{u\in\mathcal{C}_j} \mathbf{1}_{\{T_j+u\in B\}} \tag{7}$$

is the total number of events falling in B. The whole-line construction defines a genuine point process only if $N(B) < \infty$ almost surely for every bounded B [25,29,30].

For $B = [0, T]$, the relevant quantity is not merely whether an ancestral cluster survives beyond the origin but whether it contributes at least one point to the fixed window B. Define the fixed-window hitting probability

$$H_T(u) := \mathbb{P}(\mathcal{C} \cap [u, u + T] \neq \varnothing), \tag{8}$$

where $\mathcal{C} \subset \mathbb{R}_+$ denotes the set of event ages in a single cluster rooted at the origin. Immigrants born inside the window contribute a finite Poisson number of clusters with mean νT. Immigrants born before the window, at time $-u$, contribute to $[0, T]$ precisely when their age set intersects $[u, u + T]$. Therefore the number of immigrant clusters that contribute at least one point to $[0, T]$ is Poisson with mean $\nu T + \nu \int_0^\infty H_T(u)\,\mathrm{d}u$. Consequently, provided that each single cluster is locally finite almost surely, the exact local-finiteness condition for the whole-line construction is

$$\int_0^\infty H_T(u)\,\mathrm{d}u < \infty \quad \text{for every } T < \infty. \tag{9}$$

Criterion (9) determines whether the infinite-past cluster superposition defines a locally finite stationary point process. A separate question is whether, in a regime where the stationary mean intensity is infinite, this locally finite cluster process can still be identified pathwise with the marked Hawkes process defined by Eq. (1). This issue does not affect the stationarity classification below, but it is needed to ensure that the whole-line cluster construction and the conditional-intensity formulation remain equivalent beyond the usual finite-mean setting.

Proposition 1 (Consistency with the marked-Hawkes intensity). Suppose that the whole-line cluster construction is locally finite, i.e., $\int_0^\infty H_T(u)\,\mathrm{d}u < \infty$ for every finite T. Then the resulting stationary point process admits, with respect to the marked history $\mathcal{F}_{t-}$, the conditional intensity

$$\lambda(t) = \nu + \sum_{t_i<t} \kappa_i \phi(t - t_i), \tag{10}$$

where the shot-noise term is locally integrable almost surely on every bounded time interval. The random variable $\lambda(t)$ may nevertheless have infinite expectation.

To see why local finiteness is sufficient, first truncate the immigration process at time $-m$. For every finite m, the standard cluster construction gives the usual marked-Hawkes intensity $\lambda_m(t) = \nu + \sum_{t_i<t}^{m} \kappa_i \phi(t - t_i)$. As $m \to \infty$, the truncated cluster processes can be coupled monotonically. Whenever the whole-line construction is locally finite, only finitely many clusters contribute points to any bounded observation window, so the truncated point processes converge locally to the whole-line process. The same coupling gives $\lambda_m(t) \uparrow \lambda(t) = \nu + \sum_{t_i<t} \kappa_i \phi(t - t_i)$. For a bounded interval B, conditional on all parent events and their fertility marks, the direct offspring falling in B form a Poisson random measure with total conditional mean $\int_B \sum_{t_i<t} \kappa_i \phi(t - t_i)\,\mathrm{d}t$. If this quantity were infinite, then the conditional Poisson superposition would place infinitely many offspring in B almost surely, contradicting local finiteness of the cluster construction. Hence the shot-noise term is locally integrable almost surely, and the usual marked-Hawkes compensator remains well defined even though its expectation may diverge.

We now return to the central question: Under which conditions does Eq. (9) hold? Let L denote the lifetime of a single critical cluster, i.e., the elapsed time between the immigrant event at age 0 and the last event in its descendant cluster. Since $\mathcal{C} \subset \mathbb{R}_+$ denotes the set of event ages relative to the immigrant, then $L = \sup \mathcal{C}$ and its survival probability is $Q(u) := \mathbb{P}(L > u)$. A hit of the fixed observation window $[u, u + T]$ necessarily implies that the cluster survives at least until age u. Hence

$$H_T(u) \leqslant Q(u). \tag{11}$$

It follows immediately that

$$\mathbb{E}[L] = \int_0^\infty Q(u)\mathrm{d}u < \infty \tag{12}$$

is a sufficient condition for the exact local-finiteness criterion (9) and therefore for stationarity in law of the whole-line construction. This lifetime criterion is useful because $Q(u)$ is often easier to analyze than the fixed-window hitting probability $H_T(u)$. It is, however, only a one-sided criterion. The event $L > u$ merely requires the existence of at least one descendant at some age larger than u, whereas $H_T(u)$ requires

a descendant to fall specifically inside the fixed-width interval $[u, u+T]$. In a long-memory process, survival to age u may therefore be caused by a single exceptionally delayed descendant lying far beyond the window $[u, u+T]$. In such a case $Q(u)$ can be nonintegrable even though $H_T(u)$ is integrable. Thus $\mathbb{E}[L] = \infty$ does not by itself imply failure of stationarity. The relation between H_T and Q therefore determines how far the lifetime analysis can be pushed.

Three cases must be distinguished as follows:

(1) If $\int_0^\infty Q(u)\mathrm{d}u < \infty$, then Eq. (11) immediately implies $\int_0^\infty H_T(u)\,\mathrm{d}u < \infty$, and stationarity follows rigorously.

(2) If one can establish the stronger asymptotic comparability $H_T(u) \asymp Q(u)$ for fixed T, with constants that may depend on T, then a nonintegrable lifetime tail implies a nonintegrable fixed-window hitting probability, and the whole-line construction is rigorously nonstationary.

(3) In regimes where $Q(u)$ is nonintegrable but $H_T(u)$ is not asymptotically comparable to $Q(u)$, the lifetime alone does not decide the question. The fixed-window hitting probability must then be analyzed directly from its own renewal equation.

Accordingly, the classification developed below separates three levels of control: (i) a rigorous stationary regime obtained from the finite-mean lifetime condition (12); (ii) a rigorous nonstationary regime in which $H_T(u) \asymp Q(u)$ and the lifetime tail is nonintegrable; and (iii) the remaining long-memory regimes, for which stationarity is governed by the direct asymptotics of $H_T(u)$.

IV. LIFETIME TAIL AND A RIGOROUS SUFFICIENT STATIONARITY CRITERION

Let

$$Q(t) := \mathbb{P}(L > t) \sim t^{-p}, \quad t \to \infty, \tag{13}$$

possibly up to slowly varying logarithmic factors. If $p > 1$, then $\mathbb{E}[L] < \infty$, and (11) and (12) imply (9). This gives a rigorous sufficient condition for stationarity in law.

The lifetime exponent follows from the standard generating-probability formalism [31]. Let L_κ be the lifetime of a cluster initiated by a source of fixed fertility κ. Conditional on κ, the number of first-generation offspring is Poisson with mean κ. If $R(t)$ denotes the probability that one such offspring branch, including its waiting time and all its descendants, reaches beyond time t, then

$$\mathbb{P}(L_\kappa \leqslant t) = \exp[-\kappa R(t)]. \tag{14}$$

For fixed κ, $Q_\kappa(t) := \mathbb{P}(L_\kappa > t) = 1 - \exp[-\kappa R(t)] \sim \kappa R(t)$ at large t's, so the tail exponent of Q is the same as that of R.

Let $G_1(z) = \mathbb{E}[z^N]$ be the generating function of the first-generation offspring number N, after averaging over fertilities. In the mixed-Poisson representation, $G_1(z) = \mathbb{E}_\kappa[\exp[-\kappa(1-z)]]$. At criticality, $n = G_1'(1) = 1$, and

$$V(z) := G_1(1-z) + z - 1 \sim \begin{cases} B_\gamma z^\gamma, & 1 < \gamma < 2, \\ B z^2, & \gamma > 2, \end{cases} \tag{15}$$

with logarithmic corrections at $\gamma = 2$ [31]. Denoting $P(t) = 1 - Q(t)$, the probability that the descendant cluster generated by a generic event has become extinct before time t, the all-generation extinction equation is

$$P(t) = G_1((\phi * P)(t)), \tag{16}$$

and

$$R(t) = 1 - (\phi * P)(t). \tag{17}$$

Combining these two relations gives

$$R(t) - Q(t) = V(R(t)). \tag{18}$$

Let $a(t) := 1 - \int_0^t \phi(s)\,\mathrm{d}s$ be the bare waiting-time tail. For the Omori kernel $\phi(t) \sim A t^{-1-\theta}$ (2), one has $a(t) \sim (A/\theta) t^{-\theta}$. Using the small-$s$ behavior

$$1 - \widehat{\phi}(s) \sim \begin{cases} A_\theta s^\theta, & 0 < \theta < 1, \\ \bar{\tau} s, & \theta > 1, \end{cases} \quad s \to 0, \tag{19}$$

with logarithmic corrections at $\theta = 1$ and $A_\theta := (A/\theta)\Gamma(1-\theta)$, the lifetime-tail balance is

$$\mathcal{D}_\theta R(t) + V(R(t)) \simeq a(t), \tag{20}$$

where $\mathcal{D}_\theta = D_t^\theta$ for $0 < \theta < 1$, $\mathcal{D}_\theta = \bar{\tau}\, d/dt$ for $\theta > 1$, up to constants, and D_t^θ denotes the fractional derivative whose Laplace transform is s^θ. Here $\bar{\tau}$ denotes the mean offspring waiting time, $\bar{\tau} = \int_0^\infty s\phi(s)\,\mathrm{d}s$, which is finite for $\theta > 1$.

This balance (20) follows by combining the extinction equation with the temporal embedding. The branching equation gives [31]

$$Q(t) = 1 - G_1(1 - R(t)) = R(t) - V(R(t)), \tag{21}$$

where $V(z)$ is defined by (15). Substituting $R(t)$ given by (17) as $R(t) = a(t) + (\phi * Q)(t)$ in (21) yields

$$R(t) - (\phi * R)(t) = a(t) - \phi * V(R(t)).$$

At large times, $R - \phi * R$ is represented by the memory operator $\mathcal{D}_\theta R$, whose Laplace symbol is $1 - \widehat{\phi}(s)$, while $\phi * V(R) \simeq V(R)$ at leading order.

This gives the lifetime exponent defined in (13)

$$p = \begin{cases} \min(\theta/\gamma,\ 1/(\gamma-1)), & 1 < \gamma < 2, \\ \min(\theta/2,\ 1), & \gamma > 2, \end{cases} \tag{22}$$

again up to logarithmic corrections at the marginal values. Therefore

$$p > 1 \iff 1 < \gamma < 2 \quad \text{and} \quad \theta > \gamma.$$

This proves stationarity in law in the regime

$$1 < \gamma < 2, \quad \theta > \gamma, \tag{23}$$

via condition (9) with inequality (11) because $\mathbb{E}[L] < \infty$ for $p > 1$.

V. EXACT NONSTATIONARITY IN THE FINITE-MEMORY, FINITE-VARIANCE REGIME

The divergence of the mean cluster lifetime, $\mathbb{E}[L] = \infty$, does not by itself imply nonstationarity of the whole-line construction. Such a conclusion becomes possible only when the fixed-window hitting probability $H_T(u)$ is asymptotically comparable to the cluster-lifetime survival probability

$Q(u) = \mathbb{P}(L > u)$. This comparability holds in the standard finite-memory, finite-offspring-variance critical regime, where survival to a large age is generated by a genealogically active population persisting up to times of order u, rather than by an isolated descendant produced after an exceptionally long waiting time.

Consider first the regime $\theta > 2, \gamma > 2$. The fertility distribution then has finite variance and hence so does the mixed-Poisson offspring number. Indeed, if N denotes the number of direct offspring of an event, with

$$N \mid \kappa \sim \text{Poisson}(\kappa),$$

then at criticality $\mathbb{E}[\kappa] = 1$,

$$\mathbb{E}[N] = 1, \quad \text{Var}(N) = 1 + \text{Var}(\kappa) < \infty.$$

Likewise, for the Omori kernel $\phi(t) \sim At^{-1-\theta}$, the generation-time distribution has finite mean for $\theta > 1$ and finite second moment for $\theta > 2$. More specifically, if S denotes a generic offspring waiting time, then $\mathbb{P}(S > u) \sim \frac{A}{\theta} u^{-\theta}$, so that

$$u^2 \mathbb{P}(S > u) \longrightarrow 0 \quad \text{for } \theta > 2.$$

Under these conditions, the classical theory of critical general age-dependent branching processes gives an inverse-time survival law [32]. In particular, the Hawkes cluster is naturally embedded in the Crump-Mode-Jagers class, in which each individual may give birth at multiple random ages according to a reproduction point process. Then

$$\mathbb{P}\{Z(u) > 0\} \asymp \frac{1}{u}, \quad u \to \infty,$$

where $Z(u)$ denotes the number of individuals active at age u in the associated general age-dependent branching representation.

At criticality, the mean population does not grow systematically. However, conditional on the rare event of survival up to a large age u, the number of active individuals is typically of order u. Since the unconditional mean population remains of order unity, the probability of such surviving realizations must therefore decay as $1/u$. In the present setting, the same mechanism controls the tail of the cluster lifetime: survival to a large age is associated with a branching population that remains active at times of order u, rather than with a single anomalously delayed offspring. Consequently, $Q(u) \asymp \frac{1}{u}$.

The distinction between these two mechanisms of long cluster lifetime is essential. In the present finite-memory regime, the event $L > u$ is typically associated with a genealogically active population persisting up to times of order u. In the long-memory regimes considered later, the event $L > u$ can be realized by a descendant born far beyond age u, without implying any appreciable probability of observing a point in the fixed-width interval $[u, u + T]$. By contrast, in the present finite-memory regime, conditional survival to age u is associated with an active critical population in the vicinity of age u. Therefore, for every fixed $T > 0$, there exists a constant $c_T > 0$ such that asymptotically

$$\mathbb{P}(\mathcal{C} \cap [u, u + T] \neq \varnothing \mid L > u) \geqslant c_T.$$

Since trivially $H_T(u) \leqslant Q(u)$, this lower bound implies

$$H_T(u) \asymp Q(u) \asymp \frac{1}{u}, \quad u \to \infty, \tag{24}$$

where the comparison constants may depend on the fixed window length T.

It follows that

$$\int_0^\infty H_T(u)\, \mathrm{d}u = \infty.$$

Hence the exact local-finiteness condition fails, and the positive-immigration whole-line critical Hawkes construction is not stationary in law in the regime $\theta > 2, \gamma > 2$.

The cases $\theta = 2$ and $\gamma = 2$ are marginal because they lie exactly at the thresholds where the second moments cease to be finite. For $\theta = 2$, the waiting-time tail satisfies $\mathbb{P}(S > u) \sim \frac{A}{2} u^{-2}$, so that the mean waiting time remains finite but its second moment diverges logarithmically. Equivalently, $u^2 \mathbb{P}(S > u) \longrightarrow \frac{A}{2}$, rather than zero, and the standard finite-second-moment survival theorem for critical general age-dependent branching processes used above does not apply directly. Similarly, at $\gamma = 2$ the fertility distribution has a marginally divergent second moment; since the offspring number is mixed Poisson, its variance is likewise marginally divergent.

The standard finite-second-moment theorem for critical general age-dependent branching processes therefore does not apply directly at either boundary. However, the exact renewal equations can be analyzed explicitly in these marginal cases, as shown in Appendix A. The two boundaries behave differently. For $\theta = 2$ and $\gamma > 2$, the finite mean generation time still controls the leading asymptotics and one obtains $H_T(u) \asymp \frac{1}{u}$, so that the fixed-window hitting probability remains nonintegrable. By contrast, for $\gamma = 2$ and $\theta > 2$, the marginal offspring variance modifies the nonlinear term to $V(z) \sim C_\kappa z^2 \ln \frac{1}{z}$, which yields $H_T(u) \asymp \frac{1}{u \ln u}$. This is again nonintegrable. At the double-marginal point $\theta = \gamma = 2$, the lifetime tail and the fixed-window hitting probability are no longer asymptotically comparable: The exact analysis gives a slower lifetime decay, while $H_T(u) \asymp \frac{1}{u \ln u}$, which is still nonintegrable. Thus all marginal cases $\theta = 2$ and/or $\gamma = 2$ remain on the nonstationary side of the exact local-finiteness criterion, with logarithmic corrections determined by the corresponding borderline regular variation; the full derivation is given in Appendix A.

VI. FIXED-WINDOW ASYMPTOTICS FOR $1 < \gamma < 2, \theta \leqslant \gamma$ AND $\gamma > 2, \theta < 2$

A. Exact renewal equations for the fixed-window hitting probability

It remains to estimate H_T in regimes where neither $\mathbb{E}[L] < \infty$ nor the comparability $H_T(u) \asymp Q(u)$ gives a complete answer. These are

$$1 < \gamma < 2, \quad \theta \leqslant \gamma, \quad \text{and} \quad \gamma > 2, \quad \theta < 2.$$

The derivation below is a direct asymptotic treatment of the exact nonlinear renewal equation. It keeps the full convolution structure, including the end-point contributions that control the integrable regime, and thereby provides a unified and

self-consistent determination of the decay laws on both sides of the transition and of the boundary separating them. These asymptotics are obtained directly from the exact renewal equation and are not expected to be altered by a more formal existence analysis.

Let $R_T(u)$ be the probability that one first-generation offspring branch hits $[u, u+T]$. Such a hit occurs either because the offspring itself is born in the window or because it is born at time $s < u$ and its descendant cluster hits $[u-s, u-s+T]$. Therefore

$$R_T(u) = a_T(u) + (\phi * H_T)(u), \quad a_T(u) := \int_u^{u+T} \phi(s)\,\mathrm{d}s, \tag{25}$$

and, for fixed T, $a_T(u) \sim ATu^{-1-\theta}$. Conditional on fertility κ, no offspring branch hits the window with probability $\exp[-\kappa R_T(u)]$. Averaging over κ gives

$$H_T(u) = R_T(u) - V(R_T(u)). \tag{26}$$

Since $V(z) = o(z)$ as $z \downarrow 0$, $H_T(u) \sim R_T(u)$ at the level of leading powers.

B. Exact renewal identity and end-point contribution

Combining Eqs. (25) and (26) gives the exact identity

$$V(R_T(u)) = a_T(u) + (\phi * H_T)(u) - H_T(u). \tag{27}$$

This form is preferable on the integrable side of the transition because it makes the end-point contribution explicit. Standard convolution asymptotics for regularly varying integrable functions imply that end-point terms must be retained [33–35].

Assume first $0 < \theta < 1$ and

$$H_T(u) \sim C_T u^{-r}, \quad r > 1, \tag{28}$$

so that

$$M_T := \int_0^\infty H_T(s)\,\mathrm{d}s < \infty. \tag{29}$$

The convolution has two asymptotic regions. For $s \ll u$, regular variation gives

$$H_T(u-s) - H_T(u) \simeq -sH_T'(u), \quad H_T'(u) \sim -rC_T u^{-r-1}.$$

Because $\int_0^u s\phi(s)\,\mathrm{d}s \asymp u^{1-\theta}$, the corresponding bulk or local correction scales as $u^{-r-\theta}$. At the opposite end point $s \simeq u$, setting $x = u - s$ gives

$$\int_{u-o(u)}^{u} \phi(s)H_T(u-s)\,\mathrm{d}s \sim \phi(u)\int_0^\infty H_T(x)\,\mathrm{d}x = M_T\phi(u). \tag{30}$$

Since $r > 1$, $u^{-r-\theta} = o(u^{-1-\theta})$, and therefore

$$(\phi * H_T)(u) - H_T(u) \sim M_T\phi(u). \tag{31}$$

Substitution into Eq. (27) yields

$$V(R_T(u)) \sim (T + M_T)\phi(u). \tag{32}$$

For $1 < \gamma < 2$, $V(z) \sim B_\gamma z^\gamma$, hence

$$R_T(u) \sim C_T u^{-(1+\theta)/\gamma}, \quad B_\gamma C_T^\gamma = A(T + M_T). \tag{33}$$

The integrable regime also satisfies a useful exact global identity. Indeed, under the assumption $M_T = \int_0^\infty H_T(u)\mathrm{d}u < \infty$, integration of Eq. (25) over $u \geqslant 0$ and use of $\int_0^\infty \phi(s)\,\mathrm{d}s = 1$ give $\int_0^\infty R_T(u)\,\mathrm{d}u = \int_0^\infty a_T(u)\,\mathrm{d}u + \int_0^\infty H_T(u)\,\mathrm{d}u$. Since $V(R_T) = R_T - H_T$, it follows exactly that $\int_0^\infty V(R_T(u))\,\mathrm{d}u = \int_0^\infty a_T(u)\,\mathrm{d}u = \int_0^\infty \min(s, T)\phi(s)\,\mathrm{d}s = \mathbb{E}[\min(S, T)]$. The right-hand side is finite for every fixed T, since $\mathbb{E}[\min(S, T)] \leqslant T$. Thus, once H_T is integrable, the nonlinear term $V(R_T)$ is automatically integrable as well, with a total mass fixed exactly by the waiting-time distribution. This identity provides both a consistency check on the end-point asymptotics above and a useful normalization test for numerical solutions of the exact renewal equation.

Thus the end-point contribution modifies the amplitude relation but leaves the power-law exponent unchanged. More importantly, the assumption $M_T < \infty$ is self-consistent precisely when

$$\frac{1+\theta}{\gamma} > 1 \iff \theta > \gamma - 1. \tag{34}$$

Hence the boundary $\theta = \gamma - 1$ emerges directly as the self-consistency condition separating integrable from nonintegrable fixed-window hitting tails in this regime. Because the exact criterion for local finiteness of the infinite-past cluster construction is $\int_0^\infty H_T(u)\,\mathrm{d}u < \infty$ for every finite T, the condition $\theta > \gamma - 1$ identifies the stationary side of the transition: The critical Hawkes process can then exist as a locally finite, time-translation-invariant point process even though its stationary mean intensity is infinite. In other words, criticality destroys finite-mean stationarity but does not destroy the existence of a stationary law when the fixed-window hitting tail is integrable.

C. Nonintegrable side: $\theta < \gamma - 1$

If $\theta < \gamma - 1$, then the end-point candidate would have $r < 1$ and M_T is not finite. The bulk-nonlocal contribution then controls the asymptotics. For $R_T(u) \sim u^{-r}$ with $r < 1$ and $0 < \theta < 1$,

$$R_T - \phi * R_T \asymp u^{-r-\theta}, \quad V(R_T) \asymp u^{-\gamma r}. \tag{35}$$

The homogeneous balance gives

$$r + \theta = \gamma r \implies r = \frac{\theta}{\gamma - 1}. \tag{36}$$

The forcing $a_T(u) \asymp u^{-1-\theta}$ is asymptotically smaller because $\gamma\theta/(\gamma-1) < 1 + \theta$ is equivalent to $\theta < \gamma - 1$. Thus the bulk and end-point regimes meet exactly at $\theta = \gamma - 1$.

The balance (36) must also be compatible with the Laplace transform of the exact identity (27) [Eq. (B7), whose derivation does not use $\theta = \gamma - 1$], which holds for every $s > 0$ without assuming that H_T is integrable. Since $V \geqslant 0$ and $\widehat{a}_T(s) \leqslant \widehat{a}_T(0) = \mathbb{E}[\min(S, T)]$,

$$[1 - \widehat{\phi}(s)]\,\widehat{H}_T(s) \leqslant \mathbb{E}[\min(S, T)], \quad s > 0, \tag{36a}$$

so that, by Eq. (19), $\widehat{H}_T(s) = O(s^{-\theta})$ as $s \downarrow 0$, and therefore $r \geqslant 1 - \theta$. Moreover, the left-hand side of Eq. (B7) is nonnegative, so monotone convergence gives $\int_0^\infty V(R_T(u))\,\mathrm{d}u \leqslant$

$\mathbb{E}[\min(S,T)]$, and the left-hand side of Eq. (36a) tends, as $s \downarrow 0$, to $D_T := \mathbb{E}[\min(S,T)] - \int_0^\infty V(R_T(u))\,\mathrm{d}u \geqslant 0$. If $D_T > 0$, then $\widehat{H}_T(s) \simeq D_T/(A_\theta s^\theta)$, and H_T follows the linear critical (dressed Omori) law $H_T(u) \propto u^{-(1-\theta)}$ [4,36]; this is self-consistent only if $V(R_T) \asymp u^{-\gamma(1-\theta)}$ is integrable, i.e., for $\theta < (\gamma-1)/\gamma$. If $D_T = 0$, then $r > 1-\theta$ and the balance (36) applies; it satisfies $r > 1-\theta$ only for $\theta > (\gamma-1)/\gamma$. In direct space, $r+\theta > 1$ is a sign condition: for $R_T \simeq C_T u^{-r}$, the fractional term in Eq. (35) is $R_T - \phi * R_T \simeq A_\theta C_T\,[\Gamma(1-r)/\Gamma(1-r-\theta)]\,u^{-r-\theta}$, which has the negative sign required by Eq. (27) with $V(R_T) \geqslant 0$ only for $r+\theta > 1$. Hence

$$r = \max\left\{1-\theta,\ \frac{\theta}{\gamma-1}\right\}, \quad 0 < \theta < \gamma - 1, \tag{36b}$$

up to slowly varying corrections at $\theta = (\gamma-1)/\gamma$, where both branches give $r = 1/\gamma$.

Since $r < 1$ in both branches of Eq. (36b), one has $H_T(u) \sim R_T(u)$ with a nonintegrable tail, so the exact local-finiteness criterion fails and the critical whole-line Hawkes process does not exist as a locally finite stationary point process in this regime.

D. Marginal case $\theta = \gamma - 1$

At the boundary $\theta = \gamma - 1$, the two asymptotic regimes meet at the critical decay exponent $r = 1$. The leading algebraic decay alone is therefore insufficient to determine local finiteness, and the fixed-window hitting probability must be written in the marginal form

$$H_T(u) \sim \frac{L_T(u)}{u}, \tag{37}$$

with $L_T(u)$ slowly varying. As derived in Appendix B from the exact nonlinear renewal equation using a refined Laplace-Tauberian analysis, one obtains

$$L_T(u) \sim \left[\frac{A(\gamma-1)}{\gamma B_\gamma}\right]^{1/(\gamma-1)} \left[\ln\left(\frac{u}{u_0}\right)\right]^{1/(\gamma-1)}, \tag{38}$$

and therefore

$$H_T(u) \sim \left[\frac{A(\gamma-1)}{\gamma B_\gamma}\right]^{1/(\gamma-1)} \frac{[\ln(u/u_0)]^{1/(\gamma-1)}}{u}. \tag{39}$$

The logarithmic factor thus enhances, rather than suppresses, the marginal u^{-1} tail. Consequently, $\int_0^\infty H_T(u)\,\mathrm{d}u = \infty$, and the exact local-finiteness criterion fails at $\theta = \gamma - 1$. Hence the marginal point belongs to the nonstationary side of the transition: the positive-immigration, infinite-past critical Hawkes construction is not a locally finite stationary point process at $\theta = \gamma - 1$. The stationary regime therefore begins strictly above the boundary, $\theta > \gamma - 1$. The full derivation of the logarithmic correction and its amplitude is given in Appendix B.

E. Finite-mean memory: $\theta > 1$

For $\theta > 1$, the offspring clock has finite mean $\bar{\tau} = \int_0^\infty s\phi(s)\,\mathrm{d}s$. For sufficiently regular decreasing H_T, the local and end-point contributions coexist:

$$(\phi * H_T)(u) - H_T(u) \sim -\bar{\tau} H_T'(u) + M_T \phi(u). \tag{40}$$

Equation (27) therefore gives

$$V(R_T) \sim -\bar{\tau} H_T' + (T + M_T)\phi. \tag{41}$$

The end point–memory-tail balance gives $r = (1+\theta)/\gamma$, whereas the general age-dependent critical branching balance gives $r = 1/(\gamma-1)$. Hence

$$r = \min\left\{\frac{1+\theta}{\gamma}, \frac{1}{\gamma-1}\right\}, \quad 1 < \gamma < 2, \quad \theta > 1. \tag{42}$$

Both exponents exceed unity for $1 < \gamma < 2$ and $\theta > 1$.

Combining the regimes above, for $1 < \gamma < 2$ the fixed-window asymptotics are

$$H_T(u) \asymp \begin{cases} u^{-\max\{1-\theta,\,\theta/(\gamma-1)\}}, & 0 < \theta < \gamma - 1, \\ u^{-1} L_T(u), & \theta = \gamma - 1, \\ u^{-(1+\theta)/\gamma}, & \gamma - 1 < \theta \leqslant 1, \\ u^{-\min\{(1+\theta)/\gamma,\,1/(\gamma-1)\}}, & \theta > 1, \end{cases} \tag{43}$$

where the constants implicit in $\asymp$ may depend on the fixed window length T. The resulting asymptotic prediction is

$$\int_0^\infty H_T(u)\,\mathrm{d}u < \infty \iff \theta > \gamma - 1,\ 1 < \gamma < 2, \tag{44}$$

subject to the stated regular-variation assumptions. The marginal case $\theta = \gamma - 1$ is resolved in Appendix B, which gives $H_T(u) \sim C_*[\ln(u/u_0)]^{1/(\gamma-1)}/u$. This tail is nonintegrable, so equality belongs to the nonstationary side and the stationary regime begins strictly at $\theta > \gamma - 1$.

For $\gamma > 2$, $V(z) \sim Bz^2$. In the long-memory subcase $0 < \theta < 1$, the homogeneous fractional balance gives $r = \theta$, which satisfies the condition $r \geqslant 1-\theta$ of Eq. (36a) only for $\theta > 1/2$. For $0 < \theta < 1/2$, $V(R_T) \asymp H_T^2$ is integrable and the linear branch $u^{-(1-\theta)}$ applies. Hence

$$H_T(u) \asymp u^{-\max\{\theta,\,1-\theta\}}, \quad 0 < \theta < 1, \quad \gamma > 2, \tag{45}$$

up to slowly varying corrections at $\theta = 1/2$. For $1 \leqslant \theta < 2$, the finite-mean clock gives the ordinary critical age-dependent branching contribution $H_T(u) \asymp u^{-1}$, while the local forcing would decay faster. Thus

$$H_T(u) \asymp \begin{cases} u^{-\max\{\theta,\,1-\theta\}}, & 0 < \theta < 1, \\ u^{-1}, & 1 \leqslant \theta < 2, \end{cases} \tag{46}$$

with possible logarithmic corrections at the marginal values. Both cases are nonintegrable, in agreement with the exact finite-memory–finite-variance conclusion.

Table I summarizes the resulting classification. The entries marked "proved" follow either from the sufficient finite-mean lifetime condition or from the finite-memory–finite-variance comparability $H_T(u) \asymp Q(u)$. The remaining entries summarize the asymptotic prediction from the exact fixed-window renewal equation, including end-point terms on the integrable side.

VII. NUMERICAL DIAGNOSTICS FOR THE STATIONARITY BOUNDARY

We tested the fixed-window theory by solving the exact nonlinear renewal system directly rather than by simulating complete critical clusters. This is numerically advantageous

TABLE I. Critical stationarity classification from the asymptotic behavior of the fixed-window hitting probability $H_T(u)$.

Fertility regime	Memory regime	$H_T(u)$ tail	Stationary in law?
$1<\gamma<2$	$\theta>\gamma$	$H_T(u)\leqslant Q(u),\ \mathbb{E}[L]<\infty$	Yes, proved
$1<\gamma<2$	$\gamma-1<\theta\leqslant 1$	$H_T(u)\asymp u^{-(1+\theta)/\gamma},$ $\gamma-1<\theta\leqslant 1,\quad 1<\gamma<2$	Yes, asymptotically predicted
$1<\gamma<2$	$1<\theta\leqslant\gamma$	$H_T(u)\asymp u^{-r},$ $r=\min\{\frac{1+\theta}{\gamma},\frac{1}{\gamma-1}\}>1,\quad 1<\theta\leqslant\gamma$	Yes, asymptotically predicted
$1<\gamma<2$	$\theta=\gamma-1$	$\dfrac{[\ln(u/u_0)]^{1/(\gamma-1)}}{u}$	No, marginal asymptotic
$1<\gamma<2$	$0<\theta<\gamma-1$	$u^{-\max\{1-\theta,\theta/(\gamma-1)\}}$	No, asymptotically predicted
$\gamma>2$	$\theta>2$	$H_T(u)\asymp Q(u)\asymp u^{-1}$	No, proved
$\gamma>2$	$\theta=2$	$H_T(u)\asymp Q(u)\asymp u^{-1}$	No, controlled marginal asymptotic
$\gamma=2$	$\theta>2$	$H_T(u)\sim Q(u)\sim\dfrac{\bar{\tau}}{C_\kappa\, u\ln u}$	No, controlled marginal asymptotic
$\gamma=2$	$\theta=2$	$H_T(u)\asymp\dfrac{1}{u\ln u},\quad Q(u)\asymp\dfrac{1}{u\sqrt{\ln u}}$	No, controlled marginal asymptotic
$\gamma>2$	$0<\theta<1$	$u^{-\max\{\theta,1-\theta\}}$	No, asymptotically predicted
$\gamma>2$	$1\leqslant\theta<2$	u^{-1} up to slowly varying corrections	No, asymptotically predicted

because the latter have extremely broad total progeny and generate large Monte Carlo fluctuations precisely near criticality. We used

$$R_T(u)=a_T(u)+(\phi * H_T)(u),\quad H_T(u)=1-\mathbb{E}[e^{-\kappa R_T(u)}], \tag{47}$$

with the normalized Omori kernel

$$\phi(t)=\frac{\theta}{(1+t)^{1+\theta}},\quad a_T(u)=(1+u)^{-\theta}-(1+u+T)^{-\theta}, \tag{48}$$

and a Pareto fertility distribution

$$\mathbb{P}(\kappa>x)=\left(\frac{\kappa_0}{x}\right)^{\gamma},\quad x\geqslant\kappa_0,\quad \kappa_0=\frac{\gamma-1}{\gamma}, \tag{49}$$

so that $\mathbb{E}[\kappa]=1$ exactly. The calculations below use $\gamma=1.5$, $T=1$, a uniform step $\Delta u=0.25$, and $U_{\max}=8191.75$. The convolution was evaluated by a fast Fourier transform using the exact probability mass of ϕ in each grid cell, and the fixed-point iteration was continued until the maximum relative change in R_T was below 2×10^{-11}.

For $\gamma=1.5$, the theoretical boundary is

$$\theta_c=\gamma-1=0.5.$$

We therefore considered $\theta=0.35$, 0.50, and 0.70, representing the nonintegrable, marginal, and integrable sides of the transition. Figure 1 shows the resulting $H_T(u)$ together with the corresponding asymptotic forms. For $\theta=0.35$, the prediction of Eq. (36b) is

$$r=\frac{\theta}{\gamma-1}=0.70,$$

and the measured effective slopes are already very close to this value. The log-log fits give $r_{\text{eff}}=0.704$, 0.712, and 0.714 over the successive ranges $u\in[100,500]$, [500,2000], and [2000,7000].

At the marginal point $\theta=0.50$, the refined Tauberian analysis gives

$$H_T(u)\asymp\frac{(\ln u)^2}{u}$$

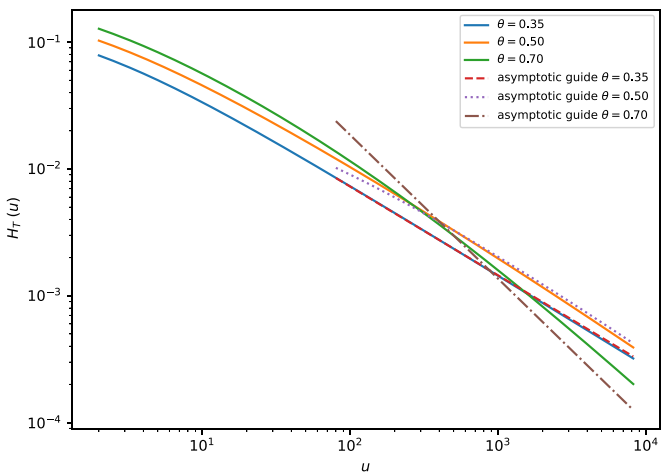


FIG. 1. Numerical solution of the exact fixed-window nonlinear renewal system defined by Eqs. (25) and (26), implemented in the equivalent form (47), with the Omori kernel (48) and the critical Pareto fertility distribution (49). The parameters are $\gamma=1.5$ and $T=1$. Solid curves show $H_T(u)$ for $\theta=0.35$, 0.50, and 0.70. Dashed, dotted, and dash-dotted guides show the predicted asymptotic forms $u^{-0.70}$, $(\ln u)^2/u$, and $u^{-1.133}$, respectively, normalized at $u=500$. The nonstationary-side exponent is reached rapidly, the marginal curve exhibits the predicted logarithmic correction, and the stationary side shows a slow crossover toward its asymptotic exponent.

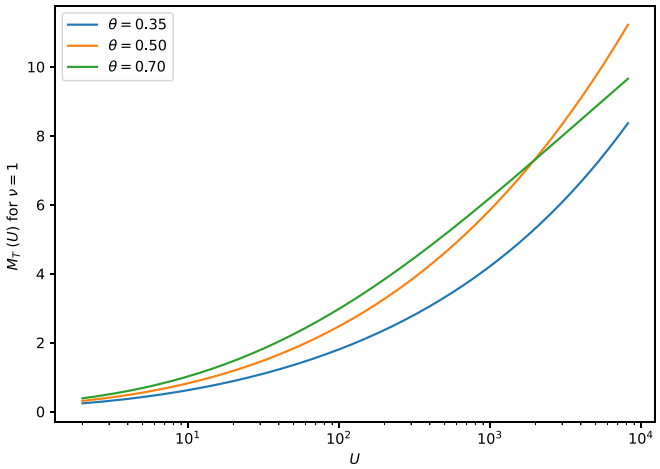


FIG. 2. Cumulative fixed-window hitting measure $M_T(U)$, defined by Eq. (50) in terms of the hitting probability $H_T(u)$ [Eq. (8)], computed from the exact nonlinear renewal system (47). The parameters are $\gamma = 1.5$, $T = 1$, $\nu = 1$, and $\theta = 0.35$, 0.50, and 0.70, as in Fig. 1. For $\theta = 0.35$, the nonintegrable hitting tail has exponent $r = \theta/(\gamma - 1) = 0.70$ [Eq. (36b)], implying algebraic growth $M_T(U) \asymp U^{1-r} = U^{0.30}$. At the marginal boundary $\theta = \gamma - 1 = 0.50$, the logarithmically enhanced hitting tail [Eq. (B31)] gives $M_T(U) \asymp (\ln U)^3$ [Eq. (B32)]. For $\theta = 0.70$, the integrable hitting tail has exponent $r = (1 + \theta)/\gamma \simeq 1.133$ [Eq. (33)], so that $M_T(U)$ approaches a finite limit, but extremely slowly: $M_T(\infty) - M_T(U) \asymp U^{1-r} \simeq U^{-0.133}$. The three curves thus illustrate the local-finiteness criterion (9) and the transition from nonstationary to stationary critical Hawkes dynamics in law.

for $\gamma = 1.5$. The numerical curve displays precisely the expected slow approach toward an inverse-time decay: The corresponding effective exponents increase from 0.714 to 0.749 and 0.774 over the same three fitting ranges. Thus the marginal curve should not be interpreted as a finite-size power law with exponent smaller than one; its progressive steepening is the signature of the logarithmically enhanced u^{-1} asymptotic form.

For $\theta = 0.70$, the predicted integrable exponent is

$$r = \frac{1+\theta}{\gamma} = \frac{1.70}{1.50} \simeq 1.133.$$

The crossover is substantially slower on this side of the transition. The effective exponents increase from 0.845 to 0.936 and 0.996 over the three ranges above. Extending this case alone to $U_{\max} \simeq 1.64 \times 10^4$ gives $r_{\text{eff}} = 1.029$ over $u \in [7000, 15\,000]$, confirming the continued drift toward the asymptotic value above unity. The slow convergence is expected because $r - 1 \simeq 0.133$ is small.

The cumulative diagnostic

$$M_T(U) := \int_0^U H_T(u)\,\mathrm{d}u \tag{50}$$

is shown in Fig. 2; since $\nu = 1$, it also equals the expected number of ancestral clusters contributed by immigrants from the preceding duration U. For $\theta = 0.35$, $M_T(U)$ grows algebraically; a fit over $U \in [500, 7000]$ gives an effective growth exponent 0.328, close to the predicted $1 - r = 0.30$. At the marginal point, $H_T(u) \asymp (\ln u)^2/u$ implies

$$M_T(U) \asymp (\ln U)^3,$$

so the cumulative number of contributing ancestral clusters diverges, although more slowly than any power. For $\theta = 0.70$, $M_T(U)$ must eventually saturate because $r > 1$, but the approach is extremely slow:

$$M_T(\infty) - M_T(U) \asymp U^{1-r} \simeq U^{-0.133}.$$

Consequently, visible saturation is not expected over only a few decades in U and is not, by itself, a reliable finite-range diagnostic close to the transition.

A more precise check on the integrable side is provided by the exact identity

$$\int_0^\infty V(R_T(u))\,\mathrm{d}u = \mathbb{E}[\min(S, T)]. \tag{51}$$

For $\theta = 0.70$ and $T = 1$, the right-hand side is known analytically from Eq. (48),

$$\mathbb{E}[\min(S, 1)] = \int_0^1 (1+s)^{-0.7}\,\mathrm{d}s = 0.77048\ldots.$$

The discretized renewal solution gives

$$\int_0^{U_{\max}} V(R_T(u))\,\mathrm{d}u = 0.77621,$$

a relative discrepancy of 0.74%. This provides an independent global validation of the numerical solution and, in particular, of the integrable end-point regime.

Taken together, the numerical results support all three asymptotic regimes. Below the boundary, H_T has the predicted nonintegrable power-law tail; at $\theta = \gamma - 1$ the logarithmic enhancement of the marginal u^{-1} law is clearly visible and implies nonstationarity; above the boundary the local slope drifts toward the predicted exponent $r > 1$, while the exact integral identity is satisfied to within a few percent. The numerical diagnostic therefore supports the precise stationarity threshold $\theta > \gamma - 1$ for $1 < \gamma < 2$, with the equality case belonging to the nonstationary side.

VIII. DISCUSSION AND PHYSICAL INTERPRETATION

At $n = 1$, each immigrant initiates a critical cascade. Criticality implies an infinite expected total progeny but not an infinite realized cascade: Under the standard branching construction, each cluster becomes extinct almost surely. The question for whole-line stationarity is therefore whether immigrants from the infinite past contribute only finitely many events to every bounded window.

The exact object controlling this question is the fixed-window hitting probability $H_T(u)$, not only the lifetime tail $Q(u)$. The lifetime criterion $\mathbb{E}[L] < \infty$ is a clean and rigorous sufficient condition, and it proves stationarity for $1 < \gamma < 2$ and $\theta > \gamma$. Conversely, in the finite-memory and finite-variance critical regime, survival to time u is produced by the ordinary critical age-dependent branching mechanism, so $H_T(u) \asymp Q(u) \asymp u^{-1}$. The exact local-finiteness condition then fails, proving nonstationarity.

The most delicate regime is the infinite-variance fertility case with $\theta < \gamma$. There, $Q(u)$ can be dominated by very

late descendants, whereas $H_T(u)$ requires a hit in a fixed observation window. On the predicted locally finite side, H_T is integrable and the convolution cannot be reduced to a purely local fractional derivative: the end-point contribution $\phi(u)\int_0^\infty H_T(s)\,\mathrm{d}s$ is of the same order as the fixed-window forcing $a_T(u)$. Retaining this term yields $V(R_T) \sim [T + M_T]\phi(u)$ and therefore $H_T(u) \sim u^{-(1+\theta)/\gamma}$. The requirement that M_T be finite is exactly $(1+\theta)/\gamma > 1$, i.e., $\theta > \gamma - 1$. Hence this boundary appears as the self-consistency boundary between an integrable end-point-dominated regime and a nonintegrable bulk or fractional regime. In the intermediate interval $\gamma - 1 < \theta \leqslant \gamma$, the mean lifetime diverges while local finiteness is asymptotically predicted.

For ETAS seismicity, the fertility exponent is $\gamma = \beta/\alpha$, where α is the productivity exponent with seismic moment and β is the moment-distribution exponent. Typical values $\beta \simeq 2/3$ and $\alpha \simeq 1/2$ give $\gamma \simeq 4/3$, and hence the time-only stationarity threshold is $\theta > \gamma - 1 = \frac{\beta-\alpha}{\alpha} \simeq \frac{1}{3}$. Empirical Omori-memory exponents range from less than 1 (which can be interpreted as the dressed Omori law $\sim 1/t^{1-\theta}$ [4,36]) to as large as 1.5. Thus, values $\theta \simeq 0.3$–0.4 [37] place the temporal ETAS near the boundary of stationary critical dynamics in law.

We can now reconcile the finite-window result (6) [28] with the present classification. The former is a genealogical statement about the total progeny generated by finitely many immigrant families, mainly controlled by the fertility exponent γ. Whole-line stationarity is instead a calendar-time embedding problem: It asks whether families born in the infinite past hit a bounded present-time window. This depends on both the memory exponent θ and the fertility exponent γ, and the correct object is the fixed-window hitting probability H_T.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

APPENDIX A: CONTROLLED ASYMPTOTICS FOR THE MARGINAL CASES $\gamma = 2$ AND $\theta = 2$

This Appendix derives the two marginal cases $\gamma = 2$ and $\theta = 2$ directly from the exact generating-function equations of the Hawkes cluster, whose results are summarized at the end of Sec. V.

1. Exact equations and notation

Let $R_T(u)$ denote the probability that one first-generation branch hits the fixed window $[u, u+T]$, and let

$$H_T(u) = R_T(u) - V(R_T(u)), \tag{A1}$$

where

$$V(z) = G_1(1-z) + z - 1. \tag{A2}$$

The exact fixed-window renewal equation is

$$R_T(u) = a_T(u) + (\phi * H_T)(u), \quad a_T(u) = \int_u^{u+T} \phi(s)\mathrm{d}s, \tag{A3}$$

so that

$$V(R_T(u)) = a_T(u) + (\phi * H_T)(u) - H_T(u). \tag{A4}$$

For an Omori kernel $\phi(u) \sim Au^{-1-\theta}$, we have, for fixed T, $a_T(u) \sim ATu^{-1-\theta}$. We denote the finite mean offspring waiting time by $\bar\tau = \int_0^\infty s\phi(s)\mathrm{d}s$. For both marginal cases treated below, $\bar\tau < \infty$.

The corresponding lifetime equations are

$$R(u) = a(u) + (\phi * Q)(u), \quad Q(u) = R(u) - V(R(u)), \tag{A5}$$

with $a(u) = \int_u^\infty \phi(s)\mathrm{d}s$. Equivalently,

$$R(u) - (\phi * R)(u) = a(u) - (\phi * V(R))(u). \tag{A6}$$

The derivations below use the standard regular-variation assumption that the asymptotic solutions are eventually monotone and sufficiently smooth for the first-order renewal expansion to apply. Under finite mean waiting time, for a regularly varying function f of negative index,

$$(\phi * f)(u) - f(u) \sim -\bar\tau f'(u), \tag{A7}$$

provided the remaining end point and higher-order terms are smaller. We verify this ordering explicitly in the marginal cases below. Thus the conclusions are controlled asymptotic consequences of the exact renewal equations; a fully abstract existence theorem could alternatively be formulated using nonlinear renewal or Tauberian theory.

2. Marginal fertility variance: $\gamma = 2$ with $\theta > 2$

Assume

$$\mathbb{P}(\kappa > x) \sim C_\kappa x^{-2}, \quad C_\kappa > 0, \tag{A8}$$

with $\mathbb{E}[\kappa] = 1$. Since

$$G_1(1-z) = \mathbb{E}[e^{-z\kappa}], \tag{A9}$$

the exact identity

$$\mathbb{E}[e^{-z\kappa}] - 1 + z\mathbb{E}[\kappa] = z\int_0^\infty (1 - e^{-zx})\mathbb{P}(\kappa > x)\mathrm{d}x \tag{A10}$$

implies, as $z \downarrow 0$,

$$V(z) \sim C_\kappa z^2 \ln(1/z). \tag{A11}$$

Indeed, the logarithm comes from the interval $1 \ll x \ll z^{-1}$, over which $1 - e^{-zx} \sim zx$ and $\mathbb{P}(\kappa > x) \sim C_\kappa x^{-2}$. Equivalently, the truncated second moment grows as

$$\mathbb{E}[\kappa^2; \kappa \leqslant x] \sim 2C_\kappa \ln x. \tag{A12}$$

Because $\theta > 2$, the waiting-time distribution has finite second moment. Hence the first-order renewal term in Eq. (A7) dominates all higher-order and end-point corrections. Moreover,

$$a_T(u) = O(u^{-1-\theta}) = o\left(\frac{1}{u^2 \ln u}\right). \tag{A13}$$

Since $V(z) = o(z)$, Eq. (A1) gives $H_T(u) \sim R_T(u)$. Substitution of Eqs. (A7) and (A11) into the exact equation (A4) yields

$$C_\kappa R_T(u)^2 \ln\frac{1}{R_T(u)} \sim -\bar\tau R_T'(u). \tag{A14}$$

Writing $Y(u) = 1/R_T(u)$ gives

$$Y'(u) \sim \frac{C_\kappa}{\bar\tau} \ln Y(u). \tag{A15}$$

Integration gives

$$\frac{Y(u)}{\ln Y(u)} \sim \frac{C_\kappa}{\bar\tau} u, \tag{A16}$$

and therefore

$$R_T(u) \sim H_T(u) \sim \frac{\bar\tau}{C_\kappa\, u \ln u}, \quad \gamma = 2, \quad \theta > 2. \tag{A17}$$

The possible $\ln\ln u$ term enters only at relative order $\ln\ln u/\ln u$ and does not modify the leading form.

The same calculation applies to the lifetime equation because, for $\theta > 2$,

$$a(u) \sim \frac{A}{\theta} u^{-\theta} = o\left(\frac{1}{u^2 \ln u}\right). \tag{A18}$$

Thus

$$Q(u) \sim \frac{\bar\tau}{C_\kappa\, u \ln u}. \tag{A19}$$

In this marginal-fertility case $H_T(u)$ and $Q(u)$ have the same leading tail. In particular,

$$\int_0^U H_T(u)\mathrm{d}u \sim \frac{\bar\tau}{C_\kappa} \ln\ln U \to \infty, \quad U \to \infty, \tag{A20}$$

so the exact whole-line local-finiteness criterion fails. Hence the positive-immigration critical process is nonstationary in law for $\gamma = 2$ and $\theta > 2$.

3. Marginal memory variance: $\theta = 2$ with $\gamma > 2$

Assume now that the fertility distribution has finite second moment, so that

$$V(z) \sim Bz^2, \quad B = \tfrac{1}{2}\mathbb{E}[\kappa^2] > 0, \tag{A21}$$

and take $\phi(u) \sim Au^{-3}$. Then

$$\mathbb{P}(S > u) \sim \frac{A}{2} u^{-2}, \tag{A22}$$

so $\bar\tau = \mathbb{E}[S] < \infty$ while $\mathbb{E}[S^2] = \infty$ only logarithmically.

The important point is that this logarithmic divergence of the second waiting-time moment does *not* modify the leading fixed-window balance. Suppose $H_T(u) \sim c_T/u$. Splitting the convolution into a local part $s = o(u)$, a bulk part $s = O(u)$ away from u, and the end point $s \simeq u$, one finds

$$(\phi * H_T)(u) - H_T(u) = -\bar\tau H_T'(u) + O\left(\frac{\ln u}{u^3}\right) \sim \frac{\bar\tau c_T}{u^2}. \tag{A23}$$

The $O(u^{-3}\ln u)$ correction contains both the logarithmically divergent truncated second moment and the far-end contribution $\phi(u)\int^u H_T(s)\mathrm{d}s$. Both are smaller by a factor $\ln u/u$ than the leading derivative term. Furthermore,

$$a_T(u) \sim ATu^{-3} = o(u^{-2}). \tag{A24}$$

Hence Eq. (A4) gives

$$BR_T(u)^2 \sim -\bar\tau R_T'(u), \tag{A25}$$

with $H_T(u) \sim R_T(u)$, and therefore

$$H_T(u) \sim R_T(u) \sim \frac{\bar\tau}{B}\frac{1}{u}, \quad \theta = 2, \quad \gamma > 2. \tag{A26}$$

Thus the marginal divergence of the waiting-time variance affects only subleading terms; the leading fixed-window hitting law remains the ordinary inverse-time critical age-dependent branching form.

The lifetime probability $Q(u)$ is slightly different because its bare forcing is one power larger than the fixed-window forcing. At $\theta = 2$, $a(u) \sim \frac{A}{2}u^{-2}$, which is of the same order as both the derivative term and $V(R)$. Writing $R(u) \sim \frac{c_Q}{u}$, the exact lifetime equation gives at leading order,

$$-\bar\tau c_Q + Bc_Q^2 = \frac{A}{2}, \tag{A27}$$

so that

$$c_Q = \frac{\bar\tau + \sqrt{\bar\tau^2 + 2AB}}{2B}. \tag{A28}$$

Consequently,

$$Q(u) \sim \frac{c_Q}{u}, \quad \theta = 2, \quad \gamma > 2. \tag{A29}$$

The memory-tail forcing therefore changes the lifetime amplitude but not the exponent. Since

$$\frac{H_T(u)}{Q(u)} \longrightarrow \frac{\bar\tau/B}{c_Q} > 0, \tag{A30}$$

the two probabilities remain asymptotically comparable. In particular, $\int^\infty H_T(u)\mathrm{d}u = \infty$, and the whole-line critical process is rigorously nonstationary at the level of this controlled renewal asymptotic for $\theta = 2$ and $\gamma > 2$.

4. Double marginal corner: $\theta = 2$ and $\gamma = 2$

The intersection of the two marginal cases is instructive. For the fixed-window problem, the forcing still satisfies $a_T(u) \sim ATu^{-3}$ and is negligible. The finite mean clock therefore balances the marginal branching nonlinearity:

$$C_\kappa R_T(u)^2 \ln\frac{1}{R_T(u)} \sim -\bar\tau R_T'(u), \tag{A31}$$

which again gives

$$H_T(u) \sim \frac{\bar\tau}{C_\kappa\, u \ln u}. \tag{A32}$$

Hence the fixed-window hitting probability is nonintegrable, with $\int^U H_T(u)\mathrm{d}u \asymp \ln\ln U$.

The lifetime equation behaves differently because the bare waiting-time tail is now $a(u) \sim \frac{A}{2}u^{-2}$. If one tried the fixed-window scale $R(u) \asymp 1/(u\ln u)$, then both the derivative and the nonlinear term would be smaller than $a(u)$. The leading lifetime balance is instead

$$C_\kappa R(u)^2 \ln\frac{1}{R(u)} \sim \frac{A}{2}u^{-2}, \tag{A33}$$

which yields

$$Q(u) \sim R(u) \sim \sqrt{\frac{A}{2C_\kappa}} \frac{1}{u\sqrt{\ln u}}. \tag{A34}$$

Thus at the double marginal point H_T and Q are *not* asymptotically comparable:

$$\frac{H_T(u)}{Q(u)} \asymp \frac{1}{\sqrt{\ln u}} \longrightarrow 0. \tag{A35}$$

This explicitly illustrates why the lifetime tail alone is not the exact stationarity criterion. Nevertheless, Eq. (A32) shows directly that H_T remains nonintegrable, so the whole-line construction is still nonstationary.

APPENDIX B: MARGINAL FIXED-WINDOW ASYMPTOTICS AT $\theta = \gamma - 1$

This Appendix resolves the slowly varying correction at the marginal boundary $\theta = \gamma - 1,\ 1 < \gamma < 2$. It is convenient to set

$$\alpha := \theta = \gamma - 1 \in (0, 1), \quad \gamma = 1 + \alpha. \tag{B1}$$

We start from the exact fixed-window equations

$$R_T(u) = a_T(u) + (\phi * H_T)(u), \tag{B2}$$

$$H_T(u) = R_T(u) - V(R_T(u)), \tag{B3}$$

where $a_T(u) = \int_u^{u+T} \phi(v)\,\mathrm{d}v$. We assume

$$\phi(u) \sim Au^{-1-\alpha}, \quad A > 0, \tag{B4}$$

$$V(z) \sim B_\gamma z^\gamma, \quad B_\gamma > 0, \quad z \downarrow 0. \tag{B5}$$

For fixed T, $a_T(u) \sim ATu^{-1-\alpha}$.

1. Exact Laplace identity

Combining Eqs. (B2) and (B3) gives

$$H_T(u) - (\phi * H_T)(u) = a_T(u) - V(R_T(u)). \tag{B6}$$

For a locally integrable non-negative function f, denote its Laplace transform by

$$\widehat{f}(s) = \int_0^\infty e^{-su} f(u)\,\mathrm{d}u, \quad s > 0.$$

Taking Laplace transforms of Eq. (B6) yields the exact identity

$$[1 - \widehat{\phi}(s)]\widehat{H}_T(s) = \widehat{a}_T(s) - \widehat{V(R_T)}(s). \tag{B7}$$

The regularly varying tail (B4) implies, by Karamata's Abelian theorem [33],

$$1 - \widehat{\phi}(s) \sim Ac_\alpha s^\alpha, \quad c_\alpha := \frac{\Gamma(1-\alpha)}{\alpha}, \quad s \downarrow 0. \tag{B8}$$

Likewise, since $a_T(u) \sim ATu^{-1-\alpha}$ and a_T is integrable,

$$\widehat{a}_T(0) - \widehat{a}_T(s) \sim ATc_\alpha s^\alpha. \tag{B9}$$

2. Marginal regular variation and the nonlinear Laplace defect

At the stationarity boundary $\theta = \gamma - 1$, the two asymptotic regimes meet at the algebraic decay exponent $r = 1$. We therefore consider the marginal regularly varying form

$$H_T(u) \sim \frac{L_T(u)}{u}, \tag{B10}$$

where L_T is positive and slowly varying. We shall show that the exact renewal equation uniquely determines the leading behavior of L_T. Define the truncated mass

$$M_T(x) := \int_{u_0}^x \frac{L_T(v)}{v}\,\mathrm{d}v, \tag{B11}$$

where $u_0 > 0$ is arbitrary and fixed. The value of u_0 affects only lower-order terms.

Because $V(z) = o(z)$ as $z \downarrow 0$, Eq. (B3) gives $R_T(u) \sim H_T(u)$. Consequently,

$$V(R_T(u)) \sim B_\gamma H_T(u)^\gamma \sim B_\gamma u^{-1-\alpha} L_T(u)^\gamma. \tag{B12}$$

The right-hand side is integrable at infinity because $\alpha > 0$ even if L_T diverges slowly. Karamata's theorem therefore gives

$$\widehat{V(R_T)}(0) - \widehat{V(R_T)}(s) \sim B_\gamma c_\alpha s^\alpha L_T(1/s)^\gamma. \tag{B13}$$

The cumulative function M_T in Eq. (B11) is slowly varying whenever it diverges. Applying the Hardy-Littlewood-Karamata-Tauberian theorem (Theorem 1.7.1 in Ref. [33]) to the non-negative measure $H_T(u)\,\mathrm{d}u$ gives

$$\widehat{H}_T(s) \sim M_T(1/s), \quad s \downarrow 0. \tag{B14}$$

In particular, $s^\alpha \widehat{H}_T(s) \to 0$. Taking $s \downarrow 0$ in Eq. (B7) therefore shows that the finite zeroth-order terms on its right-hand side cancel:

$$\widehat{V(R_T)}(0) = \widehat{a}_T(0). \tag{B15}$$

This identity is obtained here without assuming that H_T itself is integrable.

Subtracting the common limit in Eq. (B15) and using Eqs. (B8), (B9), (B13), and (B14) in Eq. (B7) gives

$$AM_T(1/s) \sim B_\gamma L_T(1/s)^\gamma - AT. \tag{B16}$$

We now show that $M_T(x)$ must diverge. Suppose, to the contrary, that

$$M_T(\infty) := \int_{u_0}^\infty \frac{L_T(v)}{v}\,\mathrm{d}v < \infty. \tag{B17}$$

Then H_T is integrable, and the end-point asymptotics derived in Sec. VI B apply. In particular,

$$(\phi * H_T)(u) - H_T(u) \sim M_T(\infty)\phi(u), \quad u \to \infty. \tag{B18}$$

Using the exact identity

$$V(R_T(u)) = a_T(u) + (\phi * H_T)(u) - H_T(u), \tag{B19}$$

together with

$$a_T(u) \sim ATu^{-1-\alpha}, \quad \phi(u) \sim Au^{-1-\alpha},$$

gives

$$V(R_T(u)) \sim A[T + M_T(\infty)]u^{-1-\alpha}. \tag{B20}$$

Since $\gamma = 1 + \alpha$ and $V(z) \sim B_\gamma z^\gamma$, it follows that

$$R_T(u) \sim C_T u^{-1}, \quad C_T = \left[\frac{A\big(T + M_T(\infty)\big)}{B_\gamma}\right]^{1/\gamma}. \tag{B21}$$

Because $V(R_T) = o(R_T)$ as $u \to \infty$,

$$H_T(u) \sim R_T(u) \sim \frac{C_T}{u}.$$

But this implies

$$\int^{\infty} H_T(u)\,\mathrm{d}u = \infty,$$

contradicting the assumption $M_T(\infty) < \infty$ in Eq. (B17). Therefore

$$M_T(x) \longrightarrow \infty, \quad x \to \infty. \tag{B22}$$

Returning to Eq. (B16), the divergent left-hand side then forces $L_T(x) \to \infty$, so that the finite forcing term AT is asymptotically negligible. Hence

$$AM_T(x) \sim B_\gamma L_T(x)^\gamma, \quad x \to \infty. \tag{B23}$$

3. Solution of the slowly varying relation

Equation (B23) determines L_T explicitly. Introduce logarithmic time $y = \ln(x/u_0)$ and

$$m_T(y) := M_T(u_0 e^y). \tag{B24}$$

By definition, $m_T'(y) = L_T(u_0 e^y)$. Hence Eq. (B23) becomes

$$Am_T(y) \sim B_\gamma [m_T'(y)]^\gamma. \tag{B25}$$

Since all quantities are positive,

$$m_T'(y) \sim \left(\frac{A}{B_\gamma}\right)^{1/\gamma} m_T(y)^{1/\gamma}. \tag{B26}$$

Let

$$q := \frac{\gamma - 1}{\gamma} = \frac{\alpha}{1 + \alpha}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}y} m_T(y)^q = q m_T(y)^{-1/\gamma} m_T'(y) \tag{B27}$$

$$\longrightarrow q\left(\frac{A}{B_\gamma}\right)^{1/\gamma}. \tag{B28}$$

Integration gives

$$m_T(y) \sim \left[\frac{\gamma - 1}{\gamma}\left(\frac{A}{B_\gamma}\right)^{1/\gamma} y\right]^{\gamma/(\gamma-1)}. \tag{B29}$$

Using Eq. (B26), or differentiating the asymptotic solution consistently through Eq. (B25), gives

$$L_T(u) \sim \left[\frac{A(\gamma - 1)}{\gamma B_\gamma}\right]^{1/(\gamma-1)} \left[\ln\left(\frac{u}{u_0}\right)\right]^{1/(\gamma-1)}. \tag{B30}$$

Combining Eqs. (B10) and (B30), we obtain

$$H_T(u) \sim C_* \frac{[\ln(u/u_0)]^{1/(\gamma-1)}}{u}, \quad C_* := \left[\frac{A(\gamma - 1)}{\gamma B_\gamma}\right]^{1/(\gamma-1)}. \tag{B31}$$

The leading amplitude is independent of the fixed window length T; the T dependence enters only through subleading terms, including the finite forcing term in Eq. (B16). This independence is a consequence of the fact that, exactly at marginality, the divergent renewal contribution dominates the finite direct-window forcing.

The accumulated fixed-window hitting mass therefore grows as

$$\int_0^U H_T(u)\,\mathrm{d}u \sim C_* \frac{\gamma - 1}{\gamma} [\ln(U/u_0)]^{\gamma/(\gamma-1)}, \quad U \to \infty. \tag{B32}$$

It diverges and indeed more strongly than the logarithmic divergence that would follow from a pure u^{-1} tail.

4. Consequence for stationarity at criticality

Recall that the exact whole-line local-finiteness criterion is

$$\int_0^{\infty} H_T(u)\,\mathrm{d}u < \infty \quad \text{for every finite } T. \tag{B33}$$

Equation (B32) shows that this criterion fails at

$$\theta = \gamma - 1, \quad 1 < \gamma < 2. \tag{B34}$$

Thus the marginal point belongs to the nonstationary side of the transition: For $\theta = \gamma - 1$, no locally finite stationary whole-line critical Hawkes process exists for the positive-immigration infinite-past cluster construction considered in the paper. Consequently, within the marginal regularly varying class analyzed here, the stationarity boundary is strictly $\theta > \gamma - 1$, which is required for integrability of the fixed-window hitting probability.

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