

Quantum diffusion of a relativistic particle in a time-dependent random potential

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We present a rigorous study of quantum diffusion of a relativistic particle subjected to a time-dependent random potential with δ correlation in time. We find that in the asymptotic time limit the particle wave packet spreads ballistically in contrast with the nonrelativistic case, which in the same situation exhibits superballistic diffusion. The relativistic suppression of wave packet diffusion is discussed in connection with statistical conservation laws that follow from relativistic dynamics.

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I. INTRODUCTION

Transport in disordered quantum systems exhibits a variety of different behaviors ranging from ballistic motion to diffusion to Anderson localization. In between also lies the more exotic type known as anomalous diffusion where the mean squared displacement increases slower than a quadratic function of time for ballistic motion, but not linearly with time as in normal diffusion. The other extreme case occurs when the disorder evolves in time. Temporal fluctuations of random potential destroy the localization and the transport rate increases even beyond the ballistic transport. This effect, which is known as stochastic acceleration in classical physics, was also predicted in an exactly solvable case of a continuum Schrödinger equation where it gives rise to cubic growth of mean squared displacement with time [1]. Recently this hypertransport has been directly demonstrated in a paraxial optical experiment [2]. The cubic behavior can also be derived from classical Langevin dynamics without the dissipative term [1] which indicates its classical nature. This is expected since the fluctuations of the potential tend to destroy the quantum coherence. In fact, the motion of classical particles in a dynamic random potential exhibits this cubic behavior only if the potential is δ correlated in time. In a potential with finite correlation time the mean squared displacement grows with different power $\frac{12}{5} = 2.4$ at intermediate time (in one dimension) [3,4], and it was predicted in Ref. [4] that the true long time asymptotic behavior is normal diffusion.

Unlike the classical problem spatial correlation of the potential also has a considerable effect in quantum dynamics. Even in a rapidly fluctuating potential (δ correlated in time) in some cases quantum effects persist for long time [5]. This persistence of quantum effects stems from statistical conservation laws that govern the quantum dynamics at all times. These conservation laws are linked to the statistical properties of the potential. For example, the average kinetic energy of a quantum particle is conserved for certain types of spatial correlations of the potential [5], while for a classical particle it increases linearly with time. As a result of this conservation, the diffusion of a quantum particle is suppressed and, instead of the cubic growth, ballistic behavior is observed.

There also exists an earlier analytical study [6,7] of a discrete model, described with a tight binding Hamiltonian on

a lattice, which leads to normal diffusion. This result is due to the microscopic length scale of the lattice which induces a large momentum cutoff. There is no such cutoff in the continuum model; therefore the particle velocity can increase indefinitely. On the other hand, indefinite acceleration of the particle in the continuum model is the result of nonrelativistic treatment of its motion. So the other feature that might affect the asymptotic behavior is the relativistic limit on the velocity. We try to answer the question of how the diffusion of the wave packet would be influenced by taking the relativistic effects into account.

Here we adopt the Dirac relativistic wave equation. Besides the fundamental interest, our motivation has also been the relevance to various electronic systems which exhibit properties that can be well described by the Dirac equation. The main example is graphene, in which electron transport is essentially governed by the Dirac equation [8]. Many years after its derivation, the Dirac equation has attracted enormous attention since the discovery of graphene in the last 10 years. Two nonintuitive predictions of the Dirac equation, Klein's paradox and Zitterbewegung, which have been inaccessible experimentally can now be tested in a condensed matter experimental setup [9,10]. Even the recent progress in trapped-ion experiments has made it possible to simulate such relativistic quantum effects [11]. Narrow-gap semiconductors [12], topological insulators, and superconductivity [13] are other examples in which the Dirac equation emerges in effective description of quantum dynamics.

II. DIRAC EQUATION AND DENSITY MATRIX FORMALISM

In $1 + 1$ dimensions in which we have 1 motional degree of freedom the Dirac equation can be expressed using two Pauli matrices, σ_x and σ_z :

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c \sigma_x \frac{\partial \psi}{\partial x} + mc^2 \sigma_z \psi + V(x, t) \psi, \quad (1)$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, c is the speed of light, and $V(x, t)$ is assumed to have a zero average, $\langle V(x, t) \rangle = 0$, a translationally invariant correlation function,

$$\langle V(x, t) V(x', t') \rangle = v_0^2 \delta(t - t') g(x - x'), \quad (2)$$

δ correlated in time, and an arbitrary correlation in space for which we will later use specific forms for $g(x)$ for explicit

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calculations. Here and below, $\langle \dots \rangle$ denotes an ensemble average and $\langle \psi | \dots | \psi \rangle$ the quantum mechanical expectation value.

We introduce the density matrix $\rho(x', x, t) = \psi^\dagger(x', t)\psi(x, t)$, which is itself a 2×2 matrix with elements $\rho_{ij}(x', x, t) = \psi_i^*(x')\psi_j(x)$, with $i, j = 1, 2$. The time evolution $\dot{\rho}(x', x, t) = \dot{\psi}^\dagger(x', t)\psi(x, t) + \psi^\dagger(x', t)\dot{\psi}(x, t)$ is obtained using Eq. (1) as follows:

$$\dot{\rho}_{11}(x', x, t) = -c \left(\frac{\partial \rho_{21}}{\partial x'} + \frac{\partial \rho_{12}}{\partial x} \right) + \frac{i}{\hbar} [V(x', t) - V(x, t)] \rho_{11}, \quad (3)$$

$$\dot{\rho}_{22}(x', x, t) = -c \left(\frac{\partial \rho_{12}}{\partial x'} + \frac{\partial \rho_{21}}{\partial x} \right) + \frac{i}{\hbar} [V(x', t) - V(x, t)] \rho_{22}, \quad (4)$$

$$\dot{\rho}_{12}(x', x, t) = -c \left(\frac{\partial \rho_{11}}{\partial x'} + \frac{\partial \rho_{22}}{\partial x} \right) + \frac{2imc^2}{\hbar} \rho_{12} + \frac{i}{\hbar} [V(x', t) - V(x, t)] \rho_{12}, \quad (5)$$

$$\dot{\rho}_{21}(x', x, t) = -c \left(\frac{\partial \rho_{11}}{\partial x'} + \frac{\partial \rho_{22}}{\partial x} \right) - \frac{2imc^2}{\hbar} \rho_{21} + \frac{i}{\hbar} [V(x', t) - V(x, t)] \rho_{21}. \quad (6)$$

In terms of new variables $\rho = \rho_{11} + \rho_{22}$, $\sigma = \rho_{11} - \rho_{22}$, $\tau = \rho_{12} + \rho_{21}$, and $\gamma = \rho_{12} - \rho_{21}$, these equations look simpler; also the quantity ρ is the probability density which is what we need for calculation of moments of the position operator:

$$\dot{\rho}(x', x, t) = -c \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \tau + \frac{i}{\hbar} [V(x', t) - V(x, t)] \rho, \quad (7)$$

$$\dot{\sigma}(x', x, t) = -c \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \gamma + \frac{i}{\hbar} [V(x', t) - V(x, t)] \sigma, \quad (8)$$

$$\dot{\tau}(x', x, t) = -c \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \rho + \frac{2imc^2}{\hbar} \gamma + \frac{i}{\hbar} [V(x', t) - V(x, t)] \tau, \quad (9)$$

$$\dot{\gamma}(x', x, t) = -c \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \sigma + \frac{2imc^2}{\hbar} \tau + \frac{i}{\hbar} [V(x', t) - V(x, t)] \sigma. \quad (10)$$

Because the quantities of interest to us are linear in density matrix, we only need the average of the density matrix over random configurations to calculate the average quantities. To obtain the equations of motion of the averaged density matrix we need to know averages like $\langle V(y, t) \rho(x', x, t) \rangle$. For a Gaussian random variable $V(x, t)$ we can use the Novikov's identity [14]:

$$\langle V(y, t) \rho(x', x, t) \rangle = \int dt'' \int dx'' \langle V(y, t) V(x'', t'') \rangle \left\langle \frac{\delta \rho(x', x, t)}{\delta V(x'', t'')} \right\rangle. \quad (11)$$

The functional derivative in the integrand can be calculated first by integrating Eq. (7) with respect to time,

$$\rho(x', x, t) - \rho(x', x, 0) = - \int_0^t c \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \tau(x', x, t') dt' + \frac{i}{\hbar} \int_0^t [V(x', t') - V(x, t')] \rho(x', x, t') dt', \quad (12)$$

and then

$$\begin{aligned} \frac{\delta \rho(x', x, t)}{\delta V(x'', t'')} &= - \int_{t''}^t c \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \frac{\delta \tau(x', x, t')}{\delta V(x'', t'')} dt' + \frac{i}{\hbar} \int_{t''}^t [V(x', t') - V(x, t')] \frac{\delta \rho(x', x, t')}{\delta V(x'', t'')} dt' \\ &\quad + \frac{i}{\hbar} \int_0^{t''} [\delta(x' - x'') - \delta(x - x'')] \delta(t' - t'') \rho(x', x, t') dt'. \end{aligned} \quad (13)$$

In the first two terms, the lower limit of integrals is changed to t'' because the density matrix depends only on potential at earlier times so the functional derivatives in the integrands vanish for $t' < t''$. The third term is equal to $\theta(t - t'') [\delta(x' - x'') - \delta(x - x'')] \rho(x', x, t)$, where $\theta(t) = 1$ for $t > 0$, $\frac{1}{2}$ for $t = 0$, and 0 for $t < 0$. Taking the limit $t'' \rightarrow t$, which we need for Eq. (11), and using Eq. (2), we have

$$\langle V(x, t) \rho(x', x, t) \rangle = \frac{i}{2\hbar} v_0^2 [g(x - x') - g(0)] \langle \rho(x', x, t) \rangle, \quad (14)$$

$$\langle V(x', t) \rho(x', x, t) \rangle = \frac{i}{2\hbar} v_0^2 [g(0) - g(x - x')] \langle \rho(x', x, t) \rangle. \quad (15)$$

By using these quantities we get the following averaged density matrix equations:

$$\frac{\partial}{\partial t} \langle \rho(x', x, t) \rangle = -c \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \langle \tau \rangle - \frac{v_0^2}{\hbar^2} [g(0) - g(x - x')] \langle \rho(x', x, t) \rangle, \quad (16)$$

$$\frac{\partial}{\partial t} \langle \sigma(x', x, t) \rangle = -c \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \langle \gamma \rangle - \frac{v_0^2}{\hbar^2} [g(0) - g(x - x')] \langle \sigma(x', x, t) \rangle, \quad (17)$$

$$\frac{\partial}{\partial t} \langle \tau(x', x, t) \rangle = -c \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \langle \rho \rangle + \frac{2imc^2}{\hbar} \langle \gamma \rangle - \frac{v_0^2}{\hbar^2} [g(0) - g(x - x')] \langle \tau(x', x, t) \rangle, \quad (18)$$

$$\frac{\partial}{\partial t} \langle \gamma(x', x, t) \rangle = -c \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \langle \sigma \rangle + \frac{2imc^2}{\hbar} \langle \tau \rangle - \frac{v_0^2}{\hbar^2} [g(0) - g(x - x')] \langle \gamma(x', x, t) \rangle. \quad (19)$$

With the change of variables $X = \frac{1}{2}(x + x')$, $Y = \frac{1}{2}(x - x')$, and $\frac{\partial}{\partial X} = \frac{\partial}{\partial x} + \frac{\partial}{\partial x'}$, $\frac{\partial}{\partial Y} = \frac{\partial}{\partial x} - \frac{\partial}{\partial x'}$, the equations are further simplified:

$$\dot{R}(X, Y, t) = -c \frac{\partial}{\partial X} T(X, Y, t) - h(Y) R(X, Y, t), \quad (20)$$

$$\dot{S}(X, Y, t) = -c \frac{\partial}{\partial Y} G(X, Y, t) - h(Y) S(X, Y, t), \quad (21)$$

$$\dot{T}(X, Y, t) = -c \frac{\partial}{\partial X} R(X, Y, t) + \frac{2imc^2}{\hbar} G(X, Y, t) - h(Y) T(X, Y, t), \quad (22)$$

$$\dot{G}(X, Y, t) = -c \frac{\partial}{\partial Y} S(X, Y, t) + \frac{2imc^2}{\hbar} T(X, Y, t) - h(Y) G(X, Y, t), \quad (23)$$

where $h(Y) = \frac{v_0^2}{\hbar^2} [g(0) - g(2Y)]$ and capital Latin letters represent corresponding Greek letters, e.g., $R(X, Y, t) = \langle \rho[x'(X, Y), x(X, Y), t] \rangle$. By applying the Laplace transform on t and the Fourier transform on X we obtain

$$[s + h(Y)] \bar{\bar{R}}(K, Y, s) = icK \bar{\bar{T}}(K, Y, s) + \bar{R}_0, \quad (24)$$

$$[s + h(Y)] \bar{\bar{T}}(K, Y, s) = icK \bar{\bar{R}}(K, Y, s) + \frac{2imc^2}{\hbar} \bar{\bar{G}}(K, Y, s) + \bar{T}_0, \quad (25)$$

$$[s + h(Y)] \bar{\bar{G}}(K, Y, s) = -c \frac{\partial}{\partial Y} \bar{\bar{S}}(K, Y, s) + \frac{2imc^2}{\hbar} \bar{\bar{T}}(K, Y, s) + \bar{G}_0, \quad (26)$$

$$[s + h(Y)] \bar{\bar{S}}(K, Y, s) = -c \frac{\partial}{\partial Y} \bar{\bar{G}}(K, Y, s) + \bar{S}_0. \quad (27)$$

Here R_0 , T_0 , G_0 , and S_0 are the initial values of the variables (at $t = 0$) which are known if the initial state of the particle is determined. For every variable we use $\bar{\bar{R}}(X, Y, s) = \int_0^\infty e^{-ts} R(X, Y, t) dt$ and $\bar{\bar{R}}(K, Y, s) = \int_{-\infty}^\infty e^{iKX} R(X, Y, t) dX$. By eliminating variables in favor of $\bar{\bar{G}}$ we get the following second-order differential equation for $\bar{\bar{G}}$:

$$\frac{\partial^2 \bar{\bar{G}}}{\partial Y^2} - \frac{h'(Y)}{s + h(Y)} \frac{\partial \bar{\bar{G}}}{\partial Y} - \frac{1}{c^2} (s + h(Y))^2 \left(1 + \frac{4m^2 c^4 / \hbar^2}{[s + h(Y)]^2 + c^2 K^2} \right) \bar{\bar{G}}(K, Y, s) = f(K, Y, s), \quad (28)$$

where

$$f(K, Y, s) = \frac{1}{c} \frac{\partial \bar{S}_0}{\partial Y} - \frac{1}{c} \frac{h'(Y)}{s + h(Y)} \bar{S}_0 + \frac{2mc}{\hbar} \frac{K[s + h(Y)]}{[s + h(Y)]^2 + c^2 K^2} \bar{R}_0 - \frac{2im}{\hbar} \frac{[s + h(Y)]^2}{[s + h(Y)]^2 + c^2 K^2} \bar{T}_0 - \frac{1}{c^2} [s + h(Y)] \bar{G}_0 \quad (29)$$

and $h'(Y) = dh(Y)/dY$. There is one requirement in order to fix the solution of this equation and that comes from square integrability of the wave function. Therefore we seek the solutions that at least vanish at infinity.

Kinematics of wave packet

So far the problem is reduced to solving the differential equation, Eq. (28). The time evolution of the wave packet can be studied through the moments of probability distribution of the position of the particle. Here we restrict ourselves to first and

second moments (multifractal behavior is also expected in the higher moments [4,5,15]). The moments of probability distribution ρ can be expressed conveniently as the derivatives of its Fourier transform in the following way:

$$\mu_n = \langle \langle \psi | \hat{x}^n | \psi \rangle \rangle = \int x^n \langle \rho(x, x, t) \rangle dx = \frac{1}{i^n} \frac{\partial^n}{\partial K^n} \bar{R}(K, Y=0, t) \Big|_{K=0}. \quad (30)$$

For $n = 0$ we simply get the normalization of the wave function at any time $\langle \langle \psi | \psi \rangle \rangle = \bar{R}(K=0, Y=0, t)$. Using Eq. (24) we have $\bar{R}(K=0, Y=0, s) = \frac{\bar{R}_0(K=0, Y=0)}{s}$, and by the inverse Laplace transform $\langle \langle \psi | \psi \rangle \rangle = \bar{R}_0(K=0, Y=0) = \langle \psi_0 | \psi_0 \rangle$, where ψ_0 is the initial state. Using the formal expression of $\bar{R}$ in terms of $\bar{G}$ and initial values, first and second moments are obtained as follows:

$$\tilde{\mu}_1 = \frac{2imc^3}{\hbar s^2} \bar{G}(K=0, Y=0, s) + \frac{c}{s^2} \bar{T}_0(K=0, Y=0) + \frac{1}{is} \frac{\partial}{\partial K} \bar{R}_0(K, Y=0) \Big|_{K=0}, \quad (31)$$

$$\tilde{\mu}_2 = \frac{4mc^3}{\hbar s^2} \frac{\partial}{\partial K} \bar{G}(K, Y=0, s) \Big|_{K=0} + \frac{2c^2}{s^3} \bar{R}_0(K=0, Y=0) - \frac{1}{s} \frac{\partial^2}{\partial K^2} \bar{R}_0(K, Y=0) \Big|_{K=0}. \quad (32)$$

The first term of each of the above equations is unknown and will be determined by solving Eq. (28). We will also be using the properties of the Laplace transform to determine the asymptotic behavior of the moments, namely, small- s behavior which corresponds to the long time behavior.

III. EXACT SOLUTION FOR FREE PARTICLE WITH INITIAL GAUSSIAN WAVE PACKET

Evolution of a Gaussian wave packet in the free Schrödinger equation is a textbook example. An initial Gaussian wave packet, $(\frac{2}{\pi\lambda^2})^{\frac{1}{4}} e^{-\frac{x^2}{\lambda^2}}$ (with zero average momentum), spreads during the time evolution but always remains Gaussian. Calculation of the moments of probability density function is therefore simple. For instance the second moment is given by $\mu_2 = \frac{\lambda^2}{4} + \frac{\hbar^2}{m^2\lambda^2} t^2$, which exhibits ballistic spreading of the wave packet. Whereas in general the motion of the wave packet in the Dirac equation is more complicated. This is due to the existence of two branches of the energy-momentum relation in the Dirac equation corresponding to positive- and negative-energy plane wave solutions. Even an initial Gaussian wave packet, which in general is a superposition of both kinds of plane waves, becomes non-Gaussian at later times, splits into two parts since positive and negative components move in opposite directions, and wiggles back and forth because of the interference of these two parts. These features have already been discussed using the numerical solution of the Dirac equation [16] and time evolution of the position operator in the Heisenberg picture [17]. However the time dependence of wave packets moving according to the Dirac equation usually cannot be determined explicitly. Here we give the closed expression of the Laplace transform of the density matrix of a free Dirac particle with an initial Gaussian wave packet.

For a free particle ($v_0 \rightarrow 0$) with a normalized initial Gaussian wave packet,

$$\psi_0 = \psi(x, t=0) = \left(\frac{2}{\pi\lambda^2}\right)^{\frac{1}{4}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{-\frac{x^2}{\lambda^2}}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad (33)$$

by inserting the corresponding initial density matrix components,

$$\bar{R}_0(K, Y) = e^{-\frac{1}{8}\lambda^2 K^2} e^{-2\frac{Y^2}{\lambda^2}}, \quad (34)$$

$$\bar{S}_0(K, Y) = (|\alpha|^2 - |\beta|^2) e^{-\frac{1}{8}\lambda^2 K^2} e^{-2\frac{Y^2}{\lambda^2}}, \quad (35)$$

$$\bar{T}_0(K, Y) = (\alpha^* \beta + \alpha \beta^*) e^{-\frac{1}{8}\lambda^2 K^2} e^{-2\frac{Y^2}{\lambda^2}}, \quad (36)$$

$$\bar{G}_0(K, Y) = (\alpha^* \beta - \alpha \beta^*) e^{-\frac{1}{8}\lambda^2 K^2} e^{-2\frac{Y^2}{\lambda^2}}, \quad (37)$$

in Eq. (28) and with some algebra, we get

$$\frac{\partial^2 \bar{G}}{\partial Y^2} - a^2 \bar{G} = (bY + d) e^{-2\frac{Y^2}{\lambda^2}}, \quad (38)$$

where $a^2 = \frac{1}{c^2} s^2 (1 + \frac{4m^2 c^4 / \hbar^2}{s^2 + c^2 K^2})$, $b = \frac{-4}{c\lambda^2} (|\alpha|^2 - |\beta|^2) e^{-\frac{1}{8}\lambda^2 K^2}$, and $d = \frac{2ms/\hbar}{s^2 + c^2 K^2} [cK + 2i\text{Re}(\alpha^* \beta)s] - 2i\text{Im}(\alpha^* \beta)s/c^2$. The general solution for the corresponding homogeneous equation of Eq. (38) is $C_1 e^{aY} + C_2 e^{-aY}$, which diverges at both $Y = \infty$ and $Y = -\infty$ (C_1 and C_2 are constants). Particular solution also has the same asymptotic behavior; therefore it is possible to determine C_1 and C_2 by requiring the complete solution to vanish at infinity. By doing so we obtain the following solution:

$$\bar{G} = \frac{\sqrt{\pi}\lambda}{16\sqrt{2}a} e^{\frac{1}{8}a^2\lambda^2} \left[(ab\lambda^2 - 4d) e^{aY} \text{erfc}\left(\frac{\lambda a}{2\sqrt{2}} + \frac{\sqrt{2}Y}{\lambda}\right) - (ab\lambda^2 + 4d) e^{-aY} \text{erfc}\left(\frac{\lambda a}{2\sqrt{2}} - \frac{\sqrt{2}Y}{\lambda}\right) \right]. \quad (39)$$

Now we are able to calculate the moments of the probability density of the position. Using Eqs. (31) and (32) we have

$$\begin{aligned} \tilde{\mu}_1 &= \frac{2c\text{Re}(\alpha^* \beta)}{s^2} - \frac{\sqrt{2\pi}\lambda mc^2 e^{\frac{\lambda^2 m^2 c^2}{2\hbar^2 s^2} + \frac{\lambda^2}{8c^2} s^2}}{s^2 \sqrt{\hbar^2 s^2 + 4m^2 c^4}} \\ &\times \left(\frac{2mc^2}{\hbar} \text{Re}(\alpha^* \beta) + \text{Im}(\alpha^* \beta)s \right) \\ &\times \text{erfc}\left(\frac{\lambda}{2\sqrt{2}c} \sqrt{s^2 + \frac{4m^2 c^4}{\hbar^2}}\right), \end{aligned} \quad (40)$$

$$\tilde{\mu}_2 = \frac{2c^2}{s^3} - \frac{2\sqrt{2\pi}\lambda m^2 c^5 e^{\frac{\lambda^2 m^2 c^2}{2\hbar^2}}}{s^3 \hbar \sqrt{\hbar^2 s^2 + 4m^2 c^4}} \times \operatorname{erfc}\left(\frac{\lambda}{2\sqrt{2}c} \sqrt{s^2 + \frac{4m^2 c^4}{\hbar^2}}\right) e^{\frac{\lambda^2}{8c^2} s^2} + \frac{\lambda^2}{4s}. \quad (41)$$

To see the long time asymptotic behavior of the moments we only need the behavior of the above expressions at small s . Keeping the most divergent terms at the limit $s \rightarrow 0$ and then by the inverse Laplace transform we obtain $\mu_1 \approx 2\operatorname{Re}(\alpha^* \beta)(1 - \sqrt{\pi} \eta \operatorname{erfc}(\eta) e^{\eta^2}) ct$ and $\mu_2 = (1 - \sqrt{\pi} \eta \operatorname{erfc}(\eta) e^{\eta^2}) c^2 t^2$, where $\eta = \frac{\lambda mc}{\sqrt{2\hbar}}$.

It is instructive to see how last two results can also be derived from representation of the position operator in the Heisenberg picture. Using the Heisenberg equation of motion one finds the velocity operator $\frac{d\hat{x}}{dt} = \frac{1}{i\hbar} [\hat{x}, H_0] = c\sigma_x$, where $H_0 = -i\hbar c \sigma_x \frac{\partial}{\partial x} + mc^2 \sigma_z$. For a massive particle the velocity operator does not commute with the Hamiltonian, so unlike the nonrelativistic case the velocity is not a constant of motion. The equation of motion for the velocity can be solved using some operator algebra and then by integrating the velocity one gets the following result for the position operator:

$$\hat{x}(t) = \hat{x}(0) + \hat{p} c^2 H_0^{-1} t + \frac{i}{2} \hbar c (\sigma_x - \hat{p} c H_0^{-1}) H_0^{-1} (e^{-2iH_0 t/\hbar} - 1), \quad (42)$$

where $\hat{p} = -i\hbar \partial/\partial x$. The third term is oscillatory in time and induces the so-called Zitterbewegung. The moments now can be obtained as the expectation values of different powers of Eq. (42) in the initial wave packet. Especially in the long time limit the first moment is dominated by the linear term in time and the second moment is dominated by the quadratic term as we obtained above. The prefactors that we obtained in μ_1 and μ_2 are actually the expectation values $\langle \psi_0 | \hat{p} c^2 H_0^{-1} | \psi_0 \rangle$ and $\langle \psi_0 | (\hat{p} c^2 H_0^{-1})^2 | \psi_0 \rangle$, respectively.

IV. PARTICLE IN A TIME-DEPENDENT RANDOM POTENTIAL

A. Massless particle

In the case of a massless particle the density matrix can be determined exactly too. Because even in the presence of a random potential, Eqs. (24) and (25) become decoupled from Eqs. (26) and (27) and we have $\tilde{R}(K, Y, s) = \frac{s+h(Y)}{[s+h(Y)]^2 + c^2 K^2} \tilde{R}_0 + \frac{icK}{[s+h(Y)]^2 + c^2 K^2} \tilde{T}_0$. The function $h(Y)$ vanishes both at $Y = 0$ and $v_0 = 0$; so in the calculation of moments of position for which we need $\tilde{R}$ at $Y = 0$, we get the massless free particle results as discussed earlier. However the effect of a random potential can be observed in correlations of the wave function at two different points ($Y \neq 0$). Further with the inverse Laplace and Fourier transforms we obtain

$$R(X, Y, t) = \frac{1}{\sqrt{2\pi}\lambda} e^{-h(Y)t} ([1 - 2\operatorname{Re}(\alpha^* \beta)] e^{-\frac{\lambda^2}{2}(X+ct)^2} + [1 + 2\operatorname{Re}(\alpha^* \beta)] e^{-\frac{\lambda^2}{2}(X-ct)^2}). \quad (43)$$

We can see that at later times the wave packet decomposes into two smaller wave packets moving in opposite directions

at the speed of light. This is actually what happens in the free Dirac equation. Here in addition we have an exponential factor, due to the random potential, representing the decay of correlations of the wave function at two points at the distance Y with the time scale $\tau_0 = \frac{1}{h(Y)}$, which depends on the spatial correlation function. We may also note that this characteristic time diverges both at $Y = 0$ and at the zero disorder limit $v_0 = 0$.

B. Massive particle

For this general case the differential equation, Eq. (28), is rather complicated to be solved analytically but the numerical solution is possible, although imposing the boundary conditions is not straight forward numerically. As we explained above the solution must vanish at infinity but numerically we can only determine the solution at a finite interval. At the boundaries we impose (we use MATHEMATICA for the numerical solution of the differential equation) a very small value for $\tilde{G}$ but we need to make sure that the interval is large enough so that the result does not depend on the boundary values. First we note that in order to calculate the moments (31) and (32) we only need $\tilde{G}$ at $K = 0$ and $\frac{\partial \tilde{G}}{\partial K}$ at $K = 0$, respectively; so first we obtain the corresponding equations of these quantities from Eq. (28), which will be relatively simpler. Moreover we are interested in the long time behavior of moments; therefore we need to know the s dependence of the Laplace transforms at $s \rightarrow 0$. We also take the short range spatial correlation $g(x) = e^{-(x/\xi)^{2r}}$, where ξ is the correlation length and r is a positive integer.

Figure 1 shows the real and imaginary parts of $\tilde{G}(K = 0, Y, s)$ for $r = 1$, $s = 0.001$, $\alpha = \sqrt{\frac{4}{5}}$, $\beta = \sqrt{\frac{1}{5}}$, and other parameters equal to 1. The real part vanishes at $Y = 0$, which is necessary because otherwise we obtain a complex value for μ_1 [see Eq. (31)]. Moreover the solution converges as s goes to 0; so from Eq. (31) and the inverse Laplace transform we have $\mu_1 = c(1 - \frac{2mc^2}{\hbar} \operatorname{Im} \tilde{G}(K = 0, Y = 0))t$.

Figure 2 shows $\tilde{G}'(Y, s) = \frac{\partial}{\partial K} \tilde{G}(K, Y, s)|_{K=0}$ and the inset is a log-log plot of $-\tilde{G}'(Y = 0, s)$ which indicates power law

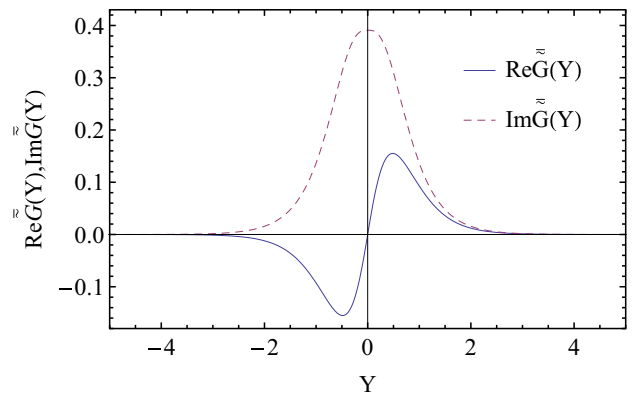


FIG. 1. (Color online) Numerical solution of Eq. (28) for $r = 1$, $K = 0$, $s = 0.001$, $\alpha = \sqrt{\frac{4}{5}}$, $\beta = \sqrt{\frac{1}{5}}$, and other parameters equal to 1. The real part is odd and therefore vanishes at $Y = 0$, and the solution is independent of s at small s .

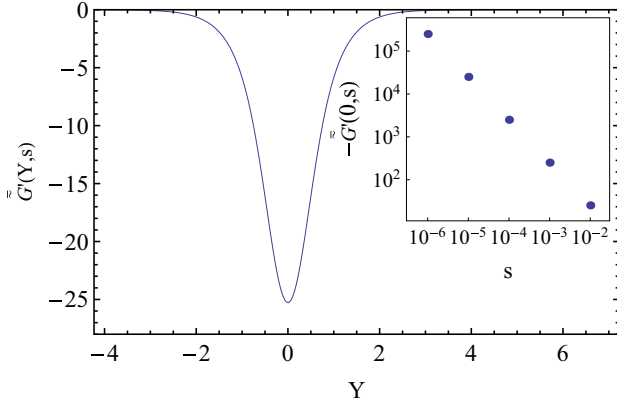


FIG. 2. (Color online) $\bar{G}'(Y, s)$ for $r = 1$, $s = 0.01$, and other parameters equal to 1. The inset is the plot of $-\bar{G}'(0, s)$ in log-log scale which shows the $\frac{1}{s}$ divergence in $s \rightarrow 0$.

singularity at $s = 0$. By fitting the inset to $s^{-\delta}$ we obtain $\delta = 1$, i.e., $\bar{G}'(Y = 0, s \rightarrow 0) = -\frac{\text{const}}{s}$; therefore the first term in Eq. (32) has also s^{-3} singularity at $s = 0$ and therefore at $t \rightarrow \infty$

$$\mu_2 \approx c^2 \left(1 + \frac{2mc}{\hbar} \lim_{s \rightarrow 0} s \bar{G}'(Y = 0, s) \right) t^2. \quad (44)$$

For comparison the corresponding nonrelativistic result (Eq. (17) of Ref. [1]), with parameters that were used here for $r = 1$, is $\mu_2 = \frac{\lambda^2}{4} + \frac{\hbar^2}{m^2 \lambda^2} t^2 + \frac{2v_0^2}{3m^2 \xi^2} t^3$. We repeated the calculations for other values of $r > 1$ and obtained the same exponent $\delta = 1$ and the same asymptotic behavior of moments.

V. SUMMARY AND DISCUSSION

To summarize, we have studied the evolution of a wave packet in the Dirac equation using the density matrix formalism. We derived exactly the density matrix for a massless particle in a time-dependent random potential as well as for a massive particle in a random time-dependent potential we obtained the asymptotic behavior of average position and mean squared displacement. In contrast to superballistic diffusion of a Schrödinger particle for the correlation function $g(x) = e^{-(x/\xi)^{2r}}$ with $r = 1$ [see Eq. (2)], our results exhibit ballistic motion in this case [Eq. (44)]. We find ballistic behavior in the long time limit for $r > 1$ as well, which coincides with nonrelativistic results.

The difference between two cases suggests the existence of another conserved quantity in the relativistic case. A non-relativistic quantum particle gains energy from the Gaussian correlated potential ($r = 1$) as its average kinetic energy is not a constant of motion. For a relativistic particle we have

$$\langle E_K \rangle = \langle \langle \psi | H_0 | \psi \rangle \rangle = \int dx \langle \psi^\dagger(x) H_0 \psi(x) \rangle, \quad (45)$$

$$= \int dx \sum_{i,j=1}^2 \langle \psi_i^*(x) (-i\hbar c \sigma_x \partial_x + mc^2 \sigma_z)_{ij} \psi_j(x) \rangle, \quad (46)$$

$$= \int dx \left[-i\hbar c \frac{\partial}{\partial x} \langle \tau(x', x, t) \rangle + mc^2 \langle \sigma(x', x, t) \rangle \right]_{x'=x}, \quad (47)$$

$$= \int dX \left[-i\hbar c \frac{1}{2} \left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y} \right) T(X, Y, t) + mc^2 S(X, Y, t) \right]_{Y=0}, \quad (48)$$

$$= \int dX \left[-\frac{i\hbar c}{2} \frac{\partial}{\partial Y} T(X, Y, t) + mc^2 S(X, Y, t) \right]_{Y=0}. \quad (49)$$

In going from Eq. (47) to Eq. (48) we have performed the change of variables $X = \frac{1}{2}(x + x')$ and $Y = \frac{1}{2}(x - x')$. In the third line, the integral of the term $\frac{\partial}{\partial X} T(X, Y, t)$ vanishes because $T(X = \pm\infty, Y, t) = 0$. Also note that in the integrands the differentiations are done first and then the values $x' = x$ ($Y = 0$) are set. Now by taking the time derivative and substitution from Eqs. (22) and (23) and using $h(0) = h'(0) = 0$, we obtain

$$\frac{d}{dt} \langle E_K \rangle = \int dX \left[-\frac{i\hbar c}{2} \frac{\partial}{\partial Y} \dot{T}(X, Y, t) + mc^2 \dot{S}(X, Y, t) \right]_{Y=0}, \quad (50)$$

$$= \int dX \left[\frac{i\hbar c^2}{2} \frac{\partial^2 R}{\partial X \partial Y} + mc^3 \frac{\partial G}{\partial Y} + \frac{i\hbar c}{2} \frac{\partial}{\partial Y} (h(Y)T) - mc^3 \frac{\partial G}{\partial Y} - mc^2 h(Y)S \right]_{Y=0}, \quad (51)$$

$$= \frac{i\hbar c^2}{2} \frac{\partial R}{\partial Y} \Big|_{X=\infty, Y=0} - \frac{i\hbar c^2}{2} \frac{\partial R}{\partial Y} \Big|_{X=-\infty, Y=0}, \quad (52)$$

$$= 0, \quad (53)$$

which shows that the average kinetic energy is conserved even for the Gaussian correlation function. The explicit derivation of the density matrix that is obtained in the massless case (Eq. 43) reveals an interesting feature. The correlation function

$\langle \psi^\dagger(x', t) \psi(x, t) \rangle$, in addition to transient time dependence due to the motion of the wave front, exhibits an exponential decay with the time scale $\tau_0 = \frac{1}{h(\frac{x-x'}{2})}$ induced by spatial correlations of a time-dependent random potential.

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