Transient and stationary characteristics of the Malthus-Verhulst-Bernoulli model with non-Gaussian fluctuating parameters

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Two generalized Verhulst equations with non-Gaussian fluctuations of the reproduction rate and the volume of resources are under analytical investigation. For the first model, using the central limit theorem, we find the asymptotic behavior of the probability distribution of population density for an arbitrary non-Gaussian colored noise with nonzero power spectral density at zero frequency. Specifically, we confirm this result in the case of Markovian dichotomous noise and examine the evolution of mean population density. For fluctuating resources with one-sided stable distribution the transient dynamics of probability density function and statistical characteristics in the steady state are obtained. As shown, the scenario of the population's evolution depends on the parameter of nonlinearity in the original stochastic equation.

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I. INTRODUCTION

The behavior of some nonlinear dynamical systems in the presence of random perturbations can be described by the Bernoulli equation

$$\frac{dx}{dt} = r(t)\, x - \beta(t)\, x^{\mu+1}, \tag{1}$$

which is a generalization of the well-known Verhulst model [1] or a logistic equation ($\mu = 1$). When applied to the population dynamics, $r(t)$ is the growth rate of the isolated population, and the nonlinear term with positive parameters μ and $\beta(t)$ is responsible for the growth restriction associated with the finite amount of food resources, the presence of predators, etc.

The Malthus-Verhulst-Bernoulli (MVB) model (1) has found a wide application in various fields of science such as population dynamics of photons in a single-mode laser [2,3], autocatalytic chemical reactions [4,5], dynamics of biological populations [6–8], self-replication of macromolecules [9], the spread of viral epidemics [10], tumor cell growth [11,12], stochastic resonance in linear systems [13,14], steady-state entropy production [15], freezing of supercooled liquids [16], and social sciences [17–19].

In most papers devoted to a consideration of this model one of the parameters $r(t)$, $\beta(t)$ in Eq. (1) was assumed to be constant and the other to be fluctuating around some mean value. Some fundamental properties of the model (1) with randomly fluctuating $r(t) = r + \xi(t)$ and $\beta(t) = \beta$ have been intensively investigated for different noise sources $\xi(t)$: white Gaussian noise (see [20] and references therein), shot noise [21,22], and Markovian dichotomous noise (see the review in [23]).

The opposite case where $r(t) = r$ and $\beta(t) = \beta + \xi(t)$ has been less studied in the literature. In Refs. [24,25] the author obtained some exact results for kinetic and stationary characteristics of the MVB equation (1) with fluctuations of $\beta(t)$ in the form of Markovian dichotomous noise and periodic perturbation with random phase.

The MVB model driven by fully correlated noises [$r(t) \sim \beta(t)$] was recently analyzed in a series of works. The authors of Ref. [26] have considered the Ornstein-Uhlenbeck process as an external perturbation and obtained exact results for the probability distribution of population density. The exact relation for the probability density function (PDF) of the Verhulst model [the case with $\mu = 1$ in (1)], driven by Markovian dichotomous noise, was found in Ref. [27]. In Ref. [28] the MVB model excited by the symmetric non-Poisson dichotomous noise was examined, and the kinetics of the PDF was found. Also several cases of white non-Gaussian noise have been analyzed in detail in Ref. [29].

In the present paper we investigate transient and stationary statistical characteristics of the MVB model of population dynamics with fully correlated parameters (non-Gaussian colored noise) and separately with fluctuations of the saturation parameter in the form of white noise with a one-sided stable distribution. The paper is organized as follows. In Sec. II the Bernoulli model with an arbitrary multiplicative non-Gaussian colored noise with nonzero power spectral density at zero frequency is analyzed. In particular, we obtain some exact results for the excitation in the form of Markovian dichotomous noise. In Sec. III we study the MVB model for the evolution of a biological population with a fluctuating volume of resources in the form of white non-Gaussian noise source with a one-sided stable distribution and derive some general expressions for transient and stationary PDFs of the population density. To find the exact analytical results we address the case of noise with the Lévy-Smirnov stable distribution in Sec. IV. Section V is devoted to concluding remarks.

II. MVB MODEL WITH FULLY CORRELATED NON-GAUSSIAN COLORED NOISES

In this section we analyze the MVB model for the population density $x(t)$ when both parameters in Eq. (1) fluctuate identically, i.e.,

$$\frac{dx}{dt} = [r + \xi(t)]\, x\, (1 - x^{\mu}). \tag{2}$$

Here r is the rate of population growth and $\xi(t)$ is a non-Gaussian noise with zero mean $\langle \xi(t) \rangle = 0$ and nonzero

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correlation time τ_{cor}. The exact solution of the stochastic equation (2) has the form

$$x(t) = \left\{1 + \frac{1 - x_0^\mu}{x_0^\mu} \exp\left[-rt - \int_0^t \xi(\tau)d\tau\right]\right\}^{-1/\mu}, \quad (3)$$

where $x_0 = x(0)$ is the initial value of the population density. According to the central limit theorem (CLT), the integral of noise $\xi(t)$ in Eq. (3) becomes a quasi-Gaussian process for sufficiently long times $t \gg \tau_{\text{cor}}$. Based on the probability conversion formula of the probability theory, from Eq. (3) one can find the following approximate expression for the PDF of the population density:

$$P(x,t) = \frac{\mu}{\sqrt{2\pi S_\xi(0)t}\; x\,(1 - x^\mu)} \times \exp\left\{-\frac{\left[rt + \ln(x^{-\mu} - 1) - \ln\left(x_0^{-\mu} - 1\right)\right]^2}{2S_\xi(0)t}\right\}, \quad (4)$$

where $S_\xi(0)$ is the power spectral density of the random process $\xi(t)$ at zero frequency, which should be nonzero, and $0 < x < 1$. In accordance with Eq. (4) for the boundaries $x = 0$ and $x = 1$ of the interval we have

$$\lim_{x\to 0^+} P(x,t) = 0, \qquad \lim_{x\to 1^-} P(x,t) = 0.$$

As an example of a non-Gaussian colored process, we further consider Markovian dichotomous noise, switching between two values $\pm\Delta$ ($\Delta < r$) with the mean rate ν. This noise, which has an exponential correlation function $K[\tau] = \Delta^2 e^{-2\nu|\tau|}$ and $S(0) = \Delta^2/\nu$, gives a good understanding of the simplest and most common physical situation: the transitions between two configurations of a system. For example, it can describe the seasonal oscillations of fertility and mortality in biological systems.

In Refs. [30,31] for the stochastic Langevin equation

$$\dot{x} = f(x) + g(x)\,\xi(t), \quad (5)$$

where $\xi(t)$ is the Markovian dichotomous noise, the following closed equation for the PDF has been obtained:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} f(x)P(x,t) + \Delta^2 \frac{\partial}{\partial x}\, g(x) \times \int_0^t e^{-\left[2\nu + \frac{\partial}{\partial x} f(x)\right](t-t')} \frac{\partial}{\partial x}\, g(x)P(x,t')\,dt'. \quad (6)$$

For sufficiently long times $t \gg 1/(2\nu)$, by substituting $t - t' = \tau$ the integro-differential equation (6) transforms into the following differential equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} f(x)P + \frac{S_\xi(0)}{2}\frac{\partial}{\partial x}\, g(x)\frac{\partial}{\partial x}\, g(x)P. \quad (7)$$

According to Eq. (5), for our system (2) $f(x) = rx(1 - x^\mu)$ and $g(x) = x(1 - x^\mu)$, and the solution of Eq. (7) has the form (4). This confirms our result obtained for an arbitrary colored noise with nonzero power spectral density at zero frequency in the approximation of large observation times $t \gg \tau_{\text{cor}}$.

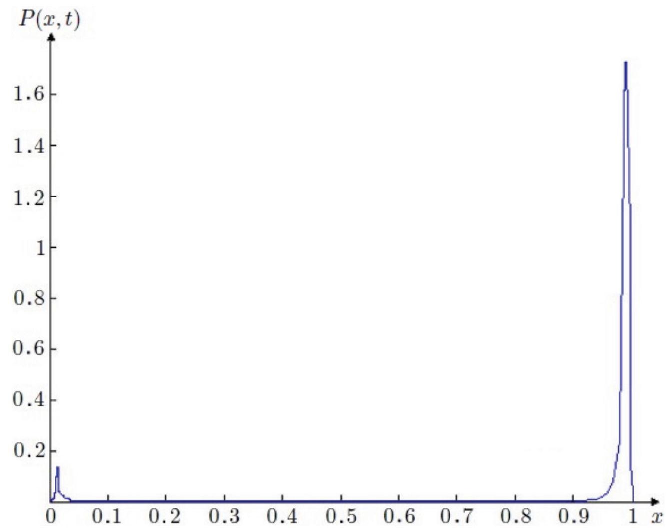

FIG. 1. (Color online) The PDF $P(x,t)$ of the population density in the case of Markovian dichotomous noise for $t = 100$ ($t \gg \tau_{\text{cor}}$). The parameters are $\tau_{\text{cor}} = 1$, $\mu = 3$, $r = 1$, $\Delta = 0.5$, $x_0 = 0.2$.

The probability distribution of the population density (4) in the case of Markovian dichotomous noise for the time $t \gg \tau_{\text{cor}} = 1/(2\nu)$ is depicted in Fig. 1.

As seen from Fig. 1, the maximum of the PDF located near the left boundary $x = 0$ is much less than the maximum located near the right boundary $x = 1$. This means that the biological population survives in the case of non-Gaussian excitation with a finite correlation time. Our conclusion is also supported by the results for the Ornstein-Uhlenbeck process $\xi(t)$ previously obtained in Ref. [26].

To find the evolution of the nth-order moment of the population density in the case of Markovian dichotomous noise we apply, as in Ref. [24], the direct averaging using the exact solution (3),

$$\langle x^n(t)\rangle = \left\langle\left(1 + \frac{1 - x_0^\mu}{x_0^\mu}\, e^{-rt - \int_0^t \xi(\tau)d\tau}\right)^{-n/\mu}\right\rangle. \quad (8)$$

In accordance with the Maclaurin expansion of an arbitrary function, we have

$$\left\langle f\left(e^{-rt - \int_0^t \xi(\tau)d\tau}\right)\right\rangle = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, e^{-krt}\left\langle e^{-k\int_0^t \xi(\tau)d\tau}\right\rangle. \quad (9)$$

In order to perform the averaging in Eq. (9) we need the exact expression for the characteristic functional (CF) of Markovian dichotomous noise $\xi(t)$

$$\Theta_t[u] = \left\langle \exp\left[i\int_0^t u(\tau)\,\xi(\tau)\,d\tau\right]\right\rangle, \quad (10)$$

where $u(t)$ is an arbitrary deterministic function. As shown in Ref. [30], this characteristic functional satisfies the following second-order differential equation:

$$\ddot{\Theta}_t[u] + \left(2\nu - \frac{\dot{u}}{u}\right)\dot{\Theta}_t[u] + \Delta^2 u^2 \Theta_t[u] = 0. \quad (11)$$

Putting, in accordance with Eqs. (9) and (10), $u(t) = ik$ in Eq. (11), solving it with the evident initial conditions

$\Theta_0[u] = 1$ and $\dot{\Theta}_0[u] = 0$, and substituting the result in Eq. (9), we arrive at

$$\left\langle f\left(e^{-rt-\int_0^t \xi(\tau)d\tau}\right)\right\rangle = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, e^{-(kr+\nu)t}\left(\cosh\sqrt{\nu^2+\Delta^2k^2}\,t + \frac{\nu}{\sqrt{\nu^2+\Delta^2k^2}}\sinh\sqrt{\nu^2+\Delta^2k^2}\,t\right). \tag{12}$$

Because

$$\lim_{t\to\infty} f\left(e^{-\int_0^t [r+\xi(\tau)]d\tau}\right) = f(0),$$

the stationary value of any order moment of the population density, in accordance with Eq. (8), equals to 1:

$$\langle x^n\rangle_{st} = \lim_{t\to\infty}\langle x^n(t)\rangle = 1, \tag{13}$$

as evidenced by Fig. 1.

As follows from Eqs. (8), (9), and (12), the evolution of nth-order moment can be written in the form of the series

$$\langle x^n(t)\rangle = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1-x_0^\mu}{x_0^\mu}\right)^k \frac{\Gamma(k+n/\mu)}{\Gamma(n/\mu)\,k!}\, e^{-(kr+\nu)t} \times\left(\cosh\sqrt{\nu^2+\Delta^2k^2}\,t + \frac{\nu}{\sqrt{\nu^2+\Delta^2k^2}}\sinh\sqrt{\nu^2+\Delta^2k^2}\,t\right). \tag{14}$$

The necessary condition of convergence for the series (14) reads $\Delta < r$, but this series converges in the absolute sense only for times

$$t > \frac{1}{r-\Delta}\ln\frac{1-x_0^\mu}{x_0^\mu}. \tag{15}$$

As can be seen from Eq. (15), only for the initial values x_0 of the population density in the range

$$\frac{1}{\sqrt[\mu]{2}} < x_0 < 1$$

do we get the evolution of nth-order moment for all times $t > 0$.

The relaxation of the mean population density ($n = 1$) for different initial values $x_0 = 0.52, 0.64, 0.8, 0.93$ and different values of the parameter of nonlinearity $\mu = 1, 1.5, 3, 9$ is shown in Fig. 2. As seen from Fig. 2, the relaxation time increases with increasing the exponent μ. All curves tend to the stationary value $\langle x\rangle_{st} = 1$ monotonically.

Using the CLT and the exact solution (3) for an arbitrary colored noise, we arrive at the result (13):

$$\langle x^n\rangle_{st} = \lim_{t\to\infty}\frac{1}{\sqrt{2\pi}\,\sigma(t)}\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-rt)^2}{2\sigma^2(t)}}\,dx}{\left(1+\frac{1-x_0^\mu}{x_0^\mu}\,e^{-x}\right)^{n/\mu}} = 1.$$

III. MVB MODEL WITH A FLUCTUATING VOLUME OF RESOURCES

Further we consider the MVB model (1) for the population density $x(t)$ with a fluctuating volume of resources (saturation

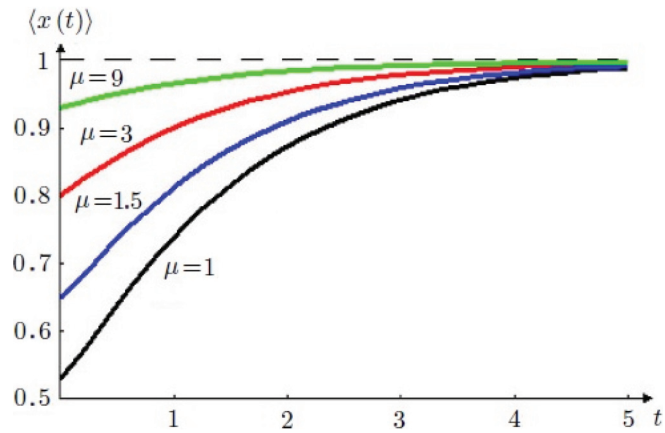

FIG. 2. (Color online) The monotonic relaxation of the mean population density to a stationary value for different initial conditions and the exponent of nonlinearity μ. The parameters are $r = 1$, $\nu = 1$, $\Delta = 0.5$.

parameter),

$$\frac{dx}{dt} = rx - \xi(t)\,x^{\mu+1}, \tag{16}$$

where $\xi(t)$ is white non-Gaussian noise with a one-sided stable probability distribution [$\xi(t) \geqslant 0$]. As is well known [32], the whole class of stable probabilistic laws $P_{\alpha,\beta}(x)$ can be found from the following characteristic function:

$$\theta_{\alpha,\beta}(k) = \exp\left\{-|k|^\alpha\, e^{\frac{i\pi\beta}{2}\operatorname{sgn}k}\right\}, \tag{17}$$

where α is the Lévy index ($0 < \alpha < 2$) and β is the asymmetry parameter ($|\beta| \leqslant 1 - |\alpha - 1|$). If $0 < \alpha < 1$ and $\beta = -\alpha$, the probability distribution is nonzero only for positive arguments. Thus, in accordance with Eq. (17), the characteristic function of noise $\xi(t)$ in Eq. (16) reads

$$\theta_\xi(k) = \theta_{\alpha,-\alpha}(\sigma k) = \exp\left\{-|\sigma k|^\alpha\, e^{-\frac{i\pi\alpha}{2}\operatorname{sgn}k}\right\}, \tag{18}$$

where σ is some scaling parameter.

Since the random process $x(t)$ is Markovian, one can obtain a closed complex integro-differential equation for its probability distribution $P(x,t)$ following Ref. [33]. However, we choose another way and will try to find the PDF of the population density directly from the stochastic equation (16).

Using the same method as in Ref. [34], we introduce the auxiliary random process

$$\eta(t) = \int_0^t e^{-\mu r(t-\tau)}\xi(\tau)\,d\tau. \tag{19}$$

Then, in this notation, the exact solution of the stochastic equation (16) takes the form

$$x(t) = \left[\mu\eta(t) + e^{-\mu rt}/x_0^\mu\right]^{-1/\mu}, \tag{20}$$

where $x_0 = x(0)$ is the initial value of population density. According to our main goal, first, we need to find the PDF of random process (19) and then apply the standard procedure for a conversion of probability distributions of random variables after nonlinear transformation (20).

Let us use the fact that any stable distribution belongs to the class of infinitely divisible distributions with the characteristic function expressed in Lévy-Khinchine form [35],

$$\Phi_{idd}(k) = \exp\left\{\int_{-\infty}^{\infty} \frac{e^{ikz}-1}{z^2}\,\rho(z)\,dz\right\}, \tag{21}$$

where $\rho(z)$ is some non-negative kernel function. Representing the logarithm of the characteristic function (18) in the integral form as in Eq. (21),

$$\ln\theta_\xi(k) = \frac{\alpha\sigma^\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{e^{ikz}-1}{z^{1+\alpha}}\,dz,$$

we get

$$\rho(z) = \frac{\alpha\sigma^\alpha}{\Gamma(1-\alpha)}\,z^{1-\alpha} \quad (z>0). \tag{22}$$

To calculate the characteristic function of the auxiliary random process (19),

$$\vartheta_\eta(k,t) = \langle e^{ik\eta(t)}\rangle = \left\langle \exp\left[i\int_0^t k e^{-\mu r(t-\tau)}\xi(\tau)\,d\tau\right]\right\rangle, \tag{23}$$

we use the general expression for the characteristic functional of white non-Gaussian noise $\xi(t)$ with nonzero mean and increments distributed by the infinitely divisible law (see Ref. [33])

$$\Theta_t[u] = \left\langle \exp\left[i\int_0^t u(\tau)\xi(\tau)\,d\tau\right]\right\rangle$$
$$= \exp\left[\int_0^t d\tau \int_{-\infty}^{\infty}\frac{e^{iu(\tau)z}-1}{z^2}\,\rho(z)\,dz\right]. \tag{24}$$

Substituting, in accordance with Eq. (23), $u(\tau) = k\,e^{-\mu r(t-\tau)}$ and the kernel function $\rho(z)$ from Eq. (22) in Eq. (24), we arrive at

$$\vartheta_\eta(k,t) = \exp\left[-\frac{(1-e^{-\mu r\alpha t})\,|k|^\alpha\,\sigma^\alpha}{\mu r\alpha}\,e^{-\frac{i\pi\alpha}{2}\,\mathrm{sgn}\,k}\right]. \tag{25}$$

By comparing Eqs. (18) and (25), we conclude that

$$\vartheta_\eta(k,t) = \theta_{\alpha,-\alpha}(k\,\sigma(t)) \tag{26}$$

and

$$\sigma(t) = \sigma\left(\frac{1-e^{-\mu r\alpha t}}{\mu r\alpha}\right)^{1/\alpha}. \tag{27}$$

The stable probability distribution of the random process $\eta(t)$, defined by Eq. (19), is derived by the reverse Fourier transform of the characteristic function (26),

$$P_\eta(y,t) = \frac{1}{\sigma(t)}\,P_{\alpha,-\alpha}\left(\frac{y}{\sigma(t)}\right), \tag{28}$$

where $P_{\alpha,-\alpha}(x)$ is the stable PDF corresponding to the characteristic function $\theta_{\alpha,-\alpha}(k)$.

Using the well-known relation of the probability theory regarding a nonlinear transformation of random variables, we derive from Eqs. (20) and (28) the final exact expression for the probability distribution of population density:

$$P(x,t) = \frac{1}{x^{\mu+1}\sigma(t)}\,P_{\alpha,-\alpha}\left[\frac{1}{\mu\sigma(t)}\left(\frac{1}{x^\mu}-\frac{e^{-\mu r t}}{x_0^\mu}\right)\right]. \tag{29}$$

Equation (29) determines the transient dynamics of the PDF from the initial state to the steady state $P_{st}(x)$ in asymptotic time ($t\to\infty$), which, according to Eq. (29), reads

$$P_{st}(x) = \frac{(\mu r\alpha)^{1/\alpha}}{x^{\mu+1}\sigma}\,P_{\alpha,-\alpha}\left(\frac{(\mu r\alpha)^{1/\alpha}}{\mu x^\mu\sigma}\right). \tag{30}$$

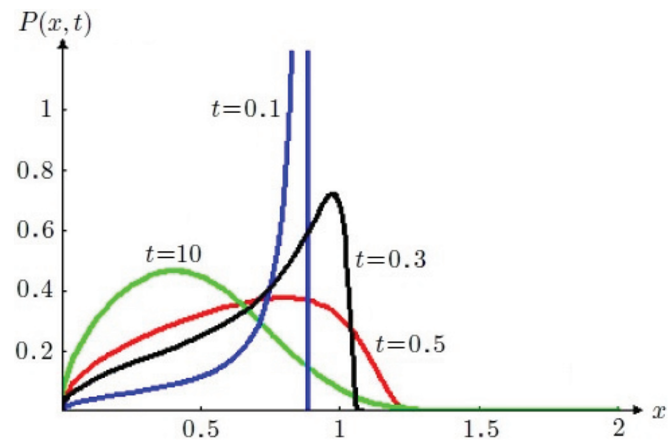

FIG. 3. (Color online) The evolution of the PDF of the population density. The values of the parameters are $\mu=3$, $x_0=0.8$, $r=1$, $\sigma=\pi$.

IV. THE CASE OF NOISE WITH THE LÉVY-SMIRNOV STABLE DISTRIBUTION

Unfortunately, our exact results (29) and (30) can be analyzed only in some special cases because stable distributions $P_{\alpha,\beta}(x)$ have complex expressions in the form of series. Further we demonstrate them with the example of white non-Gaussian noise $\xi(t)$ with a one-sided Lévy-Smirnov stable distribution ($\alpha=1/2$). Since this distribution is written in the simple analytical form (see, e.g., Ref. [32])

$$P_{1/2,-1/2}(x) = \frac{1}{2\sqrt{\pi}\,x^{3/2}}\,e^{-1/(4x)} \quad (x>0), \tag{31}$$

from Eqs. (27) and (29) we find

$$P(x,t) = \frac{1}{r}\sqrt{\frac{\sigma\mu}{\pi}}\,\frac{\left(1-e^{-\mu r t/2}\right)x^{\mu/2-1}}{\left(1-x^\mu e^{-\mu r t}/x_0^\mu\right)^{3/2}}$$
$$\times\exp\left[-\frac{\sigma x^\mu(1-e^{-\mu r t/2})^2}{\mu r^2\left(1-x^\mu e^{-\mu r t}/x_0^\mu\right)}\right]$$
$$(0<x<x_0e^{rt}). \tag{32}$$

The plots of the transient PDF (32) of the population density with the parameter of nonlinearity $\mu=3$ in Eq. (16) are shown for different times $t=0.1, 0.3, 0.5, 10$ in Fig. 3. As seen from Fig. 3, the initial distribution in the form of a δ function broadens the remaining unimodal for all times t.

According to Eq. (32), the stationary PDF ($t\to\infty$) reads

$$P_{st}(x) = \frac{1}{r}\sqrt{\frac{\sigma\mu}{\pi}}\,x^{\mu/2-1}\,e^{-\sigma x^\mu/(\mu r^2)} \quad (x>0). \tag{33}$$

The plots of the steady-state probability distribution of the population density (33) are shown in Fig. 4 for different values of the parameter of nonlinearity μ, which may be both an integer and a fraction. It is straightforward to show from

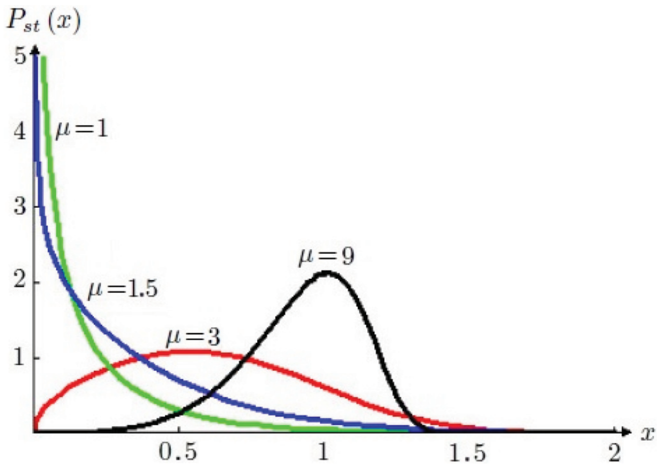

FIG. 4. (Color online) The steady-state PDF of the population density for different values of the exponent μ. The values of the parameters are $r = 1$, $\sigma = \pi$.

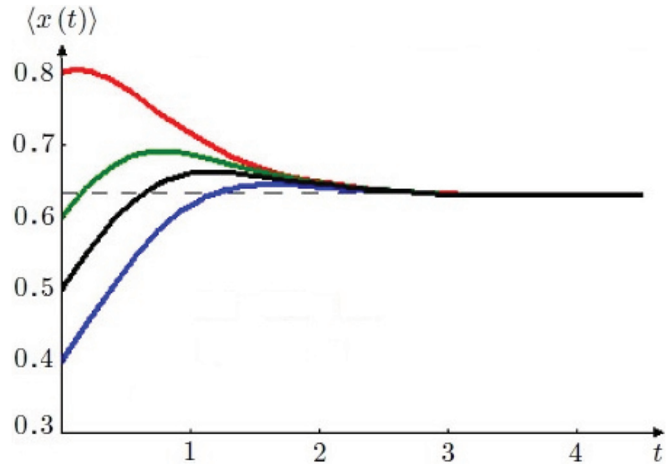

FIG. 5. (Color online) The nonmonotonic relaxation of the mean population density to the steady-state value (37) for different initial conditions. The parameters are $\mu = 3$, $r = 1$, $\sigma = \pi$.

Eq. (32) that for all times

$$\lim_{x\to 0^+} P(x,t) = \infty$$

for $0 < \mu < 2$ and

$$\lim_{x\to 0^+} P(x,t) = 0$$

for $\mu \geqslant 2$, which is clearly seen from Fig. 4 also. According to Eq. (33), if $0 < \mu < 2$, the steady-state distribution does not have extremes. At the same time, for the exponent $\mu \geqslant 2$ the most probable value of the population density is nonzero,

$$x^* = \left[\frac{(\mu-2)r^2}{2\sigma}\right]^{1/\mu}, \tag{34}$$

and the maximal value of the stationary PDF reads

$$P_{st}(x^*) = \sqrt{\frac{\mu(\mu-2)}{2\pi e}}\left[\frac{2\sigma e}{r^2(\mu-2)}\right]^{1/\mu}. \tag{35}$$

As seen from Fig. 4 and Eqs. (34) and (35), the maximum increases and shifts to the right with increasing the exponent of nonlinearity μ. For the Verhulst model ($\mu = 1$) the corresponding curve coincides with the result obtained recently in [34]. This means that in the case of $0 < \mu < 2$ fluctuations in resources with an infinite mean value [see Eq. (31)] lead to the annihilation of the population in the most likely scenario. However, for $\mu \geqslant 2$ the maximum of the steady-state distribution is not at the origin, and the biological population survives.

From Eq. (33) one can find the stationary value of the nth-order moment,

$$\langle x^n\rangle_{st} = \frac{1}{\sqrt{\pi}}\left(\frac{\mu r^2}{\sigma}\right)^{n/\mu}\Gamma\left(\frac{n}{\mu}+\frac{1}{2}\right), \tag{36}$$

and, in particular, the steady-state mean value $\langle x\rangle_{st}$,

$$\langle x\rangle_{st} = \frac{1}{\sqrt{\pi}}\left(\frac{\mu r^2}{\sigma}\right)^{1/\mu}\Gamma\left(\frac{1}{\mu}+\frac{1}{2}\right), \tag{37}$$

and the variance σ_{st}^2,

$$\sigma_{st}^2 = \frac{1}{\sqrt{\pi}}\left(\frac{\mu r^2}{\sigma}\right)^{2/\mu}\left[\Gamma\left(\frac{2}{\mu}+\frac{1}{2}\right) - \frac{1}{\sqrt{\pi}}\Gamma^2\left(\frac{1}{\mu}+\frac{1}{2}\right)\right]. \tag{38}$$

Finally, we find the evolution of the mean population density. Using Eq. (29), we arrive at

$$\begin{aligned}\langle x(t)\rangle &= \frac{1}{\sigma(t)}\int_0^{x_0 e^{rt}} P_{\alpha,-\alpha}\left[\frac{1}{\mu\sigma(t)}\left(\frac{1}{x^\mu}-\frac{e^{-\mu rt}}{x_0^\mu}\right)\right]\frac{dx}{x^\mu}\\ &= \int_0^\infty \frac{P_{\alpha,-\alpha}(z)\,dz}{\left[\mu\sigma(t)z + e^{-\mu rt}/x_0^\mu\right]^{1/\mu}}.\end{aligned} \tag{39}$$

The relaxation of the mean population density obtained by numerical calculation of the integral (39) for the Lévy-Smirnov stable distribution (31) and different initial values of the population density $x_0 = 0.4, 0.5, 0.6, 0.8$ is depicted in Fig. 5.

As can easily be seen from Fig. 5, all curves converge at large times to the value $\langle x\rangle_{st} \simeq 0.627$, which corresponds to Eq. (37) with the parameters $\mu = 3$, $r = 1$, $\sigma = \pi$, and have a nonmonotonic behavior, unlike Fig. 2 for the model (2). This interesting fact was discovered by Zygadło [24] for the MVB model (16) with excitation in the form of Markovian dichotomous noise and sinusoidal perturbation with random phase and then was confirmed by numerical simulations.

V. CONCLUSIONS

Using the central limit theorem and the exact solution, the asymptotic law for the probability distribution of population density for the Malthus-Verhulst-Bernoulli model with fluctuations of the reproduction rate in the form of non-Gaussian colored noise with nonzero power spectral density at zero frequency has been obtained. In particular, the monotonic relaxation of the mean population density to a steady-state value in the case of Markovian dichotomous noise was confirmed analytically. As was shown, asymptotically, all moments of the population density tend to 1, which indicates the survival of a biological population.

For the model with fluctuations in resources in the form of white non-Gaussian noise with a one-sided stable probability distribution we found the transient dynamics of the PDF of the population density and its momentum and probabilistic characteristics in the steady state. As has been indicated, for certain values of the parameter of nonlinearity μ the maximum of the unimodal steady-state probability distribution of the population density shifts to the right with increasing μ or with decreasing the scaling parameter σ (similar to the noise intensity). Also in this case, unlike in the previous model, we observe a nonmonotonic relaxation of the mean population density to a stationary value.

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