

Electrokinetic instability in microchannels

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The effect of geometric confinement on electroconvective instability due to nonequilibrium electro-osmotic slip at the interface of an electrolytic fluid and charge-selective solid is studied. It is shown that the topology of the marginal stability curves and the behavior of the critical parameters depend strongly on both channel geometry and dimensionless Debye length at low voltages for sufficiently deep channels, corresponding to the Rubinstein-Zaltzman instability mechanism, but that stability is governed almost entirely by channel depth for narrow channels at higher voltages. For shallow channels, it is shown that above a transition threshold, determined by both channel depth and Debye length, the low-voltage instability is completely suppressed.

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Introduction. Electroconvective (Rubinstein-Zaltzman) instability and its relationship to the overlimiting conductance of charge-selective interfaces has garnered significant attention in recent years (see Refs. [1–9]). Not only is it of practical significance for applications as diverse as desalination, fuel cells, and microfluidic analysis systems, but the strong coupling between electrodiffusion and hydrodynamics leads to very rich spatio-temporal behavior. Specifically, as the current density through the charge-selective solid exceeds the classical limiting value, the depleted region of the concentration polarization layer structure can no longer maintain local electroneutrality and an extended space-charge (ESC) region develops between the regular quasiequilibrium electric double layer and the electroneutral diffusion layer [1], the latter extending either to bulk, or to an electrode, depending on the size of the system relative to the interval of observation. Due to the nonmonotonic nature of charge density—hence gradient in the electric body force—in the ESC, the system loses mechanical stability in a way reminiscent of Benard or Marangoni convection. This instability can lead to the formation of vortex arrays which limit the extent of the depleted diffusion boundary layer adjacent to a charge-selective interface, thus in turn leading to an increase in electric current owing to an effectively decreased resistance of the depleted region. Much of the foundational theoretical work concerns open systems, i.e., relatively large electrolytic cells bounded by planar membranes or electrodes which may be considered infinite in the transverse directions; thus, in the absence of geometric inhomogeneity and prior to the threshold of instability, the fundamental problem is effectively one dimensional (1D). Such a description captures these effects well in systems with dimensions on the order of cm² or mm², predicting a voltage threshold for instability around a volt or slightly less, which has been confirmed experimentally (see Refs. [10–12]). However, a number of experimental observations of related effects in systems confined to microchannels exhibit thresholds on the order of 10's of volts and higher [12–16]. Furthermore, recent experimental [5] and numerical [6] studies show that microchannel geometry can impact the overlimiting conductance and, for sufficiently small microchannels, effectively suppress the instability at voltages as high as 40 V. This study

serves to provide a theoretical framework to bridge this gap, and in the process of doing so, also predicts unique effects due to a competition between stabilizing and destabilizing effects reminiscent of Taylor-Couette instabilities [7,8]. It is worth noting that previous work has been done regarding the effects of confinement on vortical convection in Hele-Shaw geometries [17,18]. Regarding electroconvective instability, relatively wide boundaries in the x direction have been shown to result in a wavelength selection process which ultimately chooses maximum vortex size essentially corresponding to the allowed standing perturbation modes in this direction [1,9]. This represents a study of the specific effects of narrow confinement in the z direction on electroconvective instability.

Formulation and assumptions. We consider a symmetric, binary electrolyte with a diffusivity of cations and anions $\tilde{D}$, dynamic viscosity $\tilde{\mu}$, and permittivity $\tilde{\epsilon}$, confined to a microchannel with insulating walls at $\tilde{z} = \pm\tilde{h}$ and bounded by an ideal, semiselective ion-exchange membrane surface, $\tilde{y} = 0$, and by an outer edge of the diffusion layer, $\tilde{y} = \tilde{\delta}$, with the bulk concentration $\tilde{c}_0$. Notations with tildes are used for the dimensional variables, as opposed to their dimensionless counterparts without. $\{\tilde{x}, \tilde{y}, \tilde{z}\}$ are the coordinates, where $\tilde{x}$ is directed along the membrane surface, $\tilde{y}$ is normal to the membrane surface, and $\tilde{z}$ is normal to the microchannel plane.

The flow is assumed to be Hele-Shaw, with nonconductive walls $\tilde{z} = \pm\tilde{h}$, impermeable to both kinds of ions. No-slip conditions on the walls and at the membrane surface are assumed. The potential $\tilde{\Phi}$ and concentrations $\tilde{c}^\pm$ are assumed to be constant in the $\tilde{z}$ direction, while velocity components $\tilde{u}$ and $\tilde{v}$ have a parabolic Poiseuille distribution with respect to $\tilde{z}$, $\tilde{u} = \frac{3}{2}\tilde{U}(1 - \tilde{z}^2/\tilde{h}^2)$ and $\tilde{v} = \frac{3}{2}\tilde{V}(1 - \tilde{z}^2/\tilde{h}^2)$, and the third velocity component is zero, $\tilde{w} = 0$, so the averaged velocity vector $\mathbf{U} = U\hat{i} + V\hat{j}$. Then the system of equations averaged with respect to the channel width is as follows:

$$\begin{aligned} \frac{\partial c^+}{\partial t} + U \frac{\partial c^+}{\partial x} + V \frac{\partial c^+}{\partial y} = & \frac{\partial}{\partial x} \left(c^+ \frac{\partial \Phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(c^+ \frac{\partial \Phi}{\partial y} \right) \\ & + \frac{\partial^2 c^+}{\partial x^2} + \frac{\partial^2 c^+}{\partial y^2}, \end{aligned} \quad (1)$$

$$\frac{\partial c^-}{\partial t} + U \frac{\partial c^-}{\partial x} + V \frac{\partial c^-}{\partial y} = -\frac{\partial}{\partial x} \left(c^- \frac{\partial \Phi}{\partial x} \right) - \frac{\partial}{\partial y} \left(c^- \frac{\partial \Phi}{\partial y} \right) + \frac{\partial^2 c^-}{\partial x^2} + \frac{\partial^2 c^-}{\partial y^2}, \quad (2)$$

$$\varepsilon^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = c^- - c^+, \quad (3)$$

$$\frac{\partial P}{\partial x} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{3}{h^2} U - \frac{\kappa}{\varepsilon^2} (c^+ - c^-) \frac{\partial \Phi}{\partial x}, \quad (4)$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \frac{3}{h^2} V - \frac{\kappa}{\varepsilon^2} (c^+ - c^-) \frac{\partial \Phi}{\partial y}, \quad (5)$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \quad (6)$$

The system is taken in a dimensionless form with $\tilde{\delta}$ as a characteristic length, $\tilde{\delta}^2/\tilde{D}$ as a characteristic time, $\tilde{\mu}$ as a dynamical characteristic value, and the thermodynamic potential $\tilde{\Phi}_0 = \tilde{R}\tilde{T}/\tilde{F}$ is taken as a characteristic potential and the bulk concentration $\tilde{c}_0$ as a characteristic concentration. The parameter $\kappa = \tilde{d}\tilde{\Phi}_0/\tilde{\mu}\tilde{D}$ appears as a coupling coefficient between hydrodynamics and electrostatics. It is essentially an electrokinetic Peclet number depending upon the properties of the electrolyte and not upon the properties of an imposed flow. Here velocity components U and V are averaged with respect to z , P is the averaged pressure, and ε is the dimensionless Debye length $\varepsilon = \tilde{\lambda}_D/\tilde{\delta}$ and $\tilde{\lambda}_D = \sqrt{\tilde{d}\tilde{\Phi}_0/\tilde{F}\tilde{c}_0}$, $\tilde{R}$ is the universal gas constant, $\tilde{T}$ is the absolute temperature, and $\tilde{F}$ is the Faraday constant.

The important $O(1/h^2)$ terms at $h \rightarrow 0$ originate from the wall friction. Keeping in (4) and (5) only leading order terms in h leads to Darcy's law

$$\mathbf{U} = -\frac{h^2}{3} \nabla P - \frac{\kappa h^2}{3\varepsilon^2} \rho \nabla \Phi,$$

corrected by the Coulomb force, where $\rho = c^+ - c^-$ is the space-charge density. However, we shall keep in (4) and (5) all order terms to satisfy the boundary conditions on the membrane surface, $y = 0$.

Our choice of boundary conditions at the planes $y = 0, 1$ is consistent with a simplified model of the concentration polarization layer (CPL) structure on the anodic (depleted) side of a highly charge-selective element such as a porous membrane or nanoslot—represented here by a perfectly selective membrane with a fixed, uniform cation concentration and complete anion exclusion—and bounded by a stirred bulk at $y = 1$ (see Fig. 1).

The boundary conditions at the membrane surface are as follows:

$$y = 0: \quad c^+ = p, \quad -c^- \frac{\partial \Phi}{\partial y} + \frac{\partial c^-}{\partial y} = 0, \quad (7)$$

$$U = V = 0, \quad \Phi = 0.$$

The first condition, prescribing interface concentration equal to that of the fixed charges inside the membrane, is asymptotically valid for large p and is first introduced by Rubinstein (see, for example, Ref. [3] and corresponding references) to avoid solution inside the membrane. The second condition means that the membrane is an ideally a permselective one and, hence, there is no flux for anions. The third condition is the usual

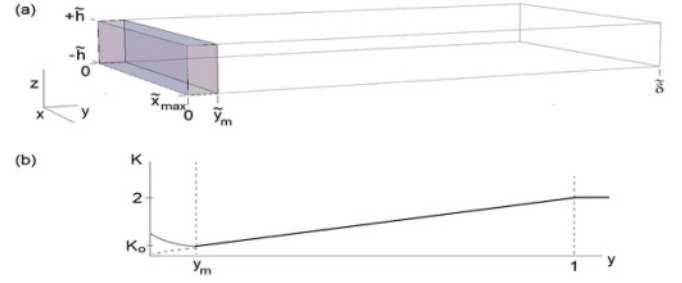


FIG. 1. (Color online) Depictions of (a) the model geometry and coordinate system (dimensional coordinates), and (b) the quasi-1D concentration profile (dimensionless coordinates.) In (a) the space-charge region containing both a macroscopic extended space-charge (ESC) region and microscopic electric double layer (EDL) are shown shaded with the outer boundary at the plane $y = y_m$. This corresponding region in (b) shows a qualitative sketch of the cation (solid gray) and anion (dashed gray) concentrations. The diffusion layer electroneutral concentration is approximately linear and shown in black.

no-slip condition and the last condition fixes zero potential at the membrane surface.

The boundary conditions at the outer edge of the diffusion layer specify both concentrations, vanishing of the velocity component V and shear stress, and ΔV specifies a drop of potential across the diffusion layer [1,2],

$$y = 1: \quad c^+ = 1, \quad c^- = 1, \quad V = 0, \quad (8)$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 0, \quad \Phi = \Delta V.$$

The problem is described by four dimensionless parameters: ΔV , ε , κ , and $h = \tilde{h}/\tilde{\delta}$. Dependence on the concentration p for the overlimiting regimes is practically absent and, hence, p is not included in the mentioned parameters. For systems confined to a small channel, in addition to the usual small parameter ε , (1)–(8), the system also has a second small parameter h .

Asymptotical solution: Nonequilibrium slip velocity in a microchannel. For $\varepsilon \rightarrow 0$ and current densities above the limiting value, the interval $0 < y < 1$ is divided into two regions: (a) a space-charge region, $0 < y < y_m$, and (b) electro-neutral region, $y_m < y < 1$. The problem (1)–(8) has a 1D steady solution, which coincides with the classical statement for a perfect semiselective membrane [1,2] and it has a well-known solution for the potential in the space-charge region,

$$\Phi = \Delta V \left[1 - \left(1 - \frac{y}{y_m} \right)^{3/2} \right], \quad (9)$$

where the potential drop in ESC is given by the following expression:

$$\frac{2\sqrt{2j}}{3\varepsilon} y_m^{3/2} = \Delta V. \quad (10)$$

If uniformity in the x direction is violated, the velocity field will be perturbed from zero. Following [2] and solving (4)–(6), taking into account $1/h$ terms, the inside ESC region yields

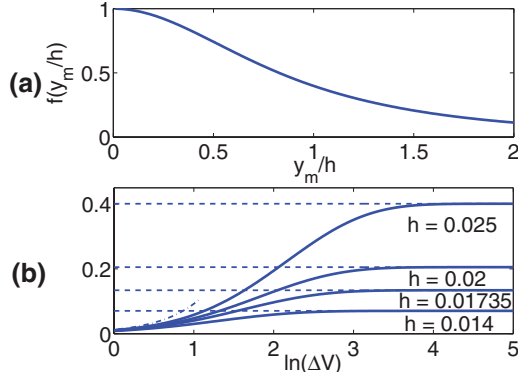


FIG. 2. (Color online) (a) Function of the confinement influence $f(y_m/h)$. Note that for sufficiently small values of y_m/h , $f \rightarrow 1$, and the slip velocity approaches the unconfined (Rubinstein-Zaltzman) case. (b) Slip velocity U_m as a function of ΔV at unit $\partial j/\partial x$ and parametrized with h at $\varepsilon = 10^{-4}$; the dashed-dotted line stands for the unconfined limit.

the slip velocity at the boundary y_m :

$$U_m = -\frac{\kappa}{8} \Delta V^2 \frac{1}{j} \frac{\partial j}{\partial x} f\left(\frac{y_m}{h}\right), \quad (11)$$

with the correction term for a microchannel,

$$f\left(\frac{y_m}{h}\right) = 2 \frac{h^2}{y_m^2} \left[\frac{1}{\sqrt{3}} \frac{h}{y_m} \tanh\left(\frac{y_m \sqrt{3}}{h}\right) - \frac{1}{\cosh\left(\frac{y_m \sqrt{3}}{h}\right)} \right].$$

Here $y_m = 9^{1/3} \varepsilon^{2/3} \Delta V^{2/3} / 2j^{1/3}$ is found from (10). (See Fig. 2.)

(a) If $y_m/h \rightarrow 0$, i.e., for very wide (essentially unconfined) channels, $f \rightarrow 1$, and the famous Rubinstein slip velocity for the phenomena of the second kind is obtained.

(b) If $\Delta V \rightarrow \infty$, $y_m/h \rightarrow \infty$ and $f(y_m/h) \rightarrow 0$, and two factors in (11) cancel each other and it turns into

$$U_m = -\frac{2\kappa}{9\sqrt{3}} \frac{h^3}{\varepsilon^2} \frac{\partial j}{\partial x}. \quad (12)$$

Solution in the electroneutral region: Stability analysis. Introduction of the stream function Ψ from the expressions $U = \partial \Psi / \partial y$ and $V = -\partial \Psi / \partial x$ along with well-known algebraic calculations turn the system (4)–(6) to the single fourth-order partial differential equation for the stream function. In the electroneutral region $c^+ = c^-$ and this equation is

$$\frac{\partial^4 \Psi}{\partial x^4} + 2 \frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Psi}{\partial y^4} - \frac{3}{h^2} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) = 0, \quad (13)$$

with the boundary conditions

$$\begin{aligned} y = y_m: \quad & \Psi = 0, \quad \frac{\partial \Psi}{\partial y} = U_m; \\ y = 1: \quad & \Psi = 0, \quad \frac{\partial^2 \Psi}{\partial y^2} = 0. \end{aligned} \quad (14)$$

Note the fourth term, equal to $3/h^2$ times the vorticity, where $h \rightarrow 0$ is a small parameter. This corresponds to a viscous shearing interaction between the fluid and the channel walls and gives rise to stability characteristics quite different than the unconfined membrane case. Far from the boundary y_m terms

$\nabla^4 \Psi$ can be dropped, but these terms become important to satisfy the boundary conditions at $y = y_m$. Because of this fact there is a thin boundary layer $O(h)$ near $y = y_m$ and there is a possibility to build up asymptotic outer and inner expansions with their following matching. However, we prefer to keep in (13) all order terms.

For the salt concentration $K = c^+ + c^-$ we have the following equations:

$$\frac{\partial K}{\partial t} + U \frac{\partial K}{\partial x} + V \frac{\partial K}{\partial y} = \frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2}. \quad (15)$$

Then, treating the edge of the space-charge region as the boundary,

$$y = y_m: \quad K = a \left(\varepsilon \frac{\partial K}{\partial y} \right)^{2/3}; \quad y = 1: \quad K = 2. \quad (16)$$

The first boundary condition is worth some discussion. In the space-charge region, which is also called a depletion region, the salt concentration K is small, $K = O(\varepsilon^{2/3})$. Outside of this region $K = O(1)$. The attempts in Ref. [1] to solve (15) with the boundary condition at $y = y_m$ taken in the first approximation, $K = 0$, led to a singularity. Physically this singularity originates from an infinite electric resistance at $K = 0$. In order to treat this singularity, higher order terms $O(\varepsilon^{2/3})$ must be included in the analysis. Such a treatment was done in Ref. [2] and the details were presented in Ref. [3]. We refer the reader to a sophisticated analysis of this work. Here we employ the boundary condition of Ref. [2], Eq. (4). The approach reduces Eqs. (1)–(3) to a single (Painlevé) differential equation for the electric field, and the constant $a \approx 0.9666$ is obtained numerically. (See Ref. [3].)

The unperturbed solution is given by

$$K_0 = j(y - y_m) + a(\varepsilon j)^{2/3}, \quad y_m = 1 - \frac{2}{j} + a\varepsilon^{2/3} j^{-1/3}.$$

At sufficiently large potential drop ΔV the 1D solution succumbs to the electroconvective instability. In contrast to the classical unconfined Rubinstein-Zaltzman instability, the viscous damping is expected to reduce this instability and become more efficient for smaller thickness h . Note that electroconvective motion in the presence of z confinement was predicted and observed in the context of electrodeposition, where it was also noted that this tended to suppress gravitational convection (see Refs. [17,18]). However, the cells were not sufficiently shallow (h on the order of mm) to have much of an effect on electroconvection in these cases.

To ascertain the threshold voltage for the onset of the vortex instability, the stability analysis is restricted by seeking the conditions for which the exponential decay is changed to the exponential growth, i.e., only those modes at a given voltage for which the exponential growth rate of the disturbance is zero, otherwise known as “marginally stable.” Therefore, assuming small neutral (time-independent) sinusoidal perturbations about this solution,

$$\begin{aligned} K &= K_0(y) + \hat{K}(y) \exp(i\alpha x), \quad \Psi = \hat{\Psi}(y) \exp(i\alpha x), \\ U_m &= \hat{U}_m \exp(i\alpha x), \end{aligned} \quad (17)$$

and linearizing (13)–(16) with respect to the perturbations turns the system into a linear ordinary differential equation system,

$$\hat{\Psi}^{IV} - 2\alpha^2 \hat{\Psi}'' + \alpha^4 \hat{\Psi} - \frac{3}{h^2}(\hat{\Psi}'' - \alpha^2 \hat{\Psi}) = 0, \quad (18)$$

$$y = y_m: \hat{\Psi} = 0, \hat{\Psi}' = \hat{U}_m; \quad y = 1: \hat{\Psi} = 0, \hat{\Psi}'' = 0, \quad (19)$$

$$\hat{K}'' - \alpha^2 \hat{K} = -i\alpha K' \hat{\Psi}, \quad (20)$$

$$y = y_m: \hat{K} = \frac{\varepsilon^{2/3}}{3} \left\{ \frac{2a}{j^{1/3}} + \frac{2^{1/3}}{j^{1/3}} \left(\frac{3}{4} \Delta V \right)^{2/3} \right\} \hat{K}'(0);$$

$$y = 1: \hat{K} = 0. \quad (21)$$

Here

$$\hat{U}_m = \hat{\Psi}' = -\frac{\kappa}{8} \frac{\Delta V^2}{j} i\alpha \hat{K}'(0) f\left(\frac{y_m}{h}\right),$$

$$y_m = \frac{3\varepsilon^{2/3}}{2 \cdot 3^{1/3} j^{1/3}} \Delta V^{2/3} \quad (22)$$

Solution of (18)–(22) gives the dispersion relation for the marginal stability curves:

$$\frac{\kappa}{8} \Delta V^2 f\left(\frac{y_m}{h}\right) = -\frac{6 \sinh^2 \alpha [\coth \alpha - \frac{\sigma}{\alpha} \coth \sigma] (1 + \alpha \coth \alpha N)}{h^2 \sinh^2 \alpha [(\sigma^2 + \alpha^2) \coth \alpha - 2\alpha \sigma \coth \sigma] - 3\alpha}, \quad (23)$$

with

$$N \equiv \frac{\varepsilon^{2/3}}{3} \left\{ \frac{2a}{j^{1/3}} + \frac{2^{1/3}}{j^{1/3}} \left(\frac{3}{4} \Delta V \right)^{2/3} \right\},$$

$$\sigma \equiv \frac{\sqrt{3}}{h} \left(1 + \frac{\alpha^2 h^2}{3} \right)^{1/2}.$$

For $h \rightarrow \infty$ we approach the usual membrane case, $\sigma \rightarrow \alpha$, $f(y_m/h) \rightarrow 1$, and (23) turns into

$$\frac{\kappa}{8} \Delta V^2 = 4 \frac{\sinh \alpha \cosh \alpha - \alpha + N\alpha(\cosh^2 \alpha - \alpha \coth \alpha)}{\sinh \alpha \cosh \alpha + \alpha - 2\alpha^2 \coth \alpha}, \quad (24)$$

which coincides with expression (33) from Ref. [2] and for $\varepsilon = 0$ with expression (94) of Ref. [1].

Results and discussion. The marginal stability curves are obtained by solution of the dispersion relation (23); results are presented in Fig. 3. For sufficiently deep microchannels, the curves approach the unconfined membrane case. This is also evident from the asymptotic expression (24) which is equivalent to the unconfined membrane case for $f \rightarrow 1$. As the microchannel depth is decreased, different behaviors become evident. As the channel depth decreases, the critical voltage rises and the region of instability becomes narrower, eventually becoming pinched on either side of the critical wave number. At a channel depth of approximately $h = h_* \approx 0.01735$, the marginal stability curve undergoes a change in topology and the pinch develops into a closed island of instability at lower

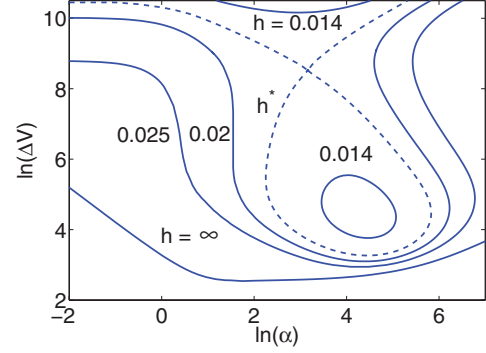


FIG. 3. (Color online) Topology of marginal stability curves for $\kappa = 0.2$ and $\varepsilon = 10^{-4}$ and for different values of the parameter h . The separatrix takes place at $h = h^* \approx 0.01735$.

critical voltages corresponding to the primary Rubinstein-Zaltzman instability with a secondary (open) instability at higher voltages. The separatrix is shown in the figure by the dashed line.

This behavior is seen in a variety of systems wherein there is a competition between multiple instability mechanisms, such as the aforementioned Taylor-Couette flow (see Ref. [7] and references therein). In the context of the present system, the physical explanation is as follows: At low vorticity, corresponding to low local fluid velocity near and slightly above the lowest critical voltage, viscous shearing between the flow and the walls at $z = \pm h$ represented by the $O(h^{-2})$ term in Eq. (13) tends to stabilize the flow via dissipation, as one would intuitively expect. For sufficiently narrow channels, this leads to the appearance of a maximally unstable critical voltage, i.e., the “top” of the island where the stabilizing effects of viscous shear begin to exceed the destabilizing effects of the forces in the extended space-charge region, represented here by the modified slip velocity. However, at higher vorticity, corresponding to higher local velocity at yet higher voltages, viscous diffusion of momentum will tend to destabilize the flow, leading to the secondary shear instability and secondary critical voltage minimum above the island. Such a dual role for viscosity is discussed in the context of Taylor-Couette flow in Ref. [8] and in the context of shearing instability in thin channels in Ref. [19].

The critical parameters of threshold of instability α_* and ΔV_* giving the critical mode and the corresponding critical voltage for this mode are determined by the condition $\partial \Delta V / \partial \alpha|_{\alpha=\alpha_*} = 0$. Plots of the critical parameters versus inverse channel depth help to clarify the nature of these changes [see Figs. 4(a) and 4(b)]. Three regimes may be identified from these results. For sufficiently deep channels, (23), in fact, coincides with (24), corresponding to the usual instability of the extended space charge, and is strongly governed by the slip velocity with $f = 1$ in (11). The system has one critical point. For sufficiently narrow channels and high voltages the system is governed by the slip velocity (12) and the system is essentially independent of concentration and governed entirely by channel geometry. The system again has a single critical point. A transitional region may be identified, which depends strongly on the Debye parameter, wherein the solution

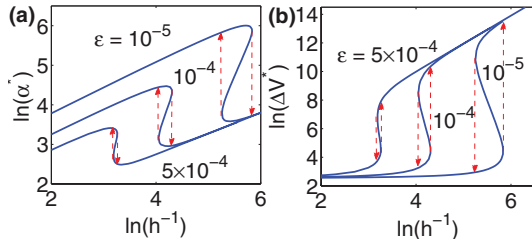


FIG. 4. (Color online) Critical wave number (a) and critical voltage (b) vs inverse channel depth for $\kappa = 0.2$ and different ε .

topology changes, corresponding to the island of instability in Fig. 3. Such topology of the critical parameters gives rise to a hysteresis: with reference to Fig. 4(a), if we were to gradually decrease the channel depth, the system would eventually jump to the lower branch of the critical curve (the jump shown by the dashed line). The jump from the lower branch to the upper on takes place in another point, for the larger confinement depth h . However, it should be noted that under typical experimental conditions, the voltage for these secondary instabilities can be

quite high, corresponding to (dimensional) applied voltages greater than 1 kV.

The general trend that decreasing microchannel depth increases the critical voltage for the onset of instability is qualitatively consistent with the observations of Yossifon *et al.* [5]. The increase in critical wave number with decreasing channel depth is consistent with the intuitive idea that the most unstable perturbations should scale with the channel dimension. The existence of a secondary instability at high voltages is also consistent with the analysis of experimental data in which the primary instability vortices are observed to disappear, and preliminary evidence suggests transient vortices occur at a later time [14].

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