Determination of the critical exponents for absorbing phase transitions in the conserved lattice gas model in three dimensions

Sang B. Lee
Department of Physics, Kyungpook National University, Daegu 702-701, Korea
(Received 3 September 2011; revised manuscript received 10 January 2012; published 17 February 2012)

The critical exponents were measured for absorbing phase transitions in the conserved lattice gas (CLG) model on a simple cubic lattice. The correlation-length exponent calculated from the dynamic exponent obtained by finite-size scaling and from the mean spreading distance was consistently found to be $\nu_\perp = 0.631 \pm 0.02$, which yields a positive specific heat exponent $\alpha = 2 - d\nu_\perp$ on a pure system. The pure fixed point should, thus, be unstable if the Harris criterion established in equilibrium systems is applicable to the nonequilibrium absorbing phase transitions of the CLG model. However, this prediction is in contradiction with recent simulation results.

DOI: 10.1103/PhysRevE.85.022101 PACS number(s): 05.70.Ln, 05.50.+q, 64.60.Ht

Nonequilibrium absorbing phase transitions (APTs) have been of great interest for more than two decades. Various models with various evolution rules have been classified into two firmly established universality classes, the directed percolation (DP) class and the parity-conserving (PC) class [1]. More recently, a new universality class was proposed by Rossi *et al*. [2] for models with a conserved field . A number of papers have been published regarding the critical behaviors of the models in this class, including the conserved lattice gas (CLG) model, the conserved threshold transfer process, and the stochastic sandpile model [3–6].

Very recently, the present author studied the critical behavior of the CLG model on diluted lattices [7]. His work was an extension of a number of papers regarding the disordered (or diluted) contact process, which was introduced by applying the Harris criterion to APTs and has attracted great attention for more than a decade [8–13]. The Harris criterion suggests that the pure fixed point is unstable if the specific-heat exponent α for a pure system is positive [14]. This implies that the amount of disorder is a relevant parameter, i.e., a small amount of disorder (diluted sites) added to the system significantly changes the critical behavior. For the contact process in the DP universality class, the specific-heat exponent obtained by the hyperscaling relation $\alpha = 2 - d\nu$ [15], ν being the correlation-length exponent in d dimensions, is known to be positive for $d < 4$ if ν is replaced by the spatial correlation-length exponent $\nu_\perp$. The critical behavior of the diluted contact process was indeed found to be different from that of the pure contact process. A similar study was also carried out for a model in the PC class [16].

For the CLG model, various values of $\nu_\perp$ have been reported on a pure lattice. In one dimension, de Oliveira obtained by a transfer matrix method $\nu_\perp = \frac{1}{2}$, which is identical to the mean-field value [17]. On the other hand, Lee and Lee obtained $\nu_\perp = 1$ by numerical simulations and claimed that the value of de Oliveira was erroneous due to a simple mistake in his calculation [18]. In two dimensions, Lübeck and Heger reported $\nu_\perp \simeq 0.80$ (which yields $\alpha > 0$) [19], whereas Lee and Lee obtained $\nu_\perp \simeq 1.01$ (yielding $\alpha \approx 0$) [20], both by numerical simulations. In three dimensions, the known value $\nu_\perp = 0.593$ [19] suggests that α is positive, while in four dimensions the mean-field value $\nu_\perp = \frac{1}{2}$ [21] yields $\alpha = 0$. Therefore, it is expected that disorder is relevant and different critical behaviors may be observed at least in three dimensions when disorder is added to the system. On the other hand, it was found by numerical simulations that the critical behavior of the CLG model was qualitatively similar to that of the pure CLG model in both two and three dimensions when the concentration of disorder was less than the critical concentration [7]. This suggests that either the correlation-length exponent $\nu_\perp$ in three dimensions might be incorrect or the Harris criterion might be invalid for APTs. It should be noted that there remains concern as to whether or not a simple replacement of ν by $\nu_\perp$ in the hyperscaling relation is acceptable, and whether the Harris criterion established in equilibrium systems is valid in nonequilibrium systems as well. [Note that there are two (spatial and temporal) correlation lengths in nonequilibrium systems, and a replacement of ν by $\nu_\perp$ is unjustified.] Therefore, it is important to accurately measure $\nu_\perp$ in three dimensions to determine if the Harris criterion is valid for APTs in the CLG model.

The purpose of this Brief Report is to provide accurate estimates of the critical exponents for the CLG model in three dimensions. It is sufficient to calculate $\nu_\perp$ for the present purpose. However, since an efficient method of directly measuring $\nu_\perp$ is unknown, it will be calculated from the known scaling relations using other critical exponents defined in APTs. The results will be confirmed by a scaling analysis.

In the CLG model, ρL^d particles are initially distributed on a d-dimensional cubic lattice of side L, and each lattice site may be occupied by at most one particle. A particle is defined as active if it has at least one particle on the nearest-neighbor sites and otherwise it is inactive. The dynamics proceeds with hopping of active particles; each of the randomly chosen active particles hops to one of the neighboring empty sites with a time increment $1/N_a$, N_a being the number of active particles at the time of hopping. The density of active particles ρ_a exhibits a power-law decay in time, $\rho_a(t) \sim t^{-\theta}$, at the critical particle density ρ_c. The steady-state density in the supercritical region, considered to be an order parameter, shows the usual power-law behavior against the distance from criticality, i.e., $\rho_{ss}(\rho) \sim (\rho - \rho_c)^\beta$. The spatial and temporal correlation lengths are also known to yield singular behaviors at ρ_c, i.e., $\xi \sim |\rho - \rho_c|^{-\nu_\perp}$ and $\tau \sim |\rho - \rho_c|^{-\nu_\parallel}$. The critical exponents defined so far are not independent and are associated with one

another via the scaling relations. The active-particle density in the vicinity of criticality may be written as a function of the ratio of t and τ:

$$\rho_a(t,\rho) = t^{-\theta}\mathcal{F}(t|\rho-\rho_c|^{\nu_\parallel}). \quad (1)$$

In the $t\to\infty$ limit, ρ_a for $\rho>\rho_c$ saturates, i.e., $\rho_a(\rho)\to\rho_{ss}(\rho)\propto(\rho-\rho_c)^{\nu_\parallel\theta}$; thus, the relation

$$\beta = \nu_\parallel\theta \quad (2)$$

is obtained. On the other hand, for a finite system of size L at criticality, the correlation length ξ cannot exceed the size of the system, i.e., $\xi\sim|\rho-\rho_c|^{-\nu_\perp}\sim L$. Substitution of $|\rho-\rho_c|\sim L^{-1/\nu_\perp}$ into Eq. (1) gives

$$\rho_a(t,L) = t^{-\theta}\mathcal{G}(t/L^z), \quad (3)$$

where

$$z = \nu_\parallel/\nu_\perp \quad (4)$$

is the dynamic exponent. In the $t\to\infty$ limit, ρ_a is no longer a function of t and becomes $\rho_a\to\rho_{ss}(L)\propto L^{-\beta/\nu_\perp}$. Therefore, the value of $\beta/\nu_\perp$ can be calculated from $\rho_{ss}(L)$ for various system sizes if ρ_a saturates for a finite-size system at criticality.

Rossi *et al.* obtained $\beta/\nu_\perp = 0.81(3)$ from a power-law plot of $\rho_{ss}(L)$ at criticality, using data averaged over samples surviving up to a predetermined time, i.e., using the surviving-sample averages on a square lattice [2]. They also calculated $\nu_\parallel$ using Eq. (4), with z calculated from the finite-size scaling plot of Eq. (3) using the all-sample average data. The value of $\nu_\parallel$ obtained in this way, however, was not consistent with the value β/θ and, based on this, the authors claimed that simple scaling in Eq. (2) was broken due to a θ anomaly [2]. The failure of similar scaling was also reported previously for stochastic sandpile models [22]. Lübeck and Misra calculated $\nu_\parallel$ using a scaling plot of the global persistence distribution for ρ_a and showed that it was consistent with the value obtained from Eq. (4) [23]. On the other hand, Lee and Lee recently claimed that the value of $\nu_\perp$ calculated from the surviving-sample averages of $\rho_{ss}(L)$ might be incorrect due to the occurrence of two competing time scales in the surviving-sample averages [20]. They calculated the exponents θ, β, $\nu_\parallel$, and z and showed that the value of $\nu_\parallel$ obtained from the global persistence distribution of ρ_a is consistent with the value β/θ when finite-size effects are correctly taken into account. They suggested $\nu_\perp = 1.01(2)$ from Eq. (4). Assuming a similar situation in one dimension (1D), the value of $\nu_\perp = 1$ obtained from the surviving-sample average data of $\rho_{ss}(L)$ may be incorrect; the correct value is $\nu_\perp = \nu_\parallel/z = 2$, which yields $\alpha\approx 0$. On the other hand, in three dimensions, the reported exponent values are $\theta = 0.745(17)$, $\beta = 0.840(12)$, $\nu_\parallel = 1.081(27)$, $\nu_\perp = 0.593$, and $z = 1.823$ [19]. With these values, the scaling relation in Eq. (2) is not satisfied while Eq. (4) holds, and $\alpha\simeq 0.22 > 0$. If the reported values of $\nu_\parallel$ and $\nu_\perp$ are considered to be incorrect for the reasons given by Lee and Lee and are recalculated using the known values of β, θ, and z, the values obtained are $\nu_\parallel\simeq 1.128$ and $\nu_\perp\simeq 0.62$, the latter yielding α much smaller but still positive. Therefore, if the Harris criterion is applicable to APTs, the disorder is relevant in three dimensions, in contradiction to the simulation results.

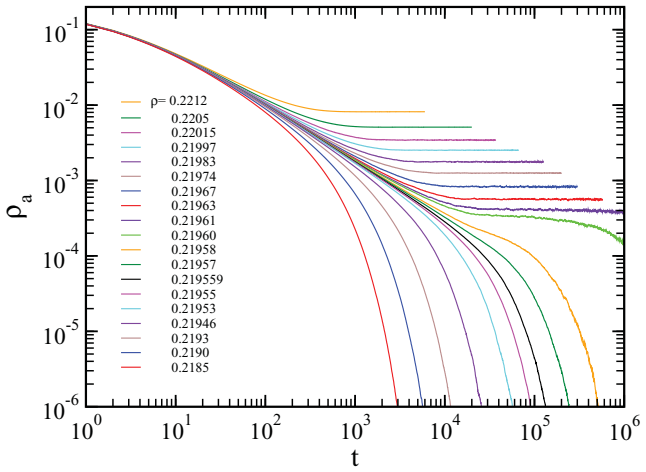

FIG. 1. (Color online) Active-particle densities ρ_a for selected particle densities ρ plotted as a function of the evolution time. The numbers in the legend are the densities of particles in the same order as the data from top to bottom.

Simulations were carried out on a cubic lattice of size $L = 400$ except when a finite-size effect was apparent, when simulations on larger systems were also carried out. The critical density was obtained, in the same way as in two dimensions [20], from the power-law plot of ρ_{ss} against $\rho-\rho_c$, assuming ρ_c as a parameter, and it was refined from the power-law behavior of $\rho_a(t)$ against t and from the degree of data collapse in both the off-critical and the finite-size scaling plots.

The active-particle densities are plotted in Fig. 1 against the evolution time. The data for $\rho < 0.2196$ decay rapidly and fall into one of the absorbing states, whereas those for $\rho > 0.2196$ apparently saturate to steady-state values. The plot of ρ_{ss} against ρ also indicated that the order parameter falls to zero at a density close to $\rho = 0.2196$, as shown in the inset of Fig. 2. From the power-law fit of the order parameter and from the best data collapse in the finite-size scaling, it is determined that $\rho_c = 0.219\,559(4)$. The critical exponents estimated from both figures were $\theta = 0.755(6)$ and $\beta = 0.848(3)$, which are very close to the values reported previously [19]. The estimates were confirmed by the off-critical scaling plot shown in Fig. 3; the data scaled with the measured values of β and θ collapsed onto two separate curves for $\rho > \rho_c$ and $\rho < \rho_c$, indicating that the measured values of β and θ are valid. The measured values yield $\nu_\parallel = 1.123(13)$.

The finite-size scaling in Eq. (3) was also examined using the data for various system sizes at ρ_c, assuming that z is a fitting parameter for data collapse. Data generated at the given ρ_c yielded the best collapse for $z = 1.78$, as shown in Fig. 4. The degree of data collapse was not sensitive enough and, accordingly, a rather larger error of ± 0.01 was determined. With this value and the value of $\nu_\parallel$ $(= \beta/\theta)$, it is found that $\nu_\perp = 0.631\pm 0.01$, which yields a small but still positive specific-heat exponent.

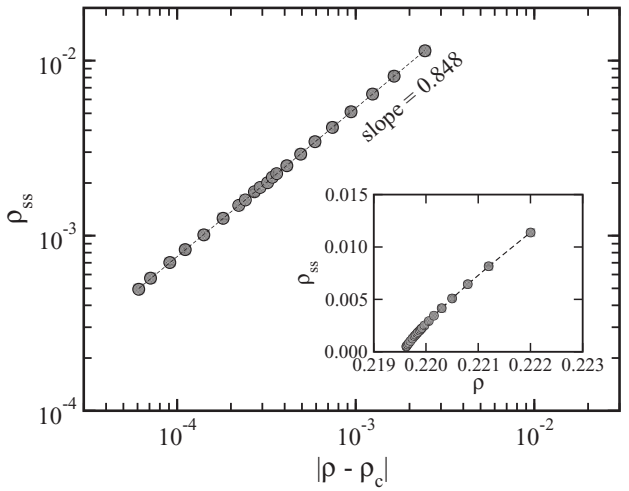

FIG. 2. Steady-state densities ρ_{ss} plotted as a function of the distance from criticality on a double logarithmic scale. The inset shows ρ_{ss} as a function of ρ on a linear scale.

Thus far, the critical exponents β, θ, and z were estimated from the active-particle densities. These exponents can alternatively be calculated by dynamic simulations, starting from one of the states very close to an absorbing state, as follows. After the system falls into one of many absorbing states, it may be perturbed by moving a particle to a randomly selected neighboring site until the moving particle and its neighboring particles become active. After resetting the evolution time to 0, the simulation begins from this perturbed state, and the survival probability $P(t)$, the number of active particles $N_a(t)$, and the spreading distance, i.e., the mean gyration radius of active particles $R(t)$, are sampled. At criticality, these quantities are known to yield power-law behaviors,

$$P(t) \sim t^{-\theta'}, \quad N(t) \sim t^{\eta}, \quad R(t) \sim t^{1/z}. \tag{5}$$

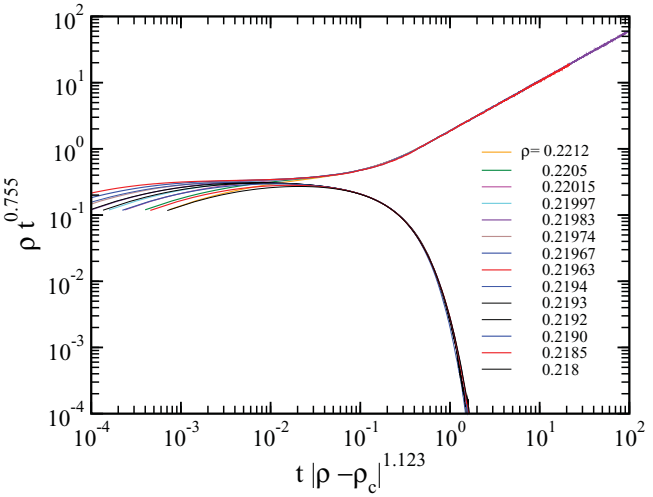

FIG. 3. (Color online) Off-critical scaling of ρ_a on a double logarithmic plot. The numbers in the legend are the densities of particles used in the scaling plot.

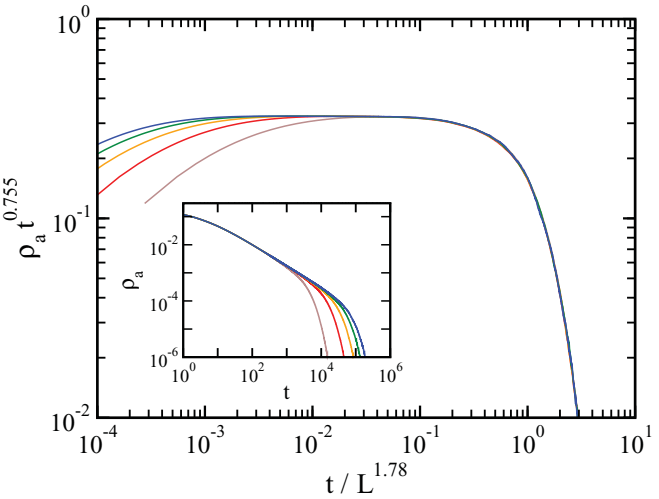

FIG. 4. (Color online) Finite-size scaling plot of ρ_a. The main plot shows the scaled active-particle densities as a function of the scaled time for systems of size, from right to left, $L = 100$, 200, 300, 400, and 500, and the inset shows ρ_a as a function of t for the same sizes as in the main plot from left to right.

Simulations were carried out on various size systems: plotted in Fig. 5 are $R(t)$ (top), $N(t)$ (middle), and $P(t)$ (bottom), each of which for a system of, from bottom to top, $L = 100$, 200, 300, 400, and 480. The dashed lines are the fitting lines over the data for $L = 480$, with the powers $\theta' = 0.773(5)$, $\eta = 0.122(3)$, and $1/z = 0.565(2)$. It should be noted that $z \simeq 1.77$ is very close to the value obtained from the finite-size scaling plot of $\rho_a(t, L)$ at criticality and the value of θ' differs by less than 3% from that of θ. The estimated values satisfy the known hyperscaling scaling relation $d/z = \theta + \theta' + \eta$ within 2% error. The results are summarized in Table I, together with the known results.

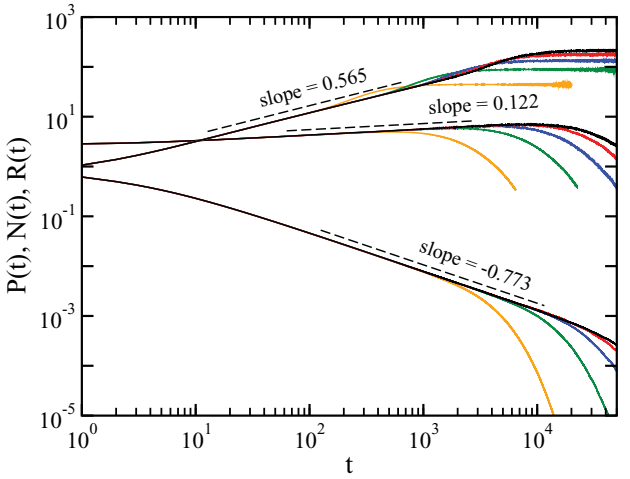

FIG. 5. (Color online) Data for $R(t)$ (top), $N(t)$ (middle), and $P(t)$ (bottom) as functions of the evolution time, obtained from a dynamic simulation, for the CLG model. In each set, the sizes of the systems are, from bottom to top, $L = 100$ (orange), 200 (green), 300 (blue), 400 (red), and 480 (black), and the dotted lines the regression fits over the data for $L = 480$.

TABLE I. Summary of the critical exponents for an APT in the CLG model.

	θ	β	$\nu_\parallel$	z	$\nu_\perp$	α
1D	$\frac{1}{4}$	1	4	2	2[a]	0
2D[b]	0.410	0.639	1.544	1.53	1.01	−0.02
3D[c]	0.745	0.840	1.081	1.823	0.593	0.221
3D[d]	0.755	0.848	1.123	1.78	0.631	0.11
4D	1	1	1	2	0.5	0

[a]Calculated by Eq. (4) using the values of $\nu_\parallel$ and z.
[b]Reference [20].
[c]Reference [19].
[d]Present work.

The measured correlation-time exponent $\nu_\parallel$ was also examined in the same way as in Ref. [23] using a scaling plot of the global persistence distribution $P_g(t)$ of ρ_a, i.e., the distribution of times for which the system persists in one of the phases, e.g., in the phase where ρ_a is smaller than the mean density $\langle\rho_a\rangle$. Near criticality, it is known to scale as

$$P_g(t) = |\rho - \rho_c|^{\nu_\parallel\theta_g}\mathcal{P}(t|\rho - \rho_c|^{\nu_\parallel}), \tag{6}$$

where θ_g is the global persistence exponent, the value of which is obtained from the power law $P_g(t) \sim t^{-\theta_g}$ at criticality. The distribution function $P_g(t)$ was calculated for two system sizes $L = 200$ and 400. It was found that the scaled data of $P_g(t)|\rho - \rho_c|^{-\nu_\parallel\theta_g}$ for various values of ρ plotted against the scaled time $t|\rho - \rho_c|^{\nu_\parallel}$ using $\theta_g = 1.52$ and $\nu_\parallel = 1.123$ fell onto a single curve for both cases of L (not shown). The scaling function was not sensitive to the size of the system, unlike the case in two dimensions [20].

In summary, various critical exponents were measured by both static and dynamic simulations for the CLG model on a simple cubic lattice. The spatial correlation-length exponent was found to be $\nu_\perp = 0.631(10)$, satisfying the inequality $\nu_\perp < 2/d$. In one and four dimensions, the equality $\nu_\perp = 2/d$ was found to be exact and, in two dimensions, it appeared to be correct although rigorous proof is still necessary. On the other hand, in three dimensions, the numerical estimates ruled out the possibility of the equality and, therefore, the specific-heat exponent was positive. This suggests, in conjunction with the Harris criterion, that the pure fixed point is unstable in three dimensions. However, an earlier study for the diluted CLG model suggested that the pure fixed point is stable [7] and, therefore, the Harris criterion appeared not to be valid for nonequilibrium APTs in the CLG model. This conclusion is in line with a recent study on nonequilibrium kinetic Ising model cellular automata (NEKIMCA) in the PC universality class, for which $\alpha > 0$ in one dimension [16]. In NEKIMCA, the disorder was introduced by a quenched random noise variable within the range $(-\epsilon,\epsilon)$, and it was found that weak disorder (small ϵ) was irrelevant while strong disorder caused continuously changing critical exponents.

This work was supported in part by the Korea Research Foundation Grant No. 2011-0007322 and by the Kyunpook National University Specialization program. The author is grateful for the support.

[1] For a review, see, e.g., J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, England, 1999); H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[2] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000).
[3] R. Pastor-Satorras and A. Vespignani, Phys. Rev. E **62**, R5875 (2000).
[4] S. S. Manna, J. Phys. A **24**, L363 (1991).
[5] D. Dhar, Physica A **263**, 4 (1999).
[6] A. Vespignani, R. Dickman, M. A. Munõz, and S. Zapperi, Phys. Rev. E **62**, 4564 (2000).
[7] S. B. Lee, Phys. Rev. E **84**, 041123 (2011).
[8] A. J. Noest, Phys. Rev. Lett. **57**, 90 (1986); Phys. Rev. B **38**, 2715 (1988).
[9] A. G. Moreira and R. Dickman, Phys. Rev. E **54**, R3090 (1996); R. Dickman and A. G. Moreira, *ibid.* **57**, 1263 (1998).
[10] J. Hooyberghs, F. Iglói, and C. Vanderzande, Phys. Rev. Lett. **90**, 100601 (2003); Phys. Rev. E **69**, 066140 (2004).
[11] T. Vojta and M. Dickison, Phys. Rev. E **72**, 036126 (2005).
[12] T. Vojta, A. Farquhar, and J. Mast, Phys. Rev. E **79**, 011111 (2009).
[13] T. Vojta and M. Y. Lee, Phys. Rev. Lett. **96**, 035701 (2006); also see M. Y. Lee and T. Vojta, Phys. Rev. E **79**, 041112 (2009).
[14] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[15] P. G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell University Press, Ithaca, NY, 1979).
[16] G. Ódor and N. Menyhárd, Phys. Rev. E **73**, 036130 (2006).
[17] M. J. de Oliveira, Phys. Rev. E **71**, 016112 (2005).
[18] S.-G. Lee and S. B. Lee, Phys. Rev. E **77**, 021113 (2008).
[19] S. Lübeck and P. C. Heger, Phys. Rev. E **68**, 056102 (2003).
[20] S. B. Lee and S.-G. Lee, Phys. Rev. E **78**, 040103(R) (2008).
[21] S. Lübeck, Phys. Rev. E **64**, 016123 (2001).
[22] A. Vespignani, R. Dickman, M. A. Munoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998); see also R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **57**, 5095 (1998).
[23] S. Lübeck and A. Misra, Eur. Phys. J. B **26**, 75 (2002).