

Staircase polygons and recurrent lattice walks

M. L. Glasser

Department of Physics and Clarkson Institute for Statistical Physics, Clarkson University, Potsdam, New York 13699-5820

E. Montaldi

Dipartimento di Fisica, Università di Milano, Via Celoria 16, 20133 Milano, Italy

(Received 22 July 1993)

In this paper we derive a direct relationship between the staircase-polygon-generating function Z_d of Guttman and Prellberg [Phys. Rev. E **47**, R2233 (1993)] and the generating function for recurrent lattice walks P_d for the simple (hyper-) cubic lattice in all dimensions d . A recursion formula is obtained for the Z_d with respect to dimension, which leads to a simplified derivation of Guttman and Prellberg's result for $d=3$, avoiding the use of the Heun function, and a derivation of their formula for $d=4$ from an integral representation is given in the Appendix.

PACS number(s): 64.60.Ak, 36.20.Ey, 75.40.Cx

INTRODUCTION

In an interesting recent paper [1], Guttman and Prellberg investigated the generating function for staircase polynomials on a d -dimensional hypercubic lattice ($d=1,2,3,\dots$). For $d=3,4$, due to the occurrence of the same Heun function as in Joyce's [2] treatment of the lattice Green functions for $d=3$, they suggested that there might be a relationship between the generating function

$$Z_d(x^2) = \sum_{k_1, \dots, k_d=0}^{\infty} \left[\begin{matrix} k_1 + \dots + k_d \\ k_1, \dots, k_d \end{matrix} \right] x^{2(k_1 + \dots + k_d)} \quad (d|x| < 1) \quad (1)$$

for squares of multinomial coefficients and the lattice Green functions

$$P_d(z) = \frac{1}{\pi^d} \int \dots \int_0^\pi \frac{dk_1 \dots dk_d}{1 - \frac{z}{d} [\cos(k_1) + \dots + \cos(k_d)]} \quad (|z| < 1) \quad (2)$$

In this Rapid Communication, we shall show that Z_d and P_d are in fact equivalent to one another through the Abel transform. In addition to some related observations, we present in the Appendix a direct evaluation of Z_d , which avoids the introduction of Heun functions.

INTEGRAL REPRESENTATIONS AND THE ABEL EQUATION

By noting the identities

$$(m!)^2 = \int_0^\infty t (t/2)^{2m} K_0(t) dt, \quad (3)$$

$$\frac{1}{z} = \int_0^\infty e^{-sz} ds, \quad (4)$$

and the representations for the modified Bessel function

$$I_0(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{(n!)^2} = \frac{1}{\pi} \int_0^\pi e^{z \cos(\theta)} d\theta, \quad (5)$$

we obtain the integral representations

$$Z_d(x^2) = \int_0^\infty t K_0(t) I_0^d(xt) dt \quad (6a)$$

and

$$P_d(z) = \int_0^\infty e^{-sz} I_0^d(sz/d) ds. \quad (6b)$$

Therefore, since

$$e^{-s} = \frac{2s}{\pi} \int_0^1 \frac{du}{u^2 \sqrt{1-u^2}} K_0(s/u), \quad (7)$$

we find the relation

$$P_d(z) = \frac{2}{\pi} \int_0^1 \frac{Z_d(u^2 z^2/d^2)}{\sqrt{1-u^2}} du. \quad (8)$$

For the parameter ranges indicated, all the integrals are absolutely convergent; sums and integrals can therefore be interchanged freely.

Equation (8) is easily found to be equivalent to the Abel integral equation [3]

$$P_d(dz^{1/2}) - 1 = \frac{1}{\pi} \int_0^z \frac{\xi^{-1/2} (Z_d(\xi) - 1)}{\sqrt{z-\xi}} d\xi \quad (z \geq 0), \quad (9)$$

where the 1's have been inserted so that the unknown function $\xi^{-1/2} (Z_d(\xi) - 1)$ vanishes as $\xi \rightarrow 0+$. The inversion of (9) yields

$$Z_d(x^2) = \frac{d}{dx} \left[x \int_0^1 \frac{u P_d(du)}{\sqrt{1-u^2}} du \right]. \quad (10)$$

Equations (8) and (10) prove that Z_d and P_d are simply different representations for the same mathematical object.

As a simple example, since [1] $Z_2(x^2) = (1-4x^2)^{-1/2}$, (8) gives

$$P_2(z) = \frac{2}{\pi} \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-z^2u^2)}} = \frac{2}{\pi} K(z), \quad (11)$$

while, since

$$\int_0^1 \frac{uK(au)}{\sqrt{1-u^2}} du = \frac{\pi}{2a} \sin^{-1} a, \quad (12)$$

(10) is also trivially satisfied. The application of this procedure to the known results for Z_3 and P_3 produces two rather fascinating integrations over the product of two

complete elliptic integrals.

Because of the equivalence of Z_d and P_d , new representations for the lattice Green functions are obtained in terms of the combinatorial quantities $S_n^{(d)}$ investigated by Guttman and Prellberg in [1]. Thus

$$P_d(z) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n! d^{2n}} S_n^{(d)} z^{2n} \quad (13)$$

with

$$S_n^{(d)} = \sum_{m_1=0}^n \sum_{m_2=0}^{m_1} \cdots \sum_{m_{d-1}=0}^{m_{d-2}} \begin{bmatrix} n \\ m_1 \end{bmatrix}^2 \cdots \begin{bmatrix} m_{d-3} \\ m_{d-2} \end{bmatrix}^2 \begin{bmatrix} m_{d-2} \\ m_{d-1} \end{bmatrix}^2. \quad (14)$$

It is also interesting to observe that

$$Z_{d+1}(x^2) = \frac{1}{\pi(1-x^2)} \int_0^\pi Z_d \left[\frac{x^2}{(1-x^2)^2} (1+2x \cos \theta + x^2) \right] d\theta. \quad (15)$$

This follows at once from the formulas $I_0^d(z) = \sum_n [(z/2)^{2n}/(n!)^2] S_n^{(d)}$, Eq. (6a), which gives

$$Z_{d+1}(x^2) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(n!)^2} S_n^{(d)} \int_0^\infty t^{2n+1} K_0(t) I_0(xt) dt, \quad (16)$$

and, in terms of Gauss's function,

$$\begin{aligned} \int_0^\infty t^{2n+1} K_0(t) I_0(xt) dt &= 4^n (n!)^2 {}_2F_1(n+1, n+1; 1; x^2) \\ &= 4^n (n!)^2 (1-x^2)^{-(2n+1)} \frac{1}{\pi} \\ &\quad \times \int_0^\pi (1+2x \cos \theta + x^2)^n d\theta. \end{aligned} \quad (17)$$

For instance, since $Z_2(x^2) = (1-4x^2)^{-1/2}$, we have

$$\begin{aligned} Z_3(x^2) &= \frac{1}{\pi} \int_0^\pi (1-6x^2-3x^4-8x^3 \cos \theta)^{-1/2} d\theta \\ &= (1+x)^{-3/2} (1-3x)^{-1/2} \\ &\quad \times {}_2F_1 \left[\frac{1}{2}, \frac{1}{2}; 1; -\frac{16x^3}{(1+x)^3(1-3x)} \right] \end{aligned} \quad (18)$$

to be compared with [1] $Z_3(x^2) = F(\frac{1}{9}, -\frac{1}{3}; 1, 1, 1, 1; x^2)$.

Furthermore, the Fuchsian differential equations given for $Z_d(x)$ with $d=2, 3, 4, 5, 6$ might be useful for writing down similar equations for the lattice Green functions in these dimensions.

ACKNOWLEDGMENT

The first author thanks Professor A. J. Guttman for a copy of his paper prior to publication and for his helpful remarks.

APPENDIX

Here we include an alternative evaluation of Z_4 , which avoids the Heun function.

For $d=4$, (6a) gives

$$Z_4(x^2) = \int_0^\infty t K_0(t) I_0^4(xt) dt. \quad (A1)$$

By using Watson's series

$$I_0^2(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{(n!)^3} x^{2n}, \quad (A2)$$

and the formula

$$\int_0^\infty t^{2m+1} K_0(t) dt = 4^m (m!)^2, \quad (A3)$$

(A1) becomes

$$Z_4(x^2) = \sum_{m,n=0}^{\infty} (4x^2)^{m+n} \frac{(\frac{1}{2})_m (\frac{1}{2})_n}{(m!n!)^3} [(m+n)!]^2. \quad (A4)$$

This expression is easily rearranged into

$$Z_4(x^2) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!} (4x^2)^k {}_4F_3 \left[\begin{matrix} -k, -k, -k, \frac{1}{2}; \\ 1, 1, \frac{1}{2}-k; \end{matrix} 1 \right]. \quad (A5)$$

Also [4]

$$\begin{aligned} &{}_4F_3 \left[\begin{matrix} -k, -k, -k, \frac{1}{2}; \\ 1, 1, \frac{1}{2}-k; \end{matrix} 1 \right] \\ &= 4^k \frac{(\frac{1}{2})_k}{k!} {}_4F_3 \left[\begin{matrix} -k/2, (1-k)/2, -k, \frac{1}{2}; \\ 1, \frac{1}{2}-k, \frac{1}{2}-k; \end{matrix} 1 \right], \end{aligned} \quad (A6)$$

so by reexpanding the hypergeometric function,

$$Z_4(x^2) = \sum_{n=0}^{\infty} (4x^2)^n \sum_{k=0}^{[n/2]} \left[\frac{(\frac{1}{2})_n}{(\frac{1}{2}-n)_k} \right]^2 4^{n-k} \frac{(-n)_k (\frac{1}{2})_k (-n/2)_k [(1-n)/2]_k 4^k}{(n!)^2 (k!)^2}. \quad (A7)$$

Again, by applying the identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} F(n, k) = \sum_{n,m=0}^{\infty} F(n+2k, k), \quad (\text{A8})$$

we obtain

$$\begin{aligned} Z_4(x^2) &= \sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_m (\frac{1}{2})_m}{(n+m)! n! (m!)^2} (-1)^m (4x^2)^{n+2m} 4^{n+m} \\ &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!} (16x^2)^n \sum_{k=0}^n \frac{(-1)^k (\frac{1}{2})_k}{(n-k)! (k!)^2} (4x^2)^k \\ &= \sum_{n=0}^{\infty} \left[\frac{(\frac{1}{2})_n}{n!} \right]^2 {}_2F_1\left(-n, \frac{1}{2}; 1; 4x^2\right) (16x^2)^n. \end{aligned} \quad (\text{A9})$$

Now, by using standard hypergeometric identities, this can be reexpressed as an Appell hypergeometric function of two variables [5]:

$$\begin{aligned} Z_4(x^2) &= \sum_{n=0}^{\infty} \left[\frac{(\frac{1}{2})_n}{n!} \right]^2 [4x^2(1+\sqrt{1-4x^2})^2]^n {}_2F_1\left[-n, -n; 1; \frac{1-\sqrt{1-4x^2}}{1+\sqrt{1-4x^2}}\right]^2 \\ &= F_4\left(\frac{1}{2}, \frac{1}{2}; 1, 1; 4x^2(1-\sqrt{1-4x^2})^2, 4x^2(1+\sqrt{1-4x^2})^2\right). \end{aligned} \quad (\text{A10})$$

Finally, by writing

$$\begin{aligned} k_+^2(1-k_-^2) &= 4x^2(1-\sqrt{1-4x^2})^2, \\ k_-^2(1-k_+^2) &= 4x^2(1+\sqrt{1-4x^2})^2, \end{aligned} \quad (\text{A11})$$

so

$$k_{\pm}^2 = \frac{1}{2} \pm 8x^2 \sqrt{1-4x^2} - \frac{1}{2}(1-8x^2) \sqrt{1-16x^2}, \quad (\text{A12})$$

and applying Bailey's theorem, we arrive at Guttman and Prellberg's formula

$$\begin{aligned} Z_4(x^2) &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k_+^2\right) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k_-^2\right) \\ &= \frac{4}{\pi^2} K(k_+) K(k_-). \end{aligned} \quad (\text{A13})$$

The key point of this evaluation is (A6). We mention

here a further ${}_4F_3$ transformation

$$\begin{aligned} {}_4F_3\left[\begin{matrix} -k, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \\ \frac{1}{2}-k, \frac{1}{2}-k, \frac{1}{2}-k; \end{matrix} 1\right] \\ = \frac{k!}{(\frac{1}{2})_k} {}_4F_3\left[\begin{matrix} -k, -k/2, (1-k)/2, \frac{1}{2}; \\ \frac{1}{2}-k, \frac{1}{2}-k, 1; \end{matrix} 1\right] \end{aligned} \quad (\text{A14})$$

[[4], p.65, Eq. (2.4.2.3)], which, together with (A6), gives

$$\begin{aligned} {}_4F_3\left[\begin{matrix} -k, -k, -k, \frac{1}{2}; \\ 1, 1, \frac{1}{2}-k; \end{matrix} 1\right] \\ = 4^k \left[\frac{(\frac{1}{2})_k}{k!} \right]^2 {}_4F_3\left[\begin{matrix} -k, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \\ \frac{1}{2}-k, \frac{1}{2}-k, \frac{1}{2}-k; \end{matrix} 1\right]. \end{aligned} \quad (\text{A15})$$

As an application, consider the fcc lattice Green function

$$I(t) = \frac{1}{\pi^3} \int \int \int_0^\pi \frac{1}{1-t(\cos x \cos y + \cos x \cos z + \cos y \cos z)} dx dy dz. \quad (\text{A16})$$

The integration over z gives

$$\frac{\pi}{1+t} (1-4u \cos^2 x / 2 \cos^2 y / 2)^{-1/2} (1-4u \sin^2 x / 2 \sin^2 y / 2)^{-1/2}, \quad (\text{A17})$$

where $u = t/(1+t)$. Therefore, we have

$$\begin{aligned} \frac{\pi^2}{4} (1+t) I(t) &= \int_0^{\pi/2} \int dx dy (1-4u \cos^2 x \cos^2 y)^{-1/2} (1-4u \sin^2 x \sin^2 y)^{-1/2} \\ &= \frac{\pi^2}{4} \sum_{r,s=0}^{\infty} \frac{(\frac{1}{2})_r (\frac{1}{2})_s}{r! s!} (4u)^{4+s} \left[\frac{(\frac{1}{2})_r (\frac{1}{2})_s}{(r+s)!} \right]^2 \\ &= \frac{\pi^2}{4} \sum_{k=0}^{\infty} \left[\frac{(\frac{1}{2})_k}{k!} \right]^3 (4u)^k {}_4F_3\left[\begin{matrix} -k, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \\ \frac{1}{2}-k, \frac{1}{2}-k, \frac{1}{2}-k; \end{matrix} 1\right], \end{aligned} \quad (\text{A18})$$

that is, recalling (A5) and (A15),

$$I(t) = \frac{1}{1+t} Z_4(u/4) = \frac{1}{1+t} \frac{4}{\pi^2} K(k_+) K(k_-), \quad (\text{A19})$$

where

$$k_{\pm}^2 = \frac{1}{2} \pm 2t(1+t)^{-3/2} - \frac{1}{2}(1-t)(1-3t)^{1/2}(1+t)^{-3/2} \quad (\text{A20})$$

[[2] Eq. (7.3)]. Thus, within the framework of the generalized hypergeometric series, the fcc lattice Green function appears to be intimately related to Z_4 .

-
- [1] A. J. Guttmann and T. Prellberg, Phys. Rev. E **47**, R2233 (1993).
 [2] G. S. Joyce, Philos. Trans. R. Soc. London **273**, 583 (1973).
 [3] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, 4th ed. (Cambridge University Press, Cambridge, 1963), p. 229.

- [4] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, 1966), p. 62, Eq. (2.4.1.2).
 [5] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge University Press, Cambridge, 1935), p. 81.