Time-series thresholding and avalanche dynamics in networks of the critical branching process

Lei Tao [1,2] Sheng-Jun Wang [1,*] and Zi-Gang Huang [3]

[1] *School of Physics and Information Technology, Shaanxi Normal University, Xi'an 710119, People's Republic of China*

[2] *Cognitive and Computational Neuroscience Lab, Psychology Department, Université de Montréal, Montreal H2V 2S9, Quebec, Canada*

[3] *School of Life Science and Technology, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China*

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Avalanche sizes and durations following power-law distributions are observed in many systems and are considered hallmarks of criticality. Time-series thresholding is a commonly used method to define avalanches, but this method is controversial. In this study, we use the time-series thresholding method to define avalanches and investigate the statistical properties of avalanches in the Kinouchi-Copelli (KC) model. We consider two definitions of avalanche size, (1) total area above threshold value reference and (2) total area above zero, and then analyze the size and duration distributions. Our results show that while avalanche size and duration obey a power-law distribution, the exponents of size and duration differ from those of the critical branching process. The scaling relation holds for avalanches defined by the first method but fails for those avalanches tdefined by the second. This study provides new insights into avalanche definition methods and avalanche distribution exponents in continuous time-series.

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I. INTRODUCTION

Scale-free distributions, or *power laws*, have been observed in biological, physical, and natural systems [1–10]. In the past two decades, a series of experimental and theoretical studies found that neuronal activity cascades, or "neuronal avalanches," exhibit power-law size distributions in experiments *in vitro* and *in vivo* [5,11–17]. These results provide evidence for the hypothesis that neural systems operate near a critical point. In particular, experimental studies have shown that the distribution exponent of avalanche size and duration are 3/2 and 2.0, respectively [5]. These exponents are the same as exponents of the critical branching process (BP) [5,18].

A branching process is a model characterized by a tree-like structure [18,19]. However, the realistic brain or complex multi-unit physical systems are embedded in a complex network, and the activity is constrained by the interaction between the elements [20]. The KC model combines the structure of complex networks and the dynamics of the branching process. The model receives external stimuli which are described by the Poisson process, thus the network activity is continuous and avalanches are absent. The KC model is used to understand the collective behavior of the neuron system [21,22], and has received significant attention since it was proposed [22–26]. Using the KC model, it has been explained that the sensitivity and dynamic range of neural networks are maximized at the critical point of a nonequilibrium phase transition [22]. Moreover, in the limit of weak perturbation, the KC model has a separation of timescales between external stimuli and network activity (TS-KC), and avalanches were obtained [27]. In TS-KC, the avalanche size follows power-law distributions and the distribution exponents are consistent with the critical branching process [18,27].

In experiments and the excitatory-inhibitory (E-I) balanced neural networks, the definition of neuronal avalanche differs from the branching process or physical model [5,28–30]. In physical models of self-organized criticality, e.g., the sandpile model, the activity is intermittent and avalanches are separated by silent periods [2]. The avalanche size is defined as the total number of sand grains that collapsed during the avalanche. In contrast, in experiments and E-I neural networks, the network activity is continuous, but the analysis of avalanches is still needed, as power-law behavior is the fingerprint of criticality. The avalanche is defined using the modified methods in which the thresholding of spatial-temporal signals or time series is needed [28–30]. The value of the threshold was selected according to the level of the network activity [28]. The start and end of an avalanche are determined by a time step with the network activity below a given threshold [28–30]. The avalanche size is defined as the area covered between the network activity curve and the threshold reference line (S) or the total area under the network activity curve (Σ) [28–30]. It was shown that, when excitatory and inhibitory are balanced, the power-law distribution of the avalanche is obtained and the exponent is 3/2 which is identical to the critical branching process [28]. Moreover, at the onset of the oscillation of neural networks, the distribution exponent of avalanche size varies continuously in a region of parameter space [30], and is not necessarily equal to that of the directed percolation universality class to which the critical branching process belongs.

The power-law distribution of "avalanche" also appeared in the time series signal of random walks (RW) [31]. In the RW process, mimicking the number of spikes, the avalanche size is defined by the area between the threshold reference line and walk trajectory curve or the overall integral of the time series

*Contact author: wangshjun@snnu.edu.cn

during the avalanche. It has been shown that the avalanche size and duration obey a power-law distribution [31], and the exponent of avalanche size Σ is 1.5 the same as BP. However, it was shown that the value of the exponent is due to the transient effect rather than criticality.

The thresholding method, as far as we know, has not been examined in a classic model of criticality, although the method has been used widely. As the KC model has an established criticality, it is an interesting question: When avalanches are defined by the time series thresholding method, do the statistical results of avalanches in the KC model exhibit the same features of the critical branching process, or are distinguishable from random walks?

In this article, we contribute an additional piece of information about the neuronal avalanches. We calculate the avalanche size and duration using the thresholding time series method and then investigate the effect of this method on the statistical results of avalanches. We show that the avalanche statistical results of the critical KC model differ from the results of the TS-KC model, critical branching process, and random walk. For the avalanche S, the distribution exponents of avalanche size and duration are smaller than the results obtained from the TS-KC model and the critical branching process. Yet, the avalanche exponents satisfy the critical scaling relation. For avalanche size Σ, the distribution exponent of avalanche size is equal to the exponent of the TS-KC model and the critical branching process, while the avalanche distribution exponents do not satisfy the critical scaling relation.

II. MODEL

We use an Erdös-Rényi undirected random network consisting of N nodes. $NK/2$ edges are assigned between randomly selected nodes in the network. K is the average number of neighbors of a node in the network, and is referred to as the average degree. In the KC model, each node has n states: $s_i = 0$, is the resting state, $s_i = 1$ is the active state, $s_i = 2, \dots, n-1$ are the refractory state. At each discrete time step, the state of nodes is updated as follows: (i) If the state of node i is 0 at time $t-1$, it can be activated by an external stimulus, and the state changes to 1 at time t. This external stimulus is modeled by a Poisson process with rate r. This means that the probability of a node receiving a stimulus is $p_r = 1 - \exp(-r\Delta t)$ at each time step [22]. In the KC model Δt is the time step, i.e., $\Delta t = 1$. (ii) If the state of node i at time t is 0 and the state of its neighbor j at time $t-1$ is 1, node i is activated by its neighbor j at time t with probability P_{ij}. P_{ij} is a random variable with uniform distribution in the interval $[0, P_{\max}]$, where $P_{\max} = 2\sigma K$ and is determined by the branching ratio σ. When the value of P_{ij} is selected, it is fixed in simulations. (iii) If $s_i \neq 0$, and node i is neither activated by an external stimulus nor by its active neighbor at time t. Its state is iterated according to the following rules: $s_i(t) = s_i(t-1) + 1$. In the networks with $n = 2$, the critical point is well characterized by the largest eigenvalue λ of the network adjacency matrix P_{ij}, and the network is critical at $\lambda = 1$ [24,25,32,33]. In this simulation, we use parameters $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1$, $r = 1/N$.

To define avalanches, we compute the instantaneous activity densities $\rho(t)$ [15,22], which is the fraction of active nodes

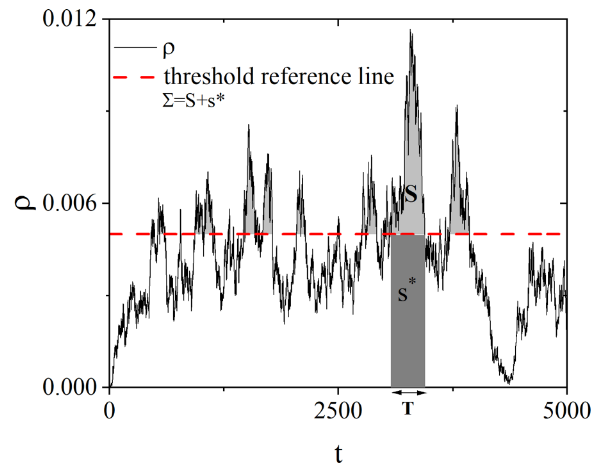

FIG. 1. The time series of average instantaneous activity density in the KC model. Red line is the arbitrarily fixed threshold employed to define avalanches. Parameter value: $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1.00$, $r = 1/N$.

at time t,

$$\rho(t) = \frac{1}{N}\sum_i \delta_{s_i,1}. \tag{1}$$

According to Refs. [28–31], when avalanches are defined by thresholding time series, there are two definition methods of avalanche size which were used in different systems. The avalanche size can be measured by the area covered between the variable and the threshold reference line, and is denoted by S. The avalanche size can also be measured by the area Σ which is the sum of S and the area s^* of the rectangle between zero and the threshold [28,31,33]. In some experimental studies, the avalanche size is calculated by the size Σ [28–30,33]. In Fig. 1, we show how avalanche duration T and size S and Σ are defined for KC model. In both cases, the avalanche duration T is defined in the same way: the number of time steps for which $\rho(t)$ is above a given threshold.

In critical states, the relationship between the average avalanche size $\langle S\rangle$ and the avalanche duration T satisfies $\langle S\rangle \sim T^{\gamma}$, and its exponent γ satisfies the following relationship:

$$\gamma = \frac{(\alpha - 1)}{(\tau - 1)}, \tag{2}$$

where α is the distribution exponent of avalanche duration and τ is the distribution exponent of the avalanche size [17,31,34,35]. The scaling relation, i.e., Eq. (2), is a stronger criterion for criticality because it would not be satisfied by noncritical models [35]. The scaling relation has been employed by several investigations [17,34,35].

III. RESULTS

A. The distribution of avalanche size and duration

Before calculating the avalanche distribution, we first select a threshold. The value of the threshold affects avalanche size and duration. When the avalanche is defined by thresholding the time series, a high threshold can divide large

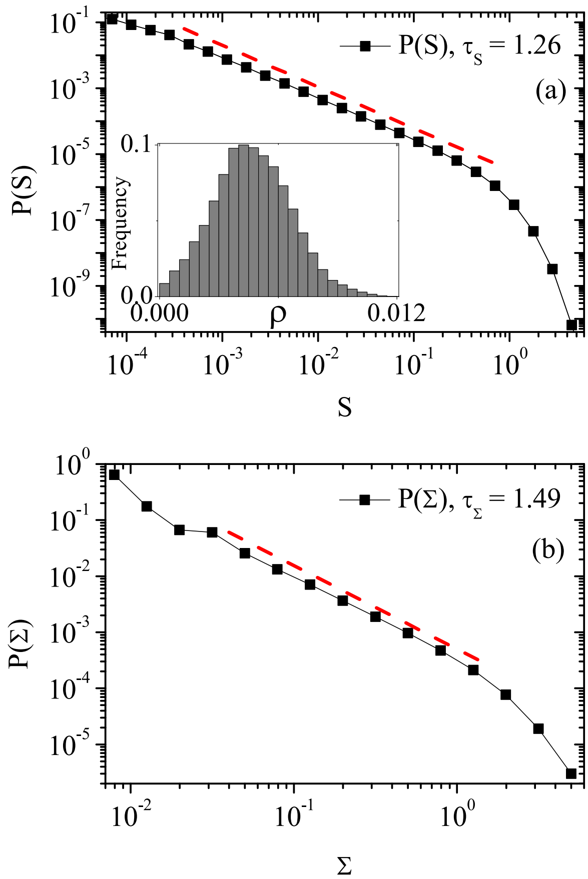

FIG. 2. Distributions of avalanche size. (a) The avalanche size S. Inset: The histogram of the average instantaneous activity density ρ. (b) The avalanche size Σ. Dashed lines: the fitting results by doing the least-squares fitting with the power-law forms. Parameter value: $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1.00$, $r = 1/N$, $V_{\rm thre} = 0.005$.

avalanches into small avalanches, while a low threshold can calculate the background signals or noise from network activities as avalanches [29]. Here, for the critical KC model of an external stimulus $r = 1/N$, the average instantaneous activity density is selected as the threshold, which is $\overline{\rho} = 0.005$.

To reduce fluctuations in the tail of distributions, we use logarithmic binning [36] whose width exponentially increases. In Fig. 2, one can note that the left of the distribution is not smooth. This is because the method of logarithmic binning affects the number of events that fall inside the bin [37]. In this model, the minimum event size depends on the time step (Δt) and the minimum instantaneous activity density ($1/N$). When time is discrete and N is finite, the value of an event size is approximately discrete, that is $\Delta t(1/N - \theta)$ for S and $\Delta t(1/N)$ for Σ. If the bin widths are not integer multiples of the minimum event size, the inappropriate division of the bin boundaries will cause errors in $P(S)$ or $P(\Sigma)$. When the length of bin is large enough, the error can be ignored. Therefore, there will be fluctuations on the left side of the distribution. When estimating the exponent of the distribution, we discard the point at the far left of the distribution.

We use the least-square method to fit the distributions and estimate the distribution exponents [28]. The red dashed line in Fig. 2 has the exponent obtained from the fitting and indicates the range of data for fitting. The distribution of avalanche size S and Σ are shown in Figs. 2(a) and 2(b), respectively. The avalanche sizes S and Σ obey a power-law distribution. The distribution exponents of avalanche size are shown in Table I. The exponent of $P(S)$ is $\tau_S = 1.26$, it is smaller than 3/2, i.e., the exponent of the BP, and the simulation result of the TS-KC model [18,27,31,38]. In fact, by mapping BP into a demographic random walk (DRW), it was shown that the small external stimulus decreases the distribution exponent of BP [38]. The decreased value of τ_S in KC model qualitatively agrees with the expectation proposed in [38]. However, we cannot obtain the exponent in the raw simulation data without the thresholding, because under the stimulus $r = 1/N$ on average one of the N neurons is activated by external stimulus at every step and the activity of the KC model is continuous. Although the stimulus is very weak in the view of studying the stimulus-response relation, it is too strong to study the ideal avalanche without thresholding.

The exponent of S is also different from a random walk. In a random walk whose time series symmetrically excurses around a mean value [31], the exponent τ_S is equal to 4/3. The symmetry of the fluctuation of the time series is necessary for the value of the distribution exponent [31]. Here, with the thresholding, we obtained that the value of $\tau_S = 1.26$ is smaller than that of random walks. The difference is due to the fact that the time series of the KC model is distinct from the symmetrical random excursion around a mean value. As shown in Fig. 1, above the threshold, the curve $\rho(t)$ is narrow peaks, while below the threshold, the curve excurses laxly. The histogram of ρ is shown in the inset of Fig. 2(a), and it is positive skewed rather than symmetric. The network activity of the KC model is asymmetric excursions around the mean value.

The distribution of avalanche size Σ is qualitatively the same as the distribution of the avalanche size S. $P(\Sigma)$ obeys a power-law distribution in the middle of the curve and turns to rapid decay on the right side, as shown in Fig. 2(b). Compared with the avalanche size S, the avalanche size Σ is larger, i.e., $\Sigma = S + s^*$ (shown in Fig. 1). The increase in avalanche size changes the range of avalanche size. It also increases the width of the bin, leading to the frequency having a larger fluctuation at the left side. In the E-I balanced network, when the thresholding method is used to define avalanches, the left side of the avalanche size distribution deviates from the power-law function, which is manifested as the left side of the avalanche size Σ distribution curve is not a straight line or the curve is not smooth in the log-log coordinates [11,28–30].

The quantitative description of the avalanche size Σ distribution is also done by estimating the distribution exponent. In the KC model with $r = 1/N$, the distribution exponent of avalanche size Σ is $\tau_\Sigma = 1.49$. Using the definition Σ, the distribution exponent of avalanche size reproduces the result of the distribution exponent of avalanche size of the BP, the TS-KC model [27,31,38], but differs from the theoretical predication of DRW, i.e., exponent decreases as external perturbation presents [38]. However, it has been shown that the results obtained using Σ may be misleading [31]. In the OU process, when the threshold is large, the exponent of the OU process is closer to 1.50 although the process is irrelevant to

TABLE I. Avalanche exponents for the critical KC model ($V_{\mathrm{thre}} = 0.005$), critical branching process (BP) [18,38], the TS-KC [27], and the random walk [31]. ± denotes the standard error of the distribution exponent.

Type	$P(T)$	$P(S)$	$\langle S\rangle \sim T$	$P(\Sigma)$	$\langle \Sigma\rangle \sim T$
0.005	1.43 ± 0.003	1.26 ± 0.002	1.64 ± 0.001	1.49 ± 0.02	1.08 ± 0.008
BP	2.00[a]	1.50[a]	2.00[a]		
TS-KC		1.50[b]			
RW	3/2[c]	4/3[c]	3/2[c]		
OU	3/2[c]			3/2[c]	1.00[c]

[a]Reference [38].
[b]Reference [27].
[c]Reference [31].

the criticality [31]. Here, the exponent τ_Σ is identical to that of the OU process.

Each avalanche has a duration (the number of time steps) that reflects the temporal properties of the evolution of avalanches. Figure 3 shows the distribution of avalanche duration $P(T)$. When the threshold is equal to the average value of $\rho(t)$, the avalanche duration obeys power-law with an exponential cutoff. The distribution exponent of avalanche duration is $\alpha = 1.43$. The exponent of the KC model is smaller than the duration distribution exponents of the BP. In BP, the exponent is determined by the statistics of the branch length, and its value is $\alpha = 2.0$ [18,31,38]. In the theory of mapping BP into DRW, the exponent is decreased by weak external perturbation [38]. The exponent α of the KC model qualitatively agrees with the predication proposed in [38]. Since the distribution exponent of avalanche duration was not studied in the literature [27], we do not compare this result with TS-KC.

We calculate the average avalanche size $\langle S\rangle$ as a function of a given duration T. Figure 4(a) shows that the relation between the average avalanche size and the duration also obeys a power-law function. Fitting the curve with $\langle S\rangle \sim T^{\gamma_S}$ yields the exponents $\gamma_S = 1.64$. Obviously, the value of the exponent is different from the theoretical expectations of BP. The exponents γ of the BP is 2.0. The DRW theory predicted that the exponent γ of BP is independent of external perturbation and remains fixed at 2.0 [38]. The KC model under the thresholding does not possess the exponents as the same as the driven BP. In symmetric random processes, e.g., the random walks or OU process, the exponent $\gamma_S = 1.50$ [31,38]. So the time series of network activity of KC model is also different from the symmetric random process.

Then, we further examine the relation between distribution exponents of avalanches using the critical scaling relation. For the definition of avalanche S, using the value of α and τ, we obtain $(\alpha - 1)/(\tau_S - 1) = 1.65 \approx \gamma_S$. Using S to measure the avalanche, the critical scaling relation is satisfied.

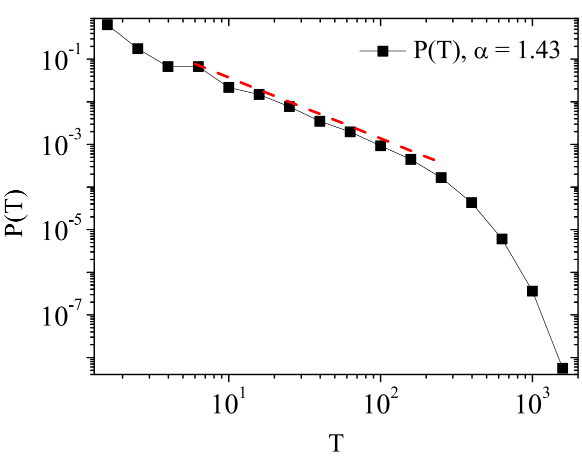

FIG. 3. Distributions of avalanche duration T. Dashed lines: the fitting results by doing the least-squares fitting with the power-law forms. Parameter value: $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1.00$, $r = 1/N$, $V_{\mathrm{thre}} = 0.005$.

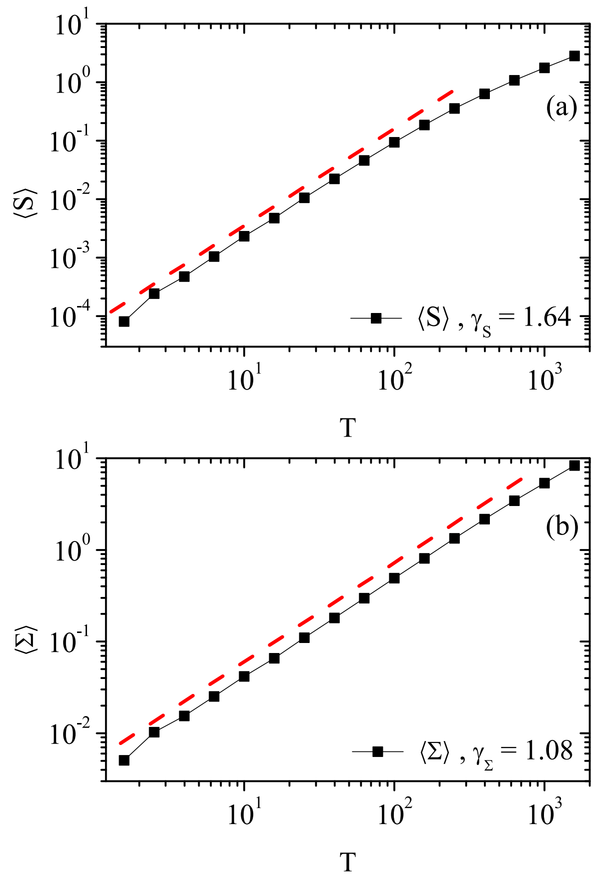

FIG. 4. The average size $\langle S\rangle$ and $\langle\Sigma\rangle$ for a given duration T. (a) The avalanche size S. (b) The avalanche size Σ. Dashed lines: the fitting result obtained by the least-squares fitting with the power-law forms. Parameter value: $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1.00$, $r = 1/N$, $V_{\mathrm{thre}} = 0.005$.

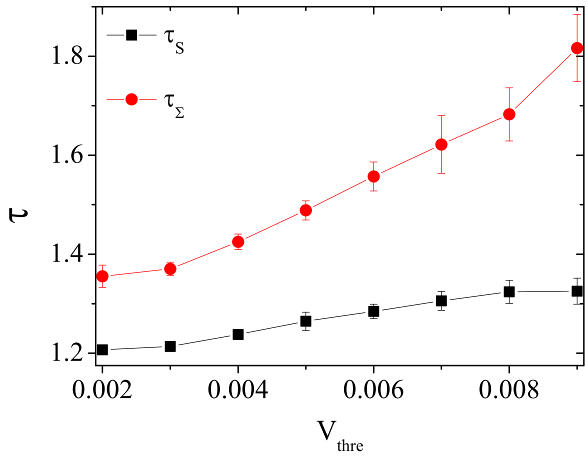

FIG. 5. In the KC model, the distribution exponent of avalanche size S and Σ changes with the threshold. The black square is the distribution exponent of S and the red circle is the distribution exponent of Σ. Parameter value: $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1.00$, $r = 1/N$, $V_{\text{thre}} = 0.002, 0.003, \ldots, 0.009$.

For the avalanche size Σ, the average size scales as $\Sigma \sim T^{\gamma_S}$, as shown in Fig. 4(b). Through fitting the curve, we obtain $\gamma_\Sigma = 1.08$, which is close to the exponent $\gamma = 1.00$ for the OU process [31]. Using Σ to measure the avalanche, critical exponents (see Table I) do not satisfy the critical scaling relation, i.e., $(\alpha - 1)/(\tau_\Sigma - 1) = 0.43/0.49 = 0.86 \leqslant \gamma_\Sigma$.

B. Effect of threshold on avalanche distribution

The effect of the threshold on avalanche exponents in a symmetrical random process has been studied in Ref. [31]. For avalanche size S, the exponents are independent of the value of the threshold. For size Σ, exponents changes with the value of the threshold. Here, we calculate the distribution of avalanche size S and Σ and their distribution exponents for different thresholds.

For different thresholds, the distribution exponent of avalanches is shown in Fig. 5. The increase in the threshold leads to an increase in the distribution exponent of avalanche size. The exponent τ_S of $P(S)$ increases from 1.20 to 1.30. The exponent is always smaller than 1.5. Although $\alpha = 1.5$ is obtained in the critical branching process or the TS-KC model [27,31,38], it cannot be reached in the thresholding time series of the KC model. When the threshold is smaller (less than 0.002), the avalanche size distribution deviates from a power-law function, and the system dynamics fail to exhibit intermittent activity. In the time series of symmetric excursion around a mean value, the exponent τ_S is independent of the threshold [31]. However, here the exponent τ_S changes with the threshold. So the network activity of the KC model is different from a random excursion around a mean value.

Figure 5 shows that the exponent of $P(\Sigma)$ increases from 1.30 to 1.80 as the threshold increases. Only when the threshold value is close to the average instantaneous activity density, the distribution exponent τ_Σ is equal to 1.50, i.e., the exponent of BP. The distribution exponent of avalanche size Σ is sensitive to the threshold, the same as the result observed in random walks [31]. The mechanism is that the area s^* affects the scaling behavior of the avalanche size Σ. As the value of the threshold increases, the contribution of s^* becomes stronger, then the effective exponent is changed.

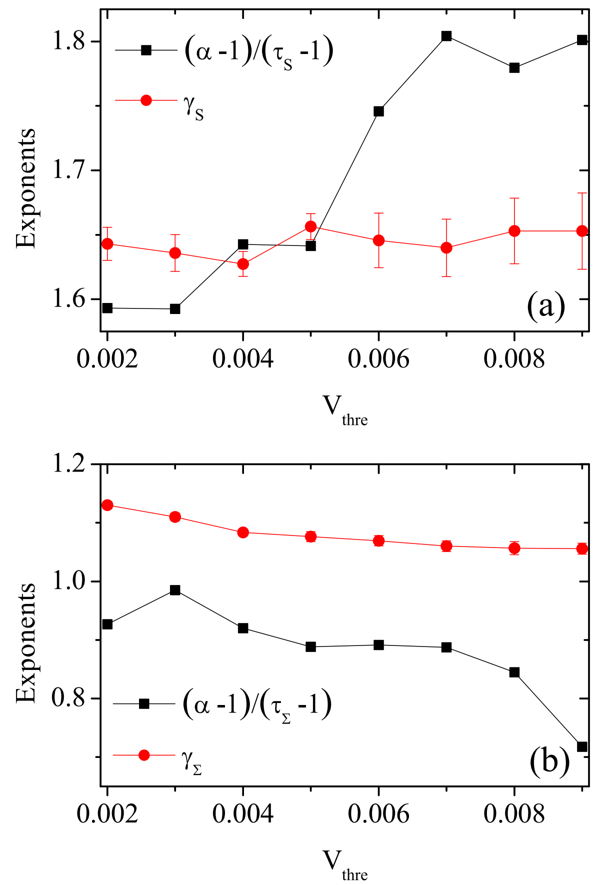

FIG. 6. Two sides of scaling relation versus the value of the threshold. (a) The avalanche size S. (b) The avalanche size Σ. Parameter value: $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1.00$, $r = 1/N$, $V_{\text{thre}} = 0.005$.

Figures 6(a) and 6(b) show the left- and right-hand sides of Eq. (2) as a function of the threshold. For the size S, the curves of $(\alpha - 1)/(\tau_S - 1)$ and γ_S intersect at the average instantaneous activity density. However, the curve $(\alpha - 1)/(\tau_\Sigma - 1)$ and γ_Σ never intersect, that is, the distribution exponent of avalanche size Σ does not satisfy the critical scaling relation, even when the threshold is changed. The critical KC model with $r = 1/N$ exhibits significant differences from E-I networks and random walks in avalanche dynamics. In E-I networks and random processes, when the avalanche size is defined as the Σ, the exponents of avalanche size and duration satisfy the critical scaling relation [30,31]. However, the critical KC model does not show this behavior.

C. Avalanche shape

Besides scale-free distribution and critical scaling relations, a further signature of criticality is the existence of universal scaling functions that capture the systems' dynamics at different scales [4,35,39–42]. We calculate the number of active nodes $N_a(t, T)$ at each time in each avalanche, T is avalanche duration and t is the time step of the dynamical evolution. Then, averaging over multiple avalanches with the same duration gives the typical shape of the avalanche shape.

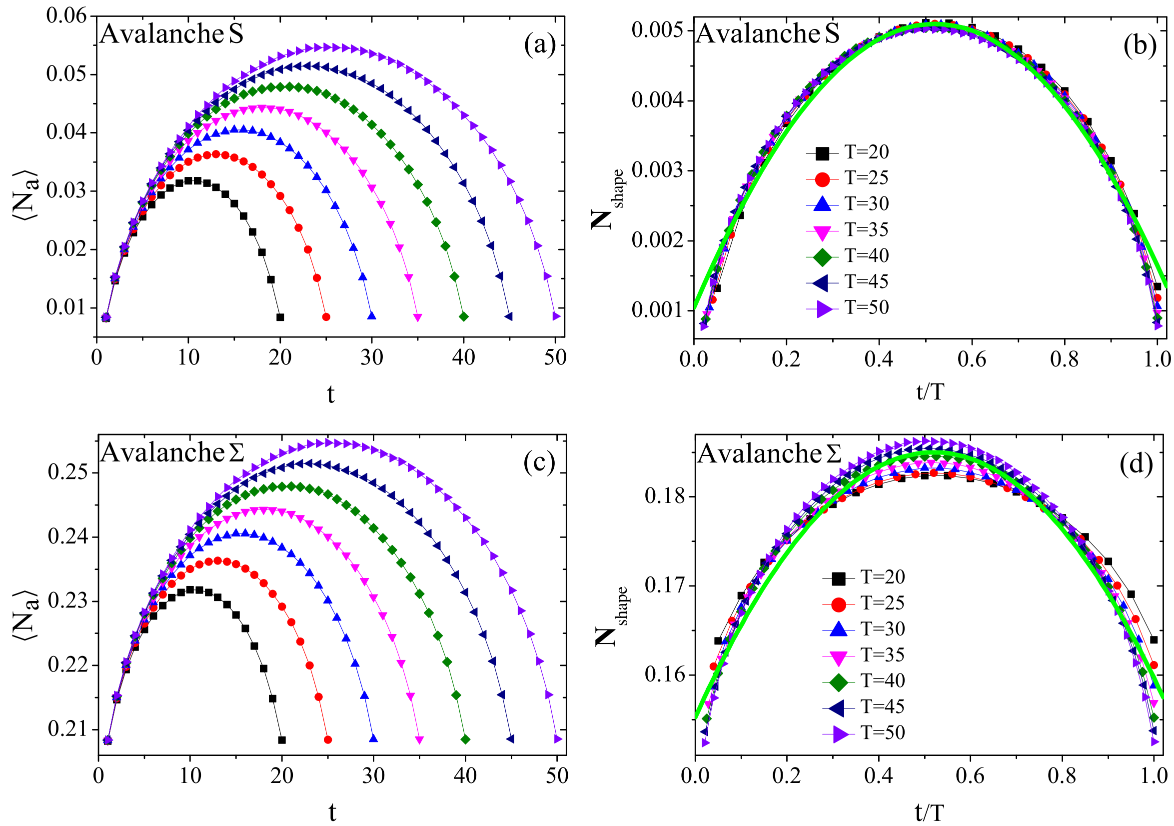

FIG. 7. At the critical regime, we calculate the relation between the avalanche duration and average avalanche size in the KC model. For different durations T, the average avalanche shape collapses onto a universal profile $\mathbf{N}_{\text{shape}} = \langle N_a \rangle / T^{(\gamma-1)}$. The green dashed line is a parabolic fit to the data. (a) and (b) is the average avalanche shape of S for different durations. (c) and (d) is the average avalanche shape of Σ for different durations. Parameter value: $N = 10^5$, $K = 10$, $n = 2$, $\lambda = 1.00$, $r = 1/N$, $V_{\text{thre}} = 0.005$.

The scaling form is

$$\langle N_a(t, T) \rangle = T^{(\gamma-1)} \mathbf{N}_{\text{shape}} \left(\frac{t}{T} \right), \tag{3}$$

where $\mathbf{N}_{\text{shape}}$ is the scale function that determines the avalanche shape [4]. The best collapse provides an independent evaluation of the exponent γ [43].

In Figs. 7(a) and 7(c), we plot the average number of active nodes $\langle N_a(t, T) \rangle$ during an avalanche as the function of the rescale time t. The profiles of avalanche S and Σ are nearly parabolic for all durations. According to Eq. (3), the avalanche profiles are rescaled, as shown in Figs. 7(b) and 7(d). The exponents γ used in collapse are from Table I. For avalanche S, the scaling behavior of the avalanche size and avalanche duration can be rescaled by the exponent $\gamma_S = 1.64$, and all rescaled curves collapse onto a universal profile [Fig. 7(b)], and the avalanche S does satisfy scale invariance. For avalanche Σ, the avalanche shapes of different durations do not collapse into a single curve when $\gamma_\Sigma = 1.08$ is used as the collapse exponent [Fig. 7(d)]. So, the avalanche Σ does not satisfy scale invariance.

IV. CONCLUSION

In this article, we use the thresholding time series method to calculate avalanches in the KC model and study the effect of the thresholding method on the statistical results of avalanches. We analyzed avalanche size and duration distributions using two definitions: S, the area above the threshold reference line, and Σ, the total area under the activity curve. Under both definitions, the distribution of avalanche size and duration were found to follow power-law functions. The distribution exponents of avalanche S satisfy the critical scaling relation, while the exponents of avalanche Σ do not satisfy the critical scaling relation. Additionally, the scaling properties of avalanche shapes revealed that S aligns with scale invariance, but Σ does not.

In the KC model, when the threshold method is used to calculate avalanches, the avalanche exponents differ from the BP and random process. Although the dynamics of the KC model are similar to those of BP and TS-KC, the distribution exponent of the avalanche size S and the relation between avalanche Σ exponents cannot reproduce the results of the BP and TS-KC process. In addition, the exponents of avalanche S and Σ are smaller than the DRW and OU process and the relation between avalanche Σ exponents does not satisfy the critical scaling relation.

Our analysis emphasizes the potential pitfalls of the thresholding method and the need for caution in interpreting results based solely on power-law distributions. It also reflects the importance of evaluating avalanche exponents and their relationships to accurately identify the critical properties of complex systems. Our study provides insights into the calculation of avalanches using threshold methods.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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