Factorization of solutions to generalized Feynman-Kac equations for jump-diffusion models

Victor E. Gluzberg [1,*] and Yuri A. Katz [2,†]
[1] *PTC, PTC drive, 1, Haifa 3490002, Israel*
[2] *Katz School of Science and Health, Yeshiva University, 205 Lexington Avenue, New York, New York 10016, USA*

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Standard affine jump-diffusion models treat shocks as increments to the system's state, whereas conventional stochastic resetting models reduce their impact to a full reset of the system's state. Neither model is adequate when diffusion along one of the state components persists, while shocks affect another component. Our study examines a linear system perturbed by two independent additive Markovian components. We demonstrate that in this model the solutions to the generalized Feynman-Kac (GFK) equations describing the expected evolution operator admit an exact factorization to solutions corresponding to separate noise components. Factorizable two-component models are applicable in the real world, particularly in systems where an endogenous background noise persists even when exogenous shocks occur. We show that this decoupling enables a comprehensive description of stylized yield-curve shapes. Furthermore, it allows for the prediction of deviations from exponential Moore's Law trajectories for novel technologies.

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I. INTRODUCTION

Jump-diffusion models have been widely employed to describe systems subject to both continuous fluctuations and discontinuous perturbations. In the standard formulation, jumps are treated as additive increments to the current state [1]. The same assumption also underlies the broader family of Lévy processes, which are characterized by stationary and independent increments. Based on these concepts, affine jump-diffusion (AJD) framework formalized by Duffie *et al.* [2] established an important class of analytically tractable models by imposing an affine structure on drift, diffusion, and jump intensity.

Despite these advances, the application of AJD models is constrained by the assumption that jump increments are identically distributed across all states. Under this assumption, the kernel of the jump term, $f(q|q')$, of the differential Chapman-Kolmogorov equation [3], i.e., the conditional probability distribution function (PDF) of the state q after a jump, given a state q' immediately prior to it, depends only on the difference $(q-q')$: $f(q|q') = h(q-q')$. However, many empirically and theoretically relevant phenomena—most notably ecological resets, catastrophic events, and market crashes—do not increment the existing state, but replace it with a new state only partially dependent on or entirely independent of the previous one. Such phenomena necessitate a fundamentally different modeling approach based on state replacement or resetting rather than simple increments. Under the assumption of Markovian behavior, these models are described by the differential Chapman-Kolmogorov equation with the jump kernel $f(q|q')$ of a general form. In the case opposite to the incremental jump model, the kernel depends only on a postjump state: $f(q|q') = h(q)$. In this model, the postjump state, q, is completely independent on the previous one, q'. When the differential Chapman-Kolmogorov equation incorporating such a jump kernel also retains a diffusion term, the corresponding process is referenced as "stochastic resetting" [4]. The simplest case is resetting with a deterministic kernel distribution: $h(q) = \delta(q - q_r)$, describing a transition to a prescribed resetting state q_r. Stochastic resetting models gained prominence following Evans and Majumdar's [5] demonstration that even a simple state resetting to a fixed point has profound effects on diffusion. Specifically, resetting generates nonequilibrium stationary states and alters large-deviation and fluctuation properties. This framework provides tractable models for intermittent search strategies and extinction-recolonization cycles; see Ref. [4].

Without diffusion, the differential Chapman-Kolmogorov equation with the state replacement kernel $f(q|q') = h(q)$ describes an *uncorrelated* purely discontinuous process (PDP) [6]. Notice that applications of the uncorrelated PDP are tracing back to the 1960s. Burshtein [7] has investigated a relaxation induced in a nonautonomous semiclassical system

$$\frac{d}{dt}x(t) = M[q(t)]x(t), \tag{1.1}$$

with an instantaneous Liouvillian (rate operator) $M[q(t)]$, which depends on a vector of exogenous classical variables $q(t)$ modeled by an uncorrelated PDP. In this model, he derived the integral equation describing the forward conditional expectation of a random realization of the evolution operator (EO) $R(t, t_0)$ that gives the solution of (1.1) as $x(t) = R(t, t_0)x(t_0)$. Subsequently, Burshtein and collaborators generalized this theory to accommodate arbitrary correlation between state variables and applied the formalism across diverse fields including magnetic and optical spectroscopy, chemical and luminescence kinetics. A significant advance came from Korst [8], who derived both forward and backward

*Contact author: vegluzberg@gmail.com
†Contact author: yuri.katz@yu.edu

kinetic equations for the conditional expectations of the EO, for a semiclassical system with $q(t)$ following either diffusion or an arbitrary PDP. Notably, in these early works the "reset" invariably targets equilibrium distributions—such as the distribution of the phase, frequency, or amplitude of laser beams, or molecular orientation and angular momentum [9]—rather than fixed values or nonequilibrium distributions.

In our recent study [6], we have connected works of Burshtein and Korst with the Feynman-Kac formalism. We derived the integrodifferential forward and backward generalized Feynman-Kac (GFK) equations describing an evolution of the respective conditionally averaged EOs for $q(t)$ following the most general continuous-time Markov process. Here, we develop a general framework for solving GFK equations for the evolution operator of the linear system (1.1) when jumps and diffusion (a) act *additively* and (b) are statistically *independent* of each other. Specifically, we consider an instantaneous scalar rate operator

$$r(t) = m + q_d(t) + q_j(t), \tag{1.2}$$

perturbed by the two latent zero-mean fluctuations: a continuous (diffusion) component $q_d(t)$ and a purely discontinuous component $q_j(t)$. By the assumption, the respective conditional single-component PDFs, $P_d(q_d, t|q_{od}, t_o)$ and $P_j(q_j, t|q_{oj}, t_o)$ evolve independently. The central theoretical result is the exact factorization of solutions to GFK equations. This factorization holds for a wide class of models that intersects but is neither contained in nor contains the class of AJD models. It enables construction of new solutions by combining already-available results for purely diffusion and purely discontinuous perturbations, bypassing the need to solve the combined integrodifferential equations directly. Importantly, in the case when $q_j(t)$ is a compound Poisson process, the assumption of statistical independence of $q_d(t)$ and $q_j(t)$ holds necessarily, even if both processes are defined on the same probability space [10].

We illustrate the application of the framework in the three relevant settings:

(1) The Ornstein-Uhlenbeck-Vasicek (OUV) diffusion with compound Poisson jumps characterized by an affine kernel, $f(q, q') = h(q-q')$. This is an example of models lying in the intersection of factorizable and affine classes. The factorization provides a transparent explanation of previously derived results [11] and allows a straightforward generalization to unequal mean-reversion constants in the diffusion and jump components. This generalization may improve calibration to processes with distinct characteristic autocorrelation times of the two components.

(2) The novel non-AJD-type model combining Cox-Ingersoll-Ross (CIR) diffusion model with gamma-distributed uncorrelated PDP and linear mean reversion in both diffusion and jump components. The factorization yields closed-form expressions for conditional and unconditional EO expectations, showing how the interplay of continuous fluctuations and state-replacement shocks generates rich term-structure behavior. When the characteristic relaxation times of these two components differ substantially while their stationary variances are of comparable magnitudes, the conditional expectation of EO reproduces the full repertoire of stylized yield-curve shapes (normal, inverted, flat, humped, inverted-humped) as a function of the latent initial state—even though unconditional expectations over the stationary distribution remain strictly monotonic.

(3) The limiting case of fast-fluctuating stationary diffusion component [12]. In this case the diffusion-related factor of the expected EO reduces to an exponential in time correction, determined by the variance and characteristic autocorrelation time of the diffusion component, without assuming any specific model of it. We apply the method to forecasting the impact of rare, long-lasting and uncorrelated exogenous shocks on the decline of expected production unit cost of emerging technologies. The fast continuous fluctuations of the instantaneous cost decline rate reflect the inherent irregularity of the technological progress, while the exogenous shocks are caused by global geopolitical events which affect the cost, but do not interrupt the technology development. Numerical examples based on recent solar PV cost data show that such shocks can however substantially flatten the expected unit cost trajectory, comparing to the Moore's exponential law (see Ref. [13] and references therein), with the effect controlled primarily by shock variance and frequency rather than background diffusion.

Note the essential and practically relevant difference between common stochastic resetting and the two-component jump-diffusion models with the uncorrelated jump component, used in the applications 2 and 3 above. In the stochastic resetting models, shocks reset the *observed* rate to a value independent of its preshock value, completely discarding the preceding diffusion path. In the two-component models, only the *latent jump component* of the rate is reset, while the diffusion component is not disturbed. Importantly, due to analytical tractability of the uncorrelated PDP model and the factorization of the two-component jump-diffusion model, the latter turns out to be analytically tractable as well. We believe that the two-component jump-diffusion models might find applications in various fields where shocks do not discard the preceding diffusion evolution of the system. Examples may include reliability theory, financial markets, population dynamics, crack propagation, and earthquake dynamics.

The paper is organized as follows. In Sec. II we formulate the central theoretical result of this work: the factorization of the conditional and unconditional expectations of EO for a linear system perturbed by two *additive, statistically independent Markov* noises (leaving some technical details and formal proof for Appendix A). In conclusion to this section, we also clarify relations between the classes of factorizable and affine models. In Sec. III we demonstrate how the GFK equation and the factorization work for a popular jump-diffusion model with mean reversion, which belongs to both classes simultaneously and which was recently solved by a different method [11]. We also demonstrate how easily in the GFK formalism this model can be generalized to account for different mean-reversion constants for the jump and diffusion components. In Sec. IV we introduce and investigate the novel jump-diffusion model, combining the gamma-distributed uncorrelated PDP with the CIR diffusion model. In Sec. V we demonstrate how the factorization works for a combination of an arbitrary fast-fluctuating perturbation and an independent PDP and apply the theory to forecasting an impact of rare exogenous shocks on the expected unit cost of an emerging technology. In

Sec. VI we briefly summarize and discuss our results. Some laborious and auxiliary calculations are provided in Appendices.

II. FACTORIZATION OF THE UNIDIRECTIONAL EXPECTED EO IN A TWO-COMPONENT NOISE MODEL

Generally, the dimensions of the instantaneous rate operator $M(t) = M[q(t)]$ in Eq. (1.1) and of the random vector of exogenous variable $q(t)$ are unrelated. Here, we consider the scalar operator $M[q(t)] = -r[q(t)]$, perturbed by a two-dimensional Markov noise. Let one component of $q(t)$ be modeled by a continuous (diffusion) process $q_d(t)$ and the other by a PDP $q_j(t)$, and let, according to Eq. (1.2),

$$r[q(t)] = m + q_d(t) + q_j(t). \tag{2.1}$$

Here m is the constant mean value of the rate $r[q(t)]$ and the latent components $q_d(t)$ and $q_j(t)$ of the two-dimensional vector $q(t) = [q_d(t), q_j(t)]^{\mathrm{T}}$ are *zero-mean*. Next, assume that, for any $t > t' \geqslant t_o$, $q_d(t)$ depends only on $q_d(t')$ and $q_j(t)$ depends only on $q_j(t')$, i.e., the conditional PDF $P(q, t|q', t')$ is factorized as $P(q, t|q', t') = P_d(q_d, t|q_d^{'}, t')P_j(q_j, t|q_j^{'}, t')$. Then the one-component conditional PDFs $P_d(q_d, t|q_{od}, t_o)$ and $P_j(q_j, t|q_{oj}, t_o)$ obey the independent one-dimensional Fokker-Planck (FP) and Kolmogorov-Feller (KF) equations, respectively. The joint conditional PDF $P(q, t|q_0, t_0)$ in the two-dimensional q-space then necessarily obeys the differential Chapman-Kolmogorov (forward and backward) equations with the following jump kernel $f(q|q')$, jump frequency $\lambda(q)$, drift vector $\mu(q)$ and diffusion tensor $D(q)$:

$$f(q|q') = f_j(q_j|q_j^{'})\delta(q_d - q_d^{'}),\ \lambda(q) = \lambda(q_j), \tag{2.2}$$

$$\mu(q) = \begin{bmatrix} \mu_d(q_d) \\ \mu_j(q_j) \end{bmatrix},\ D(q) = \begin{bmatrix} D_d(q_d) & 0 \\ 0 & 0 \end{bmatrix} \tag{2.3}$$

(see technical details in Appendix A). Let $\bar{R}(q,\ t; q_o, t_0) \equiv \mathbb{E}[R(t, t_0) \mid q(t) = q,\ q(0) = q_0]$ denote the conditional expectation of the EO $R(t, t_0)$ of the system (1.1) for given initial and terminal states. Then the functional $Q(q,\ t; q_o, t_0) \equiv \bar{R}(q,\ t; q_o, t_0)P(q,\ t|q_0, t_0)$ obeys the forward GFK equation (Ref. [6]) of the form

$$\begin{aligned}\frac{\partial}{\partial t}Q(q,\ t; q_o, t_0) = &- r(q)Q(q,\ t; q_o, t_0)\\ &+ L_{q_j}^{(\mathrm{KF})}Q(q,\ t; q_o, t_0)\\ &+ L_{q_d}^{(\mathrm{FP})}Q(q,\ t; q_o, t_0)\end{aligned} \tag{2.4}$$

and the initial condition

$$Q(q, t_0; q_o, t_0) = \delta(q - q_o). \tag{2.5}$$

Here the operators $L_{q_j}^{(\mathrm{KF})}$, $L_{q_d}^{(\mathrm{FP})}$ act on their operand as on a function of q_j or q_d, respectively. In Appendix A we show that the solution to the problems (2.4) and (2.5) has the form

$$Q(q,\ t; q_o, t_0) = e^{m(t_0 - t)}Q_d^0(q_d, t;\ q_{0d}, t_0)Q_j^0(q_j, t;\ q_{0j}, t_0), \tag{2.6}$$

where $Q_d^0(q_d, t;\ q_{0d}, t_0)$, $Q_j^0(q_d, t;\ q_{0j}, t_0)$ are solutions of isolated detrended GFK equations for pure diffusion and pure jump components:

$$\begin{aligned}&\frac{\partial}{\partial t}Q_d^0(q_d, t;\ q_{0d}, t_0) + q_dQ_d^0(q_d, t;\ q_{0d}, t_0)\\ &\quad - L_{q_d}^{(\mathrm{FP})}Q_d^0(q_d, t;\ q_{0d}, t_0) = 0,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\frac{\partial}{\partial t}Q_j^0(q_j, t;\ q_{0j}, t_0) + q_jQ_j^0(q_j, t;\ q_{0j}, t_0)\\ &\quad - L_{q_j}^{(\mathrm{KF})}Q_j^0(q_j, t;\ q_{0j}, t_0) = 0,\end{aligned} \tag{2.8}$$

with the initial conditions

$$\begin{aligned}Q_j^0(q_d, t_0;\ q_{0d}, t_0) &= \delta(q_d - q_{0d}),\\ Q_j^0(q_j, t_0;\ q_{0j}, t_0) &= \delta(q_j - q_{0j}).\end{aligned} \tag{2.9}$$

The factorization (2.6) comprises the central theoretical result of the present work.

The unidirectional conditional expectation $\bar{R}(t; q_0, t_0) \equiv \mathbb{E}[R(t, t_0) \mid q(0) = q_0]$, and the full (unconditional) expectation $\bar{R}(t, t_0) \equiv \mathbb{E}[R(t, t_0)]$ can be found from the functional $Q(q,\ t; q_o, t_0)$ by simple integration over the two-dimensional vector q and subsequent averaging over an initial distribution $\varphi_0(q_0)$ (Ref. [6]):

$$\begin{aligned}\bar{R}(t; q_0, t_0) = \int Q(q,\ t; q_o, t_0)dq,\ \bar{R}(t, t_0)\\ = \int \bar{R}(t; q_0, t_0)\varphi_0(q_0)dq_0.\end{aligned} \tag{2.10}$$

It immediately follows that the unidirectional conditional expectation of EO is also factorized as

$$\bar{R}(t; q_0, t_0) = e^{m(t_0 - t)}\bar{R}_d^0(t;\ q_{0d}, t_0)\bar{R}_j^0(t;\ q_{0j}, t_0), \tag{2.11}$$

where

$$\begin{aligned}\bar{R}_d^0(t;\ q_{0d}, t_0) = \int Q_d^0(q_d, t;\ q_{0d}, t_0)dq_d,\ \bar{R}_j^0(t;\ q_{0j}, t_0)\\ = \int Q_j^0(q_j, t;\ q_{0j}, t_0)dq_j\end{aligned} \tag{2.12}$$

are detrended unidirectional conditional expectations of EO for pure diffusion and pure jump processes, respectively. Finally, the full expectation of EO, according to Eq. (2.10), is

$$\begin{aligned}\bar{R}(t, t_0) = e^{m(t_0 - t)}\int\int \varphi_0(q_{0d},\ q_{0j})\bar{R}_d^0(t;\ q_{0d}, t_0)\\ \times \bar{R}_j^0(t;\ q_{0j}, t_0)dq_{0d}dq_{0j}.\end{aligned} \tag{2.13}$$

These results allow for reuse of established solutions for continuous (diffusion) and purely discontinuous (jump) problems to obtain expectations of the EO for various jump-diffusion models with instantaneous rate operators of the form (2.1). Clearly, if diffusion and jump components are initially statistically independent, i.e., $\varphi_0(q_0) = \varphi_{0d}(q_{0d})\varphi_{0j}(q_{0j})$, then the unconditional expectation of EO factorizes too:

$$\bar{R}(t,t_0) = e^{m(t_0-t)}\bar{R}^0_d(t,t_0)\bar{R}^0_j(t,t_0), \tag{2.14}$$

where

$$\begin{aligned}\bar{R}^0_d(t,t_0) &= \int \varphi_{0d}(q_{0d})\bar{R}^0_d(t;\, q_{0d},t_0)dq_{0d},\\ \bar{R}^0_j(t,t_0) &= \int \varphi_{0j}(q_{0j})\bar{R}^0_j(t;\, q_{0j},t_0)dq_{0j}.\end{aligned} \tag{2.15}$$

In particular, such a factorization of the unconditional expectation of EO takes place for the initial distribution coinciding with the stationary one: $\varphi_0(q_0) = \varphi(q_0) = \varphi_d(q_{0d})\varphi_j(q_{0j})$, where the stationary distributions of the jump and diffusion components satisfy the equations

$$L^{(\mathrm{KF})}_{q_j}\varphi_j(q_j) = 0,\ L^{(\mathrm{FP})}_{q_d}\varphi_d(q_d) = 0. \tag{2.16}$$

Similar to the forward GFK equation (2.4) with the initial condition (2.5), the symmetric factorization also holds for the solution of the backward GFK with the terminal condition, see details in Appendix A. Solution of backward GFK equations is important, in particular, for applications of the formalism to first-exit problems, i.e., for finding a distribution of the time before first exit from a domain $\mathcal{D}$ or/and specific sites of its boundary or complement $\mathcal{D}^c$ where the stochastic trajectory arrives at that time. However, for these applications the separation of the independent latent variables proves to be efficient only in the simplest case of a domain being a direct product $\mathcal{D} = \mathcal{D}_d \times \mathcal{D}_j$ of domains in subspaces of q_d and q_j, i.e., in the two-dimensional case, for a rectangle $a_d < q_d < b_d,\ a_j < q_j < b_j$. The core complication here consists of the necessity to switch from the original factored transition PDF $P(q,t|q_0,t_0)$ to the transition PDF $P_{\mathcal{D}}(q,t|q_0,t_0)$ of the process *killed upon exit* from $\mathcal{D}$. Though $P_{\mathcal{D}}(q,t|q_0,t_0)$ obeys the same differential Chapman-Kolmogorov equations as $P(q,t|q_0,t_0)$, it must satisfy the absorption boundary/external condition $P_{\mathcal{D}}(q,t|q_0,t_0)|_{q_0\in\mathcal{D}^c} = 0$ (for the backward equation). This condition, in general, implies loss of the factorization for $P_{\mathcal{D}}(q,t|q_0,t_0)$, though it can still admit an analytical solution in the case $\mathcal{D} = \mathcal{D}_d \times \mathcal{D}_j$. In more complicated domains, for example, $a < q_d + q_j < b$, the problem may not be analytically tractable. A more detailed treatment of this interesting topic is beyond the scope of the present work and merits a dedicated publication.

An important property of models admitting the factorization of the expected EO is worth to be noted in the context of their possible applications to finance: An existence of equivalent risk-neutral probability measures [10] for both processes q_d and q_j guarantees the existence of such a measure for the combined process too. The formal proof and implications of this fact are beyond the scope of this work.

Prior to moving on to applications, it is important to clarify the relation between the class of models admitting the factorization and the popular class of AJD models, admitting analytical solutions for the conditional expectation of EO and for moment-generating function (MGF) of the instantaneous rate $r[q(t)]$; see Refs. [2,14] and referenced therein. The affine models assume all the coefficients of the differential Chapman-Kolmogorov equation, along with $r(q)$, to have the affine structure:

$$\begin{aligned} r(q) &= r_0 + r_1 q,\quad \mu(q) = \mu_0 + \mu_1 q,\\ \sigma^2(q) &= \sigma_0^2 + \sigma_1^2 q,\quad \lambda(q) = \lambda_0 + \lambda_1 q.\end{aligned} \tag{2.17}$$

Note that our specialization (2.1) is a particular case of the first of these equations, with $r_0 = m$, $r_1 = 1$, and that the general case reduces to Eq. (2.1) by scaling the state q, without loss of generality. Of course, in the case of a multivariate state, all the constants r_1, μ_1, σ_1^2 and λ_1 should be vectors or tensors of the corresponding rank and their multiplication with q should be understood as dot-products or tensor convolutions. Another important assumption of the affine models is that the jump kernel only depends on the difference of its arguments:

$$f(q|q') = h(q - q'). \tag{2.18}$$

The class of affine models is naturally covered by GFK equations. This is demonstrated in the Appendix B. Notice, however, that the sub-class of GFK equations admitting the factorization to independent jump- and diffusion-related factors does not cover the whole class of affine models. Indeed, let us compare the assumptions (2.2) and (2.3) with (2.17) and (2.18). In the case of the two-component process $q(t) = [q_d(t), q_j(t)]^{\mathrm{T}}$, we find the following constraints to be imposed on the functions $h(y)$, $\lambda(q)$ and tensors μ_1, σ_0^2, σ_1^2 to make Eqs. (2.17) and (2.18) consistent with Eqs. (2.2) and (2.3):

$$h(y) = h\big(y_j\big)\delta(y_d),\ \lambda\big(q_d, q_j\big) = \lambda(q_j), \tag{2.19}$$

$$\mu_{1i_1i_2} = 0,\ \text{if } i_1 \neq i_2, \tag{2.20}$$

$$\sigma^2_{0i_1i_2} = 0,\ \text{if } i_1 \neq d \text{ or } i_2 \neq d, \tag{2.21}$$

$$\sigma^2_{1i_1i_2i_3} = 0,\ \text{if } i_1 \neq d \text{ or } i_2 \neq d \text{ or } i_3 \neq d. \tag{2.22}$$

To see that the class of factorizable models does not entirely belong to the class of affine models, it is sufficient to recall that the conditions (2.2) and (2.3) of factorization do not require the kernel $f_j(q_j|q_j^{'})$ to depend only on the difference $q_j - q_j^{'}$. Moreover, these conditions neither require the coefficients of the FP equation for the diffusion component nor the drift of the KF equation for the jump component to have affine or any other specific structure. The only assumption shared by both classes of models is the affine structure of the instantaneous rate, as stated by Eq. (2.1) and the first of Eqs. (2.17). It should be clear that the intersection of these classes of models is not empty, however. Indeed, if the coefficients of the FP equation for the diffusion component and the drift of the KF equation for the jump component do have the affine structure and the jump kernel only depends on the difference of its arguments, then the model is simultaneously affine *and* factorizable. In the next section we consider a simple, yet instructive example of such a model.

Hereafter in this paper, we consider only the time-homogenous case, assuming $t_0 = 0$, and omitting this parameter in the forward GFK equations and their solutions.

III. MARKOVIAN RANDOM WALK WITH DIFFUSION AND MEAN REVERSION

A. Model

Let the two-component state vector q obeys the following stochastic differential equations:

$$\frac{dq_d}{dt} = -\alpha q_d + k\zeta(t), \tag{3.1a}$$

$$\frac{dq_j}{dt} = -\alpha q_j + n(t), \tag{3.1b}$$

where $\zeta(t)$ is the generalized derivative of the standard Wiener process, i.e., the Gaussian white noise, and $n(t)$ is a white shot noise, defined as

$$n(t) = \sum_{i=-\infty}^{\infty} y_i \delta(t - t_i). \tag{3.2}$$

Here, y_i are i.i.d. random variables, obeying a distribution density of jump-sizes $h(y)$, and the sequence of t_i form a homogenous Poisson process with intensity λ. In other words, $n(t)$ is the generalized derivative of a compound Poisson process. Thus, both components experience external stochastic forcing: q_d is perturbed by a Gaussian white noise and q_j by a shot noise, with both relaxing toward 0 with the same mean-reversion constant $\alpha > 0$. If the instantaneous rate $r(t) \equiv r[q(t)]$ is determined by Eq. (2.1), then it obeys the stochastic differential equation

$$\frac{dr}{dt} = -\alpha(r - m) + k\zeta(t) + n(t), \tag{3.3}$$

which serves as the common starting point in many works, including [11]. Note that the independence of the processes $\zeta(t)$ and $n(t)$ necessarily holds, even when both are defined on the same probability space [10].

B. Solution for a given initial rate

Notice, Eqs. (3.1a) and (3.1b) are independent of each other. Hence, in this model, the unidirectional conditional expectation of the EO, $\bar{R}(t; q_0)$, admits the factorization (2.11), where the factors $\bar{R}_d^0(t;\, q_{0d})$, $\bar{R}_j^0(t;\, q_{0j})$ are given by integrals (2.12) and the functionals $Q_d^0(q_d, t;\, q_{0d})$, $Q_j^0(q_j, t;\, q_{0j})$ obey Eqs. (2.7) and (2.8) with the specialized operators $L_{q_d}^{(\mathrm{FP})}$ and $L_{q_j}^{(\mathrm{KF})}$:

$$L_{q_d}^{(\mathrm{FP})} u(q_d) \equiv \alpha \frac{\partial}{\partial q_d}[q_d u(q_d)] + D_d \frac{\partial^2}{\partial {q_d}^2} u(q_d), \tag{3.4}$$

$$L_{q_j}^{(\mathrm{KF})} u(q_j) \equiv \alpha \frac{\partial}{\partial q_j}[q_j u(q_j)] + \lambda\left[\int u(q_j^{'}) h(q_j - q_j^{'}) dq_j^{'} - u(q_j)\right], \tag{3.5}$$

where $D_d = k^2/2$. Therefore, $\bar{R}(t; q_0)$ can be obtained by simply combining the already known results for these operators, i.e., for the OUV model [15–17] and for the model of Markovian random walk with the linear drift [11] (both results were reproduced within the GFK formalism in Ref. [6]). To obtain the detrended conditional expectations of the respective EOs, we only need to drop the common factor e^{-mt} from both solutions and specialize the initial distribution for the given initial state variable, $\varphi_0(q_j) = \delta(q_j - q_{0j})$:

$$\bar{R}_d^0(t;\, q_{0d}) = \exp\left[-iq_{0d}\, p_0(\alpha, t) + A(\alpha, t)t\right], \tag{3.6}$$

$$\bar{R}_j^0(t;\, q_{0j}) = \exp\left[-iq_{0j}\, p_0(\alpha, t) - \lambda t - \lambda \int_0^{p_0(\alpha,t)} \frac{\hat{h}(p)}{i + \alpha p} dp\right]. \tag{3.7}$$

Here,

$$p_0(\alpha, t) \equiv -\frac{i}{\alpha}(1 - e^{-\alpha t}), \tag{3.8}$$

$$A(\alpha, t) \equiv \frac{k^2}{2\alpha^2} - \frac{k^2}{4\alpha^3 t}(1 - e^{-\alpha t})(3 - e^{-\alpha t}), \tag{3.9}$$

and $\hat{h}(p)$ stands for the Fourier transform of $h(y)$ (i.e., the characteristic function of the jump-size distribution, with the opposite argument, $-p$). Both factors exhibit identical dependence on the initial conditions q_{0d} and q_{0j}. Consequently, noting that $q_{0d} + q_{0j} = r(q_0) - m$, the factorization (2.11) yields

$$\bar{R}(t; q_0) = \exp\{-i[r(q_0) - m]p_0(\alpha, t) - [m + \lambda - A(\alpha, t)]t - \lambda\phi(t)\}, \tag{3.10}$$

where we denote

$$\phi(t) \equiv \int_0^{p_0(\alpha,t)} \frac{\hat{h}(p)}{i + \alpha p} dp. \tag{3.11}$$

Notably, the conditional expectation depends on the initial rate $r(q_0)$, but not on the individual initial values of the latent variables q_{d0} and q_{j0}. This enables straightforward derivation of a more useful conditional expectation for a given initial rate r_0 (rather than for a given two-component state q_0), which we denote as $\bar{R}(t|r_0)$. Indeed, considering an initial distribution of the form $\varphi_0(q_0) = \delta(r(q_0) - r_0)\psi_0(s(q_0))$, where $\psi_0(s)$ is an arbitrary PDF of the half-difference $s(q_0) \equiv \frac{q_{d0} - q_{j0}}{2}$, and noting that the Jacobian determinant $|\frac{\partial(r,s)}{\partial(q_d,q_j)}| = 1$, from Eqs. (3.10) and (2.13) we obtain

$$\bar{R}(t|r_0) = \int\int \delta(r - r_0)\psi_0(s)\bar{R}(t; q_0) dr ds = \bar{R}^{(0)}(t|r_0) e^{-\lambda[t + \phi(t)]}, \tag{3.12}$$

where

$$\bar{R}^{(0)}(t|r_0) \equiv \exp\{-i(r_0 - m)p_0(\alpha, t) - [m - A(\alpha, t)]t\}. \tag{3.13}$$

C. Discussion and extension to the stationary initial distribution

As expected, this result coincides with the one obtained by Masoliver *et al.* [11] by a different method. These authors correctly interpret the factor $\bar{R}^{(0)}(t|r_0)$ as the solution for the continuous process in the absence of jumps and hence conclude that the exponential factor in Eq. (3.12) accounts for discontinuities. Note that the initial rate value r_0 enters only the first, but not the second factor. Our derivation suggests a simple explanation of this apparent asymmetry. The "remainder" term, proportional to $(r_0 - m)$, arises from the sum of the initial values of latent state components through averaging in Eq. (3.12), and therefore does not depend on any of them individually. However, the conditional expectation $\bar{R}(t;q_0)$ for a given two-component state q_0, Eq. (3.10), naturally retains the full symmetry of the factorization (2.11).

For the stationary initial distribution, $\varphi_0(q_0)=\varphi_d(q_{0d})\varphi_j(q_{0j})$, where $\varphi_d(q_{0d})$, $\varphi_j(q_{0j})$ satisfy the equations (2.16) with the operators $L^{(\mathrm{FP})}_{q_d}$ and $L^{(\mathrm{KF})}_{q_d}$ defined by (3.4) and (3.5), we have complete factorization of the unconditional expectation of EO, Eq. (2.14). It follows from the results obtained in Ref. [6], that for the corresponding pure diffusion and pure jump models the unconditional expectations are

$$\bar{R}^0_d(t)=\hat{\varphi}_d[p_0(\alpha,\,t)]\exp\left[A(\alpha,t)t\right],\tag{3.14}$$

$$\bar{R}^0_j(t)=\hat{\varphi}_j[p_0(\alpha,\,t)]\exp\{-\lambda[t+\phi(t)]\},\tag{3.15}$$

where Fourier transforms of the stationary distributions $\varphi_d(q_{0d}),\ \varphi_j(q_{0j})$ are

$$\hat{\varphi}_d(p)=\exp\left[-\frac{k^2p^2}{4\alpha}\right],\tag{3.16}$$

$$\begin{aligned}\tilde{\varphi}_j(p)&=\exp\left[\frac{\lambda}{\alpha}\int_0^p\frac{\hat{h}(p')-1}{p'}dp'\right]\\&=\exp\left[-\frac{\lambda}{\alpha}\int_{-\infty}^{\infty}\mathrm{Ein}(ipy)h(y)dy\right],\end{aligned}\tag{3.17}$$

and $\mathrm{Ein}(z)\equiv-\sum_{k=1}^{\infty}\frac{(-z)^k}{k\cdot k!}$ denotes the entire part of the integral exponential function [18,19]. Substituting all these results into Eq. (2.14), after some algebra, we finally obtain the unconditional expectation of EO for the stationary initial distribution:

$$\begin{aligned}\bar{R}(t)=\exp\bigg\{&-mt+\frac{k^2}{2\alpha^2}\left[t-\frac{1-e^{-\alpha t}}{\alpha}\right]\\&-\lambda[t+\phi(t)+\theta(t)]\bigg\},\end{aligned}\tag{3.18}$$

where

$$\theta(t)\equiv\frac{1}{\alpha}\int_{-\infty}^{\infty}\mathrm{Ein}[ip_0(\alpha,t)y]h(y)dy.\tag{3.19}$$

These equations extend the results obtained by Masoliver *et al.* [11].

Note that, although the conditional one-component PDFs $P_d(q_d,t|q_{od},t_o)$ and $P_j(q_j,t|q_{oj},t_o)$ evolve independently by assumption, the two-component model naturally permits statistical dependence between $q_d(t)$ and $q_j(t)$. In general, one can specify an arbitrary initial joint distribution $\varphi_0(q_0)=\varphi_0(q_{0d},\,q_{0j})$, so that until a stationary distribution is established, the components q_d and q_j may remain correlated. For instance, in our derivation of the conditional expectation of EO for a given initial rate r_0, Eq. (3.12), we consider a special case of a strong correlation between the latent state components, where all possible initial states lie on the line $q_{0d}+q_{0j}=r_0-m$.

D. A generalization of the model

A simple, but essential generalization of the model (3.1) can be achieved if we discard the assumption of equal mean-reversion constants for the diffusion and jump state components:

$$\frac{dq_d}{dt}=-\alpha_dq_d+k\zeta(t),\tag{3.20a}$$

$$\frac{dq_j}{dt}=-\alpha_jq_j+n(t).\tag{3.20b}$$

It is easy to verify that, if we substitute α with α_d for all the quantities related to the diffusion component and with α_j for all of the quantities related to the jump component, then all the above results still hold, up to Eq. (3.10), which is replaced by

$$\begin{aligned}\bar{R}(t;q_0)=&\exp\big[-ip_0(\alpha_d,t)q_{d0}-ip_0\big(\alpha_j,t\big)q_{j0}\big]\\&\times\exp\bigg[-[m+\lambda-A(\alpha_d,t)]t\\&-\lambda\int_0^{p_0(\alpha_j,t)}\frac{\hat{h}(p)}{i+\alpha_jp}dp\bigg].\end{aligned}\tag{3.21}$$

For short times relative to both mean-reversion constants, $\alpha_dt\ll1$ and $\alpha_jt\ll1$, the first exponential factor in Eq. (3.21) still reduces to $\exp[-\ (r_0-m)t]$, to first approximation. However, in general, solution (3.21) no longer depends solely on the observable initial rate r_0. To find the conditional expectation for a given r_0, we must average this solution with the initial distribution $\varphi_0(q_0)=\delta(r(q_0)-r_0)\psi_0(s(q_0))$, analogous to the derivation of Eq. (3.12):

$$\begin{aligned}\bar{R}(t|r_0)&=\int\int\delta(r-r_0)\psi_0(s)\bar{R}(t;q_0)drds\\&=\exp\left[i(m-r_0)\frac{p_0(\alpha_d,t)+p_0(\alpha_j,t)}{2}\right]\\&\quad\times\int\exp\{is[p_0(\alpha_j,t)-p_0(\alpha_d,t)]\}\psi_0(s)ds\bar{R}_0(t)\\&=\exp\left[i(m-r_0)\frac{p_0(\alpha_d,t)+p_0(\alpha_j,t)}{2}\right]\\&\quad\times\ \hat{\psi}_0[p_0(\alpha_j,t)-p_0(\alpha_d,t)]\bar{R}_0(t).\end{aligned}\tag{3.22}$$

Here, $\hat{\psi}_0(p)$ is the Fourier transform of $\psi_0(s)$ and $\bar{R}_0(t)$ denotes the second factor in Eq. (3.21)—the conditional expectation for $q_0=0$: $\bar{R}_0(t)\equiv\bar{R}(t;0)$. Clearly, when $\alpha_d=\alpha_j$, Eq. (3.22) reduces to Eq. (3.12). However, if $\alpha_d\neq\alpha_j$, then the solution does depend on the PDF $\psi_0(s)$. In particular, for the stationary initial distribution, $\varphi_0(q_0)=\varphi_d(q_{0d})\varphi_j(q_{0j})$, the conditional initial PDF for a fixed initial rate r_0 is

$$\begin{aligned}\varphi_0(q_0|r_0)&=\psi_0(s(q_0))\\&=\varphi_d\left(\frac{r_0-m}{2}+s(q_0)\right)\varphi_j\left(\frac{r_0-m}{2}-s(q_0)\right),\end{aligned}\tag{3.23}$$

and hence, due to the convolution theorem,

$$\begin{aligned}\bar{R}(t|r_0)=&\exp\left[i(m-r_0)\frac{p_0(\alpha_d,t)+p_0(\alpha_j,t)}{2}\right]\\&\times\ (\hat{\varphi}_d*\hat{\varphi}_j^-)[p_0(\alpha_j,t)-p_0(\alpha_d,t)]\bar{R}_0(t).\end{aligned}\tag{3.24}$$

Here $*$ denotes the convolution, $\hat{\varphi}_j^-(p) \equiv \hat{\varphi}_j(-p)$ is the characteristic function of $\varphi_j(q_{0j})$ and the expression $[p_0(\alpha_j, t) - p_0(\alpha_d, t)]$ indicates the argument of the convolution (not a multiplicative factor). This generalization is useful when the mean reversion acts not in the system's dynamics, but is a part of the perturbations' dynamics. By allowing the diffusion and the jump components to relax with different rates α_d and α_j, one can independently adjust the characteristic autocorrelation times of the continuous fluctuations and the discontinuous shocks.

IV. CIR DIFFUSION AND UNCORRELATED PDP WITH MEAN REVERSION

The factorization of the expected EO for a two-component noise model, under conditions established in Sec. II, may be applicable in the important case of a diffusion combined with an uncorrelated PDP, when

$$f_j(q_j|q_j^{'}) = h(q_j). \tag{4.1}$$

This model describes strong shocks where the pre- and postshock values of the state variable are statistically independent. In absence of a drift, the kernel $h(q_j)$ coincides with the stationary distribution of the jump component: $\varphi_j(q_j) = h(q_j)$. This model, without diffusion and drift, was analytically investigated and quantified for the gamma distribution of the instantaneous rate $r_j = m_j + q_j$,

$$h(q_j) = \frac{1}{\Gamma(\gamma_j)\theta_j^{\gamma_j}} r_j^{\gamma_j - 1} e^{-\frac{r_j}{\theta_j}}, \tag{4.2}$$

where $\theta_j \equiv \frac{m_j}{\gamma_j}$, in Ref. [20]. Such a choice of the distribution $h(q_j)$ is adequate for many real-world phenomena (ranging from a luminescence quenching to financial interest yield curves) where the instantaneous rate remains positive. The model was subsequently generalized to include linear mean-reversion drift

$$\mu_j(q_j) = -\alpha_j q_j \tag{4.3}$$

in Ref. [6]. The conditional expectation of the detrended EO for this model takes the form of an inverse Laplace transform:

$$\bar{R}_j^0(t; q_{0j}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\tilde{\delta}(s + \lambda; q_{0j})}{1 - \lambda \tilde{h}(s + \lambda)} e^{st} ds, \tag{4.4}$$

where $\tilde{\delta}(s; q_0)$ and $\tilde{h}(s)$ are the Laplace transforms of $e^{-ip_0(\alpha_j, t)q_{0j}}$ and $\hat{h}[p_0(\alpha_j, t)]$, respectively, $\hat{h}(p)$ denotes the Fourier transform of the kernel $h(q_j)$ and the integration path should go around all singularities of the integrand on the right.

A. Transition process and the stationary distribution of uncorrelated PDP

The solution (4.4) for the gamma kernel (4.2) was quantified in Ref. [6] only in terms of the long-run limit $\rho_{j\infty} \equiv \lim_{t\to\infty} \rho_j(t; q_{0j})$ of the instantaneous decline rate $\rho_j(t; q_{0j}) \equiv -\frac{d}{dt} \ln \bar{R}_j(t; q_{0j})$. Note that this limit does not depend on the initial condition q_{0j}. To illustrate the transition process $\bar{R}_j(t; q_{0j})$ in the short and medium term and, in particular, the impact of mean reversion, Fig. 1 presents families of the integral decline rate curves $Y_j(t; q_{0j}) \equiv -\frac{1}{t} \ln \bar{R}_j(t; q_{0j})$ for different values of α_j, for the three characteristic initial deviations q_{0j} from the mean rate. These curves are calculated by numeric integration of the Laplace transform $\tilde{h}(s)$ and the inverse Laplace transform (4.4). For the latter the integration path in complex plane is shaped so that to minimize the integrand phase variation. Below, for brevity, we sometimes use the term "yield" for the integral decline rate. Clearly, every yield curve $Y_j(t; q_{0j})$, as well as the instantaneous rate $\rho_j(t; q_{0j})$ curve, originates from the initial rate $m_j + q_{0j}$ at $t = 0$ and tends to the limit rate $\rho_{j\infty}$ upon $t \to \infty$. Variation of the mean-reversion constant α_j has the twofold effect: the higher α_j, the lower the limit rate $\rho_{j\infty}$ and the shorter the transition process. The decrease of the limit rate $\rho_{j\infty}$ with growth of α_j is consistent with the fast-motional limit, because both the variance and the characteristic autocorrelation time of the stationary process decrease with the growth of α_j, as was shown in Ref. [6].

Another important impact of mean reversion on the uncorrelated PDP model is that its stationary distribution $\varphi_j(q_j)$ no longer coincides with the kernel $h(q_j)$. The analytical connection between $\varphi_j(q_j)$ and $h(q_j)$ in general case is derived from the differential Chapman-Kolmogorov equation in Appendix C. Figure 2 illustrates this through a family of the cumulative distribution functions (CDF) of the normalized rate r_j/m_j, found for the gamma kernel (4.2) by the numeric integration.

B. GFK equation for CIR diffusion model and its solution

A natural further generalization of the uncorrelated PDP model is achieved by adding a diffusion component to the noise such that the rate $r[q(t)]$, Eq. (2.1), remains strictly positive. If the sum $r_j = m_j + q_j$ is distributed over a positive semiaxis, we must impose a lower bound on the domain of the diffusion component: $q_d \geqslant -m_d = m_j - m$. Thus, the sum $r_d = m_d + q_d$ also take only positive values. A well-known diffusion model satisfying this restriction is the CIR model [21], which combines the linear mean-reversion drift with the diffusion coefficient that is linear in the domain $r_d > 0$ and vanishes at $r_d = 0$:

$$\mu_d(q_d) = \alpha_d q_d, \ \sigma_d^2(q_d) = k^2(m_d + q_d). \tag{4.5}$$

The forward GFK equation (2.7) for the detrended diffusion-related factor of the functional $Q(q,\ t; q_o)$ then takes the form

$$\begin{aligned}\frac{\partial}{\partial t} Q_d^0(q_d, t; q_{0d}) = &- q_d Q_d^0(q_d, t; q_{0d}) \\ &+ \alpha_d \frac{\partial}{\partial q_d}\left[q_d Q_d^0(q_d, t; q_{0d})\right] \\ &+ \frac{k^2}{2} \frac{\partial^2}{\partial^2 q_d}\left[(m_d + q_d) Q_d^0(q_d, t; q_{0d})\right].\end{aligned} \tag{4.6}$$

Applying the Fourier transform over the state variable q_d to both sides of this equation and the initial condition (2.9) reduce the problem to a first-order PDE and further to an ODE along its characteristics. This allows for straightforward

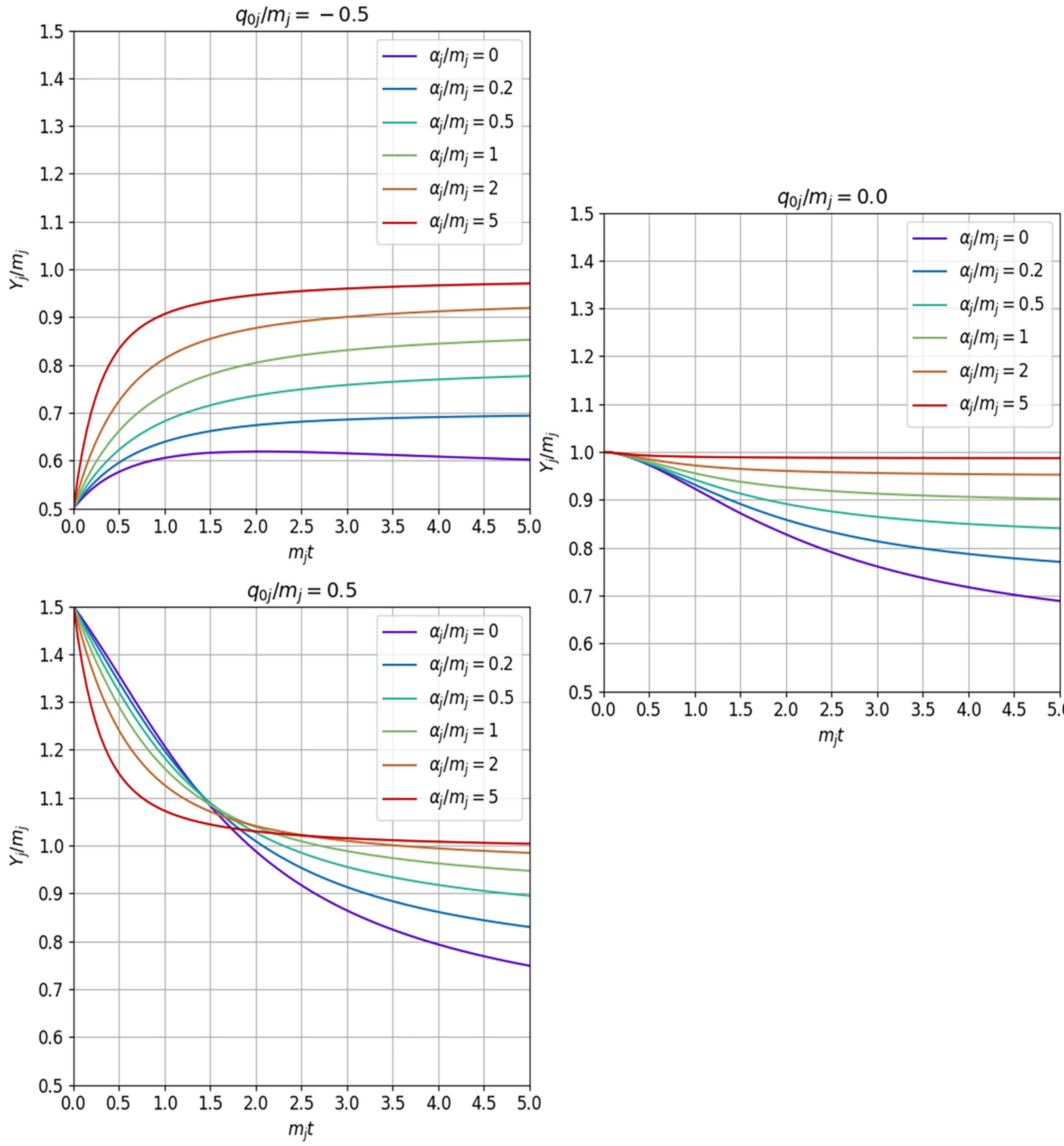

FIG. 1. Normalized yield curves, $Y_j(t; q_{0j})/m_j$, for the gamma-distributed uncorrelated PDP, as function of $m_j t$ with different initial states and mean-reversion constants; $\gamma_j = 1,\ \lambda/m_j = 1$.

determination of the Fourier transform $\hat{Q}_d^0(p, t; q_{0d})$, similar to the OUV model treatment in Ref. [6]. Integration (2.12) then reduces to evaluating the obtained Fourier transform at the zero wave number $p = 0$, yielding the expectation of the (detrended) unidirectional conditional EO:

$$\bar{R}_d^0(t; q_{0d}) = \left[\frac{2\nu}{(\nu + \alpha_d) + (\nu - \alpha_d)e^{-\nu t}}\right]^{\gamma_d} \times \exp\left[-\frac{2(1 - e^{-\nu t})(m_d + q_{0d})}{(\nu + \alpha_d) + (\nu - \alpha_d)e^{-\nu t}} + \left(m_d - \gamma_d \frac{\nu - \alpha_d}{2}\right)t\right], \tag{4.7}$$

where $\nu = \sqrt{\alpha_d^2 + 2k^2}$, and $\gamma_d = \frac{2\alpha_d m_d}{k^2}$ gives the shape parameter of the gamma distribution, that is the stationary solution of the KF equation with the coefficients (4.5). Evidently, the final result for the diffusion-related factor $\bar{R}_d(t; q_{0d}) = \bar{R}_d^0(t; q_{0d})e^{-m_d t}$ coincides, up to notations, with the one, known from Ref. [21] and also derived by a different method in Ref. [17].

C. Combination of CIR diffusion with uncorrelated PDP

Substituting $\bar{R}_j^0(t; q_{0j})$ and $\bar{R}_d^0(t; q_{0d})$ from Eqs. (4.4) and (4.7) into Eqs. (2.11) and (2.13), we obtain the conditional expectation of EO for a given initial state $(q_{0d},\ q_{0j})$ and, respectively, unconditional expectation for an arbitrary initial distribution $\varphi_0(q_{0d},\ q_{0j})$, for the combination of uncorrelated

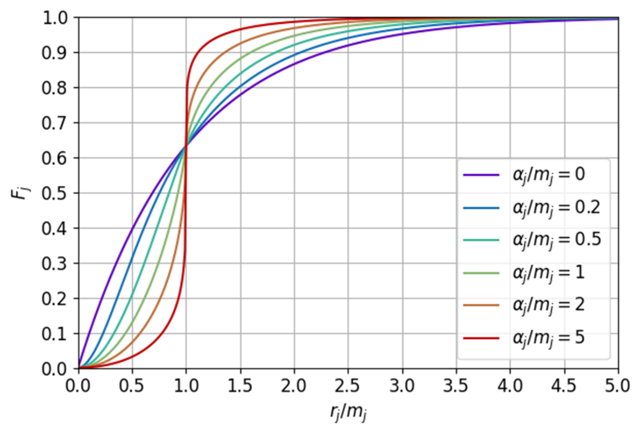

FIG. 2. Family of the stationary CDF of the normalized rate r_j/m_j for the gamma-distributed uncorrelated PDP for different mean-reversion constants; $\gamma_j = 1,\ \lambda/m_j = 1$.

PDP with mean reversion and the CIR diffusion model. Both $\bar{R}_j^0(t; q_{0j})$ and $\bar{R}_d^0(t; q_{0d})$ in Eqs. (4.4) and (4.7) have exponential asymptotic upon $t \to \infty$, with the long-run decline rates $\rho_{j\infty}^0$ and $\rho_{d\infty}^0$ independent of the initial state. Indeed, it is shown in Ref. [6], that $\rho_{j\infty}^0 = \lambda - s_0^{'}$, where $s_0^{'}$ is the rightmost root of the equation

$$1 - \lambda \tilde{h}(s_0^{'}) = 0. \tag{4.8}$$

On the other hand, directly from Eq. (4.7) we have

$$\rho_{d\infty}^0 = \gamma_d \frac{\nu - \alpha_d}{2} - m_d. \tag{4.9}$$

Therefore, the long-run decline rate of the combined process, for any initial condition, is

$$\rho_\infty = m + \rho_{j\infty}^0 + \rho_{d\infty}^0 = \rho_{j\infty} + \rho_{d\infty}, \tag{4.10}$$

where

$$\rho_{j\infty} = \lambda + m_j - s_0^{'}, \quad \rho_{d\infty} = \gamma_d \frac{\nu - \alpha_d}{2} = \frac{2m_d}{1 + \sqrt{1 + 2\frac{k^2}{\alpha_d^2}}}. \tag{4.11}$$

The latter equality follows from the definitions of the parameters ν and γ_d, introduced in Eq. (4.7). Further analysis of the dependence of $\rho_{j\infty}$ on the jump model parameters can be found in Ref. [6]. The expression for $\rho_{d\infty}$, naturally, coincides, up to notations, with that derived and analyzed by Farmer *et al.* [17], who solved the same problem by a different method.

Factorization (2.11) also facilitates investigation of the transition process $\bar{R}(t; q_0)$ in terms of either the instantaneous rate $\rho(t;\, q_0) \equiv -\frac{d}{dt} \ln \bar{R}(t; q_0)$ or the yield $Y(t; q_0) \equiv -\frac{1}{t} \ln \bar{R}(t; q_0)$. In both cases the result reduces to the sum of the same characteristics for the separate jump and diffusion components:

$$\rho(t;\, q_0) = \rho_j(t;\, q_{0j}) + \rho_d(t;\, q_{0d}),\ Y(t; q_0) = Y_j(t; q_{0j}) + Y_d(t; q_{0d}). \tag{4.12}$$

The same applies to the full expectation $\bar{R}(t)$, if the initial diffusion and jump components are statistically independent, Eq. (2.14). In particular, for the stationary initial distribution $\varphi_0(q_0) = \varphi(q_0) = \varphi_d(q_{0d})\varphi_j(q_{0j})$, we have

$$\rho(t) \equiv -\frac{d}{dt} \ln \bar{R}(t) = \rho_j(t) + \rho_d(t),\ Y(t) \equiv -\frac{1}{t} \ln \bar{R}(t) = Y_j(t) + Y_d(t), \tag{4.13}$$

where

$$\rho_j(t) \equiv m_j - \frac{d}{dt} \ln \bar{R}_j^0(t) = m_j - \frac{d}{dt} \ln \int \varphi_j(q_{0j}) \bar{R}_j^0(t;\, q_{0j}) dq_{0j}, \tag{4.14}$$

$$Y_j(t) \equiv m_j - \frac{1}{t} \ln \bar{R}_j^0(t) = m_j - \frac{1}{t} \ln \int \varphi_j(q_{0j}) \bar{R}_j^0(t;\, q_{0j}) dq_{0j}, \tag{4.15}$$

and similarly for $\rho_d(t)$ and $Y_d(t)$. Note that these instantaneous and integral rates characterize the full expectation of the EO for the stationary initial distribution and should be distinguished from *empirically* averaged rates obtained by averaging the observed rates over various initial conditions [22]. Such empirically averaged rates correspond to the rates calculated for the *conditional* EO expectation and *then* averaged over the stationary initial distribution:

$$\bar{\rho}_j(t) \equiv \int \varphi_j(q_{0j}) \rho_j(t;\, q_{0j}) dq_{0j} = m_j - \int \varphi_j(q_{0j}) \frac{d}{dt} \ln \bar{R}_j^0(t;\, q_{0j}) dq_{0j}, \tag{4.16}$$

$$\bar{Y}_j(t) \equiv \int \varphi_j(q_{0j}) Y_j(t;\, q_{0j}) dq_{0j} = m_j - \int \varphi_j(q_{0j}) \frac{1}{t} \ln \bar{R}_j^0(t;\, q_{0j}) dq_{0j}, \tag{4.17}$$

and similarly for the empirically averaged rates of the diffusion component, $\bar{\rho}_d(t)$ and $\bar{Y}_d(t)$. Because of concavity of the logarithmic function, due to Jensen's inequality, we necessarily have $\bar{Y}_j(t) \geqslant Y_j(t)$, $\bar{Y}_d(t) \geqslant Y_d(t)$ and hence $\bar{Y}(t) \geqslant Y(t)$.

D. Numeric examples and discussion

Figure 3 presents examples of the curves $\bar{Y}_j(t)$, $Y_j(t)$, $\bar{Y}_d(t)$, and $Y_d(t)$, obtained by numerical integration, except for the latter, which admits a simple analytical expression (derived in Appendix D). All these curves are monotonically decreasing. Therefore, the same applies to the curves $\bar{Y}(t)$, $Y(t)$ for the combined process.

Results (4.12)–(4.17), implied by factorizations (2.11) and (2.14), lead to an important prediction: When characteristic relaxation times of the conditionally averaged decline rates differ substantially between the diffusion and jump components, the term-structure of the combined rates may reveal nonmonotonicity. This is illustrated by examples of the integral rate curves in Fig. 4. In Appendix E we propose a specific definition of these characteristic times and derive them for the CIR diffusion and the gamma-distributed uncorrelated PDP:

$$T_d = 2\alpha_d^{-1},\ T_j = 2(\lambda + \alpha_j)^{-1}. \tag{4.18}$$

For both models and for any initial state, the tangent line to the rate curve at the origin (depicted by dashed lines in

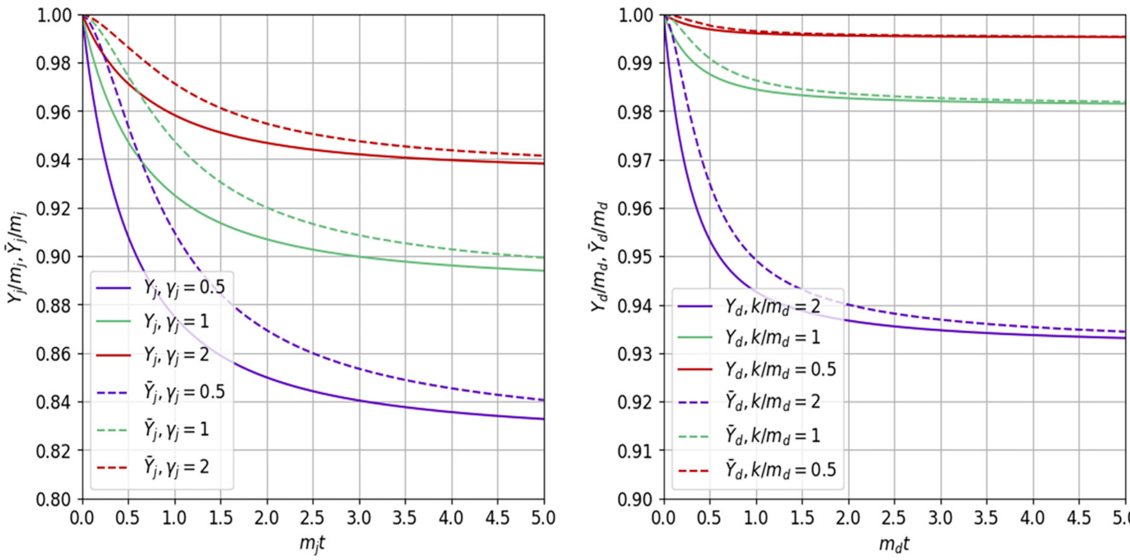

FIG. 3. Yield curves for the stationary initial condition. Left plot: the normalized unconditional expectation, $Y_j(t)/m_j$ and empirical average of conditional expectation, $\bar{Y}_j(t)/m_j$ as functions of $m_j t$, for gamma-distributed uncorrelated PDP with different shape parameter γ_j; $\alpha_j/m_j = 1$, $\lambda/m_j = 1$. Right plot: the normalized unconditional expectation, $Y_d(t)/m_d$ and empirical average of conditional expectation, $\bar{Y}_d(t)/m_d$ as functions of $m_d t$, for the CIR model with different values of k/m_d; $\alpha_d/m_d = 5$.

Fig. 4) intersects the time axis at the point T_d or T_j, respectively. The model parameters in Fig. 4 are chosen so that $T_d/T_j = 1/5$. At the same time, standard deviations of the stationary distributions of the diffusion and jump components are commensurable: $\sigma_\infty^{(d)}/m_d = k/\sqrt{2\alpha_d m_d} \cong 0.447$, $\sigma_\infty^{(j)}/m_j = (\sigma_h/m_j)/(1+2\alpha_j/\lambda) \cong 0.408$ (the formula for $\sigma_\infty^{(j)}$ was derived in Ref. [6]; σ_h denotes the standard deviation of the jump kernel distribution). Consequently, varying the initial state $q_0 = (q_{0d}, q_{0j})$, we obtain various shapes of the rate curves for a combined process—monotonically increasing or decreasing, nearly flat, or revealing distinct maxima or minima at mid-term.

One may assume that the observed yield curves reflect the conditional discount factor expectations for the latent initial state, $(q_{0d}, q_{0j}) \equiv (r_{0d} - m_d, r_{0j} - m_j)$, *as if* it were known, rather than only the observed sum, $r_0 - m = q_{0d} + q_{0j}$. Under this assumption, the jump-diffusion model with an uncorrelated PDP describes the full variety of stylized yield curve shapes—from the normal (upward sloping) to inverted (downward sloping), including flat, humped and inverted humped (see the bottom plot in Fig. 4). However, practical application of this model to financial markets requires a more thorough analysis which is beyond the scope of this work.

V. FAST-MOTIONAL LIMIT FOR A DIFFUSION-DRIVEN COMPONENT

In the previous sections, we use factorization to solve the GFK equations within specific models for both diffusion $q_d(t)$ and jump $q_j(t)$ components. We now demonstrate how this approach can work without specifying a particular diffusion model, under the assumption that the diffusion-driven component represents fast fluctuations. More precisely, we assume the stationarity of $q_d(t)$ and the condition

$$\sigma_d T_d \ll 1, \tag{5.1}$$

where σ_d is the standard deviation of $q_d(t)$ and T_d—its characteristic autocorrelation time, defined as

$$T_d \equiv \frac{1}{C_d(0)} \int_0^\infty C_d(t)dt. \tag{5.2}$$

Here, $C_d(T) \equiv E[q_d(t)q_d(t+T)]$ is the autocovariance function (so that $C_d(0) = \sigma_d^2$). The condition (5.1) enables application of the nonmodel stochastic perturbation theory [12]. To the first nonvanishing order, i.e., with the accuracy $O(\sigma_d T_d)^2$, this theory yields the detrended EO factor determined solely by the variance σ_d^2 and the characteristic time T_d:

$$\bar{R}_d^0(t) = e^{\sigma_d^2 T_d t}. \tag{5.3}$$

When such fast fluctuations are combined with a statistically independent PDP, we can substitute this result directly into Eq. (2.14) and thus, for the stationary initial condition, obtain the fully averaged EO

$$\bar{R}(t) = e^{-r_e t} \bar{R}_j^0(t), \tag{5.4}$$

where $r_e \equiv m - \sigma_d^2 T_d$ is the effective rate accounting for the fast fluctuations around the mean rate m. The negative correction $r_e - m = -\sigma_d^2 T_d$ reflects the Jensen's inequality, applied to the concave function $e^{-\bar{r}_t t}$ of the averaged rate $\bar{r}_t = t^{-1}\int_0^t [m + q(t')]dt'$; see Ref. [20]. Note that, due to the condition (5.1), this correction remains relatively small even when the standard deviation of the fast fluctuations is comparable with the mean rate:

$$\frac{\sigma_d^2 T_d}{m} = \sigma_d T_d \frac{\sigma_d}{m} \ll \frac{\sigma_d}{m}. \tag{5.5}$$

The second, jump-related factor, $\bar{R}_j^0(t)$, in Eq. (5.4) in general case determines a deviation of the expected EO trajectory from the simple exponential decline.

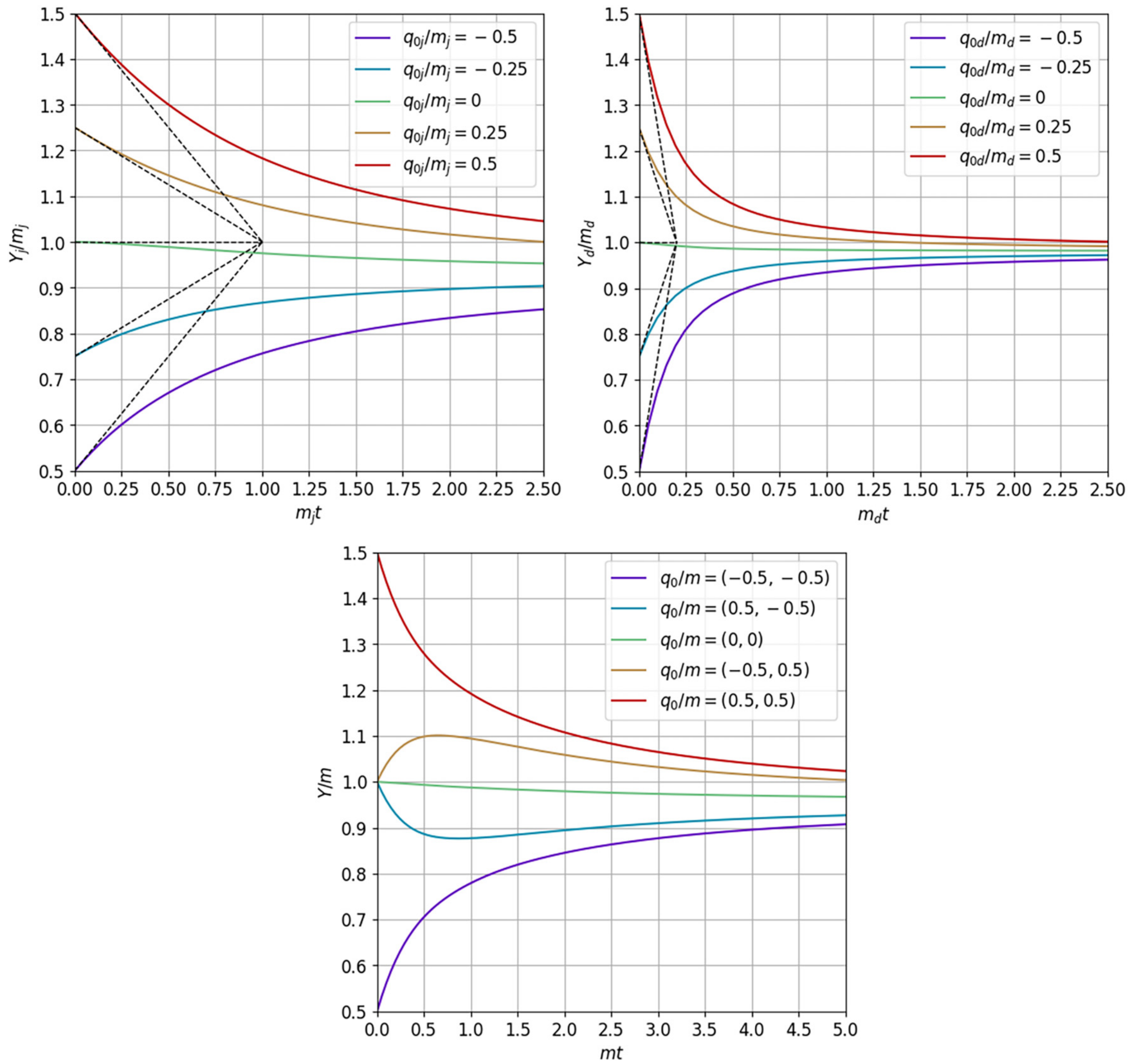

FIG. 4. Comparison of transition processes in the gamma-distributed uncorrelated PDP, diffusion (CIR) and combined jump-diffusion models. Left upper plot: $Y_j(t; q_{0j})/m_j$ as function of $m_j t$, for different initial states $q_{0j}/m_j; \alpha_j/m_j = 1,\ \lambda/m_j = 1, \gamma_j = 2$. Right upper plot: $Y_d(t; q_{0d})/m_d$ as function of $m_d t$, for different initial states $q_{0d}/m_d; \alpha_d/m_d = 10,\ k/m_d = 2$. Bottom plot: $Y(t; q_0)/m$ as function of mt, for the combined model with $m_j/m = m_d/m = 0.5$ and different initial states. See comments in text.

A. Case of gamma-distributed uncorrelated PDP

Consider, for example, an uncorrelated PDP without mean-reversion. Its combination with fast fluctuations enables a practically interesting generalization of the stochastic model of Moore's law suggested in Ref. [13]. This study demonstrated that, in the limits of exponential growth of production by an emerging technology, the expected unit cost declines exponentially (the generalized Moore's law), with the effective rate determined by Jensen's correction: $r_e = m - \sigma_d^2 T_d$. But what happens if, along with the frequent and effectively weak fluctuations, the instant rate undergoes rare, but long-lasting perturbations, due to global exogenous factors, such as changes in government policy or international conflicts? An uncorrelated PDP can model such perturbations. To minimize the number of model parameters, we do not assume the PDP to have mean reversion. Let the support of distribution $h(q_j)$ in Eq. (4.1) be bounded from below by $-m$, preventing long periods of negative rates. We can reuse results from the previous section by assuming the sum $(m + q_j)$ to obey the gamma distribution with the mean m:

$$h(q_j) = \frac{1}{\Gamma(\gamma)\theta^\gamma}(m + q_j)^{\gamma-1} e^{-\frac{m+q_j}{\theta}}. \tag{5.6}$$

Here, $\theta = m/\gamma$ and we drop the subscript j at the distribution shape and scale parameters, since we do not need now to distinguish them from parameters of a distribution of the diffusion component. The detrended EO expectation, Eq. (4.4), in absence of mean reversion, for the stationary initial distribution reduces to (Refs. [20,6])

$$\bar{R}_j^0(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\tilde{h}(s+\lambda)}{1 - \lambda\tilde{h}(s+\lambda)} e^{st} ds. \tag{5.7}$$

For the specific kernel distribution (5.6), we have

$$\tilde{h}(s) = \frac{(s-m)^{\gamma-1}}{\theta^\gamma} e^{\frac{s-m}{\theta}} \Gamma\left(1-\gamma,\ \frac{s-m}{\theta}\right), \tag{5.8}$$

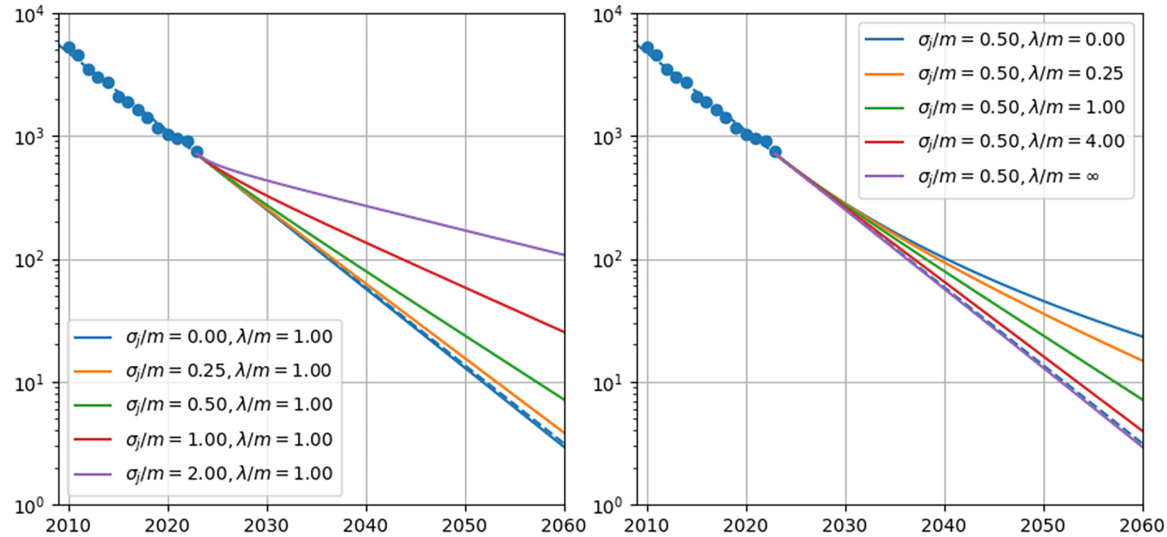

FIG. 5. Expected unit cost in the combined uncorrelated-PDP and fast-fluctuations model (\$/Wdc, 2022 USD), along with the historical data. Dashed line—the log-linear regression of observations; solid blue line in the left pane—fast-fluctuations model without jumps.

where $\Gamma(\nu,\ x)$ is the incomplete gamma function. Substituting this expression into Eq. (5.7) and the latter one into Eq. (5.4), we obtain the expected EO for the stationary initial distribution of the uncorrelated PDP in the presence of independent fast fluctuations of the instantaneous rate.

B. Practical application example

As an example of practical applications of this model, consider a possible impact of strong exogenous shocks on the expected future cost of installed solar PV plants. Our forecast is based on the latest available data from Ref. [23]. Simple analysis of the instantaneous rate fluctuations and their autocorrelation function yields the following estimations (with 80% confidence intervals, estimated under the assumption of normality of fluctuations): $m = (0.150 \pm 0.024)$ year^{-1}, $\sigma_d = (0.069^{+0.063}_{-0.040})$ year^{-1}, $T_d = (0.37^{+0.78}_{-0.37})$ year. Despite a wide confidence interval for the characteristic autocorrelation time, the observations are consistent with assumption (5.1): $\sigma_d T_d < 0.132 * 1.15 = 0.152$, with probability $P \geqslant 0.99$ (assuming the errors of σ_d and T_d to be statistically independent). Jensen's correction is also confidently much less than the mean rate: $\sigma_d^2 T_d \lesssim 0.02$ year^{-1}. The term structure of expected future costs for a set of hypothetical values of the parameters of the jump component process, along with the observations, are presented on Fig. 5. The term structure is calculated according to Eq. (5.4), with the use of the last observation as the initial mean price:

$$\bar{y}(t) = y(2023)e^{-r_e(t-2023)}\bar{R}_j^0(t - 2023). \tag{5.9}$$

Here, t represents the calendar year, and the exponential factor corresponds to the generalized Moore's law. Plots in Fig. 5 are obtained using the same numerical integration method for $\bar{R}_j^0(t)$ as in Ref. [20].

Notice that an impact of shocks vanishes only in two limiting cases: for zero variance of the jump component, $\sigma_j^2/m^2 = m/\gamma = 0$ (i.e., for the infinitely high shape parameter, $\gamma = \infty$) or for infinitely high jump frequency, $\lambda/m = \infty$. In the first case the shifted gamma distribution (5.6) reduces to the deterministic distribution, $\delta(q_j)$. The second case corresponds to infinitely fast fluctuations caused by shocks themselves, with a vanishing Jensen' correction. In all other cases, shocks lead to significantly higher expected costs. The effect is most sensitive to the variance of shocks: if the ratio σ_j/m grows without bound, the decline rate of the expected cost approaches zero so rapidly that, at every finite time, the expected cost tends to its initial value (at 2023). This rate also decreases with a decrease of shocks frequency. However, in the limit $\lambda/m \to 0$, the expected costs at finite times remain below the initial value, declining with time according to a power law. This behavior is known as the static limit; for details, see Ref. [20].

VI. SUMMARY AND DISCUSSION

We apply GFK formalism to prove that for linear systems perturbed by (a) additive and (b) independent diffusion and jump noise components the detrended conditional expectation of EO exactly factorizes to the detrended expectations of EOs for *separate* diffusion and jump perturbations. Moreover, unconditional expectation of EO, averaged over an initial distribution with statistically independent diffusion and jump components, admit the same factorization too. In particular, due to independent evolution of the components, this always works at the *equilibrium*, i.e., for the stationary initial distribution. These results allow for reuse of already known solutions for pure diffusion and pure jump models, to obtain expectations of the EO for jump-diffusion models where both factors act additively.

We clarify the relationship between the factorizable two-component models and the widely used AJD models. The intersection of these classes is nonempty and productive. For example, our approach provides a clear physical explanation for the apparently "asymmetric" factorization observed in recent studies of OUV diffusion in the presence of Markovian random walks (affine jumps). Factorization allows to demonstrate that what appeared to be an asymmetry is simply the natural result of averaging over initial states. It also permits

TABLE I. Comparison of factorizable two-component models with standard AJD and stochastic resetting models.

Feature	Standard AJD	Stochastic resetting	Factorizable two-component noise
Jump kernel constraints	Incremental: $f(q\|q') = h(q-q')$	State replacement: $f(q\|q') = h(q)$	None assumed in general
Drift/diffusion/intensity constraints	Must be affine (linear) functions of state	None assumed in general	None assumed in general
Additiveness of diffusion and jumps	Assumed	Precluded	Assumed
Independence of diffusion and jumps	Implied by Lévy-Itô decomposition	Assumed	Assumed
What persists across a jump	Prejump state, incremented	Nothing, full state discarded	Diffusive component
NESS	Possible (if the intensity depends on state or kernel is asymmetric)	Inevitable	Possible (if the intensity depends on state or kernel is asymmetric)
Solution method(s)	Reduction to Riccati ODEs by exponential-affine transform; method of characteristics for Fourier transform of GFK equation	Reduction to pure diffusion problem by renewal equation; direct solution for specific $h(q)$	Exact factorization into separately solved pure-diffusion and pure-jump EO factors
Canonical examples	Merton [1]; Duffie–Pan–Singleton [2]; CIR with affine jumps	Evans-Majumdar [5]; resetting to a fixed point	OUV + random walk (Sec. III, intersects AJD); CIR + gamma-distributed uncorrelated PDP (Sec. IV, novel)

a straightforward extension to unequal mean-reversion constants in the diffusion and jump components.

Table I compares the class of factorizable two-component models with AJD models and stochastic resetting models, highlighting their similarities and differences.

We illustrate the efficiency of our framework with two specific applications. First, we introduce a novel non-AJD model combining CIR diffusion with gamma-distributed uncorrelated PDP. This model is well suited for systems where the instantaneous rate remains strictly positive but is subject to exogenous shocks, while the diffusion path persists. The jump component reflects an impact of exogenous shocks. In the context of mathematical finance, such shocks to interest rate are related to a fast regime shift, e.g., a central bank is suddenly resetting its target inflation rate, or a market crash occurs due to pandemic. However, the underlying macroeconomic cycle continues evolving from where it was. The factorizable two-component model captures this: the jump component resets, while diffusion remains unaffected. Standard stochastic resetting would discard the entire history of economic development at each event, which contradicts economic reality. Factorization implies that both instantaneous and integral decline rates are additive. When the characteristic relaxation times of the diffusion- and jump-related factors differ significantly, the model can reproduce the full spectrum of stylized yield curve shapes—including normal, "inverted," "humped," and "inverted-humped."

Second, we consider the frequent and effectively weak background fluctuations of the instant rate in the presence of rare exogenous shocks. We applied the nonmodel stochastic perturbation theory to compute the factor corresponding to the fast-fluctuating component. When such fast fluctuations are combined with a statistically independent PDP, we can apply factorization to simplify the solution of the problem. As an example of practical applications of this model, we consider a possible impact of strong exogenous shocks on the future cost of installed solar PV plants, which is expected to follow the stochastic Moore's law; see Ref. [13] and references therein. In this context, the gamma-distributed uncorrelated PDP component represents abrupt policy shifts, technological breakthroughs, or geopolitical events that affect the instantaneous decline rate, without stopping the underlying technological development. Numerical examples based on recent solar PV cost data show that exogenous shocks can cause significant departures from the exponential Moore's Law.

The factorization technique developed in this work admits a straightforward generalization to the case of multivariate components $q_1(t) = (q_{11}(t), \cdots q_{1n_1}(t))^{\mathrm{T}}$, $q_2(t) = (q_{21}(t), \cdots q_{2n_2}(t))^T$ of arbitrary stochastic nature, provided both are Markovian and statistically independent of each other, and the instantaneous rate is additive:

$$r[q(t)] = m + d_1[q_1(t)] + d_2[q_2(t)]. \tag{6.1}$$

Here $q(t) = \left[\begin{smallmatrix} q_1(t) \\ q_2(t) \end{smallmatrix}\right]$ and the functions $d_1(q_1),\ d_2(q_2)$ may be arbitrary (not necessarily linear), but having zero expectations for the stationary distributions of $q_1(t)$ and $q_2(t)$, respectively, so that the sum $d_1[q_1(t)] + d_2[q_2(t)]$ still represents the detrended rate. Clearly, if a multivariate component and the corresponding d function, in their turn, have the same structure, then conditional expectations of EO can be recursively expanded to three or more factors, each obeying an independent GFK equation of a reduced dimension.

Factorizable models can also serve as an initial approximation for cases with a weak nonlinear coupling of noise components like

$$r[q(t)] = m + q_d(t) + q_j(t) + \varepsilon q_d(t)q_j(t). \quad (6.2)$$

Such a coupling can also be interpreted as a weak *dependence* between the components, when one of them is slightly modulated by the other: $q_j(t) + \varepsilon q_d(t)q_j(t) = q_j(t)(1 + \varepsilon q_d(t))$, or, alternatively, $q_d(t) + \varepsilon q_d(t)q_j(t) = q_d(t)(1 + \varepsilon q_j(t))$ (depending on which component is considered "fast" and which "slow"). A weak dependence of another nature emerges if we slightly deviate from the structural constraints (2.2), (2.3), for example, admitting a dependency of the diffusion coefficient on the jump component, or a dependency of the jump intensity on the diffusion component:

$$D_d = D_d(q_d) + \varepsilon D_d^{(1)}(q_j),\ \lambda = \lambda(q_j) + \varepsilon\lambda^{(1)}(q_d). \quad (6.3)$$

It is important to note that the two-dimensional GFK equations for the expected EO still hold in all the models like Eqs. (6.2) and (6.3), for an *arbitrary* value of the parameter ε. Only factorization results do no longer hold for $\varepsilon \neq 0$, because equation (2.4) for such models acquires an additional term of a form like $\varepsilon L_q^{(1)} Q(q,\ t; q_o, t_0)$, where the operator $L_q^{(1)}$ acts on *both* components. For a small parameter, $|\varepsilon| \ll 1$, the approximate solution can be built by the regular perturbation method for linear integrodifferential equations. Seeking the solution in the form

$$Q(q,\ t; q_o, t_0) = e^{m(t_0-t)} Q_d^0(q_d, t; q_{0d}, t_0) Q_j^0(q_j, t; q_{0j}, t_0)$$
$$\times (1 + \varepsilon C_1(q, t;\ q_0, t_0) + \varepsilon^2 C_2(q, t;\ q_0, t_0) + \cdots), \quad (6.4)$$

for the first-order correction it is easy to come to the equation

$$\frac{\partial C_1}{\partial t} = \frac{\left[L_{q_j}^{(\mathrm{KF})}, C_1\right] Q_j^0}{Q_j^0} + \frac{\left[L_{q_d}^{(\mathrm{FP})}, C_1\right] Q_d^0}{Q_d^0} + \frac{L_q^{(1)} Q_j^0 Q_d^0}{Q_j^0 Q_d^0}, \quad (6.5)$$

where all the arguments are omitted for brevity, and $[\ \cdot, C_1]$ denotes the commutator of a specified operator with the operator of multiplying by $C_1(q)$. Initial conditions (2.5) and (2.9) imply the initial conditions $C_n(q, t_0;\ q_0, t_0) = 0$, for all $n = 0,\ 1,\ \cdots$. For some specific forms of the operators $L_q^{(1)}$, $L_{q_j}^{(\mathrm{KF})}$ and $L_{q_d}^{(\mathrm{FP})}$, equation (6.5) with this initial condition might be more tractable than the original exact GFK equation. The same approach is applicable, of course, to the backward GFK equation too.

We believe that the factorization property offers a versatile analytical bridge between the classical AJD tractability, the growing literature on stochastic resetting, and applications requiring robust treatment of rare but transformative exogenous shocks. It should be stressed that the conditions (a) and (b) delimit the scope of these results. Factorization does not survive a jump intensity that depends on the diffusion component, a diffusion coefficient modulated by the jump component, a correlation between the components induced by a common latent factor, or a multiplicative term in the rate. Neither does it extend automatically to functionals other than the exponential one, nor to first-exit problems in domains that are not direct products of subdomains in the separate latent coordinates, as discussed at the end of Sec. II. The present work should therefore be read as an exact solution to GFK equations for expected EO of a linear system perturbed by a Markov process with statistically independent and additive diffusion and jump components. Two of the departures listed above indicate how they might be treated perturbatively.

A remark on empirical implementation is in order. Applying a two-component model of the form of Eq. (2.1) to data might require that the diffusive and the purely discontinuous contributions to the observed rate first be separated. This is a well-studied problem in its own right: realized bi-power variation and the associated jump tests of Barndorff-Nielsen and Shephard [24], and the threshold and truncation estimators reviewed by Aït-Sahalia and Jacod [25], consistently split the quadratic variation of a discretely sampled path into its continuous and jump parts. Once such a separation has been performed, the two components can then be calibrated independently, with univariate estimators already available for pure-diffusion and for pure-jump models. The corresponding limitation is that this two-step procedure is justified only insofar as the independence condition (b) holds. Residual dependence between the extracted components—e.g., a systematic association between shock arrivals and the local variance of the background fluctuations—would invalidate the factorization, so a diagnostic test of that dependence should precede any predictive use of the model. For the solar PV record used above the sample is too short for such a test to be informative, and we regard a systematic empirical study as a natural direction for follow-up work.

The views expressed in this paper are those of the authors and do not necessarily represent the views of PTC or Yeshiva University.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: TWO-COMPONENT JUMP-DIFFUSION MODELS AND FACTORIZATION OF GFK SOLUTIONS

In the assumptions stated in the first paragraph of Sec. II, the one-component conditional PDFs $P_d(q_d, t|q_{od}, t_o)$ and $P_j(q_j, t|q_{oj}, t_o)$ obey the one-dimensional Fokker-Planck (FP) and Kolmogorov-Feller (KF) equations, respectively. The corresponding forward operators, in the time-homogenous case, are defined as follows:

$$L_{q_d}^{(\mathrm{FP})} u(q_d) \equiv -\frac{\partial}{\partial q_d}[\mu_d(q_d)u(q_d)] + \frac{\partial^2}{\partial^2 q_d}[D_d(q_d)u(q_d)] \quad (\mathrm{A1})$$

and

$$\begin{aligned} L_{q_j}^{(\mathrm{KF})} u(q_j) \equiv & -\frac{\partial}{\partial q_j}[\mu_j(q_j)u(q_j)] \\ & + \int \lambda(q_j')f_j(q_j|q_j')u(q_j')dq_j' \\ & - \lambda(q_j)u(q_j), \end{aligned} \quad (\mathrm{A2})$$

where $D_d(q_d)$ denotes the diffusion coefficient, $\mu_d(q_d)$ and $\mu_j(q_j)$ are drifts of the continuous and discontinuous states, and $\lambda(q_j)$ is the jump intensity (average frequency). In this model, the joint conditional PDF $P(q,t|q_0,t_0)$ in the two-dimensional q-space obeys the forward differential Chapman-Kolmogorov equation

$$\frac{\partial}{\partial t}P(q,t|q_o,t_0) = L_q P(q,t|q_o,t_0), \tag{A3}$$

where the operator L_q is defined (for any suitable function $u(q)$) as follows:

$$\begin{aligned} L_q u(q) \equiv &- \nabla_q \cdot [\mu(q)u(q)] + \nabla_q \nabla_q^{\mathrm{T}} \cdot [D(q)u(q)] \\ &+ \int \lambda(q')f(q|q')u(q')dq' - \lambda(q)u(q), \end{aligned} \tag{A4}$$

$$f(q|q') = f_j(q_j|q_j')\delta(q_d - q_d'),\ \lambda(q) = \lambda(q_j), \tag{A5}$$

$$\mu(q) = \begin{bmatrix} \mu_d(q_d) \\ \mu_j(q_j) \end{bmatrix},\ D(q) = \begin{bmatrix} D_d(q_d) & 0 \\ 0 & 0 \end{bmatrix}. \tag{A6}$$

Here, the dot operator denotes contraction over all state coordinate indices (scalar product in case of vectors). Equivalently, Eq. (A3) can be written as

$$\frac{\partial}{\partial t}P(q,t|q_o,t_0) = L_{q_j}^{(\mathrm{KF})}P(q,t|q_o,t_0) + L_{q_d}^{(\mathrm{FP})}P(q,t|q_o,t_0). \tag{A7}$$

Following Ref. [6], it is straightforward to see that in this case, the forward GFK equation for the functional $Q(q,\ t;q_o,t_0) \equiv \bar{R}(q,\ t;q_o,t_0)P(q,\ t|q_0,t_0)$, where $\bar{R}(q,\ t;q_o,t_0) \equiv \mathbb{E}[R(t,t_0) \mid q(t) = q,\ q(0) = q_0]$ denotes the conditional expectation of the EO $R(t,t_0)$ of the system (1.1) for given initial $q_0 = q(t_0)$ and terminal $q = q(t)$ states, takes the form

$$\begin{aligned} \frac{\partial}{\partial t}Q(q,\ t;q_o,t_0) = &- r(q)Q(q,\ t;q_o,t_0) \\ &+ L_{q_j}^{(\mathrm{KF})}Q(q,\ t;q_o,t_0) \\ &+ L_{q_d}^{(\mathrm{FP})}Q(q,\ t;q_o,t_0). \end{aligned} \tag{A8}$$

The initial condition for this equation is

$$Q(q,t_0;q_o,t_0) = \delta(q - q_o). \tag{A9}$$

The unidirectional conditional expectation $\bar{R}(t;q_0,t_0) \equiv \mathbb{E}[R(t,t_0) \mid q(0) = q_0]$ and the full (unconditional) expectation $\bar{R}(t,t_0) \equiv \mathbb{E}[R(t,t_0)]$ can be found from a solution to this problem by a simple integration over the two-dimensional vector q and subsequent averaging over an initial distribution $\varphi_0(q_0)$ (see Ref. [6]):

$$\begin{aligned} \bar{R}(t;q_0,t_0) = \int Q(q,\ t;q_o,t_0)dq,\ \bar{R}(t,t_0) \\ = \int \bar{R}(t;q_0,t_0)\varphi_0(q_0)dq_0. \end{aligned} \tag{A10}$$

Let the auxiliary one-dimensional functionals $Q_d^0(q_d,t;\ q_{0d},t_0)$ and $Q_j^0(q_j,t;\ q_{0j},t_0)$ satisfy the equations

$$\begin{aligned} \frac{\partial}{\partial t}Q_d^0(q_d,t;\ q_{0d},t_0) + q_d Q_d^0(q_d,t;\ q_{0d},t_0) \\ - L_{q_d}^{(\mathrm{FP})}Q_d^0(q_d,t;\ q_{0d},t_0) = 0, \end{aligned} \tag{A11}$$

$$\begin{aligned} \frac{\partial}{\partial t}Q_j^0(q_j,t;\ q_{0j},t_0) + q_j Q_j^0(q_j,t;\ q_{0j},t_0) \\ - L_{q_j}^{(\mathrm{KF})}Q_j^0(q_j,t;\ q_{0j},t_0) = 0, \end{aligned} \tag{A12}$$

with the initial conditions

$$\begin{aligned} Q_j^0(q_d,t_0;\ q_{0d},t_0) &= \delta(q_d - q_{0d}), \\ Q_j^0(q_j,t_0;\ q_{0j},t_0) &= \delta(q_j - q_{0j}). \end{aligned} \tag{A13}$$

It is easy to verify that the product

$$Q(q,\ t;q_o,t_0) = e^{m(t_0-t)}Q_d^0(q_d,t;\ q_{0d},t_0)Q_j^0(q_j,t;\ q_{0j},t_0) \tag{A14}$$

gives the solution to the problems (A8) and (A9), with $r[q(t)]$ defined by Eq. (2.1). Functionals $Q_d^0(q_d,t;\ q_{0d},t_0)$ and $Q_j^0(q_j,t;\ q_{0j},t_0)$ represent the detrended diffusion- and jump-related factors of $Q(q,\ t;q_o,t_0)$. It immediately follows that the unidirectional conditional expectation of EO, Eq. (A10), is also factorized as

$$\bar{R}(t;q_0,t_0) = e^{m(t_0-t)}\bar{R}_d^0(t;\ q_{0d},t_0)\bar{R}_j^0(t;\ q_{0j},t_0), \tag{A15}$$

where the *detrended* factors

$$\begin{aligned} \bar{R}_d^0(t;\ q_{0d},t_0) &= \int Q_d^0(q_d,t;\ q_{0d},t_0)dq_d, \\ \bar{R}_j^0(t;\ q_{0j},t_0) &= \int Q_j^0(q_j,t;\ q_{0j},t_0)dq_j \end{aligned} \tag{A16}$$

correspond to EO of pure diffusion and pure jump processes, respectively. Finally, the full expectation of EO, according to Eq. (A.10), is

$$\begin{aligned} \bar{R}(t,t_0) = e^{m(t_0-t)}\int\int \varphi_0(q_{0d},\ q_{0j})\bar{R}_d^0(t;\ q_{0d},t_0) \\ \times \bar{R}_j^0(t;\ q_{0j},t_0)dq_{0d}dq_{0j}. \end{aligned} \tag{A17}$$

Clearly, if initially diffusion and jump components are statistically independent, i.e., $\varphi_0(q_0) = \varphi_{0d}(q_{0d})\varphi_{0j}(q_{0j})$, then the unconditional expectation of EO factorizes too:

$$\bar{R}(t,t_0) = e^{m(t_0-t)}\bar{R}_d^0(t,t_0)\bar{R}_j^0(t,t_0), \tag{A18}$$

where

$$\begin{aligned} \bar{R}_d^0(t,t_0) &= \textstyle\int \varphi_{0d}(q_{0d})\bar{R}_d^0(t;\ q_{0d},t_0)dq_{0d}, \\ \bar{R}_j^0(t,t_0) &= \textstyle\int \varphi_{0j}(q_{0j})\bar{R}_j^0(t;\ q_{0j},t_0)dq_{0j}. \end{aligned} \tag{A19}$$

Such a factorization of the unconditional expectation of EO takes place for the initial distribution coinciding with the stationary one: $\varphi_0(q_0) = \varphi(q_0) = \varphi_d(q_{0d})\varphi_j(q_{0j})$. Indeed, it follows from the differential Chapman-Kolmogorov equation (A7) that it has a stationary solution $\varphi(q) = \varphi_d(q_d)\varphi_j(q_j)$, provided the normalized solutions $\varphi_d(q_d)$, $\varphi_j(q_j)$ to equations

$$L_{q_j}^{(\mathrm{KF})}\varphi_j(q_j) = 0,\ L_{q_d}^{(\mathrm{FP})}\varphi_d(q_d) = 0 \tag{A20}$$

exist.

Similarly to the forward GFK equation (A8) with the initial condition (A9), the symmetric factorization of solutions also holds for the backward GFK equation [6]:

$$\begin{aligned}\frac{\partial}{\partial t_0}Q(q,\ t;\ q_o, t_0) = &-r(q_0)Q(q,\ t;\ q_o, t_0)\\ &- L^{*(\mathrm{KF})}_{q_j}Q(q,\ t; q_o, t_0)\\ &- L^{*(\mathrm{FP})}_{q_d}Q(q,\ t; q_o, t_0),\end{aligned}\tag{A21}$$

with the terminal condition

$$Q(q,\ t;\ q_0, t) = \delta(q - q_o).\tag{A22}$$

Here the adjoint operators are

$$L^{*(\mathrm{FP})}_{q_{0d}}u(q_{0d}) \equiv \mu_d(q_{0d})\frac{\partial}{\partial q_{0d}}u(q_{0d}) + D_d(q_{0d})\frac{\partial^2}{\partial^2 q_{0d}}u(q_{0d})\tag{A23}$$

and

$$\begin{aligned}L^{*(\mathrm{KF})}_{q_{0j}}u(q_{0j}) \equiv{}& \mu_j(q_{0j})\frac{\partial}{\partial q_{0j}}u(q_{0j}) + \lambda(q_{0j})\\ &\times \left[\int f_j(q_{0j}^{'}|q_{0j})u(q_{0j}^{'})dq_{0j}^{'} - u(q_{0j})\right].\end{aligned}\tag{A24}$$

Let the functionals $Q^0_d(q_d, t;\ q_{0d}, t_0)$ and $Q^0_j(q_j, t;\ q_{0j}, t_0)$ now satisfy the backward equations

$$\begin{aligned}&\frac{\partial}{\partial t_0}Q^0_d(q_d, t;\ q_{0d}, t_0) - q_{0d}Q^0_d(q_d, t;\ q_{0d}, t_0)\\ &\quad + L^{*(\mathrm{FP})}_{q_{0d}}Q^0_d(q_d, t;\ q_{0d}, t_0) = 0,\end{aligned}\tag{A25}$$

$$\begin{aligned}&\frac{\partial}{\partial t_0}Q^0_j(q_j, t;\ q_{0j}, t_0) - q_{0j}Q^0_j(q_j, t;\ q_{0j}, t_0)\\ &\quad + L^{*(\mathrm{KF})}_{q0_j}Q^0_j(q_j, t;\ q_{0j}, t_0) = 0,\end{aligned}\tag{A26}$$

with the terminal conditions

$$\begin{aligned}Q^0_d(q_d, t;\ q_{0d}, t) &= \delta(q_d - q_{0d}),\\ Q^0_j(q_j, t;\ q_{0j}, t) &= \delta(q_j - q_{0j}).\end{aligned}\tag{A27}$$

Then, similarly to the consideration above, we have

$$Q(q,\ t; q_o, t_0) = e^{m(t_0-t)}Q^0_d(q_d, t;\ q_{0d}, t_0)Q^0_j(q_j, t;\ q_{0j}, t_0).\tag{A28}$$

Taking into account that
$Q(q,\ t; q_o, t_0) \equiv \bar{R}(q,\ t; q_o, t_0)P(q,\ t|q_0, t_0)$,
$Q^0_d(q_d, t;\ q_{0d}, t) \equiv \bar{R}^0_d(q_d,\ t; q_{0d}, t_0)P_d(q_d, t|q_{od}, t_o)$,
$Q^0_j(q_j, t;\ q_{0j}, t) \equiv \bar{R}^0_j(q_j,\ t; q_{0j}, t_0)P_j(q_j, t|q_{oj}, t_o)$, and
$P(q, t|q', t') = P_d(q_d, t|q_d^{'}, t')P_j(q_j, t|q_{oj}, t_o)$,
it follows that

$$\bar{R}(q,\ t; q_o, t_0) = e^{m(t_0-t)}\bar{R}^0_d(q_d, t;\ q_{0d}, t_0)\bar{R}^0_j(q_j, t;\ q_{0j}, t_0).\tag{A29}$$

Integrating this equation by q_o, we obtain the backward unidirectional conditional expectation:

$$\begin{aligned}\bar{R}(q,\ t; t_o) &\equiv \mathbb{E}[R(t, t_0)\mid q(t) = q]\\ &= e^{m(t_0-t)}\bar{R}^0_d(q_d, t;\ t_0)\bar{R}^0_j(q_j, t;\ t_0),\end{aligned}\tag{A30}$$

where $\bar{R}^0_d(q_d, t;\ t_0)$, $\bar{R}^0_j(q_j, t;\ t_0)$ denote the corresponding conditional expectations of the detrended EOs for pure diffusion and pure jump processes, respectively. Symmetrically to Eq. (A15), this expression gives the expected EO for a deterministic terminal condition, $q(t)$, averaged over the initial states. For any given terminal distribution $\varphi_t(q)$, it allows for finding the unconditional expectation at the initial time:

$$\begin{aligned}\bar{R}(t, t_0) = e^{m(t_0-t)}&\int\int \varphi_t(q_d,\ q_j)\bar{R}^0_d(q_d, t;\ t_0)\\ &\times \bar{R}^0_j(q_j, t;\ t_0)dq_d dq_j.\end{aligned}\tag{A31}$$

Clearly, if the components q_d and q_j are independent in the terminal distribution, $\varphi_t(q) = \varphi_{td}(q_d)\varphi_{tj}(q_j)$ (e.g., for the stationary distribution), then this expression implies the factorization of the unconditional expectation too, see Eq. (A18).

APPENDIX B: SOLUTION TO THE FORWARD GFK EQUATION FOR THE GENERAL AFFINE MODEL

Under the assumptions (2.17) and (2.18), the forward GFK equation [6], for the functional $Q(q,\ t; q_o, t_0)$, takes the form

$$\begin{aligned}\frac{\partial}{\partial t}Q(q,\ t; q_o, t_0) = &-r(q)Q(q,\ t; q_o, t_0) + L_{0q}Q(q,\ t; q_o, t_0)\\ &+ L_{1q}Q(q,\ t; q_o, t_0),\end{aligned}\tag{B1}$$

where the operators L_{0q}, L_{1q} are defined as

$$\begin{aligned}L_{0q}u(q) \equiv &-\nabla_q[\mu_0 u(q)] + \frac{1}{2}\nabla^2_q\left[\sigma_0^2 u(q)\right]\\ &+ \int \lambda_0 u(q')h(q - q')dq' - \lambda_0 u(q),\end{aligned}\tag{B2a}$$

$$\begin{aligned}L_{1q}u(q) \equiv &-\nabla_q[\mu_1 q u(q)] + \frac{1}{2}\nabla^2_q\left[\sigma_1^2 q u(q)\right]\\ &+ \int \lambda_1 q' u(q')h(q - q')dq' - \lambda_1 q u(q).\end{aligned}\tag{B2b}$$

Applying the Fourier transform over the state q to both sides of Eq. (B1), we get

$$\begin{aligned}\frac{\partial}{\partial t}\hat{Q}(p,\ t; q_o, t_0) = &G_0(p)\hat{Q}(p,\ t; q_o, t_0)\\ &+ iG_1(p)\frac{\partial \hat{Q}(p,\ t; q_o, t_0)}{\partial p},\end{aligned}\tag{B3}$$

where, for both indices $\nu = 0, 1$,

$$G_\nu(p) \equiv -r_\nu - i\mu_\nu p - \frac{\sigma_\nu^2 p^2}{2} + \lambda_\nu[\hat{h}(p) - 1],\tag{B4}$$

and $\hat{Q}(p,\ t; q_o, t_0)$ and $\hat{h}(p)$ are the Fourier transforms of $Q(q,\ t; q_o, t_0)$ and $h(y)$. The first-order equation (B3) reduces to the ODE

$$\frac{d}{dt}\hat{Q}(p(t),\ t; q_o, t_0) = G_0(p(t))\hat{Q}(p(t),\ t; q_o, t_0),\tag{B5}$$

where the full derivative $\frac{d}{dt}$ is taken along a characteristic curve, determined by the ODE

$$\frac{dp(t)}{dt} = -iG_1(p). \tag{B6}$$

Let $p(t') = i\beta_s(t', t)$ be a solution of this ODE in the interval $t_0 \leqslant t' \leqslant t$, obeying the boundary condition $p(t) = is$, for a given real parameter s. The solution of ODE (B5) for this characteristic curve gives

$$\hat{Q}(is, t; q_o, t_0) = \hat{Q}(i\beta_s(t_0), t_0; q_o, t_0)e^{\alpha_s(t_0,t)}, \tag{B7}$$

where

$$\alpha_s(t_0, t) = \int_{t_0}^{t} G_0(i\beta_s(t', t))dt'. \tag{B8}$$

The initial condition (2.5) implies that $\hat{Q}(i\beta_s(t_0, t), t_0; q_o, t_0) = e^{\beta_s(t_0,t)q_o}$ and hence we finally get

$$\hat{Q}(is, t; q_o, t_0) = e^{\alpha_s(t_0,t)+\beta_s(t_0,t)q_o}. \tag{B9}$$

This result comprises the workhorse relation which yields analytical (explicit or in quadratures) solutions for all the affine models. To present it in the conventional form, notice that the functions $\alpha_s(t_0, t)$ and $\beta_s(t_0, t)$ obey the following ODEs and boundary conditions:

$$\frac{d\alpha_s(t_0, t)}{dt_0} = -G_0(i\beta_s(t_0, t)),\ \alpha_s(t, t) = 0, \tag{B10}$$

$$\frac{d\beta_s(t_0, t)}{dt_0} = -G_1(i\beta_s(t_0, t)),\ \beta_s(t, t) = s. \tag{B11}$$

Both functions $G_\nu(p)$, Eq. (B4), take real values at the imaginary axes and hence solutions $\alpha_s(t_0, t)$ and $\beta_s(t_0)$ are real-valued functions. They are given by the integrals

$$\int_{s}^{\beta_s(t_0,t)} \frac{ds'}{G_0(is')} = t - t_0 \tag{B12}$$

and (B8). For some specific kernel distributions $h(y)$ these integrals admit explicit analytical expressions [14,11]. Furthermore, at the left-hand side of Eq. (B9) we have

$$\begin{aligned}\hat{Q}(is, t; q_o, t_0) &= \int Q(q,\ t; q_o, t_0)e^{sq}dq \\ &= \int \bar{R}(q,\ t; q_o, t_0)P(q,\ t|q_0, t_0)e^{sq}dq \\ &= E\left\{\exp\left[-\int_{t_0}^{t} r(q(t'))dt' + sq\right]\middle| q_o, t_0\right\},\end{aligned} \tag{B13}$$

where $E\{\cdot|q_o, t_0\}$ denotes the conditional expectation for the initial condition $q(t_0) = q_o$. Thus, from the functional $\hat{Q}(is, t; q_o, t_0)$ one can immediately get the conditional EO expectation $\bar{R}(t; q_o, t_0) = \hat{Q}(0, t; q_o, t_0)$, as well as the MGF of the instantaneous rate $E[e^{sq}] = \hat{Q}(is, t; q, t)$. It is obvious now that our solution of GFK equation for the Fourier transform of the functional $Q(q,\ t; q_o, t_0)$ precisely reproduces the conclusion of Proposition 1 of Duffie *et al.* [2]. The premise of this Proposition presumes existence of unique solutions to the problems (B10), (B11), along with some additional conditions, sufficient for the process

$$\Psi_{t_0} \equiv \exp\left[-\int_0^{t_0} r(q(t'))dt' + \alpha_s(t_0, t) + \beta_s(t_0, t)q(t_0)\right]$$

to be a martingale, for $0 < t_0 \leqslant t$. In our formalism, as follows from the derivation above, the boundedness of $\hat{h}(is)$ in an interval $s_{\min} < s < s_{\max}$ and existence of a unique solution to Eq. (B11) such that $s_{\min} < \beta_s(t') < s_{\max}$ for all $t_0 \leqslant t' \leqslant t$ are sufficient for the result (B9) to hold. Clearly, if $s_{\min} < 0$, then these conditions require the kernel distribution $h(y)$ to have a light enough left tail; if $s_{\max} > 0$, then the same requirement applies for the right tail.

APPENDIX C: STATIONARY DISTRIBUTION OF UNCORRELATED PDP WITH MEAN REVERSION

The first of the equations (2.16), for the kernel (4.1), drift (4.3), and $\lambda(q_j) =$ const, takes the form

$$\alpha_j \frac{d}{dq_j}[q_j\varphi_j(q_j)] + \lambda[h(q_j) - \varphi_j(q_j)] = 0. \tag{C1}$$

When $\alpha_j = 0$, this immediately gives $\varphi_j(q_j) = h(q_j)$. It is also easy to check that, for $\alpha_j > 0$, the solution of ODE (C1), satisfying the boundary conditions $\lim_{q_j\to\pm\infty} \varphi_j(q_j) = 0$, is

$$\varphi_j(q_j) = \begin{cases} -\Lambda q_j^{\Lambda-1} \int_{-\infty}^{q_j} q_j'^{-\Lambda} h(q_j')dq_j', & \text{, if } q_j < 0, \\ \Lambda q_j^{\Lambda-1} \int_{q_j}^{\infty} q_j'^{-\Lambda} h(q_j')dq_j', & \text{if } q_j > 0, \end{cases} \tag{C2}$$

where $\Lambda \equiv \lambda/\alpha_j$. Changing the integration variable to $x = q_j'/q_j$, we can rewrite this solution in a shorter form:

$$\varphi_j(q_j) = \Lambda \int_1^{\infty} x^{-\Lambda} h(xq_j)dx. \tag{C3}$$

From this form, changing the order of integration, we also derive the CDF

$$\Phi_j(q_j) \equiv \int_{-\infty}^{q_j} \varphi_j(q_j')dq_j' = \Lambda \int_1^{\infty} x^{-\Lambda-1} H(xq_j)dx, \tag{C4}$$

where $H(q_j)$ is the CDF of the kernel distribution $h(q_j)$. This result verifies that $\lim_{q_j\to\infty} \Phi_j(q_j) = \lim_{q_j\to\infty} H(q_j) = 1$. Note that, due to the mean value theorem, the relation (C4) implies: $\Phi_j(q_j) = H(x_*(q_j)q_j)$, where $x_*(q_j) \geqslant 1$ and the equality is achieved at $q_j = 0$. Informally, this means that the curve $\Phi_j(q_j)$ results from a nonlinear horizontal compression of the curve $H(q_j)$ so that the quantile corresponding to the zero mean value remains at its place. In the limit $\alpha_j \to \infty$, i.e., $\Lambda \to 0$, we have $x_*(q_j) \to \infty$ for any $q_j \neq 0$, that is, $\Phi_j(q_j)$ tends to the deterministic distribution, localized at point $q_j = 0$.

APPENDIX D: UNCONDITIONAL EXPECTATION OF EO FOR THE STATIONARY INITIAL DISTRIBUTION IN THE CIR MODEL

The stationary CIR process obeys the gamma distribution [21]:

$$\varphi_d(q_d) = \frac{1}{\Gamma(\gamma_d)\theta_d^{\gamma_d}} r_d^{\gamma_d-1} e^{-\frac{r_d}{\theta_d}}, \tag{D1}$$

where $r_d = m_d + q_d$, $\gamma_d = \frac{2\alpha_d m_d}{k^2}$, and $\theta_d = \frac{m_d}{\gamma_d} = \frac{k^2}{2\alpha_d}$. Substitute $\varphi_{0d}(q_{0d}) = \varphi_d(q_{0d})$ and $\bar{R}_d^0(t;\, q_{0d})$ from Eq. (4.7) into the first of integrals (2.15) and note that

$$\bar{R}_d^0(t;\, q_{0d}) = \bar{R}_d^0(t;\, -m)e^{-\beta(t)(m_d+q_{0d})}, \tag{D2}$$

where

$$\beta(t) \equiv \frac{2(1-e^{-\nu t})}{(\nu+\alpha_d)+(\nu-\alpha_d)e^{-\nu t}}. \tag{D3}$$

Then, changing the integration variable to $x = m_d + q_{0d}$, we obtain

$$\bar{R}_d^0(t) = \frac{\bar{R}_d^0(t;\, -m)}{\Gamma(\gamma_d)\theta_d^{\gamma_d}}\int_0^\infty x^{\gamma_d-1}e^{-\frac{x}{\theta_d}-\beta(t)x}dx$$
$$= \frac{\bar{R}_d^0(t;\, -m)}{[1+\beta(t)\theta_d]^{\gamma_d}}. \tag{D4}$$

APPENDIX E: CHARACTERISTIC TIMES OF RELAXATION OF THE CONDITIONALLY EXPECTED DECLINE RATE IN THE CIR AND UNCORRELATED PDP MODELS

Let us define the characteristic time of relaxation of the integral decline rate $Y_i(t; q_{0i})$, where the index i stands for either CIR (d) or uncorrelated PDP (j) model, as

$$T_i \equiv -\frac{Y_i(0; q_{0i}) - m_i}{Y_i'(0; q_{0i})}, \tag{E1}$$

where $'$ denotes the derivation by time. Of course, generally, this ratio might depend on the initial state: $T_i = T_i(q_{0i})$. By the definition of the integral rate $Y_i(t; q_{0i})$, we have

$$Y_i(t; q_{0i}) = \frac{1}{t}\int_0^t \rho_i(t';\, q_{0i})dt'$$
$$= \rho_i(0;\, q_{0i}) + \rho_i'(0;\, q_{0i})\frac{t}{2} + O(t^2), \tag{E2}$$

where $\rho_i(t;\, q_{0i}) \equiv -\frac{d}{dt}\ln\bar{R}_i(t; q_{0i})$ is the instantaneous expected decline rate. Therefore, $T_i = 2\mathcal{T}_i$, where $\mathcal{T}_i$ is the similarly defined characteristic time of relaxation of the instantaneous decline rate:

$$\mathcal{T}_i \equiv \frac{\rho_i(0;\, q_{0i}) - m_i}{\rho_i'(0;\, q_{0i})} = -\frac{(\ln\bar{R}_i)' + m_i}{(\ln\bar{R}_i)''}. \tag{E3}$$

Here, both time derivatives are taken at the initial time $t = 0$ and the argument q_{0i} is omitted for brevity. A note is needed here, to avoid a confusion: Eqs. (E1) and (E3) take the differences between the initial rate and the mean instantaneous rate m_i, rather than the long-term limit $\rho_{i\infty}$, which is known to be below the mean. The advantage of these definitions, as we will see soon, is that they guarantee existence of finite limits $\lim_{q_{0i}\to 0} T_i(q_{0i})$ and $\lim_{q_{0i}\to 0} \mathcal{T}_i(q_{0i})$. However, they imply that these times only characterize how fast the conditionally expected rates approach the mean instantaneous rate, in the short term, but do not account for a time required for approaching the long-term limit (which time generally might have a different order of magnitude).

Application of Eq. (E3) to the EO expectation $\bar{R}_d^0(t; q_{0d})$ in the CIR model, Eq. (4.7), though somewhat bulky, is straightforward and, with taking into account that $\gamma_d = \frac{2\alpha_d m_d}{k^2}$, leads to the simple result, independent of the initial state q_{0d}:

$$\mathcal{T}_d = \alpha_d^{-1}. \tag{E4}$$

To apply the formula (E3) to the uncorrelated PDP model, consider the Laurent expansion of the Laplace transform of $\bar{R}_j^0(t; q_{0j})$, Eq. (4.4), at $s = \infty$:

$$\tilde{R}_j^0(s; q_{0j}) = \frac{\tilde{\delta}(s+\lambda;\, q_{0j})}{1-\lambda\tilde{h}(s+\lambda)}$$
$$= \tilde{R}_j^{0'}s^{-1} + \frac{1}{2}\tilde{R}_j^{0''}s^{-2} + \frac{1}{6}\tilde{R}_j^{0'''}s^{-3} + O(s^{-4}), \tag{E5}$$

where $'$ at the Laplace transform $\tilde{R}_j^0$ denotes derivation by $x = s^{-1}$, all the derivatives are taken at $x = 0$ and the argument q_{0j} is omitted for brevity. Due to the initial value theorem, it follows

$$\bar{R}_j^0 = \tilde{R}_j^{0'} = 1,\ \bar{R}_j^{0'} = \tfrac{1}{2}\tilde{R}_j^{0''},\ \bar{R}_j^{0''} = \tfrac{1}{6}\tilde{R}_j^{0'''}, \tag{E6}$$

where $\bar{R}_j^0$ and its derivatives are taken at $t = 0$ and the argument q_{0j} is also omitted for brevity. Therefore,

$$\rho_j(0; q_{0j}) - m_j = -\frac{\bar{R}_j^{0'}}{\bar{R}_j^0} = -\frac{1}{2}\tilde{R}_j^{0''}, \tag{E7a}$$

$$\rho_j'(0; q_{0j}) = -\frac{\bar{R}_j^{0''}}{\bar{R}_j^0} + \left(\frac{\bar{R}_j^{0'}}{\bar{R}_j^0}\right)^2 = -\frac{1}{6}\tilde{R}_j^{0'''} + \frac{1}{4}\tilde{R}_j^{0''2}. \tag{E7b}$$

It only remains to calculate the derivatives $\tilde{R}_j^{0''}$ and $\tilde{R}_j^{0'''}$. For this purpose, we make use of the following expansions established in Gluzberg and Katz [6]:

$$\tilde{h}(s) = \sum_{n=0}^\infty \mu_n\frac{(-1)^n}{s(s+\alpha_j)\cdots(s+n\alpha_j)},$$
$$\tilde{\delta}(s;\, q_{0j}) = \sum_{n=0}^\infty q_{0j}^n\frac{(-1)^n}{s(s+\alpha_j)\cdots(s+n\alpha_j)}, \tag{E8}$$

where $\mu_n \equiv \int q^n h(q)dq$ is the n-th moment of the kernel distribution. Substituting here $s = x^{-1}$, we have

$$\tilde{h}(s+\lambda) = \sum_{n=0}^\infty \mu_n\frac{(-1)^n x^{n+1}}{(1+\lambda x)(1+\lambda_1 x)\cdots(1+\lambda_n x)}, \tag{E9a}$$

$$\tilde{\delta}(s;\, q_{0j}) = \sum_{n=0}^\infty q_{0j}^n\frac{(-1)^n x^{n+1}}{(1+\lambda x)(1+\lambda_1 x)\cdots(1+\lambda_n x)}, \tag{E9b}$$

where $\lambda_n \equiv \lambda + n\alpha_j$. The first three derivatives of these series by x, at $x = 0$ are

$$\tilde{h} = 0,\ \tilde{h}' = 1,\ \tilde{h}'' = -2\lambda,\ \tilde{h}''' = 6(\lambda^2 + \sigma_h^2), \tag{E10a}$$

$$\tilde{\delta} = 0,\ \tilde{\delta}' = 1,\ \tilde{\delta}'' = -2(\lambda + q_{0j}),\ \tilde{\delta}'''$$
$$= 6[\lambda^2 + (2\lambda + \alpha_j)q_{0j} + q_{0j}^2]. \tag{E10b}$$

Here, it is taken into account that $\mu_0 = 1$, $\mu_1 = 0$, and $\sigma_h^2 \equiv \mu_2$ stands for the variance of the kernel distribution. Thus, differentiating $\hat{R}_j^0(s; q_{0j})$, Eq. (E5), by x and substituting (E.10), we find

$$\tilde{R}_j^{0''} = \tilde{\delta}'' + 2\lambda = -2q_{0j}, \tag{E11a}$$
$$\tilde{R}_j^{0'''} = \tilde{\delta}''' + 3\lambda\tilde{\delta}'' + 3\lambda\tilde{h}'' + 6\lambda^2$$

$$= 6\left[(\lambda + \alpha_j)q_{0j} + q_{0j}^2\right]. \tag{E11b}$$

Substituting this into Eq. (E.7), finally, from Eq. (E3) we obtain

$$T_j = (\lambda + \alpha_j)^{-1}. \tag{E12}$$

This remarkably simple result is independent of the initial state q_{0j} and a shape of the kernel distribution $h(q)$. It reflects the fact that the relaxation of the conditionally averaged decline rate in the uncorrelated PDP model is caused by the shocks and the mean-reversion drift. Accordingly, the rate of the relaxation is determined by the sum of the two key parameters: the average shock frequency λ and the mean-reversion constant α_j. The difference between results (E4) and (E12) reflects the fact that in the CIR model the mean reversion is the only cause of the relaxation.

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