Letter

Largest connected component in duplication-divergence growing graphs with symmetric coupled divergence

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The largest connected component in duplication-divergence growing graphs with symmetric coupled divergence is studied. Finite-size scaling reveals a phase transition occurring at a divergence rate δ_c. The δ_c found is close to the locus of zero in Euler characteristic of finite-size graphs known to reflect the proximity of the largest connected component transition. A close correspondence with the vanishing of a scaling relation exponent for moments of the vertex degree distribution is shown, with such a scaling relation that generalizes a known form for duplication-divergence model graphs. The role of noninteracting vertices in shaping this transition with their presence or absence in duplication is also considered through a particular relation which would result in the two cases being comparable. The findings have relevancy for bond percolation in these growing graph models.

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Introduction. Sequentially growing network models are paradigmatic for the understanding of how the structure of complex networks emerges [1–5], and for studying what principles underlying growth and evolution lead to the emergence of their structural characteristics [6–8]. Among these models, duplication-divergence models are based on the growth principle of duplication, according to which a randomly chosen vertex i is duplicated into a copy vertex i' with the same edges of i; divergence refers to probabilistic loss of duplicate edges [9]. Some sophistications extend duplication-divergence models by adding edges other than those that are duplicated through, e.g., dimerization (an edge between i and i') [10–13], mutation (edges between i' and other vertices in the graph) [14–16], and vertex deletion [17]. Most of these graphs aim at modeling structural characteristics of biological networks (e.g., pairwise protein interaction networks [10,14–16], gene networks [18,19]), the world-wide web [20,21], scientific citation networks [22], and online social networks grown by vertex copying [8,12,23]. Duplication-divergence is among network growth processes that may admit a limiting power-law dependence of the vertex degree distribution [24], and the emergence of proportional preference in the growth process (see, e.g., [22,25,26]), as in Ref. [27].

Complete asymmetric divergence typically refers to duplicate edges loss only by vertex i' [28]. Yet, the divergence process can also affect duplicate edges of i and i' with different probabilities (asymmetric divergence) or with equal probability (symmetric divergence) [28]. In the latter case, one can distinguish between two cases. (i) Coupled divergence: given a duplicate edge pair $\{(i, j), (i', j)\}$, only one edge can be lost while the other is retained; (ii) uncoupled divergence: both edges of the pair can be lost [13,29]. In Ref. [28], the divergence asymmetry rate σ allows generalizing the complete asymmetric divergence ($\sigma = 1$), and the coupled symmetric divergence ($\sigma = 1/2$), as well as intermediate configurations between these two limit cases [28]. The model of Ref. [28] also considers the possibility of presence (with parameter $d = 0$) or absence ($d = 1$) of noninteracting vertices among vertices for possible duplication, yet including them in the graph when the divergence process yields a noninteracting vertex.

For some of these growing network models, prior studies showed structural phase transitions. These transitions concern the model with complete asymmetric divergence with divergence probability $\delta = 1$ and nonzero mutation probability showing an infinite-order percolation transition [14], reminiscent of a Berezinskii-Kosterlitz-Thouless transition [30]; the network growth by vertex copying showing distinction between sparse and dense networks [8,12]; the complete asymmetric divergence model with inclusion of noninteracting vertices and dimerization rates, with changes in network topology signaling a largest connected component transition [31]; and the symmetric coupled divergence with dimerization and mutation suggesting a sharp change in the average connected component size [32] for the models in Refs. [16, 10].

In Ref. [31], for the complete asymmetric divergence model with noninteracting vertices, zeros of the Euler characteristic of finite-size graphs were considered indicative for the largest connected component transition, with loci $\delta_{c,\xi}$ of such zeros close to critical values δ_c of divergence probability that may be expected from a percolation transition. In this respect, an understanding of these structural changes in the minimal model of duplication-divergence with symmetric coupled divergence has rarely been deepened.

Here, the duplication-divergence model with symmetric coupled divergence is studied by focusing on the largest connected component transition through both the Euler

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characteristic of finite-size graphs to infer $\delta_{c,\xi}$, and finite-size scaling to infer δ_c and the scaling behavior of the relative size of the largest connected component, also considering how noninteracting vertices may change such transition loci. This Letter is organized as follows: the duplication-divergence model with symmetric coupled divergence is introduced as a special case of the model in Ref. [28], emphasizing model assumptions and the vertex degree distribution; results are then shown and discussed; finally, concluding remarks summarize the main findings.

Model. For a coupled divergence process, in a growth iteration, each duplicate edge pair (i.e., $i \bullet\!\!-\!\!-\!\!\bullet i'$) resulting from duplication has the following probabilities of transitioning to the configuration indicated in parentheses on the left-hand side of these equations

$$\mathcal{P}(\ i \bullet\!\!-\!\!-\!\!\bullet i'\ ) = \sigma(1-\delta) + (1-\sigma)(1-\delta) = 1-\delta, \tag{1a}$$

$$\mathcal{P}(\ i\bullet\ \bullet\!\!-\!\!\bullet i'\ ) = (1-\sigma)\delta, \quad \mathcal{P}(\ i \bullet\!\!-\!\!\bullet\ \bullet i'\ ) = \sigma\delta, \tag{1b}$$

with $\delta, \sigma \in [0,1]$ the divergence probability and the divergence asymmetry rate [28], with mutually exclusive events of edge loss from i or i'. Equations (1) sum up to 1, and the corresponding events cover the entire sample space with intriguing symmetry properties. When coupled divergence is symmetric, $\sigma = 1/2$ in (1b).

A generic growth iteration t can be enunciated as follows: a vertex i, chosen uniformly at random, either among interacting vertices (when $d = 1$), or among all vertices including noninteracting ones (when $d = 0$), is duplicated into a vertex i' having the same edges of vertex i; for each pair $\{(i,j),(i',j)\}$ of duplicate edges, only one of the two edges of the pair is lost with probability δ. This means that one chooses if to remove an edge of the pair (removal occurring with probability δ), and where to remove it: either from i with probability $1/2$ or from i' with same probability $1/2$, but not from both. This divergence process is symmetric (because of an equal probability of losing an edge for i and for i'), and coupled [because removing (i,j) implies that (i',j) is not removed, and vice versa]; see Fig. 1.

For $d = 1$, $k \geqslant 1$, asymptotically assuming linear scaling for the number of k-degree vertices with the number of interacting vertices (plausible for sparse graphs), the expected stationary vertex degree distribution n_k reads

$$\hat{\mu} n_k = (1-\delta)[(k-1)n_{k-1} - k n_k] + \mathcal{M}_k, \tag{2}$$

with $\mathcal{M}_k = 2\sum_{s=k}^{\infty} \binom{s}{k} q^k (1-q)^{s-k} n_s$, where q may be considered equal to $(1-\delta)/2$, or $1-\delta/2$: the former emphasizes the concept of subfunctionalization in gene duplication that inspired the symmetric divergence scenario [33]; the latter reflects more formally the aforementioned growth iteration and it is the one considered hereafter (to deepen their relation, see Sec. V of the Supplemental Material [34]); resulting differences may be absorbed by a proper regularization. Noticing that for large k the summand of $\mathcal{M}_k$ is sharply peaked around k/q and then n_s is substituted with its value at k/q [35]; with these assumptions, (2) admits a solution of the form $n_k \sim k^{-\gamma}$. To ensure consistency of Eq. (2) with the choice of $q = 1-\delta/2$: $\hat{\mu} = 2\sum_k n_k[1-(\delta/2)^k]$. Having a normalized n_k such that $\sum_{k\geqslant 1} n_k = 1$, $\hat{\mu}$ can be written as $\mu + 1 = \sum_k n_k[1-2(\delta/2)^k] + 1$, and (2) becomes

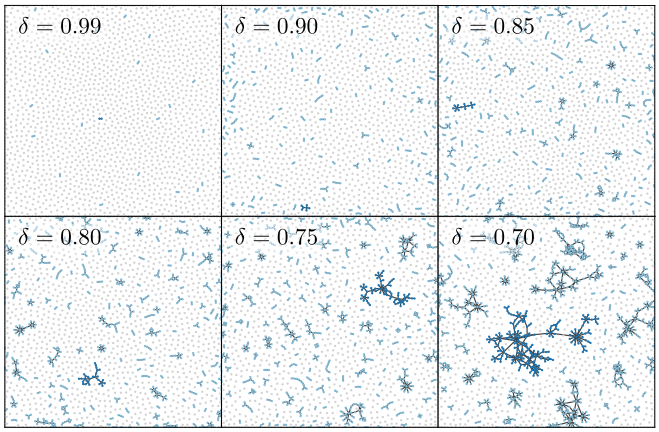


FIG. 1. Depiction of realizations of the generalized model [28] ($\sigma = 1/2$ and $d = 1$) for various δ (upper left of each panel). Each panel is a snapshot from one simulation at $t = 2 \times 10^3$ vertices; brightest dots are noninteracting vertices (vertex degree $k = 0$), darkest dots are vertices of the largest connected component.

$$(\mu+1)n_k = (1-\delta)[(k-1)n_{k-1} - k n_k] + \mathcal{M}_k, \tag{3}$$

where μ is the expected rate of increase of interacting vertices (vertices with at least one edge). Given an initial graph with $t_0 = 2$ connected vertices, and being t the total number of vertices (interacting plus noninteracting vertices) of the growing graph but also a discrete time variable counting the number of iterations ($\Delta t = 1$) as in Ref. [28], one has $\Delta N(\delta,t)/\Delta t = \sum_{k=1}^{\infty} n_k[1-(\delta\sigma)^k - (\delta-\delta\sigma)^k]$, with $(\delta\sigma)^k + (\delta-\delta\sigma)^k$ being proportional to the probability of getting a noninteracting vertex either from i' (first term of the sum) or from i (second term of the sum); $\sigma = 1/2$ yields $\Delta N(\delta,t)/\Delta t \propto \sum_k k^{-\gamma}[1-2(\delta/2)^k]$, which can give μ in (3). When $t \gg t_0$, it turns out that the first term of the series is preponderant (see Sec. I in the Supplemental Material [34]) and, according to Refs. [9,28], the proportionality is substituted with a prefactor of 2, yielding to consider in the large k limit, the following:

$$N(\delta,t) \simeq 2(1-\delta)t, \tag{4}$$

then $\mu := 2(1-\delta)$ is here assumed. With this μ in (3), a transcendental equation relating γ and δ reads

$$\gamma = 3 + \frac{1}{1-\delta} - \frac{2^{2-\gamma}}{1-\delta}(2-\delta)^{\gamma-1}, \tag{5}$$

which admits two branches of solutions, one is a trivial solution $\gamma = 2$, and a different nontrivial branch of solutions increasing monotonically with δ, e.g., $\gamma = 3$ at $\delta = 2-\sqrt{2}$, diverging asymptotically for δ approaching 1. Note that, due to symmetry of sample space, regularization of (5) yields $\gamma = 3 - 2^{2-\gamma}(1-\delta)^{\gamma-2}$ as in Ref. [28] when $\sigma = 1/2$, describing its essence without diverging terms that emerge when $\delta \to 1$.

Without stationarity assumption in the rate equation for n_k, one can begin by considering the expected number of vertices with degree k when the graph has a total number of vertices t, denoted by $N_k(t)$, and the proportion of k-degree vertices as $n_k(t) = N_k(t)/t$, and then, $\frac{\partial N_k(t)}{\partial t} = t\frac{\partial n_k(t)}{\partial t} + n_k(t)$. $N := N(\delta,t)$ being the number of vertices

with degree $k \geqslant 1$, when $N \neq t$ but generically $N = \mu t$, it follows that $\mu \frac{\partial N_k}{\partial N} = N \frac{\partial n_k}{\partial N} + \mu n_k$, where here $n_k(N) := N_k(N)/N$ (with $k \geqslant 1$) defines the expected vertex degree distribution normalized such that $\sum_{k\geqslant 1} N_k = N$ and $\sum_{k\geqslant 1} n_k = 1$, due to variable change $t \to N$. A rate equation for the evolution of n_k reads

$$N\frac{\partial n_k}{\partial N} + \mu n_k = (1-\delta)[(k-1)n_{k-1} - k n_k] + \mathcal{M}_k - n_k, \tag{6}$$

where $\mathcal{M}_k$ is that of (2) with $q = 1 - \delta/2$. On the right-hand side of (6), when multiplied by $(1-\delta)$, the two terms in squared parentheses are, respectively, (i) the contribution of adjacent vertices of a vertex with degree $k-1$ chosen for duplication when the duplicate edge is not lost (a gain term for n_k), and (ii) the contribution of adjacent vertices of a k-degree vertex chosen for duplication when the duplicate edge is not lost (a loss term for n_k). $\mathcal{M}_k$ is a gain term for n_k due to all possible removal of $s-k$ edges due to divergence, where $q^k(1-q)^{s-k}$ reflects the binomial choice with k edges retained and $s-k$ edges lost; the factor 2 in $\mathcal{M}_k$ accounts for such a removal process for i and i'. The term $-n_k$ represents the situation in which vertex i, when initially has vertex degree k, disappears from the count of k-degree vertices, replaced by its current vertex degree. Note that i and i' are indistinguishable after duplication due to the coupled symmetric nature of the divergence process. One can also recast (6) via a continuum approximation

$$N\frac{\partial n_k}{\partial N} + (\mu+1)n_k = (\delta-1)\frac{\partial (k n_k)}{\partial k} + \mathcal{M}_k, \tag{7}$$

which yields (3) in the stationary limit assumption $\partial n_k/\partial N = 0$. With a nonstationary assumption, l moments of n_k are computed by summing over all k and multiplying by k^l (see Sec. IV in the Supplemental Material [34]), which yields

$$M_l(N,\delta) \propto N^{\Phi(l,\delta,\mu)}, \tag{8a}$$

$$\Phi(l,\delta,\mu) = l(1-\delta) + 2(1-\delta/2)^l - (\mu+1). \tag{8b}$$

Equations (8) are a generalized form of that provided in Refs. [7,10] (which emerges exactly with $\mu = 1$ where dimerization is also considered). The nonlinearity of $\Phi(l,\delta)$ reflects the multifractal nature [7,36] of this model for the symmetric coupled duplication-divergence case when considering nonstationarity of n_k. The $\mu := 2(1-\delta)$ would yield a constant average vertex degree, and M_2 would diverge for $\delta < 2 - \sqrt{2}$, agreeing with the transcendental equation (5) with $\gamma = 3$ at $\delta = 2 - \sqrt{2}$ found with the hypothesis of stationarity in the thermodynamic limit. A peculiar feature of μ is that it acts as a gauge fixing offset ensuring stationarity of M_1 (constant average vertex degree), with a relative shift of the order of moments (which are relative to M_1). Calculating $\partial\Phi/\partial l$, expanding the resulting nonlinear terms, and calculating the curvature $\frac{\partial}{\partial\delta^2}(\partial\Phi/\partial l)|_{\delta\to 0} = l - 1/2$, suggests that fractional moments may be relevant, which is expected given the multifractality [indeed, one will see that $E(\delta,t)$ with t fixed for $d=1$, shown in Fig. 2 for $t = 1024$, is calculated through the 1.5th moment].

The mean number of edges $E(\delta,t)$ for the model with $d=0$, with an initial graph with two connected vertices ($t_0 = 2$, $E_0 = 1$) as in Ref. [28] (see Sec. II in the Supplemental Material [34]) reads as

$$E(\delta,t) = E_0 \frac{\Gamma(t_0)\Gamma[2(1-\delta)+t]}{\Gamma[t_0 + 2(1-\delta)]\Gamma(t)}, \tag{9}$$

with $\Gamma(\cdot)$ the Euler Gamma function. Note that, for increasingly large t, the model with $d=0$ shows the growth pattern of $E(\delta,t)$ versus $N(\delta,t)$ of the model with $d=1$ (see inset of Fig. S1 in Sec. I in the Supplemental Material [34]), yet described by Eq. (9) with t (as shown in Ref. [28]). The relevant difference between the model with $d=0$ and with $d=1$ stands in the mean number of edges given a fixed t, due to the nonuniform probability distribution of choosing a vertex for duplication, which is nonzero only for vertices with degree $k \geqslant 1$ when $d=1$.

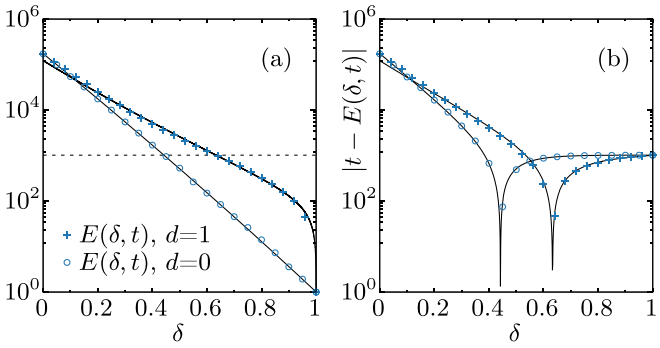


FIG. 2. In (a), plots of $E(\delta,t)$ for the model with $d=0$ and $d=1$ (see legend) with $t=1024$ (dashed horizontal line). In (b), the respective $|t - E(\delta,t)|$ for the model with $d=0$ and for the model with $d=1$ showing a locus of singularity, respectively, at $\delta_{c,\xi} \approx 0.442$ and $\delta_{c,\xi} \approx 1 - e^{-1}$. Points shown are obtained through averaging over 10^3 simulations; solid curves are obtained from (9) for $d=0$ and from the moment distribution for $d=1$.

Euler characteristic. In Ref. [31], it has been shown that topological transitions can be found through the Euler characteristic of a simplicial complex, and for the special case of a duplication-divergence graph with complete asymmetric divergence ($\sigma = 1$). The Euler characteristic is defined as $-\sum_n (-1)^n \kappa_n$, where κ_n is the total number of n cliques. Then, as in Ref. [31], for the special case of a duplication-divergence graph, which does not include n cliques with $n \geqslant 3$, the Euler characteristic assumes the form $t - E(\delta,t)$ (with t the total number of vertices in the graph), and the Euler entropy is the natural logarithm of its absolute value, $\ln(|t - E(\delta,t)|)$. Thus, zeros of Euler characteristic correspond to singularities in Euler entropy, with a singularity locus $\delta_{c,\xi}$ arising at the formation of n cycles and expected to be close to a critical value δ_c, where the largest connected component transition occurs [31]. For finite-sized graphs, Fig. 2 shows a $\delta_{c,\xi} \approx 0.442$ (see Sec. II in the Supplemental Material [34]) that agrees with the result in Ref. [31] for complete asymmetric divergence ($\sigma = 1$) although here, instead, it is found for the symmetric coupled divergence case ($\sigma = 1/2$) of the model with $d=0$. Euler entropy of finite-size graphs with $t = 1024$ and $d=1$ is close to $\delta_{c,\xi} \approx 1 - e^{-1}$ [see Fig. 2(b)]. Due to a different number of noninteracting vertices, (4) may not directly hold for the model with $d=0$, nonetheless, the mean number of edges of interacting vertices can be matched

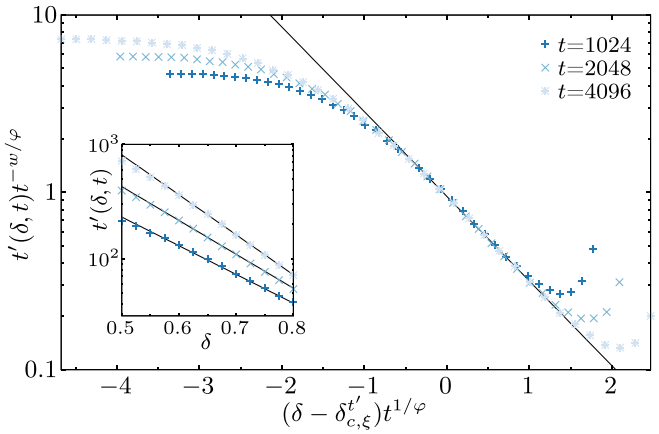


FIG. 3. Finite-size scaling for $t'(\delta,t)$ with scaling collapse for δ approximately in [0.55,0.75]. The linear behavior in log-linear plot suggests the exponential form (shown for visual reference), $a\times\exp[-b(\delta-\delta^{t'}_{c,\xi})t^{1/\varphi}]$ with $a\approx 0.95$, $b\approx 1.05$ which, when unscaled, describes a subset of points of $t'(\delta,t)$ vs δ (see inset); points are averages over 3×10^3 simulations.

across the case of $d=0$ and $d=1$ (see the inset of Fig. S1 in the Supplemental Material [34]), and one can get the corresponding t (as if the model was with $d=1$) from the model with $d=0$, through the following relation (with $\delta\neq 1$):

$$t'(\delta,t)\simeq N(\delta,t)/2(1-\delta). \tag{10}$$

Note that t' is a δ-dependent function and not a fixed value as in the calculation of the Euler entropy shown in Fig. 2. It is suggested that $t'(\delta,t)$ also has a scaling form for increasing t in a subset of δ values where curves for various t collapse on the same function. The ansatz considered here for such a scaling of $t'(\delta,t)$ is

$$t'(\delta,t)=t^{w/\varphi}h[(\delta-\delta^{t'}_{c,\xi})t^{1/\varphi}], \tag{11}$$

with w, φ, and $h(\cdot)$, respectively, two unknown exponents, and a scaling function. Different curves for various t collapse on the same curve for $\delta\in[0.55,0.75]$ approximately (see Fig. 3), with $\delta^{t'}_{c,\xi}=0.638\pm 0.006$, $w=2.831\pm 0.095$, and $\varphi=4.177\pm 0.367$, which is indeed the range of δ values where $\delta_{c,\xi}$ is expected in the model with $d=1$. Intriguingly, the transformation $t'(\delta,t)$ on the model with $d=0$ can be leveraged to get an Euler entropy curve different from the one estimated numerically for the model with $d=1$, yet exhibiting nearly the same singularity locus of the Euler entropy for finite-sized graphs, e.g., $\delta^{t'}_{c,\xi}\approx\delta_{c,\xi}$ with $d=1$, $t=1024$, see Figs. 4(b) and 2(b). The value of $\delta_{c,\xi}$ considers finite-sized graphs and it may be only indicative of the largest connected component transition; thus, in the following, finite-size scaling for the relative size of the largest connected component is tackled to consider δ_c in infinite-sized graphs from finite-size behavior [37,38].

Finite-size scaling. An ansatz for the probability of a vertex to belong to the th largest connected component $P^{(n)}(\delta,t)$, with $P_\infty:=P^{(n=1)}$ for the largest (with $n=2$ the second largest, $n=3$ the third largest,...) written as

$$P^{(n)}(\delta,t)=t^{\zeta/\nu}f^{(n)}[(\delta-\delta_c)t^{1/\nu}], \tag{12}$$

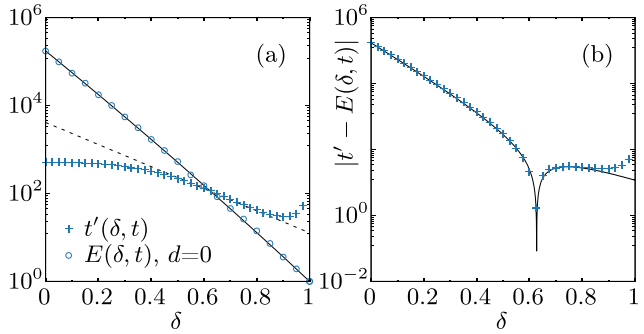


FIG. 4. In (a), $E(\delta,t)$ as in Fig. 2(a), $t'(\delta,t)$ as in the inset of Fig. 3 with $t=1024$. In (b), Euler entropy curve assuming $t'(\delta,1024)$ in the model with $d=0$, with $\delta\in[0.55,0.75]$ from Fig. 3 extended for visual reference to $\delta\in[0,1]$; the locus of zero of $|t'-E(\delta,t)|$ is near the one for $d=1$ in Fig. 2(d).

where ζ, ν, and $f^{(n)}(\cdot)$ are, respectively, two unknown exponents and a scaling function on which one would expect scaling collapse of $P^{(n)}(\delta,t)$ for various sizes t, i.e., various graph order in the language of graph theory. Denoting $P_\infty:=P_\infty(\delta,t)=P^{(1)}(\delta,t)$, and its susceptibility

$$\chi(\delta,t)=t(\langle P^2_\infty\rangle-\langle P_\infty\rangle^2), \tag{13}$$

which characterizes the intensity of fluctuation about the mean of the order parameter, then the following scaling ansatz for the susceptibility is written

$$\chi(\delta,t)=t^{\psi/\nu}g[(\delta-\delta_c)t^{1/\nu}], \tag{14}$$

with ψ an unknown exponent, and $g(\cdot)$ a scaling function. For the largest connected component, the value δ_c found here is close to the locus $\delta_{c,\xi}$: in Fig. 5, the collapse on the same curve is obtained for $\zeta=-0.033\pm 0.006$, $\nu=9.634\pm 0.069$, and $\delta_c=0.6\pm 0.002$ (formal uncertainties affected by discretization effects of δ with $\Delta\delta=0.025$), and ψ determined by satisfying the relation $\psi/\nu=1+2\zeta/\nu$ (scaling is in terms of "volume," total number of vertices t). Note that, from Eq. (1), $p=1-\delta$ would link what here studied to a bond percolation

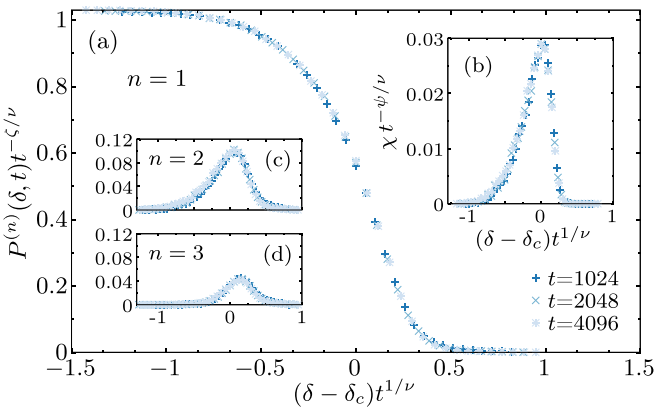


FIG. 5. Scaling collapse of $P_\infty(\delta,t):=P^{(n=1)}(\delta,t)$ in (a), and $\chi(\delta,t)$ in (b), respectively, from Eqs. (12) and (14). Scaling collapse for $P^{(n=2)}(\delta,t)$ and $P^{(n=3)}(\delta,t)$, respectively, shown in (c) and (d), see also Sec. III in the Supplemental Material [34]; all points shown were obtained by averaging over 3×10^3 simulations.

TABLE I. Values of δ (exact for $l = 2$, numerical otherwise) at which $\Phi(l, \delta) = 0$ (l is in the column header), with $\mu = 2(1 - \delta)$ in (8); when $\mu = 1$ in (8) one has a $\Phi = 0$ exactly at $\delta = 1/2$ and $\delta = 4 - 2\sqrt{3}$, respectively, for $l = 1$ and $l = 2$.

	$l = 1.5$	$l = 2$	$l = 2.5$	$l = 3$	$l = 3.5$
δ	0.562	$2 - \sqrt{2}$	0.610	0.635	0.661

on growing graphs while the exponents may be reminiscent of those of an explosive transition with trivial exponents $\zeta = 0$ and $\psi/\nu = 1$ according to relations between exponents in [38,39], with some plausible analogy to jamming [40]; yet, the estimated exponent ζ is nonzero and of the order of 10^{-2} (extremely small), thus only $\zeta \approx 0$ and $\psi/\nu \approx 1$, which also suggests considering this transition as continuous [41,42].

The critical value δ_c is found within the region where exponents of n_k moments vanish (see Table I). As shown in Fig. 6(a), ψ/ν describes the scaling of peaks χ^* of $\chi(\delta, t)$, for various t, and $\langle s\rangle^*$ of the weighted average connected component size $\langle s\rangle$, the latter defined as $\langle s\rangle \propto \sum_{s=1}^{\infty} s^2 C_s(\delta, t) - s_\infty^2$, with $C_s(\delta, t)$ and s_∞, respectively, the expected number of connected components of size s and size of the largest connected component. Indeed, at the critical point, one would expect $\chi(\delta_c, t) \sim t^{\psi/\nu}$ (see also Sec. III in the Supplemental Material [34]), provided that the estimates of ψ and ν are correct [37]. As discussed for the Euler entropy, the role of noninteracting vertices appears to be crucial in changing the locus of the critical value in topological transitions of the duplication-divergence graph model with symmetric coupled divergence. From such a consideration, one can further define the following observable $\langle s'\rangle = \sum_{s>1} sC_s(\delta, t)/\sum_{s>1} C_s(\delta, t)$, it being an unconventional average connected component size in which noninteracting vertices ($s = 1$) have not been considered in the sum. Then, one can consider (with $\delta \neq 1$) $\chi'(\delta, t) = (\langle s'^2\rangle - \langle s'\rangle^2)/N(\delta, t)$, proportional to the fluctuation about the mean quantity $\langle s'\rangle$. Then for χ', a different scaling ansatz is written

$$\chi'(\delta, t) = t^{\varsigma/\nu'} \ell[(\delta - \delta_{c'}) t^{1/\nu'}], \tag{15}$$

with ς, ν' two unknown exponents and $\ell(\cdot)$ a scaling function on which one may expect collapse of curves for various t with the correct choice of the exponents and of $\delta_{c'}$. The estimated exponent is $\nu' = 6.185 \pm 0.002$, with the scaling collapse occurring for $\varsigma = 6.827 \pm 0.002$; see Fig. 6(b). The value

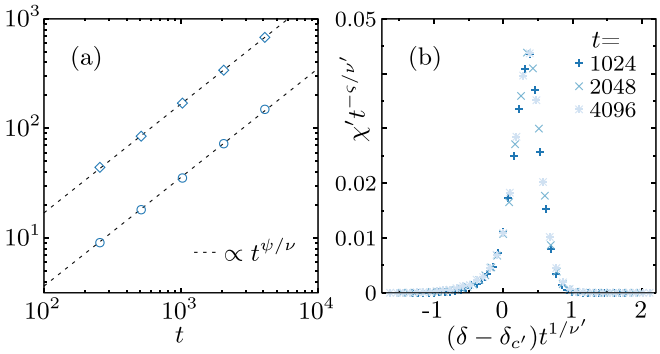


FIG. 6. In (a), scaling with t of χ^* ($\diamond$) and $\langle s\rangle^*$ ($\circ$) for increasing sizes t: 256, 512, 1024, 2048, and 4096. Dashed lines are visual references for the scaling $t^{\psi/\nu}$. In (b), scaling collapse of $\chi'(\delta, t)$, with $\delta_{c'}$ that precedes $\delta_{c,\xi}$ shown in Fig. 2(b) for $d = 0$; χ' quantifies the fluctuation about $\langle s'\rangle$.

of $\delta_{c'} = 0.425 \pm 0.001$ slightly anticipates the value $\delta_{c,\xi} \approx 0.442$ found for the model with $d = 0$. It is noteworthy that the estimated $\delta_{c'}$ precedes $1/2$ which it may occur presumably with, e.g., slow dimerization, mutation rates [32], yet it is also worth noting that noninteracting vertices have a relevant role in shaping the transition studied. While in the model with $d = 1$ at the critical point δ_c there might be a scaling $\langle s\rangle \propto t^{\psi/\nu}$ [Fig. 6(a)], further research would further characterize it as the focus here was mainly on the largest connected component. Cases with general divergence asymmetry rates σ are also yet to be deepened.

Conclusion. The largest connected component in the duplication-divergence network model with symmetric coupled divergence ($\sigma = 1/2$) plausibly emerges at a divergence rate δ_c found via finite-size scaling here with close correspondence with a generalized nonstationary (multifractal) form of vertex degree distribution moments, nearly agreeing with loci of zeros of Euler characteristics in finite-size graphs. Noninteracting vertices presence or absence in duplication was suggestive to change loci, making them comparable via a proper relation. The findings contribute to enhancing knowledge of evolving networks with multiple connected components.

Data availability. The data that support the findings of this article are not publicly available. The data are available from the author upon reasonable request.

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