Angular momentum of an unconfined charged Brownian particle in a static magnetic field within the framework of the generalized Langevin equation

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(Received 19 February 2026; revised 29 May 2026; accepted 7 August 2026; published 24 August 2026)

We investigate the angular momentum and kinetic energy of a classical charged Brownian particle in a static magnetic field within the framework of the generalized Langevin equation with Ornstein–Uhlenbeck noise. The mean kinetic energy retains its equilibrium value for all times after the external field is switched on. An analytical expression for the mean time-dependent angular momentum is derived and supported by high-precision numerical calculations. We show that, in the long-time limit, the mean angular momentum approaches a finite nonzero value. While this result might appear to contradict the Bohr–van Leeuwen theorem, we demonstrate that the system considered here does not correspond to a canonical equilibrium ensemble due to the absence of spatial confinement. In particular, although the velocity distribution thermalizes, the position of the particle remains unbounded, and the partition function over position space is not normalizable. The obtained nonzero angular momentum should therefore be interpreted as a property of a nonequilibrium asymptotic dynamical regime within the Langevin description rather than as a violation of equilibrium statistical mechanics. The present results provide a transparent analytical reference for stochastic dynamics of charged particles in magnetic fields and highlight the role of unbounded motion and noncommuting limits in such systems.

DOI: 10.1103/297f-hxh9

I. INTRODUCTION

One of the basic results that can be found in any textbook on magnetic phenomena is the Bohr−van Leeuwen (BvL) theorem [1,2]. This theorem, in its modern formulation [3], states that, in classical physics, and in thermal equilibrium, the net magnetization of a system of charged particles is zero. The magnetic field modifies the particle trajectories but does not change the total energy of the system, which is what determines the Boltzmann statistical distribution describing the thermal equilibrium. At the same time, it is understood that the BvL theorem breaks down if the system of classical charged particles uniformly rotates (as an equilibrium state) after turning on the magnetic field [4] or whenever one takes into account the internal magnetic fields produced by moving particles [5]. The theorem has also been questioned in relation to the Brownian motion (BM) of charged particles in a magnetic field [6] if described by the classical Langevin equation (LE) of motion [7]. However, the work [6] was soon criticized in [8,9], and the existence of diamagnetism predicted in [6] was refuted. The validity of the BvL theorem implicitly requires that the spatial degrees of freedom are properly normalized, which is typically ensured by confinement or finite volume [10–13], and now the BvL theorem is considered valid for Brownian particles [14,15].

In the present work, we consider a different situation: a single charged Brownian particle (BP) moving in an unbounded plane under the influence of a magnetic field. Within the Langevin description, the velocity distribution of the particle thermalizes, but the spatial distribution remains unbounded and does not correspond to a normalizable equilibrium ensemble. As a consequence, the assumptions underlying the BvL theorem are not fulfilled in this system. In this work, we are interested in whether the memory in the particle dynamics affects the mean angular momentum (and thus the magnetic moment) of BPs at long times after the application of a constant magnetic field. We assumed that the particle dynamics can be described by the generalized Langevin equation (GLE) [16], valid both for micrometer-sized BPs but also for atoms or molecules of the liquid itself. The averaging of the time correlation functions describing the particle dynamics is accomplished using the stochastic force (the colored thermal noise, not the white one as in the classical LE) included in the GLE and satisfying the fluctuation-dissipation theorem (FDT) [16]. As the basic theory in which the GLE is derived, the popular Zwanzig–Caldeira–Leggett (ZCL) particle-bath model can be chosen [17–19] if the bath is assumed neutral. The equations of motion are supplemented with the magnetic force acting on a charged BP. We have obtained the full-time dependence of the angular momentum of the BP after turning on the magnetic field and have come to its nonzero value at infinite time. It is discussed that this result does not violate the BvL theorem since the assumptions underlying this theorem are not satisfied in an unbounded stochastic system.

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II. ANGULAR MOMENTUM OF A CHARGED BROWNIAN PARTICLE

We consider a system of a charged BP of mass m and carrying charge Q immersed in a neutral bath. The system is in equilibrium up to the moment $t = 0$ when the external constant magnetic field $\mathbf{B} = (0, 0, B)$ is switched on. We restrict ourselves to the dynamics of the BP in the plane x, y, since its movement along the axis z is not affected by the applied magnetic field. It is assumed that the BM, in the absence of external fields, is described by the GLE [16]. The GLE in the case when the BP is charged and is under the influence of an external magnetic field was considered in Refs. [20,21], assuming, within the ZCL model, that the bath particles are charged as well. Here, a simpler situation of a neutral bath will be studied. Then the equations of motion of the GLE type for the BP derived in [20] simplify to two equations,

$$m\dot{\upsilon}_x(t) = QB\upsilon_y(t) - \int_0^t \upsilon_x(t')\Gamma(t-t')dt' + f_x(t),$$
$$m\dot{\upsilon}_y(t) = -QB\upsilon_x(t) - \int_0^t \upsilon_y(t')\Gamma(t-t')dt' + f_y(t), \quad (1)$$

where $\upsilon_x(t) = \dot{x}(t)$ and $\upsilon_y(t) = \dot{y}(t)$ are the projections of the velocity of the BP on the axis x and y, respectively. Here, $\Gamma(t)$ is known as the memory kernel (or memory function), and $f_x(t)$ and $f_y(t)$ are the projections of the random thermal force. These zero-mean forces differ only by changing x to y and, when it is possible, they will be denoted as $f(t)$. They are statistically independent and, assuming the stationarity of the system, related to the memory function by the FDT [16],

$$\langle f(t')f(t'')\rangle = k_B T\Gamma(|t'-t''|), \quad (2)$$

where $\langle\cdot\rangle$ means the statistical averaging. Depending on the system parameters and the distribution of the bath internal frequencies, $\Gamma(t)$ can be very different [18,22]. In what follows, we will use the exponentially decaying memory function $\Gamma(t) = (\xi/\tau)\exp(-t/\tau)$ corresponding to the thermal force called the Ornstein−Uhlenbeck noise, with ξ being the friction coefficient of the Stokes force when the noise is white [$\Gamma(t) = 2\xi\delta(t)$] and the integral in (1) becomes $-\xi\upsilon_x(t)$. The parameter τ has the sense of the relaxation time of the random force f.

We are interested in the time behavior of the mean magnetic moment $\langle M_z(t)\rangle$ of the BP in the applied magnetic field and mainly its value at $t\to\infty$. For this purpose, since the magnetic moment is proportional to the angular momentum $mL(t)$, in the following, we will focus on calculating the quantity $L(t) = x(t)\dot{y}(t) - y(t)\dot{x}(t)$, its average value, and $\langle L(t)\rangle$ at infinite times.

To determine the positions and velocities of the BP, it is suitable to apply the Laplace transform (LT) $\mathcal{L}\{\varphi(t)\} = \tilde{\varphi}(s) = \int_0^\infty \varphi(t)e^{-st}dt$ [the inverse LT will be denoted as $\mathcal{L}^{-1}\{\tilde{\varphi}(s)\} = \varphi(t)$] to Eqs. (1) and a similar equation for $\upsilon_y(t)$, which allows transforming the integrodifferential equations to algebraic ones. With the rules for the operations of the LT [23], the solution of these equations can be written in the form

$$\tilde{x}(s) = \frac{x(0)}{s} + m\dot{x}(0)\tilde{\Phi}(s) + m\dot{y}(0)\tilde{\Psi}(s) + \tilde{\Phi}(s)\tilde{f}_x(s) + \tilde{\Psi}(s)\tilde{f}_y(s), \quad (3)$$

$$\tilde{\upsilon}_x(s) = m\dot{x}(0)s\tilde{\Phi}(s) + m\dot{y}(0)s\tilde{\Psi}(s) + s\tilde{\Phi}(s)\tilde{f}_x(s) + s\tilde{\Psi}(s)\tilde{f}_y(s), \quad (4)$$

where

$$\tilde{\Phi}(s) = \frac{ms^2 + s\tilde{\Gamma}(s)}{[ms^2 + s\tilde{\Gamma}(s)]^2 + (QBs)^2}, \quad (5)$$

$$\tilde{\Psi}(s) = \frac{QBs}{[ms^2 + s\tilde{\Gamma}(s)]^2 + (QBs)^2}, \quad (6)$$

with, for the chosen memory function, $\tilde{\Gamma}(s) = \xi/(\tau s + 1)$. Similar solutions take place for $\tilde{y}(s)$ and $\tilde{\upsilon}_y(s)$ if x is replaced by y and the terms with $\tilde{\Psi}(s)$ are written with the opposite sign. Now we have to invert these solutions and, in the time domain average, the result for $L(t) = x(t)\upsilon_y(t) - y(t)\upsilon_x(t)$. Taking into account Eqs. (3) and (4) and equations for $\tilde{y}(s)$ and $\tilde{\upsilon}_y(s)$, the independence of the different projections of the thermal force, and the fact that among the products of the initial values and velocities only $\langle x^2(0)\rangle = \langle y^2(0)\rangle$ and $\langle\dot{x}^2(0)\rangle = \langle\dot{y}^2(0)\rangle = k_BT/m$ could be nonzero, we arrive at the following terms that could contribute to $\langle L(t)\rangle$ (the terms with x and y contribute equally):

$$\langle L_0(t)\rangle = -2\langle x(0)\mathcal{L}^{-1}\{s\tilde{\Psi}(s)\tilde{f}_x(s)\}\rangle,$$
$$\langle L_1(t)\rangle = 2m^2\langle\dot{x}^2(0)\rangle[\mathcal{L}^{-1}\{s\tilde{\Phi}(s)\}\mathcal{L}^{-1}\{\tilde{\Psi}(s)\} - \mathcal{L}^{-1}\{\tilde{\Phi}(s)\}\mathcal{L}^{-1}\{s\tilde{\Psi}(s)\}],$$
$$L_2(t) = 2[\mathcal{L}^{-1}\{s\tilde{\Phi}(s)\tilde{f}(s)\}\mathcal{L}^{-1}\{\tilde{\Psi}(s)\tilde{f}(s)\} - \mathcal{L}^{-1}\{\tilde{\Phi}(s)\tilde{f}(s)\}\mathcal{L}^{-1}\{s\tilde{\Psi}(s)\tilde{f}(s)\}]. \quad (7)$$

To calculate the mean values, one can, with no loss of generality, place the origin of the Cartesian axes at the initial point $x(0) = y(0) = 0$, so $\langle L_0(t)\rangle = 0$, and it is not necessary to consider it. The term $\langle L_1(t)\rangle$ can be rewritten as

$$\langle L_1(t)\rangle = 2mk_BT[\Phi'(t)\Psi(t) - \Phi(t)\Psi'(t)], \quad (8)$$

where it was used that $\langle\dot{x}^2(0)\rangle = k_BT/m$ due to equipartition, and $\Phi(0) = \Psi(0) = 0$, so $\mathcal{L}\{\Phi'(t)\} = s\tilde{\Phi}(s)$ and $\mathcal{L}\{\Psi'(t)\} = s\tilde{\Psi}(s)$ [23]. At $t\to\infty$, $\langle L_1(t)\rangle \to 0$. This can be proved with use of the expressions for $\Phi(t)$ and $\Psi(t)$, see Sec. III and the details of their obtaining in Appendix and Supplemental Material, Part II [24]. The time dependence of the mean angular momentum $m\langle L(t)\rangle$ is thus determined by $\langle L(t)\rangle = \langle L_1(t)\rangle + \langle L_2(t)\rangle$, while its value at $t\to\infty$ is given by $\langle L_2(t)\rangle$:

$$\langle L_2(t)\rangle = \frac{2k_BT\xi}{\tau}\int_0^t dt'\int_0^t dt''[\Psi(t')\Phi'(t'') - \Psi'(t')\Phi(t'')]\exp\frac{-|t'-t''|}{\tau}. \quad (9)$$

Here, we have used the convolution theorem [23] and the FDT (2), according to which the time correlation function of the random force is

$$\langle f(t')f(t'')\rangle = \frac{\xi k_BT}{\tau}\exp\frac{-|t'-t''|}{\tau}. \quad (10)$$

Remark. We will derive the formulas for positive values m, Q, B, ξ, and τ. It is evident that changing the sign of the value Q will change the sign of the functions $\tilde{\Psi}(s)$, $\Psi(t)$, and also

$\Psi'(t)$. This leads to the change of the sign of the mean value $\langle L(t)\rangle$ and its limit as $t\to\infty$.

An alternative treatment of generalized Langevin dynamics with memory is based on a Markovian embedding procedure, in which auxiliary variables are introduced so the resulting stochastic dynamics becomes Markovian in an extended phase space. Such an approach allows the construction of corresponding Fokker–Planck equations and has recently been applied to charged Brownian motion in magnetic fields [25]. The present work instead follows a direct analytical route based on the LT of the GLE. While both approaches are formally related, the LT method used here provides explicit time-dependent expressions for the displacement, velocity, angular momentum, and kinetic energy without introducing additional variables. We also emphasize that the physical setting considered in Ref. [25] differs from the present work in that the particle is confined and driven by an additional nonequilibrium noise violating the FDT, whereas our model describes an unconfined particle coupled to a thermal bath satisfying the fluctuation-dissipation theorem.

In the following Secs III–V, we give a short extraction of the results of the analytical calculations of functions $\Phi(t)$ and $\Psi(t)$, and then the angular momentum and kinetic energy of the BP given in Appendix and supported by extensive high-precision analytical and numerical calculations) presented in detail in the Supplemental Material [24].

III. INVERSE LAPLACE TRANSFORM OF THE FUNCTIONS $\tilde{\Phi}(s)$ AND $\tilde{\Psi}(s)$

As seen from Eqs. (8) and (9), the key role in obtaining the mean angular momentum $\langle L(t)\rangle=\langle L_1(t)\rangle+\langle L_2(t)\rangle$ in the time domain is played by the inverse LTs of functions $\tilde{\Phi}(s)$ and $\tilde{\Psi}(s)$ from Eqs. (5) and (6). The auxiliary functions $\Phi(t)$ and $\Psi(t)$ are searched for using the decomposition in simple fractions of their LTs:

$$\tilde{\Phi}(s)=\sum_{i=1}^{5}\frac{a_i}{s-s_i},\ \tilde{\Psi}(s)=\sum_{i=1}^{5}\frac{b_i}{s-s_i}. \tag{11}$$

It was used here that $\tilde{\Phi}(s)$ and $\tilde{\Psi}(s)$ have five distinct roots s_i. From Eqs. (11), one can immediately obtain the wanted functions in the time domain. However, obtaining the roots and the coefficients a_i and b_i is an involved task presented in Appendix and in more detail in [24]. It is shown there that $s_5=0$ and the other roots are complex and have negative real parts. Also, $s_3=s_1^*$ and $s_4=s_2^*$. The coefficient of the partial fraction decomposition can be determined straightforwardly by the substitution of roots after subtracting the root factors, or, formally,

$$a_i=[(s-s_i)\tilde{\Phi}(\mathrm{s})]|_{s=s_i},\ b_i=[(s-s_i)\tilde{\Psi}(\mathrm{s})]|_{s=s_i}, \tag{12}$$

$i=1,2$. We also have $a_3=a_1^*$, $a_4=a_2^*$, $b_3=b_1^*$, $b_4=b_2^*$, and

$$a_5=\frac{\xi}{(QB)^2+\xi^2},\quad b_5=\frac{QB}{(QB)^2+\xi^2}. \tag{13}$$

Now, from Eqs. (11), we can write the searched function in the complex form [23]

$$\Phi(t)=a_1e^{s_1t}+a_2e^{s_2t}+a_1^*e^{s_1^*t}+a_2^*e^{s_2^*t}+a_5. \tag{14}$$

It can be shown that $b_k=-ia_k$, $k=1,2$. Hence,

$$\Psi(t)=-ia_1e^{s_1t}-ia_2e^{s_2t}+ia_1^*e^{s_1^*t}+ia_2^*e^{s_2^*t}+b_5. \tag{15}$$

It is also possible to write functions $\Phi(t)$ and $\Psi(t)$ in real form; however, the computation of the integrals in (9) is much simpler in the complex form.

IV. MEAN ANGULAR MOMENTUM AND ITS $t\to\infty$ LIMIT

So, to calculate the mean value $\langle L(t)\rangle$ in Eqs. (8) and (9), we deal with the integral:

$$I_{12}(c_1,c_2)=\int_0^t\int_0^t dt'dt''\exp[c_1(t-t')]\times\exp[c_2(t-t'')]\exp\frac{-|t'-t''|}{\tau}, \tag{16}$$

where c_1 and c_2 are complex constants. It is possible to use $c_1=0$ or $c_2=0$ (not simultaneously). Indeed, we are first interested in the limit $\lim\limits_{t\to\infty}\langle L(t)\rangle$. To find a formula for such a limit, it is sufficient to use the limit

$$\lim_{t\to\infty}I_{12}(c_1,c_2)=\frac{\tau[\tau(c_1+c_2)-2]}{(c_1+c_2)(\tau c_1-1)(\tau c_2-1)}, \tag{17}$$

which is derived using the conditions $\tau>0$, $\mathrm{Re}(s_k)<0$, $k=1,2,3$, and 4.

The integrals $I_{12}(c_1,c_2)$ were calculated for given values of t using the computer algebra system Maple. After many analytical integral calculations and using the numerical tests with high precision, the full mean angular momentum was obtained (see [24], Eq. (45) in Part IV):

$$\begin{aligned}m[\langle L_1(t)\rangle+\langle L_2(t)\rangle]=-2mk_BT\{&b_5[1-\cos(\mathrm{Im}(s_2)t)e^{\mathrm{Re}(s_2)t}]\\&-a_5\sin(\mathrm{Im}(s_2)t)e^{\mathrm{Re}(s_2)t}\\&+2\mathrm{Im}(a_1)[\cos(\mathrm{Im}(s_1)t)e^{\mathrm{Re}(s_1)t}\\&-\cos(\mathrm{Im}(s_2)t)e^{\mathrm{Re}(s_2)t}]\\&+2\mathrm{Re}(a_1)[\sin(\mathrm{Im}(s_1)t)e^{\mathrm{Re}(s_1)t}\\&-\sin(\mathrm{Im}(s_2)t)e^{\mathrm{Re}(s_2)t}]\}.\end{aligned} \tag{18}$$

.

Its $t\to\infty$ limit is determined by the coefficient b_5 from Eqs. (13),

$$\lim_{t\to\infty}m\langle L(t)\rangle=-2k_BTmb_5=-\frac{2k_BTm}{\xi}\frac{QB/\xi}{(QB/\xi)^2+1} \tag{19}$$

Here, $2k_BT/\xi$ is the Einstein diffusion coefficient of the BP in the plane x, y when the external field is absent. In particular, for $B\approx0$ and fixed ξ, we get $\lim_{t\to\infty}\langle L(t)\rangle\approx-2k_BTQB/\xi^2$ and $\lim_{t\to\infty}\langle L(t)\rangle=0$ if $B\to\infty$.

Figure 1 shows the relative value $\langle L(t)\rangle$ divided by the $\lim_{t\to\infty}\langle L(t)\rangle$ as a function of t/τ for $Q=1.6\times10^{-19}$C, and the rest of the fixed parameters are estimated as for a methane molecule in water at room temperature, $\xi=2.0\cdot10^{-12}$ kg/s, $\tau=4.0\cdot10^{-14}$ s, and $m=3.0\cdot10^{-26}$. Figure 2 shows that the mean value $\langle L(t)\rangle$ converges to the limit very fast in an exponentially oscillatory manner.

For the details of the calculation of $m\langle L(t)\rangle$ and its $t\to\infty$ limit, see [24], Part IV.

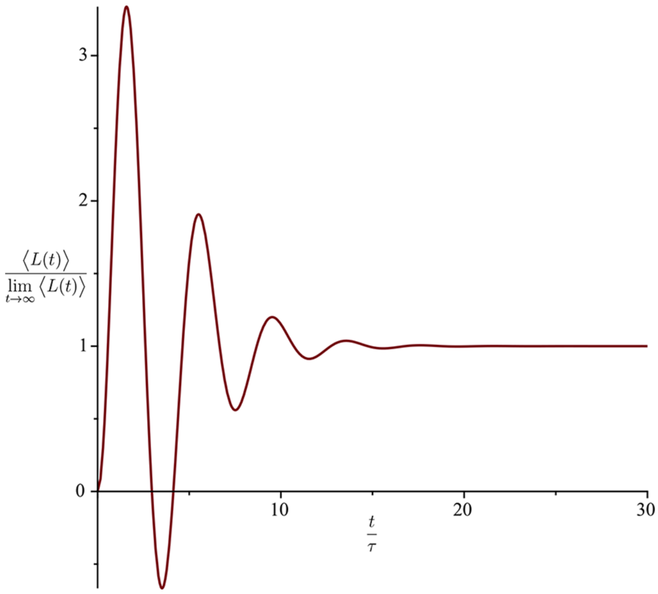

FIG. 1. Mean value $\langle L(t)\rangle$ divided by the $\lim_{t\to\infty}\langle L(t)\rangle$ as a function of time.

V. MEAN KINETIC ENERGY

Here, we present the results of the calculation of the mean kinetic energy of the BP,

$$\langle E(t)\rangle = m\big[\langle\upsilon_x^2(t)\rangle + \langle\upsilon_y^2(t)\rangle\big]/2, \tag{20}$$

$\langle\upsilon_x^2(t)\rangle = \langle\upsilon_y^2(t)\rangle$. With the use of the solution (4) for $\tilde{\upsilon}_x(s)$ and a similar one for $\tilde{\upsilon}_y(s)$, and skipping the terms that, after averaging, give zero (as it was done in the calculation of the mean angular momentum), Eq. (20) can be rewritten in the form

$$\begin{aligned}\langle E(t)\rangle &= m[A(t) + B(t) + C(t) + D(t)]\\ &= \langle E_1(t)\rangle + \langle E_2(t)\rangle,\end{aligned} \tag{21}$$

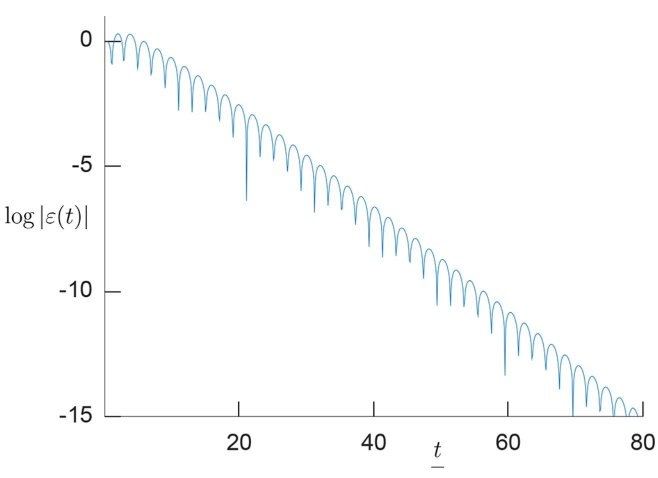

FIG. 2. Logarithm of the absolute value of the relative difference between $\langle L(t)\rangle$ and $\lim_{t\to\infty}\langle L(t)\rangle$, $\varepsilon(t) = (\langle L(t)\rangle - \lim_{t\to\infty}\langle L(t)\rangle)/\lim_{t\to\infty}\langle L(t)\rangle$.

where

$$\begin{aligned}A(t) &= m^2\langle\dot{x}^2(0)\rangle(\mathcal{L}^{-1}\{s\tilde{\Phi}(s)\})^2,\\ B(t) &= m^2\langle\dot{y}^2(0)\rangle(\mathcal{L}^{-1}\{s\tilde{\Psi}(s)\})^2,\\ C(t) &= \langle(\mathcal{L}^{-1}\{s\tilde{\Phi}(s)\tilde{f}_x(s)\})^2\rangle,\\ D(t) &= \langle(\mathcal{L}^{-1}\{s\tilde{\Psi}(s)\tilde{f}_y(s)\})^2\rangle.\end{aligned} \tag{22}$$

The term

$$\langle E_1(t)\rangle = A(t) + B(t) = m^2k_BT[(\Phi'(t))^2 + (\Psi'(t))^2] \tag{23}$$

disappears at long times. The explicit formula for $\langle E_1(t)\rangle$ is obtained in [24], Part V, Eq. (48):

$$\begin{aligned}\langle E_1(t)\rangle = 4m^2k_BT\{&|a_1|^2|s_1|^2e^{2\mathrm{Re}(s_1)t} + |a_2|^2|s_2|^2e^{2\mathrm{Re}(s_2)t}\\ &+ 2e^{-t/\tau}[(\mathrm{Re}(a_1s_1)\mathrm{Re}(a_2s_2) + \mathrm{Im}(a_1s_1)\mathrm{Im}(a_2s_2))\\ &\times\cos(\Sigma_-t/\sqrt{2}m\tau) + (\mathrm{Re}(a_1s_1)\mathrm{Im}(a_2s_2)\\ &- \mathrm{Im}(a_1s_1)\mathrm{Re}(a_2s_2))\sin(\Sigma_-t/\sqrt{2}m\tau)]\}.\end{aligned} \tag{24}$$

The term

$$\langle E_2(t)\rangle = m[C(t) + D(t)] \tag{25}$$

that contributes to the $t\to\infty$ limit is determined by the integral

$$\begin{aligned}\langle E_2(t)\rangle = \frac{k_BTm\xi}{\tau}\int_0^t dt'\int_0^t dt''[\Phi'(t')\Phi'(t'')\\ + \Psi'(t')\Psi'(t'')]\exp\frac{-|t'-t''|}{\tau}.\end{aligned} \tag{26}$$

The formula for $\langle E_2(t)\rangle$ is given in [24], Part V, Eq. (49):

$$\begin{aligned}\langle E_2(t)\rangle =& \frac{4mk_BT\xi}{\tau}\left\{\frac{|a_1|^2|s_1|^2[\mathrm{Re}(s_1)+1/\tau]}{|s_1+1/\tau|^2\mathrm{Re}(s_1)}e^{2\mathrm{Re}(s_1)t}\right.\\ &+ \frac{|a_2|^2|s_2|^2[\mathrm{Re}(s_2)+1/\tau]}{|s_2+1/\tau|^2\mathrm{Re}(s_2)}e^{2\mathrm{Re}(s_2)t}\\ &\left.+2\mathrm{Re}\left[\frac{a_1s_1a_2^*s_2^*(s_1+s_2^*+2/\tau)}{(s_1+1/\tau)(s_2^*+1/\tau)(s_1+s_2^*)}e^{(s_1+s_2^*)t}\right]\right\}\\ &+ k_BT,\end{aligned} \tag{27}$$

with the limit $\lim_{t\to\infty}\langle E_2(t)\rangle = k_BT$. Moreover, it is proven in [24] that the mean energy (20) is $\langle E(t)\rangle = \langle E_1(t)\rangle + \langle E_2(t)\rangle = k_BT$ at any time $t > 0$, i.e., the same as it was in equilibrium up to $t = 0$ in accordance with the equipartition theorem. The dependencies of the parts of the mean energy on the time are shown in Fig. 3.

VI. CONCLUSIONS

In conclusion, we have considered the classical BM of a charged particle in a neutral bath when this system is placed in a constant magnetic field. The dynamics of the BP is described by the GLE, which naturally incorporates memory effects. The solution yields a nonzero mean angular momentum of the BP (and thus a nonzero mean diamagnetic moment) in the long time limit. This result should be interpreted with care as a property of a nonequilibrium asymptotic regime of the unconfined Brownian particle rather than as a steady state.

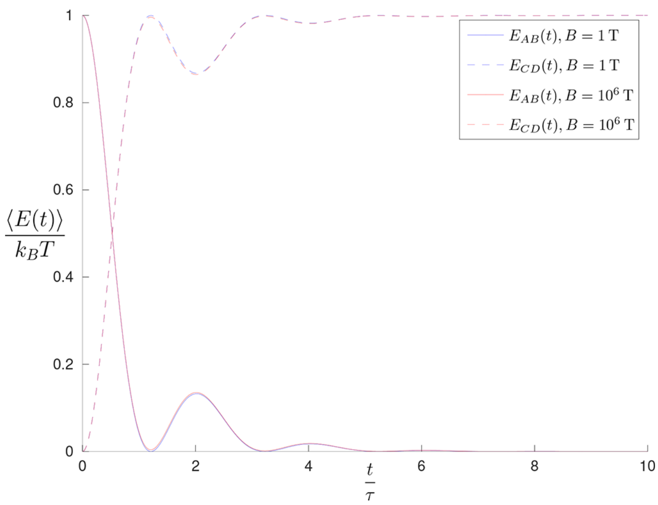

FIG. 3. Time dependence of the parts $\langle E_1(t)\rangle$ (24) and $\langle E_2(t)\rangle$ (27) giving in sum the full mean kinetic energy (21) $\langle E(t)\rangle = k_BT$. The parameters are the same as those used in Fig. 1.

While the velocity distribution thermalizes and quantities such as the mean angular momentum and kinetic energy approach constant values, the position distribution continues to broaden indefinitely because the particle undergoes unbounded diffusion. Consequently, the full phase-space distribution never becomes stationary and does not correspond to a normalizable canonical equilibrium ensemble. The obtained nonzero angular momentum therefore does not represent a violation of the Bohr–van Leeuwen theorem but rather reflects the nonequilibrium nature of the asymptotic dynamics in an unconfined system. The mean kinetic energy of the particle remains equal to k_BT for all times after the magnetic field is switched on. Although the velocity degrees of freedom thermalize and satisfy the equipartition theorem, the absence of spatial confinement prevents the system from reaching a canonical equilibrium state. The physical origin of the effect lies in persistent correlations between the particle position and velocity induced by the magnetic field and stochastic forcing. These correlations lead to a finite angular momentum even at long times. In contrast, if the particle is confined (for example, by a harmonic potential), the system reaches a proper equilibrium state, and the BvL theorem is recovered. This highlights the crucial role of confinement in such systems as discussed in several papers, in which the magnetic moment of a charged Brownian particle was calculated within the framework of the ordinary memoryless Langevin equation with dissipation, classical [10,11,12] or quantum [13]. Importantly, it was shown in these papers that taking the limit of long time and vanishing confinement cannot be interchanged. Within our approach, the result for the angular momentum at $t \to \infty$ is the same as in the memoryless case [26] and also does not depend on the relaxation time of the random force τ, but the time dependence is different as a consequence of the memory in the particle dynamics.

Another interesting problem is connected with our counterintuitive finding that, according to the simple formula proposed by us, the angular momentum (and thus magnetic moment) of the BP tends to zero as the magnetic field B approaches infinity. This is in accordance with the behavior of the mean square displacement of the particle (see [27] for the ordinary Langevin theory and [28] for the GLE with Ornstein–Uhlenbeck memory), which becomes zero at any time t if B is extremely strong (at long times $\mathrm{MSD}(t) \approx 4k_BT\xi t/[\xi^2 + (QB)]^2$): the BP is pushed to the initial point where it was before turning on the magnetic field.

In typical Brownian systems, the magnitude of the obtained effect is extremely small, as can be confirmed by order-of-magnitude estimates. So, for a micron-sized Brownian particle ($a \approx 1\,\mu\mathrm{m}$) in water ($\eta \approx 10^{-3}\mathrm{Pa}\cdot\mathrm{s}$), the friction coefficient can be estimated using the Stokes formula $\gamma = 6\pi\eta a$. At room temperature ($T \approx 300\mathrm{K}$), and assuming a strong magnetic field of 10 T and a charge Q equivalent to several elementary charges, Eq. (19) yields a value of of the angular momentum of approximately $10^{-36}\mathrm{kgm}^2/\mathrm{s}$. The effect is therefore unlikely to be directly observable. However, it may become more pronounced in systems with lower damping or stronger magnetic fields, such as charged particles in rarefied gases or plasma environments. For a discussion of a demanding problem of experimental realization of the studied classical system, we refer to Ref. [14].

The present analysis highlights the importance of carefully distinguishing between equilibrium and nonequilibrium situations in stochastic dynamics and clarifies the role of unbounded motion in systems subjected to magnetic fields. Future work may extend this approach to confined systems, interacting particles, or more general memory kernels to further elucidate the interplay between stochastic dynamics and magnetic response.

ACKNOWLEDGMENTS

This work was supported by the Scientific Grant Agency of the Slovak Republic through Grant No. VEGA 1/0197/26. All Maple calculations were performed using the MLIT License Management System. The authors thank the anonymous referees for their constructive comments, which helped to improve the clarity of this work.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: CALCULATION OF THE MEAN ANGULAR MOMENTUM AND KINETIC ENERGY

The purpose of this appendix is to provide an explanation clarifying the way of obtaining the basic article formulas. The detailed derivation of the results is given in the Supplemental Material [24], where every step of calculation is presented in a controllable way.

1. Auxiliary functions $\Phi(t)$ and $\Psi(t)$

The generalized Langevin Eqs. (1) are solved using the Laplace transform, which converts the integrodifferential equations of motion into algebraic ones. The solution can be expressed in terms of two auxiliary functions $\tilde{\Phi}(s)$ and $\tilde{\Psi}(s)$

given in Eqs. (5) and (6). These formulae can be rewritten as follows:

$$\tilde{\Phi}(s) = \hat{\Phi}(S) = \frac{\tau^2 S(S^2 - S + \tau\xi/m)}{m(S-1)D(S)}, \quad (A1)$$

$$\tilde{\Psi}(s) = \hat{\Psi}(S) = \frac{QB\tau^3 S^2}{m^2(S-1)D(S)}, \quad (A2)$$

where $S = \tau s + 1$ and

$$D(S) = S^4 - 2S^3 + \left[1 + \frac{2\xi\tau}{m} + \left(\frac{QB\tau}{m}\right)^2\right]S^2 - \frac{2\xi\tau}{m}S + \left(\frac{\xi\tau}{m}\right)^2$$

has a form enabling to find its roots analytically, and to determine the partial fractions decomposition of both functions $\hat{\Phi}(S)$ and $\hat{\Psi}(S)$. The solution of the equation $D(S) = 0$ gives the roots S_i [24], Part II, from which

$$s_{1,2} = \frac{-\sqrt{2}m \pm \Sigma_+}{2\sqrt{2}m} + i\frac{\sqrt{2}QB\tau \pm \Sigma_-}{2\sqrt{2}m\tau} \quad (A3)$$

and $s_{3,4} = s^*_{1,2}$. Here,

$$\Sigma_- = \sqrt{\Sigma - \Pi},\ \Sigma_+ = \sqrt{\Sigma + \Pi} = 2QBm\tau/\Sigma_-,$$
$$\Sigma = [(QB\tau)^2 + m(\sqrt{m} - 2\sqrt{\xi\tau})]^{1/2} \times [(QB\tau)^2 + m(\sqrt{m} + 2\sqrt{\xi\tau})]^{1/2},$$
$$\Pi = m^2 - 4m\xi\tau - (QB\tau)^2.$$

It can be proved that the real parts of these roots are negative (if $\xi\tau/m > 0$) [24]. This was used in the calculation of mean values of the angular momentum and kinetic energy.

The coefficients a_i and b_i in Eqs. (11) are obtained using the concealment method [24] according to Eqs. (12). They are of a rather complicated form ([24], Eq. (35)), except a_5 and b_5 shown in Eqs. (13).

The functions $\Phi(t)$ and $\Psi(t)$ have the form

$$\Phi(t) = 2\mathrm{Re}[V(t)] + a_5,\ \Psi(t) = 2\mathrm{Im}[V(t)] + b_5, \quad (A4)$$

where $V(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_1^* e^{s_1^* t} + a_2^* e^{s_2^* t}$. Thus, only four coefficients are needed to express the auxiliary functions $\Phi(t)$ and $\Psi(t)$, Eqs. (14) and (15). Their time derivatives are

$$\Phi'(t) = 2\mathrm{Re}[W(t)],\ \Psi'(t) = 2\mathrm{Im}[W(t)],$$

where $W(t) = V'(t)$.

At $t = 0$, we have

$$\Phi(0) = 2\mathrm{Re}(a_1) + 2\mathrm{Re}(a_2) + a_5 = 0,$$

$$\Psi(0) = 2\mathrm{Im}(a_1) + 2\mathrm{Im}(a_2) + b_5 = 0.$$

2. Calculation of the mean angular momentum

The mean values of the angular momentum and kinetic energy are calculated using many integrals of the following type, cf. Eq. (16):

$$I_{12}(c_1, c_2) = \int_0^t \int_0^t dt_1 dt_2 e^{c_1 t_1 + c_2 t_2} \exp\frac{-|t_1 - t_2|}{\tau}$$
$$= \frac{\exp[(c_1 - 1/\tau)t]}{(c_1 - 1/\tau)(c_2 + 1/\tau)} - \frac{\exp[(c_2 - 1/\tau)t]}{(c_1 + 1/\tau)(c_2 - 1/\tau)} + \frac{[(c_1 + c_2) + 2/\tau]\exp[(c_1 + c_2)t]}{(c_1 + 1/\tau)(c_2 + 1/\tau)(c_1 + c_2)} + \frac{c_1 + c_2 - 2/\tau}{(c_1 - 1/\tau)(c_2 - 1/\tau)(c_1 + c_2)}. \quad (A5)$$

The cases $c_1 = 0$ or $c_2 = 0$ are possible, however, $|c_1| + |c_2| > 0$. Because c_1 and c_2 are some nonzero roots s or conjugate values $s*$ with negative real values, the limit of the right side of (A5) at $t \to \infty$ is equal to the last term (all exponential terms vanish).

The mean value of the angular momentum $m\langle L(t)\rangle = m(\langle L_1(t)\rangle + \langle L_2(t)\rangle)$ is determined by the functions $\langle L_1(t)\rangle$ and $\langle L_2(t)\rangle$ from Eqs. (8) and (9), respectively. Using Eqs. (A4), $\langle L_1(t)\rangle$ can be expressed as

$$\langle L_1(t)\rangle = 8mk_BT \cdot \mathrm{Re}\left[-iV(t)W^*(t) + \frac{(QB + i\xi)W(t)}{2[(QB)^2 + \xi^2]}\right] \quad (A6)$$

and

$$\langle L_2(t)\rangle = \frac{4\xi k_BT}{\tau}\int_0^t dt' \int_0^t dt''\{[\mathrm{Re}W(t - t'')] \times [2\mathrm{Im}V(t - t') + b_5] - [\mathrm{Im}W(t - t')] \times [2\mathrm{Re}V(t - t'') + a_5]\}e^{-|t' - t''|/\tau}. \quad (A7)$$

After substituting the functions $V(t)$ and $W(t)$ in (A6), we obtain the final result for the t dependence of $\langle L_1(t)\rangle$ [24]. The full-time result for $\langle L_2(t)\rangle$ requires calculating many double integrals of the type of (A5), leading in summary to 160 single integrals arising when $V(t)$ and $W(t)$ are put in (A7). All details of the calculations supported by precise numerical checks are given in [24], Part II. Despite the involved calculations, the final result for $m\langle L(t)\rangle$ shown in Eq. (45) (Eq. (18) in the main text) is quite simple, with the exceedingly simple $t \to \infty$ limit, Eq. (19). This result shows that the long-time angular momentum is finite and determined by the competition between magnetic and dissipative effects. It also highlights that the asymptotic behavior does not depend on the detailed structure of the memory kernel but rather on general features of the Langevin dynamics.

3. Calculation of the mean kinetic energy

The resulting formulae (24) and (27) for the mean kinetic energy of the BP have been obtained in the following way. First, the part of the kinetic energy $\langle E_1(t)\rangle$ (23) can be expressed through the derivations of the auxiliary functions $\Phi(t)$ and $\Psi(t)$ as (see [24], Part V)

$$\langle E_1(t)\rangle = m^2 k_BT[\Phi'(t)]^2 + [\Psi'(t)]^2 = 4m^2 k_BT|W(t)|^2. \quad (A8)$$

The part of the kinetic energy $\langle E_2(t)\rangle$ that does not disappear at $t \to \infty$ is

$$\langle E_2(t)\rangle = \frac{4k_B T m\xi}{\tau}\int_0^t dt' \int_0^t dt'' \mathrm{Re}[W(t')W(t'')]e^{-\frac{t'-t''}{\tau}}. \tag{A9}$$

After the substitution of $W(t)$ in (A8), we obtain Eq. (24) for $\langle E_1(t)\rangle$ in the main text. The calculation of $\langle E_2(t)\rangle$ requires the use of eight integrals of the type (A5). Collecting all terms in $\langle E_1(t)\rangle$ and $\langle E_2(t)\rangle$, explicitly given in [24], we have proved that all exponential functions in $\langle E(t)\rangle = \langle E_1(t)\rangle + \langle E_2(t)\rangle$ vanish and the mean energy is equal to the $t \to \infty$ limit of $\langle E_2(t)\rangle$ only, $\langle E(t)\rangle = 1\, k_B T$.

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