Quantum geometry effects in quantum field theory: Hamiltonian constraint generates gravity-matter entanglement in spherically symmetric loop quantum gravity

Gaoping Long[1] and Cong Zhang[2,*]

[1]*College of Physics and Optoelectronic Engineering, Jinan University, Guangzhou 510632, Guangdong, China*

[2]*School of Physics and Astronomy, Key Laboratory of Multiscale Spin Physics, Ministry of Education, Beijing Normal University, Beijing 100875, China*

(Received 15 February 2026; accepted 29 July 2026; published 19 August 2026)

In this paper, we address a foundational challenge in quantum field theory on curved spacetime by developing a consistent framework in spherically symmetric loop quantum gravity. We introduce a methodology for defining meaningful superpositions of quantum geometry and matter states near the semiclassical apparent horizon. This is achieved by identifying a restricted subspace of the gravitational phase space, which ensures unitary equivalence among Fock representations of a scalar field across different quantum geometries. Within the resulting well-defined state space, we derive weak solutions to the quantum Hamiltonian constraint of general relativity near the semiclassical apparent horizon. Furthermore, we generalize the Hartle-Hawking vacuum state to this quantum geometric framework. The resulting state exhibits the inherent entanglement between geometry and matter, which arises from the quantum Hamiltonian constraint of general relativity. This work establishes a principled framework for studying geometry-matter entanglement and offers new insights into the quantum foundations of the black hole information paradox.

DOI: 10.1103/wl57-t1lv

I. INTRODUCTION

The black hole information paradox remains one of the most profound challenges at the interface of quantum mechanics (QM) and general relativity (GR). Specifically, while black holes are among the most significant predictions of GR, Hawking's discovery of thermal radiation from evaporating black holes suggests that information about the initial state of matter may be lost during evaporation—thereby violating the unitarity of QM [1–8]. A central development in addressing this paradox is the derivation of the Page curve, which describes the time evolution of the entanglement entropy of Hawking radiation under the assumption of unitary evaporation [9–11]. The Page curve implies that, at late times, the entropy of the radiation is suppressed as it becomes purified by correlations with the remaining black hole interior—a process that necessarily requires entanglement between the radiation field and the underlying spacetime geometry. This insight strongly suggests that a complete resolution of the information paradox must account for the quantum nature of geometry itself. In a full quantum gravity framework, the total state of the system may take the form of an entangled superposition

$$|\Psi\rangle = \sum_{\mathfrak{g},\phi} c_{\mathfrak{g},\phi} |\mathfrak{g}\rangle \otimes |\phi\rangle, \tag{1}$$

where $|\mathfrak{g}\rangle$ denotes a quantum state of spacetime geometry and $|\phi\rangle$ a state of the matter field. However, while the Page curve has been successfully reproduced using semiclassical gravitational path integrals and replica wormhole techniques [8,11–13], these approaches do not yet provide a dynamical unitary evolution of a state like (1) within a concrete Hilbert space of quantum gravity. More fundamentally, the very definition of the product states $|\mathfrak{g}\rangle \otimes |\phi\rangle$ remains ambiguous: what does it mean to define a quantum field on a quantum geometry [14,15]?

Historically, quantum field theory on curved spacetime (QFTCS) sidesteps this issue by treating geometry as a fixed classical background $\mathfrak{g}$, on which matter states $|\phi\rangle_{\mathfrak{g}}$ are constructed [16–18]. This approach is perturbative in nature, since the backreaction of the energy-momentum tensor $\hat{T}_{\mu\nu}$ of the quantum matter field on the classical geometry is ignored or effectively treated via the effective Einstein equation $G_{\mu\nu} = 8\pi G\langle\hat{T}_{\mu\nu}\rangle$. However, from the perspective of quantum gravity, distinct matter states generically correspond to distinct geometric configurations, even if their classical limits coincide. In particular, these microscopic geometric distinctions may encode the microstates responsible for black hole entropy [19–24].

*Contact author: cong.zhang@bnu.edu.cn

Consequently, when considering the entanglement between geometry and matter, the QFTCS prescription—building all $|\phi\rangle$ on a single classical $\mathfrak{g}$—becomes ill defined, as it conflates physically distinct quantum geometries [25].

To construct the entangled state (1) rigorously, three interrelated obstacles must be overcome:

(1) A well-defined Hilbert space $\mathcal{H}_{\mathrm{g}}$ for quantum geometry, containing states $|\mathfrak{g}\rangle$ that admit a semiclassical interpretation;
(2) A consistent formulation of the matter Hilbert space $\mathcal{H}^{\phi}$ on each quantum geometry $|\mathfrak{g}\rangle$, such that the superpositions of the product states $|\mathfrak{g}\rangle \otimes |\phi\rangle$ across different $|\mathfrak{g}\rangle$ are mathematically meaningful;
(3) The product states $|\mathfrak{g}\rangle \otimes |\phi\rangle$ must satisfy the quantum constraints of GR, reflecting its nature as a totally constrained Hamiltonian system [26,27].

In this work, we address these challenges within the framework of loop quantum gravity (LQG)—a nonperturbative, background-independent approach to quantum gravity that provides a rigorously defined kinematical Hilbert space for geometry [28–33]. LQG has already yielded microscopic derivations of black hole entropy through counting of boundary states [34–42], and its polymer quantization scheme resolves the classical singularity in symmetry-reduced models, yielding bouncing cosmology models and regular black hole interiors [43–57]. Specifically, we work in the spherically symmetric model of gravity and employ semiclassical coherent states $|\mathfrak{g}\rangle$ in loop quantum theory for gravitational quantum states, with $|\mathfrak{g}\rangle$ being sharply peaked at classical phase space points and satisfying the Ehrenfest property [58–62]. To tackle the second obstacle, we recognize that the Fock representation of a scalar field depends on the background geometry. For arbitrary $|\mathfrak{g}\rangle$ and $|\mathfrak{g}'\rangle$, the corresponding Fock spaces $\mathcal{H}^{\phi}_{\mathfrak{g}}$ and $\mathcal{H}^{\phi}_{\mathfrak{g}'}$ are generally unitarily inequivalent, rendering superpositions like (1) ill defined [17]. We resolve this by identifying a restricted subspace $\mathcal{U}_{v,v'}$ of the gravitational phase space, in which the Bogoliubov transformation between Fock spaces $\mathcal{H}^{\phi}_{\mathfrak{g}}$ and $\mathcal{H}^{\phi}_{\mathfrak{g}'}$ satisfies the Shale-Stinespring criterion (i.e., the β coefficients form a Hilbert-Schmidt operator) [17,63]. In this restricted subset $\mathcal{U}_{v,v'}$, a common Fock space $\mathcal{H}^{\phi}$ exists, enabling a quantum state space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$ in which superpositions like (1) are well defined. The third obstacle stems from the fact that GR, in its Hamiltonian formulation, is governed entirely by the ADM constraints—the diffeomorphism and Hamiltonian constraints [26,27]. Hence, physical states must lie in the kernel of quantum constraint operators. Rather than seeking strong solutions (which remain elusive even in reduced models), we adopt a weak constraint approach. In detail, we require that matrix elements of the quantum Hamiltonian constraint vanish between selected coherent–Fock states. This yields a family of approximate physical states that are inherently entangled between quantum geometries and matter.

Our results show that weak solutions to the quantum Hamiltonian constraint, which are constructed from gravitational coherent states $|\mathfrak{g}\rangle$ and scalar Fock states $|\phi\rangle$, naturally encode entanglement between geometry and matter. Specifically, when the coherent state $|\mathfrak{g}\rangle$ is peaked at a semiclassical geometry containing an apparent horizon, the admissible Fock modes (defined in a modified tortoise coordinate) are correlated with the fluctuating wave packets on the gravitational phase space. Crucially, this correlation is not imposed by hand, but emerges as a direct consequence of the imposition of the constraint of GR. Moreover, we consider a generalized Hartle-Hawking vacuum on quantum geometry by superposing the weak solutions of the quantum Hamiltonian constraint, which takes precisely the entangled form (1). Generally, this work provides a concrete realization of how the constraints of GR enforce geometry-matter entanglement, and it may offer a first-principles foundation for understanding the quantum origin of the Page curve and the unitary evaporation of black holes.

This paper is organized as follows. In Sec. II A, we will review the classical theory of the gravitationally spherically symmetric model coupled to a massless scalar field in its Hamiltonian formulation; Then, in Sec. II B, we will consider the quantum scalar field on a classical geometry and highlight a fundamental issue inherent in QFT on a fixed classical background geometry. To tackle this issue, we will turn to quantum gravity in Sec. III. Specifically, in Sec. III A, we will discuss the unitary equivalence among Fock representations for the scalar field across different quantum geometries; then, the well-defined weak solution space of the quantum Hamiltonian constraint will be given in Sec. III B; Further, in Sec. III C, we will establish a generalized Hartle-Hawking vacuum on quantum geometry by superposing weak solutions of the quantum Hamiltonian constraint and analyze its properties. Finally, we conclude in Sec. IV with a discussion of our results and future directions.

II. SPHERICALLY SYMMETRIC MODEL OF GRAVITY COUPLING TO MASSLESS SCALAR FIELD

A. Classical theory

The spherically symmetric model of gravity provides us an effective and simplified framework to study GR and its quantization theory. By carrying out the spherically symmetric reduction in the connection formulation of GR, the degrees of freedom of phase space are eliminated and the constraint equations are simplified [51,64–67]. Now, let us first give the basic elements of the spherically symmetric model of gravity in the connection formulation.

The connection formulation of the spherically symmetric GR contains a spatial three-manifold $\Sigma = \mathbb{X} \times \mathbb{S}^2$ carrying the $su(2)$-valued Ashtekar-Barbero connection field A^i_a and its conjugate momentum field E^b_j, where $\mathbb{X}$ denotes a 1D manifold and $\mathbb{S}^2$ is the two-sphere. Let us introduce the coordinates (x, θ, φ) on Σ which adapt to its topology. Then, the phase space $\mathcal{P}^{\mathrm{g}}$ of the gravitational field is coordinatized by the reduced fields $(K_x, E^x, K_\varphi, E^\varphi)$ defined on the quotient manifold $\Sigma/SU(2) \cong \mathbb{X}$ [64,68]. The Ashtekar-Barbero variables (A^i_a, E^b_j) in terms of the reduced variables $(K_x, E^x, K_\varphi, E^\varphi)$ can be given as [69,70]

$$\begin{aligned} A^i_a\tau_i \mathrm{d}x^a &= K^i_a\tau_i \mathrm{d}x^a + \Gamma^i_a\tau_i \mathrm{d}x^a, \\ E^a_i\tau^i\partial_a &= E^x \sin\theta\tau_3\partial_x + E^\varphi \sin\theta\tau_1\partial_\theta + E^\varphi\tau_2\partial_\varphi, \end{aligned} \tag{2}$$

where $\tau_j = -i\sigma_j/2$ with σ_j being the Pauli matrices, Γ^i_a is the spin connection compatible with the densitized triad E^a_i, and K^i_a is the extrinsic curvature 1-form given by

$$K^i_a\tau_i \mathrm{d}x^a = K_x\tau_3 \mathrm{d}x + K_\varphi\tau_1 \mathrm{d}\theta + K_\varphi \sin\theta\tau_2 \mathrm{d}\varphi. \tag{3}$$

The spatial metric q_{ab} related to E^a_i by $\tilde{\tilde{q}}^{ab} = E^a_i E^b_j \delta^{ij}$ with $q^{ac}q_{cb} = \delta^a_b$ and $\tilde{\tilde{q}}^{ab}$ being double densitized. The line element of q_{ab} on Σ can be expressed in terms of the canonical variables $(K_x, E^x, K_\varphi, E^\varphi)$ as

$$ds^2 = \frac{(E^\varphi)^2}{|E^x|}dx^2 + |E^x|(d\theta^2 + \sin^2\theta d\varphi^2). \tag{4}$$

Moreover, the extrinsic curvature K_{ab} of Σ is given by $K_{ab} = \sqrt{q}^{-1} q_{bc}K^i_a E^c_i$ with q being the determinant of q_{ab}. The nonvanishing Poisson brackets between the phase space variables are

$$\begin{aligned} \{K_x(x), E^x(x')\} &= 2G\delta(x - x'), \\ \{K_\varphi(x), E^\varphi(x')\} &= G\delta(x - x'), \end{aligned} \tag{5}$$

where G is the gravitational constant.

The Hamiltonian formulation of a massless scalar field in the spherical symmetric model is based on the phase space $\mathcal{P}^\phi$, which is composed by the fields (ϕ, π_ϕ) defined on the quotient manifold $\mathbb{X}$, and equipped with the nonvanishing Poisson brackets,

$$\{\phi(x), \pi_\phi(x')\} = \delta(x - x'). \tag{6}$$

Then, the total phase space of the spherically symmetric model of gravity coupled to the massless scalar field is given as $\mathcal{P}^{\mathrm{g}} \times \mathcal{P}^\phi$. Moreover, this system is totally constrained and is governed by two sets of constraints: the diffeomorphism constraint H_x and the Hamiltonian constraint H. They are expressed as

$$H_x = H^{\mathrm{g}}_x + H^\phi_x \approx 0 \tag{7}$$

and

$$H = H^{\mathrm{g}} + H^\phi \approx 0, \tag{8}$$

respectively, where [69,70]

$$\begin{aligned} H^{\mathrm{g}} &= -2\frac{E^\varphi}{\partial_x E^x}\partial_x M + 2\frac{\sqrt{E^x}K_\varphi}{\partial_x E^x}H^{\mathrm{g}}_x, \\ H^{\mathrm{g}}_x &= \frac{1}{2G}(-K_x\partial_x E^x + 2E^\varphi\partial_x K_\varphi) \end{aligned} \tag{9}$$

with $M = \frac{\sqrt{E^x}}{2G}(1 + (K_\varphi)^2 - \frac{(\partial_x E^x)^2}{4(E^\varphi)^2})$ being the Misner-Sharp mass [71], and

$$H^\phi = \frac{1}{2}\left(\frac{\pi^2_\phi}{\sqrt{E^x}E^\varphi} + \frac{(E^x)^{3/2}}{E^\varphi}(\partial_x\phi)^2\right), \qquad H^\phi_x = \pi_\phi\partial_x\phi. \tag{10}$$

Note that the gravitational part H^{g} of Hamiltonian constraint is just the standard Hamiltonian constraint of GR, which is formulated as a combination of two terms. These two terms are proportional to the total derivative of Misner-Sharp mass and the gravitational part of diffeomorphism constraint, respectively. These constraints form a first class constraint system, with the corresponding constraint algebra being given by

$$\{H[\mathcal{N}], H[\mathcal{M}]\} = H_x\left[\frac{E^x}{(E^\varphi)^2}(\mathcal{N}\partial_x\mathcal{M} - \mathcal{M}\partial_x\mathcal{N})\right], \tag{11}$$

$$\{H[\mathcal{N}], H_x[\mathcal{M}_x]\} = -H[\mathcal{M}_x\partial_x\mathcal{N}], \tag{12}$$

and

$$\{H_x[\mathcal{N}_x], H_x[\mathcal{M}_x]\} = H_x[\mathcal{N}_x\partial_x\mathcal{M}_x - \mathcal{M}_x\partial_x\mathcal{N}_x], \tag{13}$$

where $\mathcal{N}$, $\mathcal{M}$ are the lapse functions and $\mathcal{N}_x$, $\mathcal{M}_x$ are shift vectors.

To define the apparent horizon in the spherically symmetric model of gravity, let us express the null outward expansion of two-spheres in terms of the canonical variables $(K_x, E^x, K_\varphi, E^\varphi)$. In the following part of this article, we choose the gauge $E^\varphi > 0$ and $E^x > 0$ without loss of generality. Then, for the two-sphere S defined by $x =$ constant with outward unit normal

$$S^a = \frac{\sqrt{E^x}}{E^\varphi}\left(\frac{\partial}{\partial x}\right)^a,$$

the null outward expansion of S in the spherically symmetric model can be expressed as [72–74]

$$\begin{aligned} \Theta_{\mathrm{out}} &= D_a S^a + K_{ab}S^a S^b - K \\ &= \frac{1}{\sqrt{E^x}E^\varphi}\partial_x(E^x) - \frac{2}{\sqrt{E^x}}K_\varphi, \end{aligned} \tag{14}$$

where D_a is determined by $D_c q_{ab} = 0$ and $K \coloneqq K_{ab}q^{ab}$.

Now, let us turn to consider the solutions of the constraints. To obtain the quantum physical states, one needs to quantize the phase space $\mathcal{P}^{\mathrm{g}} \times \mathcal{P}^{\phi}$ to get the corresponding Hilbert space $\mathcal{H}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$, and then solve the quantum versions of the diffeomorphism constraint H_x and the Hamiltonian constraint H. In this paper, we focus only on the quantum solution of Hamiltonian constraint H. The first step to seek the quantum solution is introducing the smearing version of Hamiltonian constraint H. Recall that the gravitational part H^{g} of Hamiltonian constraint is expressed as a combination of two terms, which are proportional to the total derivative of Misner-Sharp mass and the gravitational part of diffeomorphism constraint, respectively. This expression of H^{g} suggests us to reformulate the total Hamiltonian H as

$$\begin{aligned} H &= -2\frac{E^{\varphi}}{\partial_x E^x}\partial_x M + 2\frac{\sqrt{E^x}K_{\varphi}}{\partial_x E^x}H_x^{\mathrm{g}} + H^{\phi} \\ &= -2\frac{E^{\varphi}}{\partial_x E^x}\left(\partial_x M - \frac{\sqrt{E^x}}{E^{\varphi}}K_{\varphi}H_x^{\mathrm{g}} - \frac{\partial_x E^x}{2E^{\varphi}}H^{\phi}\right) \\ &=: -2\frac{E^{\varphi}}{\partial_x E^x}\tilde{H}. \end{aligned} \tag{15}$$

Then, we consider the smeared version of $\tilde{H}$, which reads

$$\tilde{H}(v,v') := \int_{x(v)}^{x(v')} dx \tilde{H} = (M_{v'} - M_v) - \tilde{H}_{\phi}(v,v'), \tag{16}$$

where v, v' are two arbitrary points on the one-dimensional manifold $\mathbb{X}$, and $\tilde{H}_{\phi}(v,v')$ is defined by

$$\tilde{H}_{\phi}(v,v') := \int_{x(v)}^{x(v')} dx \left(\frac{\sqrt{E^x}}{E^{\varphi}}K_{\varphi}H_x^{\mathrm{g}} + \frac{\partial_x E^x}{2E^{\varphi}}H^{\phi}\right). \tag{17}$$

One can see that the Hamiltonian constraint equations $H = 0$ are solved if and only if the equations

$$\tilde{H}(v,v') = (M_{v'} - M_v) - \tilde{H}_{\phi}(v,v') = 0 \tag{18}$$

are solved for all $v, v' \in \mathbb{X}$. In the following part of this article, we will focus on the solution of the quantum version of Eq. (18).

To carry out a further discussion, let us fix the gauge by introducing the areal radius R as

$$E^x = R^2, \tag{19}$$

where R is a certain field on $\mathbb{X}$ and $R(x) > 0$ is a monotonically increasing function. Now, R and x provide two coordinate systems respectively. In particular, the evolution along the coordinates R and x are relational to the geometric variable $\sqrt{E^x}$ and $R^{-1}(\sqrt{E^x})$ respectively, and $R(x)$ is the corresponding coordinate transformation. Then, let us solve the diffeomorphism constraint $H_x = 0$, which leads to

$$K_x = \frac{2E^{\varphi}\partial_x K_{\varphi} + 2G\pi_{\phi}\partial_x \phi}{\partial_x E^x}. \tag{20}$$

Now, the remaining dynamical variables contain only the pairs $\mathfrak{g} = (K_{\varphi}, E^{\varphi})$ and (ϕ, π_{ϕ}), which formulate a reduced phase space $\bar{\mathcal{P}}^{\mathrm{g}} \times \mathcal{P}^{\phi}$ with $\mathfrak{g} \in \bar{\mathcal{P}}^{\mathrm{g}}$.

B. Issue of quantum field theory on classical geometry

The quantum theory of the scalar field on a specific classical spacetime geometry can be established by effectively solving Eq. (18). In other words, one can fix the geometry components and promote the scalar field component as the quantum counterpart in Eq. (18), which leads to the effective Hamiltonian constraint equation

$$M_{v'}(\mathfrak{g}) - M_v(\mathfrak{g}) = \frac{\langle \phi' | \widehat{\tilde{H}_{\phi}}(\mathfrak{g}; v, v') | \phi \rangle}{\langle \phi' | \phi \rangle} \tag{21}$$

where $\mathfrak{g} = \{E^{\varphi}(\mathfrak{g}), E_{\varphi}(\mathfrak{g})\}$ is a point in the reduced phase space $\bar{\mathcal{P}}^{\mathrm{g}}$, $M_{v'}(\mathfrak{g}), M_v(\mathfrak{g})$ are the values of $M_{v'}, M_v$ at the phase space point $\mathfrak{g}$, respectively, and $\widehat{\tilde{H}_{\phi}}(\mathfrak{g}; v, v')$ is defined by

$$\begin{aligned} \widehat{\tilde{H}_{\phi}}(\mathfrak{g}; v, v') = \int_v^{v'} dx \Bigg(& R(x)\left(\frac{K_{\varphi}}{E^{\varphi}}\bigg|_{\mathfrak{g}}\right) H_x^{\mathrm{g}}(\mathfrak{g}) \\ & + \left(\frac{R(x)\partial_x R(x)}{E^{\varphi}(x)}\bigg|_{\mathfrak{g}}\right) \hat{H}^{\phi}(\mathfrak{g}) \Bigg) \end{aligned} \tag{22}$$

with $H_x^{\mathrm{g}}(\mathfrak{g})$ being the value of H_x^{g} at the phase space point $\mathfrak{g}$ and $\hat{H}^{\phi}(\mathfrak{g})$ being defined by

$$\begin{aligned} \hat{H}^{\phi}(\mathfrak{g}) := \frac{1}{2}\Bigg(& \left(\frac{1}{R(x)E^{\varphi}(x)}\bigg|_{\mathfrak{g}}\right)(\hat{\pi}_{\phi}(x))^2 \\ & + \left(\frac{(R(x))^3}{E^{\varphi}(x)}\bigg|_{\mathfrak{g}}\right)(\partial_x \hat{\phi})^2 \Bigg). \end{aligned} \tag{23}$$

Then, for the given spatial (intrinsic and extrinsic) geometry $\mathfrak{g}$, the quantum scalar field in the region $x(v) < x < x(v')$ must satisfy Eq. (21).

There is no guarantee that the matter Hamiltonian $\int_v^{v'} dx \left(\frac{R(x)\partial_x R(x)}{E^{\varphi}(x)}\Big|_{\mathfrak{g}}\right)\hat{H}^{\phi}(\mathfrak{g})$ can be well represented in the Fock space based on the plane wave $e^{\mathrm{i}kx}$ for an arbitrary coordinate system x. To find a specific coordinate, we reformulate the matter Hamiltonian as

$$\begin{aligned} & \int_v^{v'} dx \left(\frac{R(x)\partial_x R(x)}{E^{\varphi}(x)}\bigg|_{\mathfrak{g}}\right)\hat{H}^{\phi}(\mathfrak{g}) \\ & = \frac{1}{2}\int_v^{v'} dx \left(\frac{R^2(x)\partial_x R(x)}{(E^{\varphi}(x))^2}\bigg|_{\mathfrak{g}}\right)\left(\frac{\hat{\pi}_{\phi}^2(x)}{R^2(x)} + R^2(x)(\partial_x \hat{\phi})^2\right). \end{aligned} \tag{24}$$

This formulation of matter Hamiltonian can be well represented in the Fock space based on the plane wave $e^{\mathbf{i}kx}$, if $R(x)$ satisfies

$$R^2(x)\partial_x R(x) = (E^\varphi(x))^2 \tag{25}$$

and

$$x(v') - x(v) \to \infty. \tag{26}$$

Notice that E^φ has density one and thus $E^\varphi(R) = \frac{\partial x}{\partial R} E^\varphi(x)$. Then, Eq. (25) becomes

$$R^2(x) = \partial_x R(x)(E^\varphi(R))^2, \tag{27}$$

and Eq. (26) becomes

$$x(v') - x(v) = \int_v^{v'} dx = \int_v^{v'} dR \frac{(E^\varphi(R))^2}{R^2} \to \infty. \tag{28}$$

From now on, let us focus on a specific collection $\mathcal{S}_{v,v'}$ of $\mathfrak{g}$, which ensures Eqs. (27) and (28) have a solution in the interval $R(v) < R < R(v')$. This collection $\mathcal{S}_{v,v'}$ is given by

$$\mathcal{S}_{v,v'} \equiv \Bigg\{ \mathfrak{g} \in \mathcal{P}^{\mathrm{g}} | \Theta_{\mathrm{out}}(\mathfrak{g})|_v = 0, K_\varphi(\mathfrak{g})|_{R(v)<R<R(v')} = 0, \int_v^{v'} dR \frac{(E^\varphi(R))^2}{R^2} \to \infty \Bigg\}, \tag{29}$$

where the null outward expansion satisfying $\Theta_{\mathrm{out}}(\mathfrak{g})|_v = 0$ means an apparent horizon at $R = R_{\mathrm{h}} = R(v)$, and $K_\varphi(\mathfrak{g})|_{R(v)<R<R(v')} = 0$ is a gauge choice with respect to the Hamiltonian constraint. For an arbitrary spatial (intrinsic and extrinsic) geometry $\mathfrak{g} \in \mathcal{S}_{v,v'}$, we can solve Eqs. (27) and (28) by fixing the coordinate x as the tortoise coordinate $x^*_{\mathfrak{g}}$, with $x^*_{\mathfrak{g}}(R)$ being given by

$$x^*_{\mathfrak{g}}(R) = \delta^*_{\mathfrak{g}} - \int_R^{R(v')} dx^*_{\mathfrak{g}} = \delta^*_{\mathfrak{g}} - \int_R^{R(v')} dR \frac{(E^\varphi(R))^2}{R^2}, \quad 0 < \delta^*_{\mathfrak{g}} < +\infty. \tag{30}$$

It is easy to see that $x^*_{\mathfrak{g}}$ satisfies the boundary condition

$$x^*_{\mathfrak{g}}(v) \to -\infty, \qquad x^*_{\mathfrak{g}}(v') = \delta^*_{\mathfrak{g}}. \tag{31}$$

Let us then start to represent the matter Hamiltonian $\int_v^{v'} dx (\frac{\partial_x E^x}{2E^\varphi}|_{\mathfrak{g}}) \hat{H}^\phi(\mathfrak{g})$ as a well-defined operator in the Fock space based on the plane wave $e^{\mathbf{i}kx^*_{\mathfrak{g}}}$. The nonvanishing quantum commutator corresponding to (6) in the tortoise coordinate $x^*_{\mathfrak{g}}$ is given by

$$[\hat{\phi}(x^*_{\mathfrak{g}}), \hat{\pi}_\phi(x'^*_{\mathfrak{g}})] = \mathbf{i}\hbar\delta(x^*_{\mathfrak{g}} - x'^*_{\mathfrak{g}}). \tag{32}$$

One can define the annihilation operator $a^{\mathfrak{g}}_k$ as

$$a^{\mathfrak{g}}_k \coloneqq \lim_{L\to\infty} \int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} \frac{1}{\sqrt{2\omega_k}} \left(R \cdot \omega_k \hat{\phi}(\check{x}^*_{\mathfrak{g}}) + \mathbf{i} \frac{\hat{\pi}_\phi(\check{x}^*_{\mathfrak{g}})}{R} \right) \tag{33}$$

where $\omega_k \coloneqq |k|$, $R = R(\check{x}^*_{\mathfrak{g}})$ and

$$\check{x}^*_{\mathfrak{g}} = x^*_{\mathfrak{g}} + L - \delta^*_{\mathfrak{g}}. \tag{34}$$

Correspondingly, the Hermitian conjugate gives the creation operator $a^{\mathfrak{g},\dagger}_k$ as

$$a^{\mathfrak{g},\dagger}_k \coloneqq \lim_{L\to\infty} \int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}} e^{\mathbf{i}k\check{x}^*_{\mathfrak{g}}} \frac{1}{\sqrt{2\omega_k}} \left(R \cdot \omega_k \hat{\phi}(\check{x}^*_{\mathfrak{g}}) - \mathbf{i} \frac{\hat{\pi}_\phi(\check{x}^*_{\mathfrak{g}})}{R} \right). \tag{35}$$

Then, the operators $\hat{\phi}(\check{x}^*_{\mathfrak{g}})$ and $\hat{\pi}_\phi(\check{x}^*_{\mathfrak{g}})$ in the region $-\infty < x^*_{\mathfrak{g}} < \delta^*_{\mathfrak{g}}$ can be given as

$$\hat{\phi}(\check{x}^*_{\mathfrak{g}}) \overset{L\to\infty}{=} \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} \left(\frac{e^{\mathbf{i}k\check{x}^*_{\mathfrak{g}}}}{R} a^{\mathfrak{g}}_k + \frac{e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}}}{R} a^{\mathfrak{g},\dagger}_k \right) \tag{36}$$

and

$$\hat{\pi}_\phi(\check{x}^*_{\mathfrak{g}}) \overset{L\to\infty}{=} (-\mathbf{i}) \int \frac{dk}{2\pi} \sqrt{\frac{\omega_k}{2}} R (e^{\mathbf{i}k\check{x}^*_{\mathfrak{g}}} a^{\mathfrak{g}}_k - e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} a^{\mathfrak{g},\dagger}_k) \tag{37}$$

where we used

$$\lim_{L\to\infty} \int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}} e^{\mathbf{i}k\check{x}^*_{\mathfrak{g}}} = 2\pi\delta(k). \tag{38}$$

Furthermore, by using (32), one can check that

$$[a^{\mathfrak{g}}_k, a^{\mathfrak{g},\dagger}_{k'}] \overset{L\to\infty}{=} 2\pi\hbar\delta(k-k'), \quad [a^{\mathfrak{g}}_k, a^{\mathfrak{g}}_{k'}] \overset{L\to\infty}{=} [a^{\mathfrak{g},\dagger}_k, a^{\mathfrak{g},\dagger}_{k'}] \overset{L\to\infty}{=} 0. \tag{39}$$

With this algebra between $a^{\mathfrak{g}}_k$ and $a^{\mathfrak{g},\dagger}_{k'}$ established, one can define the vacuum state (Boulware-like) $|0_{\mathrm{B}}\rangle_{\mathfrak{g}}$ in the region $-\infty < x^*_{\mathfrak{g}} < \delta^*_{\mathfrak{g}}$ by

$$a^{\mathfrak{g}}_k |0_{\mathrm{B}}\rangle_{\mathfrak{g}} = 0, \quad \forall\ k. \tag{40}$$

Correspondingly, the one-particle state can be defined as

$$|k\rangle_{\mathfrak{g}} \coloneqq a^{\mathfrak{g},\dagger}_k |0_{\mathrm{B}}\rangle_{\mathfrak{g}} \tag{41}$$

with the inner product

$$\langle k'|k\rangle_{\mathfrak{g}} = \delta(k'-k). \tag{42}$$

Then, one can introduce the Hilbert space $\mathcal{H}^{\phi}_{\mathfrak{g}}$ of the field ϕ spanned by acting on the vacuum with all possible combinations of $a^{\mathfrak{g},\dagger}_k$, which reads

$$\mathcal{H}^{\phi}_{\mathfrak{g}} \coloneqq \{|0_{\mathrm{B}}\rangle_{\mathfrak{g}}, a^{\mathfrak{g},\dagger}_k|0_{\mathrm{B}}\rangle_{\mathfrak{g}}, a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g},\dagger}_{k'}|0_{\mathrm{B}}\rangle_{\mathfrak{g}}, a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g},\dagger}_{k'} a^{\mathfrak{g},\dagger}_{k''}|0_{\mathrm{B}}\rangle_{\mathfrak{g}}, \ldots\}. \tag{43}$$

We make two remarks on the construction of the Fock space $\mathcal{H}^{\phi}_{\mathfrak{g}}$. First, Eq. (34) can be regarded as a coordinate transformation introducing the modified tortoise coordinate $\check{x}^*_{\mathfrak{g}}$. This modified tortoise coordinate $\check{x}^*_{\mathfrak{g}}$ ensures that the region $-\infty < x^*_{\mathfrak{g}} < \delta^*_{\mathfrak{g}}$ can be represented as $-\infty < \check{x}^*_{\mathfrak{g}} < +\infty$ in the limit $L \to \infty$. Hence, the Fock space $\mathcal{H}^{\phi}_{\mathfrak{g}}$ can be constructed in the region $-\infty < x^*_{\mathfrak{g}} < \delta^*_{\mathfrak{g}}$ based on the mode $e^{\mathbf{i}k\check{x}^*_{\mathfrak{g}}}$. Second, notice that the construction of $\mathcal{H}^{\phi}_{\mathfrak{g}}$ is valid for all of $\mathfrak{g} \in \mathcal{S}_{v,v'}$. However, the Fock spaces $\mathcal{H}^{\phi}_{\mathfrak{g}}$ constructed on different $\mathfrak{g}$ may not be equivalent; in other words, one cannot ensure unitary equivalence $\mathcal{H}^{\phi}_{\mathfrak{g}} \cong \mathcal{H}^{\phi}_{\mathfrak{g}'}$ for arbitrary $\mathfrak{g} \neq \mathfrak{g}'$.

Now, the matter Hamiltonian $\int_v^{v'} d\check{x}^*_{\mathfrak{g}} \left(\frac{R(\check{x}^*_{\mathfrak{g}})\partial_{\check{x}^*_{\mathfrak{g}}} R(\check{x}^*_{\mathfrak{g}})}{E^{\varphi}(\check{x}^*_{\mathfrak{g}})}\Big|_{\mathfrak{g}}\right) \times \hat{H}^{\phi}(\mathfrak{g})$ can be represented in the Fock space $\mathcal{H}^{\phi}_{\mathfrak{g}}$ as

$$\begin{aligned}
&\int_v^{v'} d\check{x}^*_{\mathfrak{g}} \left(\frac{R(\check{x}^*_{\mathfrak{g}})\partial_{\check{x}^*_{\mathfrak{g}}} R(\check{x}^*_{\mathfrak{g}})}{E^{\varphi}(\check{x}^*_{\mathfrak{g}})}\Big|_{\mathfrak{g}}\right)\hat{H}^{\phi}(\mathfrak{g}) \\
&= \frac{1}{2}\int_v^{v'} d\check{x}^*_{\mathfrak{g}} \left(\frac{\hat{\pi}^2_{\phi}(\check{x}^*_{\mathfrak{g}})}{R^2(\check{x}^*_{\mathfrak{g}})} + R^2(\check{x}^*_{\mathfrak{g}})(\partial_{\check{x}^*_{\mathfrak{g}}}\hat{\phi})^2\right) \\
&\overset{L\to\infty}{=} \int \frac{dk}{2\pi}\frac{\omega_k}{2}(a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g}}_k + a^{\mathfrak{g}}_k a^{\mathfrak{g},\dagger}_k),
\end{aligned} \tag{44}$$

where we used

$$\int_v^{v'} d\check{x}^*_{\mathfrak{g}} \frac{\hat{\pi}^2_{\phi}(\check{x}^*_{\mathfrak{g}})}{R^2(\check{x}^*_{\mathfrak{g}})} \overset{L\to\infty}{=} -\int \frac{dk}{2\pi}\frac{\omega_k}{2}(a^{\mathfrak{g}}_k a^{\mathfrak{g}}_{-k} - a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g}}_k - a^{\mathfrak{g}}_k a^{\mathfrak{g},\dagger}_k + a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g},\dagger}_{-k}), \tag{45}$$

$$\int_v^{v'} d\check{x}^*_{\mathfrak{g}} R^2(\check{x}^*_{\mathfrak{g}})(\partial_{\check{x}^*_{\mathfrak{g}}}\hat{\phi})^2 \overset{L\to\infty}{=} -\int \frac{dk}{2\pi}\frac{\omega_k}{2}(-a^{\mathfrak{g}}_k a^{\mathfrak{g}}_{-k} - a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g}}_k - a^{\mathfrak{g}}_k a^{\mathfrak{g},\dagger}_k - a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g},\dagger}_{-k}), \tag{46}$$

and the large R approximation

$$\begin{aligned}
\partial_{\check{x}^*_{\mathfrak{g}}}\hat{\phi}(\check{x}^*_{\mathfrak{g}}) &\overset{L\to\infty}{=} \int \frac{dk}{2\pi}\frac{\mathbf{i}k}{\sqrt{2\omega_k}}\left(\frac{e^{\mathbf{i}k\check{x}^*_{\mathfrak{g}}}}{R}a^{\mathfrak{g}}_k - \frac{e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}}}{R}a^{\mathfrak{g},\dagger}_k\right) - \frac{1}{R}\frac{\partial R}{\partial \check{x}^*_{\mathfrak{g}}}\hat{\phi}(\check{x}^*_{\mathfrak{g}}) \\
&\approx \int \frac{dk}{2\pi}\frac{\mathbf{i}k}{\sqrt{2\omega_k}}\left(\frac{e^{\mathbf{i}k\check{x}^*_{\mathfrak{g}}}}{R}a^{\mathfrak{g}}_k - \frac{e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}}}{R}a^{\mathfrak{g},\dagger}_k\right).
\end{aligned} \tag{47}$$

Further, by choosing the normal ordering the operators, one has

$$\int_v^{v'} d\check{x}^*_{\mathfrak{g}} \left(\frac{R(\check{x}^*_{\mathfrak{g}})\partial_{\check{x}^*_{\mathfrak{g}}} R(\check{x}^*_{\mathfrak{g}})}{E^{\varphi}(\check{x}^*_{\mathfrak{g}})}\Big|_{\mathfrak{g}}\right)\hat{H}^{\phi}(\mathfrak{g}) \overset{L\to\infty}{=} \int \frac{dk}{2\pi}\omega_k a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g}}_k. \tag{48}$$

Next, by using Eq. (48), the right-hand side of Eq. (21) can be represented in the Hilbert space $\mathcal{H}^{\phi}_{\mathfrak{g}}$ directly, which leads to

$$M_{v'}(\mathfrak{g}) - M_v(\mathfrak{g}) = \frac{\int \frac{dk}{2\pi}\omega_k \langle\phi'|a^{\mathfrak{g},\dagger}_k a^{\mathfrak{g}}_k|\phi\rangle}{\langle\phi'|\phi\rangle} \tag{49}$$

for $\mathfrak{g} \in \mathcal{S}_{v,v'}$ and $L \to \infty$. The solution of Eq. (49) is given by

$$|\phi\rangle = |\vec{k}_{\vec{n}}(\mathfrak{g})\rangle \coloneqq \left(\prod_k \frac{(a^{\mathfrak{g},\dagger}_k)^{n_k}}{(\mathcal{N})^{n_k}\sqrt{n_k!}}\right)\Big|0_{\mathrm{B}}\Big\rangle, \tag{50}$$

where $\vec{n} = (n_{k_1}, n_{k_2}, n_{k_3}, \ldots)$ with the sequence $k_1 < k_2 < k_3 < \ldots$ composed of all possible values in the spectrum of k, $n_{k_1}, n_{k_2}, n_{k_3}, \ldots \in \{0, 1, 2, 3, \ldots\}$ denote the particle number in modes $k_1, k_2, k_3, \ldots, \mathcal{N} \coloneqq \frac{1}{\sqrt{2\pi\hbar\delta(k-k')|_{k=k'}}}$ is a normalization constant and $\vec{k}_{\vec{n}}(\mathfrak{g})$ is defined by

$$\vec{k}_{\vec{n}}(\mathfrak{g}) \coloneqq (\overbrace{(k_1, \ldots, k_1)}^{n_{k_1}\text{-tuple } k_1}, \overbrace{(k_2, \ldots, k_2)}^{n_{k_2}\text{-tuple } k_2}, \overbrace{(k_3, \ldots, k_3)}^{n_{k_3}\text{-tuple } k_3}, \ldots) \tag{51}$$

with $(n_{k_1}, n_{k_2}, n_{k_3}, \ldots)$ and $(k_1, k_2, k_3, \ldots)$ satisfying

$$\begin{aligned}
M_{v'}(\mathfrak{g}) - M_v(\mathfrak{g}) &= n_{k_1}\omega_{k_1(\mathfrak{g})} + n_{k_2}\omega_{k_2(\mathfrak{g})} + n_{k_3}\omega_{k_3(\mathfrak{g})} \\
&\quad + \cdots \eqqcolon \omega_{\vec{k}_{\vec{n}}(\mathfrak{g})}.
\end{aligned} \tag{52}$$

Here, we would like to emphasize that, in order to get the Eqs. (48) and (49), the modified tortoise coordinate $\check{x}^*_{\mathfrak{g}} = x^*_{\mathfrak{g}} + L - \delta^*_{\mathfrak{g}}$ is used with the limit $L \to \infty$. These constructions ensure that the Fock states $|\vec{k}_{\vec{n}}\rangle$ defined in this

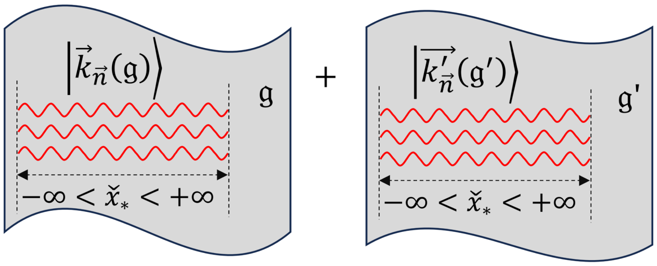

FIG. 1. Fock states $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ and $|\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$ (illustrated by red wavy lines) constructed on the classical geometries $\mathfrak{g}$ and $\mathfrak{g}'$ (illustrated by grey surfaces), respectively. The superposition state $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle + |\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$ is ill defined because of the following two reasons; first, $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ and $|\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$ are constructed on different classical geometries $\mathfrak{g}$ and $\mathfrak{g}'$ so that they belong to different Hilbert spaces $\mathcal{H}^{\phi}_{\mathfrak{g}}$ and $\mathcal{H}^{\phi}_{\mathfrak{g}'}$, while one cannot guarantee a unitary equivalence $\mathcal{H}^{\phi}_{\mathfrak{g}} \cong \mathcal{H}^{\phi}_{\mathfrak{g}'}$ for arbitrary $\mathfrak{g} \neq \mathfrak{g}'$. Second, $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ and $|\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$ must be associated to the classical geometries $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, by the Hamiltonian constraint, while the classical geometries can not be superposed.

coordinate $\check{x}^*_{\mathfrak{g}}$ are the eigenstate of the operator $\widehat{\tilde{H}_{\phi}}(\mathfrak{g}; v, v')$ on the right-hand side of Eq. (21). In fact, this modified tortoise coordinate can only be defined near a horizon up to now, and thus we only solve a subset of the effective Hamiltonian constraint (21).

However, the solution (50) of the effective Hamiltonian constraint equation (21) reveals a fundamental issue inherent in QFT formulated on a fixed classical geometry. In detail, for a given spatial geometry $\mathfrak{g}$, the solutions of Eq. (21) are certain Fock states $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ constructed in the modified tortoise coordinate $\check{x}^*_{\mathfrak{g}}$ and satisfying Eq. (52). Consider two such Fock states $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ and $|\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$, each solving the Hamiltonian constraint (21) and built upon distinct classical geometries $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, since $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ and $|\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$ are physical states in the sense of weakly solving the Hamiltonian constraint (21). Hence, one could expect that their linear superposition

$$|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle + |\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle \tag{53}$$

also represent a valid physical state. However, this superposition is, in fact, ill defined, and thus highlights a core limitation of the QFT on a fixed classical background geometry, see the illustration in Fig. 1.

In detail, the illness of this superposition is caused by two interrelated reasons:

(i) The underlying superposition of classical geometry: Each Fock state $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ is inextricably tied to its background geometry $\mathfrak{g}$ through the Hamiltonian constraint (21). Hence, the superposition of states $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ and $|\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$ on distinct background geometries causes the underlying superposition of classical geometry, which leads to this superposition being ill defined.

(ii) Inequivalent Fock space: The states $|\vec{k}_{\vec{n}}(\mathfrak{g})\rangle$ and $|\vec{k}'_{\vec{n}}(\mathfrak{g}')\rangle$ are defined with respect to different classical geometries $\mathfrak{g}$ and $\mathfrak{g}'$ and therefore reside in different Fock spaces $\mathcal{H}^{\phi}_{\mathfrak{g}}$ and $\mathcal{H}^{\phi}_{\mathfrak{g}'}$, respectively. However, one cannot guarantee an unitary equivalence $\mathcal{H}^{\phi}_{\mathfrak{g}} \cong \mathcal{H}^{\phi}_{\mathfrak{g}'}$ for arbitrary $\mathfrak{g} \neq \mathfrak{g}'$, which leads the superposition of states from these distinct spaces mathematically undefined.

Our result reveals a key conceptual shortcoming of QFT on a fixed classical spacetime: it cannot consistently accommodate quantum superpositions that involve changes of the background geometry. Resolving this issue typically requires moving beyond fixed backgrounds—toward a fully quantum theory of gravity, where geometry itself becomes a dynamical quantum variable.

III. QUANTUM SCALAR FIELD ON SEMICLASSICAL GEOMETRY

A. Fock representations on quantum geometry

The issue of QFT on a fixed classical geometry suggests us to consider QFT on the quantum geometry of spacetime. Specifically, to tackle the illness of superposition (53) proposed in 2.2, one needs to consider the quantum states that satisfy the following two conditions:

(i) The background geometry sector should be described by some quantum states, i.e., the coherent states $|\mathfrak{g}\rangle$ of spatial geometry sharply peaked at the phase space point $\mathfrak{g}$, and now the Fock space $\mathcal{H}^{\phi}_{\mathfrak{g}}$ can be defined on the semiclassical geometry $|\mathfrak{g}\rangle$.

(ii) The superposition of the product states $|\mathfrak{g}, \phi\rangle \coloneqq |\mathfrak{g}\rangle \otimes |\phi\rangle$ should be defined in a certain state space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$ with $|\mathfrak{g}\rangle \in \tilde{\mathcal{H}}_{\mathrm{g}}$ and $|\phi\rangle \in \mathcal{H}^{\phi}$; especially, the space $\mathcal{H}^{\phi}$ must satisfy the unitary equivalence $\mathcal{H}^{\phi} \cong \mathcal{H}^{\phi}_{\mathfrak{g}}$ of representations for arbitrary $|\mathfrak{g}\rangle \in \tilde{\mathcal{H}}_{\mathrm{g}}$, so that the superposition of such states $|\mathfrak{g}, \phi\rangle$ is well defined.

Now, let us start by constructing the product state $|\mathfrak{g}, \phi\rangle$ and the certain state space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$ step by step. We first focus on the gravitational sector. The quantization of the gravitational sector of this system can be realized by following the standard loop quantization method for the spherically symmetric model, which leads to the Hilbert space $\mathcal{H}_{\mathrm{g}}$ of the gravitational sector [75,76]. Then, the coherent state in loop quantum theory can be constructed correspondingly (see the details in Appendix A). The resulting coherent states $|\mathfrak{g}\rangle \in \mathcal{H}_{\mathrm{g}}$ sharply peaked at the phase space point $\mathfrak{g}$; Specifically, the coherent states distinguish the classical geometries $\mathfrak{g}$ and $\mathfrak{g}'$ by the exponentially decayed inner product $\langle \mathfrak{g}|\mathfrak{g}'\rangle$, which satisfies

$$\langle \mathfrak{g}|\mathfrak{g}'\rangle \sim \exp(|\mathfrak{g} - \mathfrak{g}'|^2/t) \tag{54}$$

with $|\mathfrak{g}-\mathfrak{g}'|$ being the distance of the points $\mathfrak{g}$ and $\mathfrak{g}'$ in the phase space, and $t \propto G\hbar$ being a controlled parameter labeling the width of the wave packet of the coherent state. Moreover, the sharply peaked coherent states $|\mathfrak{g}\rangle$ satisfy the Ehrenfest property, which is given as

$$\langle\mathfrak{g}|\hat{O}|\mathfrak{g}'\rangle = \langle\mathfrak{g}|\mathfrak{g}'\rangle O(\mathfrak{g}')(1+\mathcal{O}(t)), \tag{55}$$

where $O(\mathfrak{g}')$ is a regular functional on $\mathcal{P}^{\mathrm{g}}$ and $\hat{O}$ is the operator corresponding to $O(\mathfrak{g}')$. Consequently, the coherent state $|\mathfrak{g}\rangle$ has a well-behaved semiclassical property and therefore can be regarded as the semiclassical state.

Let us then turn to the scalar field sector to construct a whole product space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi} \subset \mathcal{H}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$. For a given semiclassical coherent state $|\mathfrak{g}\rangle \in \mathcal{H}_{\mathrm{g}}$ with $\mathfrak{g} \in \mathcal{S}_{v,v'}$, we can introduce the corresponding Fock space $\mathcal{H}^{\phi}_{\mathfrak{g}}$ on the geometry $\mathfrak{g}$ directly. However, the Fock space $\mathcal{H}^{\phi}_{\mathfrak{g}}$ is associated with certain semiclassical geometry described by $|\mathfrak{g}\rangle$, and one cannot ensure a unitary equivalence $\mathcal{H}^{\phi}_{\mathfrak{g}} \cong \mathcal{H}^{\phi}_{\mathfrak{g}'}$ of representations for arbitrary $|\mathfrak{g}\rangle \neq |\mathfrak{g}'\rangle$. Hence, at this stage the Hilbert space $\mathcal{H}^{\phi}$ cannot be defined as $\mathcal{H}^{\phi} \equiv \mathcal{H}^{\phi}_{\mathfrak{g}}$. Nevertheless, we can focus on the subspace $\mathcal{U}_{v,v'} \subset \mathcal{S}_{v,v'}$, which ensures the unitary equivalence

$$\mathcal{H}^{\phi}_{\mathfrak{g}} \cong \mathcal{H}^{\phi}_{\mathfrak{g}'}, \quad \forall\ \mathfrak{g},\mathfrak{g}' \in \mathcal{U}_{v,v'}. \tag{56}$$

Correspondingly, one has the space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$ with

$$\tilde{\mathcal{H}}_{\mathrm{g}} \coloneqq \{|\mathfrak{g}\rangle \in \mathcal{H}_{\mathrm{g}} | \mathfrak{g} \in \mathcal{U}_{v,v'}\} \quad \text{and} \quad \mathcal{H}^{\phi} \equiv \mathcal{H}^{\phi}_{\mathfrak{g}} \cong \mathcal{H}^{\phi}_{\mathfrak{g}'}, \quad \forall\ \mathfrak{g},\mathfrak{g}' \in \mathcal{U}_{v,v'}. \tag{57}$$

The remaining task is to find the subspace $\mathcal{U}_{v,v'} \subset \mathcal{S}_{v,v'}$ that satisfies Eq. (56). Notice that $\mathcal{H}^{\phi}_{\mathfrak{g}}$ and $\mathcal{H}^{\phi}_{\mathfrak{g}'}$ are generated by $(a^{\mathfrak{g}}_k, a^{\mathfrak{g},\dagger}_k)$ and $(a^{\mathfrak{g}'}_k, a^{\mathfrak{g}',\dagger}_k)$ respectively, and thus one has the Bogoliubov transformation mapping the annihilation operators $a^{\mathfrak{g}'}_k$ to

$$a^{\mathfrak{g}}_k = \int \frac{dj}{2\pi}(\alpha_{kj}(\mathfrak{g}',\mathfrak{g}) a^{\mathfrak{g}'}_j + \beta_{kj}(\mathfrak{g}',\mathfrak{g}) a^{\mathfrak{g}',\dagger}_j). \tag{58}$$

Recalling Eqs. (33), (36), and (37), one can immediately get

$$\alpha_{kk'}(\mathfrak{g}',\mathfrak{g}) = \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\left(\lim_{L\to\infty}\int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} e^{\mathbf{i}k'\check{x}^*_{\mathfrak{g}'}}\right) + \frac{1}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}\left(\lim_{L\to\infty}\int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}'} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} e^{\mathbf{i}k'\check{x}^*_{\mathfrak{g}'}}\right) \tag{59}$$

and

$$\beta_{kk'}(\mathfrak{g}',\mathfrak{g}) = \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\left(\lim_{L\to\infty}\int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} e^{-\mathbf{i}k'\check{x}^*_{\mathfrak{g}'}}\right) - \frac{1}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}\left(\lim_{L\to\infty}\int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}'} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} e^{-\mathbf{i}k'\check{x}^*_{\mathfrak{g}'}}\right). \tag{60}$$

Then, by using the Shale-Stinespring criterion, one has the following result;

(i) The unitary equivalence $\mathcal{H}^{\phi}_{\mathfrak{g}} \cong \mathcal{H}^{\phi}_{\mathfrak{g}'}$ holds if and only if the Bogoliubov transformation mapping the annihilation operators $a^{\mathfrak{g}'}_k$ to

$$a^{\mathfrak{g}}_k = \int \frac{dj}{2\pi}(\alpha_{kj}(\mathfrak{g}',\mathfrak{g}) a^{\mathfrak{g}'}_j + \beta_{kj}(\mathfrak{g}',\mathfrak{g}) a^{\mathfrak{g}',\dagger}_j) \tag{61}$$

can be implemented by a unitary operator $U_{\mathfrak{g},\mathfrak{g}'}$ on the Fock space, i.e., there exists a unitary $U_{\mathfrak{g}',\mathfrak{g}}$ such that $a^{\mathfrak{g}}_k = U_{\mathfrak{g}',\mathfrak{g}} a^{\mathfrak{g}'}_k U^{\dagger}_{\mathfrak{g}',\mathfrak{g}}$ for all k; moreover, such a unitary $U_{\mathfrak{g}',\mathfrak{g}}$ exists if and only if the matrix of Bogoliubov coefficients $\beta_{kj}(\mathfrak{g}',\mathfrak{g})$ is a Hilbert-Schmidt operator, namely,

$$\int dkdj |\beta_{kj}(\mathfrak{g}',\mathfrak{g})|^2 < \infty. \tag{62}$$

Now, the subspace $\mathcal{U}_{v,v'} \subset \mathcal{S}_{v,v'}$ that satisfies Eq. (56) can be defined by

$$\mathcal{U}_{v,v'} \coloneqq \left\{\mathfrak{g}'' \in \mathcal{S}_{v,v'} \middle| \int dkdj \middle|\beta_{kj}(\mathfrak{g}',\mathfrak{g})|^2 < \infty,\ \forall\ \mathfrak{g}',\mathfrak{g} \in \mathcal{U}_{v,v'}\right\}. \tag{63}$$

Moreover, further discussion of the subspace $\mathcal{U}_{v,v'} \subset \mathcal{S}_{v,v'}$ based on an example is given in the Appendix B.

B. The weak solution of quantum Hamiltonian constraint

Now, the illness of superposition (53) proposed in 2.2 disappears in the quantum state space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$ defined by (57). Hence, one can consider the totally quantized version of Eq. (18), which reads

$$\langle \hat{M}_{v'} \rangle - \langle \hat{M}_{v} \rangle = \langle \widehat{\tilde{H}^{\delta}_{\phi}}(v, v') \rangle, \tag{64}$$

where $\hat{M}_v, \hat{M}_{v'}$ is the operator of $M_v, M_{v'}$, $\widehat{\tilde{H}^{\delta}_{\phi}}(v, v')$ is the operator of $\tilde{H}^{\delta}_{\phi}(v, v')$, and $\langle \hat{P} \rangle$ represents the expectation value of an operator $\hat{P}$. For an arbitrary state $|\mathfrak{g}, \phi\rangle \in \tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$, Eq. (64) can be written as

$$\frac{\langle \mathfrak{g}', \phi' | \hat{M}_{v'} | \mathfrak{g}, \phi \rangle}{\langle \mathfrak{g}', \phi' | \mathfrak{g}, \phi \rangle} - \frac{\langle \mathfrak{g}', \phi' | \hat{M}_{v} | \mathfrak{g}, \phi \rangle}{\langle \mathfrak{g}', \phi' | \mathfrak{g}, \phi \rangle} = \frac{\langle \mathfrak{g}', \phi' | \widehat{\tilde{H}^{\delta}_{\phi}}(v, v') | \mathfrak{g}, \phi \rangle}{\langle \mathfrak{g}', \phi' | \mathfrak{g}, \phi \rangle}. \tag{65}$$

Furthermore, by using the property (55) of the coherent state $|\mathfrak{g}\rangle$, one has

$$\langle \mathfrak{g}', \phi' | (\hat{M}_{v'} - \hat{M}_{v}) | \mathfrak{g}, \phi \rangle = \langle \mathfrak{g}', \phi' | \mathfrak{g}, \phi \rangle (M_{v'}(\mathfrak{g}) - M_{v}(\mathfrak{g}))(1 + \mathcal{O}(t)) \tag{66}$$

and

$$\langle \mathfrak{g}', \phi' | \widehat{\tilde{H}^{\delta}_{\phi}}(v, v') | \mathfrak{g}, \phi \rangle = \langle \phi' | \widehat{\tilde{H}^{\delta}_{\phi}}(\mathfrak{g}; v, v') | \phi \rangle \langle \mathfrak{g}' | \mathfrak{g} \rangle (1 + \mathcal{O}'(t)), \tag{67}$$

where $M_{v'}(\mathfrak{g}), M_{v}(\mathfrak{g})$ and $\widehat{\tilde{H}^{\delta}_{\phi}}(\mathfrak{g}; v, v')$ are the values of $M_{v'}, M_v$ and $\widehat{\tilde{H}^{\delta}_{\phi}}(v, v')$ at the phase space point labeled by $\mathfrak{g}$, with $\widehat{\tilde{H}^{\delta}_{\phi}}(\mathfrak{g}; v, v')$ being given by Eq. (22).

Now, Eq. (64) can be simplified as

$$M_{v'}(\mathfrak{g}) - M_{v}(\mathfrak{g}) = \frac{\langle \phi' | \widehat{\tilde{H}^{\delta}_{\phi}}(\mathfrak{g}; v, v') | \phi \rangle}{\langle \phi' | \phi \rangle} \tag{68}$$

at the leading order of t. One can see that Eq. (68) is identical to Eq. (21). Hence, by recalling the solution (50) of Eq. (21), a weak solution of the quantum constraint (64) can be given by

$$|\mathfrak{g}, \phi\rangle = |\mathfrak{g}, \vec{k}_{\vec{n}}(\mathfrak{g})\rangle \equiv |\mathfrak{g}\rangle \otimes |\vec{k}_{\vec{n}}(\mathfrak{g})\rangle \tag{69}$$

where $\vec{k}_{\vec{n}}(\mathfrak{g})$ is given by Eq. (51) for the specific $\mathfrak{g}$. Equivalently, one can also express the solution (69) as

$$|\mathfrak{g}, \phi\rangle = |\mathfrak{g}(\vec{k}_{\vec{n}}), \vec{k}_{\vec{n}}\rangle \equiv |\mathfrak{g}(\vec{k}_{\vec{n}})\rangle \otimes |\vec{k}_{\vec{n}}\rangle \tag{70}$$

where

$$\vec{k}_{\vec{n}} := (\overbrace{(k_1, \ldots, k_1)}^{n_{k_1}\text{-tuple } k_1}, \overbrace{(k_2, \ldots, k_2}^{n_{k_2}\text{-tuple } k_2}, \overbrace{(k_3, \ldots, k_3)}^{n_{k_3}\text{-tuple } k_3}, \ldots) \tag{71}$$

with $\mathfrak{g}(\vec{k}_{\vec{n}}) \in \mathcal{U}_{v,v'}$ being the geometry satisfying

$$M_{v'}(\mathfrak{g}(\vec{k}_{\vec{n}})) - M_{v}(\mathfrak{g}(\vec{k}_{\vec{n}})) = n_{k_1}\omega_{k_1} + n_{k_2}\omega_{k_2} + n_{k_3}\omega_{k_3} + \cdots =: \omega_{\vec{k}_{\vec{n}}} \tag{72}$$

for specific $(n_{k_1}, n_{k_2}, n_{k_3}, \ldots)$ and $(k_1, k_2, k_3, \ldots)$.

More generally, the weak solutions of the quantum Hamiltonian constraint (64) in space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$ are given by the linear combinations of such states (69) or (70), e.g., the well-defined superposition state

$$c|\mathfrak{g}(\vec{k}_{\vec{n}})\rangle \otimes |\vec{k}_{\vec{n}}\rangle + c'|\mathfrak{g}(\vec{k}'_{\vec{n}})\rangle \otimes |\vec{k}'_{\vec{n}}\rangle + c''|\mathfrak{g}(\vec{k}''_{\vec{n}})\rangle \otimes |\vec{k}''_{\vec{n}}\rangle + \cdots \in \tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}. \tag{73}$$

In other words, the weak solution of the quantum Hamiltonian constraint (64) in the space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$ spans a subspace $(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$ defined by

$$(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'} := \{|\mathfrak{g}(\vec{k}_{\vec{n}})\rangle \otimes |\vec{k}_{\vec{n}}\rangle \in \tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi} | M_{v'}(\mathfrak{g}(\vec{k}_{\vec{n}})) - M_{v}(\mathfrak{g}(\vec{k}_{\vec{n}})) = \omega_{\vec{k}_{\vec{n}}}\}. \tag{74}$$

We make two remarks on the states in $(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$.

(1) Notice that we only considered the specific v and v' in above calculations. Hence, the states in $(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$ are just some weak solutions of a subset of the quantum Hamiltonian constraints (64) at the order of t.

(2) Although the states in $(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$ are just some weak solutions of a subset of the quantum Hamiltonian constraints (64) at the order of t, it does indicate the entanglement between the quantum matter field and geometry. Especially, the weak solutions in $(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$ can be regarded as the physical states of quantum gravity in the specific finite region $R(v) \leq R \leq R(v')$ approximately; hence, focusing on this region, it is reasonable to require that all of the states in QFTCS should be effectively approximate states of the linear combination of physical states in $(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$. In the next Section, we discuss the Hartle-Hawking vacuum as an example.

C. Generalized Hartle-Hawking vacuum on quantum geometry

As mentioned above, the issue of QFTCS can be resolved by considering the state in $(\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$.

Specifically, once we consider the states in QFTCS in the region $R(v) < R < R(v')$, we should make the substitution

$$|\vec{k}_{\vec{n}}\rangle_{\mathfrak{g}} \to |\mathfrak{g}(\vec{k}_{\vec{n}}), \vec{k}_{\vec{n}}\rangle \in (\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}, \tag{75}$$

where $|\vec{k}_{\vec{n}}\rangle_{\mathfrak{g}}$ defined in the Appendix C is the Fock state constructed on some fixed classical geometry $\mathfrak{g} \in \mathcal{S}_{v,v'}$. As an example, one can construct the generalized Hartle-Hawking vacuum on quantum geometry by using substitution (75). In detail, on the fixed classical geometry $\mathfrak{g} \in \mathcal{S}_{v,v'}$, the local Hartle-Hawking vacuum in the region $R(v) < R < R(v')$ in QFTCS is given as (see details in the Appendix C)

$$|0_{\mathrm{HH}}\rangle_{\mathfrak{g}} = \sum_{\vec{n}} \left(\prod_{k} (1 - e^{-2\pi \xi_{\mathfrak{g}}^{-1} \omega_k})^{1/2} e^{-\pi n_k \omega_k / \xi_{\mathfrak{g}}} \right) \Big| \vec{k}_{\vec{n}}\rangle_{\mathfrak{g}}. \tag{76}$$

By using substitution (75), one obtains the generalized Hartle-Hawking vacuum on the quantum geometry, which reads

$$|0_{\mathrm{HHQG}}\rangle \coloneqq \sum_{\vec{n}} \Bigg(\prod_{k \in \mathbb{R}} (1 - \exp(-2\pi\omega_k/\xi_{\mathfrak{g}(\vec{k}_{\vec{n}})}))^{1/2} \times \exp(-\pi n_k \omega_k / \xi_{\mathfrak{g}(\vec{k}_{\vec{n}})}) \Bigg) \Big| \mathfrak{g}(\vec{k}_{\vec{n}}), \vec{k}_{\vec{n}}\rangle. \tag{77}$$

Specifically, $|0_{\mathrm{HHQG}}\rangle$ is the generalized Hartle-Hawking vacuum on quantum geometry in the specific finite region $R(v) < R < R(v')$, where the apparent horizon for each $\mathfrak{g}(\vec{k}_{\vec{n}})$ is located at $R = R(v)$, $|\mathfrak{g}(\vec{k}_{\vec{n}})\rangle \otimes |\vec{k}_{\vec{n}}\rangle \in (\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi})^{\mathrm{sol.}}_{v,v'}$ is the physical state of quantum gravity, and $\xi_{\mathfrak{g}(\vec{k}_{\vec{n}})}$ is an undetermined parameter.

It is necessary to further discuss the state $|0_{\mathrm{HHQG}}\rangle$. First, by comparing $|0_{\mathrm{HHQG}}\rangle$ with $|0_{\mathrm{HH}}\rangle$, one can see that the Fock states $|\vec{k}_{\vec{n}}\rangle$ in the combination (77) are entangled with the quantum geometry state $|\mathfrak{g}(\vec{k}_{\vec{n}})\rangle$, while there is no such entanglement in the combination (76). Especially, this entanglement in the state $|0_{\mathrm{HHQG}}\rangle$ comes from weakly solved Hamiltonian constraints (64). Second, the state $|0_{\mathrm{HHQG}}\rangle$ is a quantum state for both spacetime geometry and the scalar field. Thus, it is different from any vacuum states in the QFT on a fixed classical spacetime. In fact, the state $|0_{\mathrm{HHQG}}\rangle$ is not defined as an eigenstate of any annihilation operator. In order to determine the parameter $\xi_{\mathfrak{g}(\vec{k}_{\vec{n}})}$ in Eq. (77), let us require that the state $|0_{\mathrm{HHQG}}\rangle$ should get back to the local Hartle-Hawking vacuum $|0_{\mathrm{HH}}\rangle_{\mathfrak{g}_c}$ in the semiclassical limit of the quantum geometry; In other words, the parameter $\xi_{\mathfrak{g}(\vec{k}_{\vec{n}})}$ should be determined by the semiclassical consistency conditions

$$\lim_{\epsilon(\mathfrak{g}_c, \mathfrak{g}(\vec{k}_{\vec{n}})) \to 0} A_k^{\mathfrak{g}_c} |0_{\mathrm{HHQG}}\rangle = 0, \tag{78}$$

where $\epsilon(\mathfrak{g}_c, \mathfrak{g}(\vec{k}_{\vec{n}})) \equiv \sqrt{1 - |\langle \mathfrak{g}_c | \mathfrak{g}(\vec{k}_{\vec{n}})\rangle|^2}$, $A_k^{\mathfrak{g}_c}, A_k^{\mathfrak{g}_c,\dagger}$ are defined by Eq. (C21) on the classical geometry $\mathfrak{g}_c$, with $\mathfrak{g}_c$ given by the expectation values

$$\mathfrak{g}_c = \left\{ \frac{\langle 0_{\mathrm{HHQG}} | \hat{E}^{\varphi} | 0_{\mathrm{HHQG}}\rangle}{\langle 0_{\mathrm{HHQG}} | 0_{\mathrm{HHQG}}\rangle}, \frac{\langle 0_{\mathrm{HHQG}} | \hat{K}_{\varphi} | 0_{\mathrm{HHQG}}\rangle}{\langle 0_{\mathrm{HHQG}} | 0_{\mathrm{HHQG}}\rangle} \right\}. \tag{79}$$

In fact, one can notice that the coefficient $\exp(-\pi n_k \omega_k / \xi_{\mathfrak{g}(\vec{k}_{\vec{n}})})$ in the superposition (77) is exponentially suppressed with n_k going large. Thus, the calculations of the expectation values in (79) are dominated by the terms in (77) with small n_k. For the case that $\mathfrak{g}_c$ corresponds to the geometry of a large mass black hole, the difference between $\mathfrak{g}_c$ and each $\mathfrak{g}(\vec{k}_{\vec{n}})$ with small n_k in Eq. (77) is just some small perturbations. Hence, one has the estimation $\xi_{\mathfrak{g}(\vec{k}_{\vec{n}})} \approx \xi_{\mathfrak{g}_c} = \frac{1}{2} \partial_R \big(\frac{R^2}{(E^{\varphi}|_{\mathfrak{g}_c})^2} \big) \big|_{R = R_{\mathrm{h},\mathfrak{g}_c}}$ with $R = R_{\mathrm{h},\mathfrak{g}_c}$ being the apparent horizon in the geometry $\mathfrak{g}_c$.

One can consider another type of generalized Hartle-Hawking vacuum state $|0_{\mathrm{HH}}, \mathfrak{g}_c\rangle \coloneqq |0_{\mathrm{HH}}\rangle_{\mathfrak{g}_c} \otimes |\mathfrak{g}_c\rangle$, which also satisfies some semiclassical consistency conditions

$$A_k^{\mathfrak{g}_c} |0_{\mathrm{HH}}, \mathfrak{g}_c\rangle = 0, \tag{80}$$

and

$$\mathfrak{g}_c = \left\{ \frac{\langle 0_{\mathrm{HH}}, \mathfrak{g}_c | \hat{E}^{\varphi} | 0_{\mathrm{HH}}, \mathfrak{g}_c\rangle}{\langle 0_{\mathrm{HH}}, \mathfrak{g}_c | 0_{\mathrm{HH}}, \mathfrak{g}_c\rangle}, \frac{\langle 0_{\mathrm{HH}}, \mathfrak{g}_c | \hat{K}_{\varphi} | 0_{\mathrm{HH}}, \mathfrak{g}_c\rangle}{\langle 0_{\mathrm{HH}}, \mathfrak{g}_c | 0_{\mathrm{HH}}, \mathfrak{g}_c\rangle} \right\}. \tag{81}$$

Note that both states $|0_{\mathrm{HH}}, \mathfrak{g}_c\rangle$ and $|0_{\mathrm{HHQG}}\rangle$ give the local Hartle-Hawking vacuum $|0_{\mathrm{HH}}\rangle_{\mathfrak{g}_c}$ in the semiclassical limit of the quantum geometry by some semiclassical consistency conditions. Nevertheless, they possess fundamentally different physical interpretations at the quantum geometric level; In fact, the state $|0_{\mathrm{HHQG}}\rangle$ weakly solves the quantum Hamiltonian constraints (64), and these constraints naturally encode the entanglements between field excitations and the underlying quantum geometry; However, it is obvious that the state $|0_{\mathrm{HH}}, \mathfrak{g}_c\rangle$ fails to satisfy the quantum Hamiltonian constraints (64) and contains no geometry-matter entanglement. This distinction highlights a core advantage of the quantum gravity framework: physical constraints not only select permissible states but also determine their intrinsic entanglement structure. The construction of $|0_{\mathrm{HHQG}}\rangle$ embodies the paradigm shift from "QFT on curved spacetime" to "QFT with quantum spacetime," where geometry is no longer a fixed background but a physical degree of freedom that participates equally with matter fields in quantum dynamics.

IV. CONCLUSIONS AND DISCUSSION

In this paper, we present a consistent framework for describing quantum scalar fields on semiclassical quantum geometries, resolving a foundational inconsistency in QFT on curved spacetime. We model the background geometry with semiclassical coherent states from loop quantum gravity. In particular, we consider the coherent states whose labels are restricted to a subspace $\mathcal{U}_{v,v'}$ of the phase space of geometry, on which the Fock representations of the scalar field are unitarily equivalent. This restriction enables the construction of the space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$, in which superpositions of the geometry-matter product states become well defined. Moreover, we obtain weak solutions to the quantum Hamiltonian constraint of GR in space $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$. Consequently, general physical states—defined as the linear combination of these weak solutions in $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$—are inherently entangled. We also construct the generalized Hartle-Hawking vacuum on quantum geometry by superposing the weak solutions in $\tilde{\mathcal{H}}_{\mathrm{g}} \otimes \mathcal{H}^{\phi}$, which produces the state $|0_{\mathrm{HHQG}}\rangle$. Unlike its QFTCS counterpart, this vacuum is not defined with respect to a fixed background. More importantly, the state $|0_{\mathrm{HHQG}}\rangle$ exhibits the inherent entanglement between geometry and matter, arising from the fundamental principles of quantum gravity.

Several important directions remain open for future investigation.

(1) The current construction of the subspace $\mathcal{U}_{v,v'}$ relies on spherical symmetry and a specific region near the apparent horizon. It is essential to investigate whether analogous subspaces exist in more general settings.

(2) Our construction provides kinematical states that satisfy the quantum Hamiltonian constraint weakly. A complete description of black hole evaporation would require a notion of time evolution, potentially via relational observables or a path-integral formulation over histories of entangled geometry-matter configurations.

(3) It would be valuable to compute corrections to the Hawking spectrum arising from the quantum-geometric structure of $|0_{\mathrm{HHQG}}\rangle$ by considering the case where $\mathfrak{g}_{\mathrm{c}}$ corresponds to the geometry of a black hole with small mass, which could provide observational signatures of Planck-scale physics.

(4) The built-in geometry-matter entanglement provides a natural framework for studying the black hole information paradox. Future work should explore whether unitarity is preserved under full quantum gravitational evolution, with entanglement playing a key role in information recovery.

(5) The construction presented in this article relies on the coherent states of quantum geometry, owing to their well-behaved semiclassical properties. One may be interested in the dynamical behavior of these states, as well as the scenarios in the absence of coherent states. Indeed, the dynamics of coherent states in quantum geometry have been explored using the coherent-state path integral [75,77,78]. Extending this approach to quantum gravity coupled to a quantum scalar field represents the next step of our research. Furthermore, since coherent states form a complete set in the Hilbert space of quantum geometry 61,79]], our construction is applicable to arbitrary quantum geometry states via expansion in the coherent state basis. In fact, the generalized Hartle-Hawking vacuum obtained here is inherently a superposition of coherent states rather than a single one; this superposition structure is the key to capturing general states and generating the entanglement between quantum geometry and the scalar field.

In summary, our work demonstrates that, by treating geometry as quantum variables, we resolve the foundational inconsistencies of QFTCS on a fixed background and uncover a richer structure in which spacetime and matter coevolve as entangled quantum entities. This result not only resolves a long-standing conceptual issue, but also provides a new framework for understanding the black hole information paradox and the nature of quantum spacetime.

ACKNOWLEDGMENTS

This work is supported by the project funded by the National Natural Science Foundation of China (NSFC) (with Grants No. 12405062, No. 12275022, and No. 12505055), and “the Fundamental Research Funds for the Central Universities” (with Grants No. 21624340 and No. 2253100010). G. L. is supported by the Guangdong Basic and Applied Basic Research Foundation with Grant No. 2026A1515012166.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: QUANTIZATION OF GRAVITATIONAL FIELD AND THE SEMICLASSICAL SPACETIME GEOMETRY

With gauge fixing (19) and(20), the dynamical variables remaining for the geometry sector contain only the canonical pair $(K_{\varphi}, E^{\varphi})$. The loop quantization of the reduced phase space $\bar{\mathcal{P}}^{\mathrm{g}}$ coordinatized by $(K_{\varphi}, E^{\varphi})$ gives the kinematic Hilbert space of quantum gravity in the spherical model [51]. This space consists of colored network states that encode quantum geometry, represented by cylindrical functions defined on some graphs γ. For a given graph γ on the manifold $\mathbb{X}$, we associate the variable K_x with each elementary edge and the scalar K_{φ} with each vertex. Then, the Hilbert space $\mathcal{H}_{\mathrm{g}}^{\gamma}$ associated with γ is spanned by colored network states on γ, which take the form

$$T_{\gamma,\vec{\mu}}(K_{\varphi}) = \prod_{v\in\gamma} \exp\left(\mathbf{i}\mu_v K_{\varphi}(v)\right), \tag{A1}$$

where $\mu_v \in \mathbb{R}$ is the color of the vertex v, and $\vec{\mu} = (\ldots, \mu_{v_{\iota-1}}, \mu_{v_\iota}, \mu_{v_{\iota+1}}, \ldots)$ denote all the colors associated with the edges and vertices of the graph. The inner product among colored network states on γ is given by

$$\langle \gamma, \vec{\mu} | \gamma, \vec{\mu}' \rangle = \delta_{\vec{\mu}, \vec{\mu}'}, \tag{A2}$$

where $|\gamma, \vec{\mu}\rangle$ represents the state function $T_{\gamma,\vec{\mu}}$. The fundamental operators in $\mathcal{H}_{\mathrm{g}}^{\gamma}$ associated with $E_v^{\varphi} \coloneqq \lim_{\epsilon\to 0} \int_{x(v)-\epsilon}^{x(v)+\epsilon} dx E^{\varphi}$ are given by

$$\hat{E}_v^{\varphi} |\gamma, \vec{\mu}\rangle = G\hbar \mu_v |\gamma, \vec{\mu}\rangle. \tag{A3}$$

The fundamental operators in $\mathcal{H}_{\mathrm{g}}^{\gamma}$ associated with $N_{v,\rho}^{\pm} \coloneqq \exp(\pm \mathbf{i}\rho K_{\varphi}(v)), 0 < \rho \in \mathbb{R}$ are given by

$$\hat{N}_{v,\rho}^{\pm} |\gamma, \vec{\mu}\rangle = |\gamma, \vec{\mu}_{v,\pm\rho}\rangle, \tag{A4}$$

where $\vec{\mu}_{v_\iota, \pm\rho} \coloneqq (\ldots, \mu_{v_{\iota-1}}, \mu_{v_\iota} \pm \rho, \mu_{v_{\iota+1}}, \ldots)$.

The semiclassical geometry in this quantum theory is described by the coherent state. For an arbitrary given $\vec{\rho} = (\ldots, \rho_{v_{\iota-1}}, \rho_{v_\iota}, \rho_{v_{\iota+1}}, \ldots)$ and $\vec{\beta} = (\ldots, \beta_{v_{\iota-1}}, \beta_{v_\iota}, \beta_{v_{\iota+1}}, \ldots)$ with $0 \le \beta_v < \rho_v$, we have the subspace $\mathcal{H}_{\gamma}^{\vec{\beta}} \subset \mathcal{H}_{\mathrm{g}}^{\gamma}$ defined by $\mathcal{H}_{\gamma}^{\vec{\beta}} \coloneqq \{ |\gamma, \vec{j}, \vec{\mu} = \vec{\beta} \pm \vec{n} \cdot \vec{\rho} \rangle | \ \forall \ \vec{n} = (\ldots, n_{v_{\iota-1}}, n_{v_\iota}, n_{v_{\iota+1}}, \ldots), n_v \in \mathbb{N} \}$. The coherent state in $\mathcal{H}_{\gamma}^{\vec{\beta}}$ can be established as [40,58–62,80–82]

$$T_{\gamma,\vec{n}^0,\vec{\xi}^0}(K_{\varphi}) = \prod_{v\in\gamma} \sum_{n_v} \exp\left(\frac{-(n_v - n_v^0)^2}{t/\rho_v^2} \right) \exp\left(\mathbf{i}(\beta_v + n_v \rho_v)(K_{\varphi}(v) - \xi_v^0) \right). \tag{A5}$$

Denoted by $|\mathfrak{g}_\gamma\rangle \in \mathcal{H}_{\mathrm{g}}^{\gamma}$ the normalized version of the coherent state $T_{\gamma,\vec{n}^0,\vec{\xi}^0}(K_{\varphi})$ with

$$\mathfrak{g}_\gamma \coloneqq \{ (E_v^{\varphi,0}, K_{\varphi}^0(v)) | v \in \gamma \}, \tag{A6}$$

where $K_{\varphi}^0(v) = \xi_v^0$ and $E_v^{\varphi,0} = (\beta_v + n_v^0 \rho_v) G\hbar$. It can be verified the following three properties of $|\mathfrak{g}_\gamma\rangle$. First, the overlap of $|\mathfrak{g}_\gamma\rangle$ is estimated by

$$\begin{aligned} \langle \mathfrak{g}_\gamma | \mathfrak{g}'_\gamma \rangle \simeq \prod_v \exp & \left(-\mathbf{i} \left(\beta_v + \frac{n_0(v) + n'_0(v)}{2} \rho \right) (\xi^0(v) - \xi'^0(v)) \right) \\ & \cdot \exp\left(-\frac{\rho_v^2}{2t} (n_0(v) - n'_0(v))^2 - \frac{t}{8} (\xi^0(v) - \xi'^0(v))^2 \right); \end{aligned} \tag{A7}$$

second, the expectation values of the basic operators in $|\mathfrak{g}_\gamma\rangle$ are given by

$$\langle \mathfrak{g}_\gamma | \hat{N}_{v,\rho_v}^{\pm} | \mathfrak{g}_\gamma \rangle \approx \exp(\pm \mathbf{i} \rho_v \xi_v^0), \qquad \langle \mathfrak{g}_\gamma | \hat{E}_v^{\varphi}/(G\hbar) | \mathfrak{g}_\gamma \rangle \approx \beta_v + n_v^0 \rho_v \tag{A8}$$

with the corresponding uncertainty being given by

$$\Delta \langle \mathfrak{g}_\gamma | \hat{N}_{v,\rho_v}^{\pm} | \mathfrak{g}_\gamma \rangle \propto \frac{|\rho_v|}{\sqrt{t}}, \qquad \Delta \langle \mathfrak{g}_\gamma | \hat{E}_v^{\varphi}/(G\hbar) | \mathfrak{g}_\gamma \rangle \propto \sqrt{t}; \tag{A9}$$

third, $|\mathfrak{g}_\gamma\rangle$ satisfies the Ehrenfest property as

$$\langle \mathfrak{g}_\gamma | \hat{O}_\gamma | \mathfrak{g}'_\gamma \rangle = \langle \mathfrak{g}_\gamma | \mathfrak{g}'_\gamma \rangle O_\gamma(\mathfrak{g}'_\gamma)(1 + \mathcal{O}(t)), \tag{A10}$$

where $O_\gamma(\mathfrak{g}'_\gamma)$ is a regular functional of $\{(E_v^{\varphi}, N_{v,\rho_v}^{\pm}) | v \in \gamma\}$, and $\hat{O}_\gamma$ is the operator corresponding to $O_\gamma(\mathfrak{g}'_\gamma)$. These properties of the coherent state $|\mathfrak{g}_\gamma\rangle$ give a semiclassical and discrete description of the spacetime geometry.

Let us consider the semiclassical and continuum description of the spacetime geometry. We denote by μ_γ the coordinate distance of the adjacent vertices in γ. We can consider the above construction in the limit $\mu_\gamma \to 0$, and we have

$$\mathfrak{g} \underset{\mu_\gamma \to 0}{=} \mathfrak{g}_\gamma, \ |\mathfrak{g}\rangle \underset{\mu_\gamma \to 0}{\coloneqq} |\mathfrak{g}_\gamma\rangle. \tag{A11}$$

Correspondingly, we have

$$\bar{\mathcal{P}}^{\mathrm{g,qua}} \underset{\mu_\gamma \to 0}{=} \bar{\mathcal{P}}_{\gamma}^{\mathrm{g}}, \qquad \mathcal{H}_{\mathrm{g}} \underset{\mu_\gamma \to 0}{\coloneqq} \mathcal{H}_{\mathrm{g}}^{\gamma}; \tag{A12}$$

and

$$O \underset{\mu_\gamma \to 0}{=} O_\gamma, \qquad \hat{O} \underset{\mu_\gamma \to 0}{\coloneqq} \hat{O}_\gamma, \tag{A13}$$

where $\bar{\mathcal{P}}^{\mathrm{g,qua}} \ni \mathfrak{g}$, $\bar{\mathcal{P}}^{\mathrm{g}}_{\gamma} \ni \mathfrak{g}_\gamma$, O is the regular functional on $\bar{\mathcal{P}}^{\mathrm{g,qua}}$, and the phase space $\bar{\mathcal{P}}^{\mathrm{g,qua}}$ is a quantum generalization of $\bar{\mathcal{P}}^{\mathrm{g}}$ with $\bar{\mathcal{P}}^{\mathrm{g}}$ being a dense subspace of $\bar{\mathcal{P}}^{\mathrm{g,qua}}$. In particular, the configurations of the phase space $\bar{\mathcal{P}}^{\mathrm{g,qua}}$ include the discontinuous geometric fields, since these fields are given by a continuum limit of discrete geometries [83–86]. Finally, in the limit $\mu_\gamma \to 0$, the overlap (A7) and the Ehrenfest property (A10) of $|\mathfrak{g}_\gamma\rangle$ are given as

$$\langle \mathfrak{g}|\mathfrak{g}'\rangle \sim \exp(|\mathfrak{g}-\mathfrak{g}'|^2/t) \tag{A14}$$

and

$$\langle \mathfrak{g}|\hat{O}|\mathfrak{g}'\rangle = \langle \mathfrak{g}|\mathfrak{g}'\rangle O(\mathfrak{g}')(1+\mathcal{O}(t)), \tag{A15}$$

respectively, which are just the Eqs. (54) and (55) used in Sec. III A.

APPENDIX B: AN EXAMPLE OF THE UNITARY BOGOLIUBOV TRANSFORMATION (61)

Recall the subspace $\mathcal{U}_{v,v'} \subset \mathcal{S}_{v,v'}$ defined by

$$\mathcal{U}_{v,v'} \coloneqq \left\{ \mathfrak{g}'' \in \mathcal{S}_{v,v'} \middle| \left| \int dk dk' |\beta_{kk'}(\mathfrak{g}',\mathfrak{g})\right|^2 < \infty,\ \forall\ \mathfrak{g}',\mathfrak{g} \in \mathcal{U}_{v,v'} \right\} \tag{B1}$$

and the Bogoliubov coefficients $\beta_{kk'}(\mathfrak{g}',\mathfrak{g})$ given by

$$\beta_{kk'}(\mathfrak{g}',\mathfrak{g}) = \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}} I_{\mathfrak{g}} - \frac{1}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}} I_{\mathfrak{g}'} = \frac{1}{2} I_{\mathfrak{g}}\left(\sqrt{\frac{\omega_k}{\omega_{k'}}} - \sqrt{\frac{\omega_{k'}}{\omega_k}}\right) + \frac{1}{2}\sqrt{\frac{\omega_{k'}}{\omega_k}}(I_{\mathfrak{g}} - I_{\mathfrak{g}'}) \tag{B2}$$

with $I_{\mathfrak{g}} \coloneqq \lim_{L\to\infty} \int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} e^{-\mathbf{i}k'\check{x}^*_{\mathfrak{g}'}}$ and $I_{\mathfrak{g}'} \coloneqq \lim_{L\to\infty} \int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}'} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} e^{-\mathbf{i}k'\check{x}^*_{\mathfrak{g}'}}$. One can introduce a set of $\mathfrak{g}_M \in \mathcal{S}_{v,v'}$ parametrized by M with $R(v) = 2GM_v \leq 2GM < R(v')$; specifically, it is given by

$$(E^\varphi)^2|_{\mathfrak{g}_M} = \begin{cases} \frac{R^2}{1-2GM/R}, & R = R(v'), \\ \frac{R^2}{1-2GM_v/R}, & R(v) < R < R(v'). \end{cases} \tag{B3}$$

Then, an example of the space $\mathcal{U}_{v,v'}$ can be given as

$$\mathcal{U}_{v,v'} \equiv \{\mathfrak{g} = \mathfrak{g}_M \in \mathcal{S}_{v,v'} | R(v) \leq 2GM < R(v')\}. \tag{B4}$$

It is necessary to verify whether $\int dkdj|\beta_{kj}(\mathfrak{g}',\mathfrak{g})|^2 < \infty$ holds for arbitrary $\mathfrak{g}',\mathfrak{g}\in\mathcal{U}_{v,v'}$. Let us consider arbitrary $\mathfrak{g}' = \mathfrak{g}_{M'}, \mathfrak{g} = \mathfrak{g}_M \in \mathcal{U}_{v,v'}$ with $M < M'$, one has

$$\frac{\partial \check{x}^*_{\mathfrak{g}}}{\partial \check{x}^*_{\mathfrak{g}'}} = \frac{(E^\varphi)^2|_{\mathfrak{g}}}{(E^\varphi)^2|_{\mathfrak{g}'}} = \begin{cases} \text{bounded and positive real number}, & R = R(v'), \\ 1, & R(v) < R < R(v'). \end{cases} \tag{B5}$$

One can choose the boundary condition that ensures $\check{x}^*_{\mathfrak{g}} = \check{x}^*_{\mathfrak{g}'}$ in $R(v) < R < R(v')$, and notice that the point $R = R(v')$ is located at $\check{x}^*_{\mathfrak{g}} \to +\infty$ in the limit $L \to \infty$. Then, it is direct to get

$$I_{\mathfrak{g}} = \lim_{L\to\infty} \int_{-L}^{+L} d\check{x}^*_{\mathfrak{g}} e^{-\mathbf{i}k\check{x}^*_{\mathfrak{g}}} e^{-\mathbf{i}k'\check{x}^*_{\mathfrak{g}'}} = 2\pi\delta(k+k'). \tag{B6}$$

Furthermore, it is direct to verify that the Bogoliubov coefficients $\beta_{kj}(\mathfrak{g}',\mathfrak{g})$ for our chosen $\mathfrak{g}' = \mathfrak{g}_{M'}, \mathfrak{g} = \mathfrak{g}_M$ satisfy

$$\int dkdk'|\beta_{kk'}(\mathfrak{g}',\mathfrak{g})|^2 = 0. \tag{B7}$$

Now, we can conclude that $\int dkdj|\beta_{kj}(\mathfrak{g}',\mathfrak{g})|^2 < \infty$ holds for arbitrary $\mathfrak{g}',\mathfrak{g}\in\mathcal{U}_{v,v'}$, so $\mathcal{U}_{v,v'}$ defined by (B4) is an example of the space $\mathcal{U}_{v,v'}$ exactly.

It is necessary to emphasize that the space $\mathcal{U}_{v,v'}$ given by (B4) contains the geometry fields that are discontinuous at v'. It seems that the space $\mathcal{U}_{v,v'}$ is not a subspace of the classical reduced phase space $\bar{\mathcal{P}}^{\mathrm{g}}$, since $\bar{\mathcal{P}}^{\mathrm{g}}$ contains no discontinuous geometry field. In fact, the loop representation extends our consideration to the enlarged phase space $\bar{\mathcal{P}}^{\mathrm{g,qua}}$ defined by (A12), which contains discontinuous geometry field configuration [83,85]. In particular, the space $\mathcal{U}_{v,v'}$ given by (B4) is a subspace of $\bar{\mathcal{P}}^{\mathrm{g,qua}}$ exactly.

APPENDIX C: LOCAL HARTLE-HAWKING VACUUM ON CLASSICAL GEOMETRY

Consider the spherical spacetime metric generated by the phase space configuration $\mathfrak{g} \in \mathcal{S}_{v,v'}$, which reads

$$ds^2 = -N^2 dt^2 + \frac{(E^\varphi)^2}{E^x} dx^2 + E^x(d\theta^2 + \sin^2\theta d\varphi^2). \quad \text{(C1)}$$

One can fix the spacetime coordinate by $N = \frac{\sqrt{E^x}}{E^\varphi}$ and $E^x = R^2$. Then, we denote $h_\mathfrak{g}(R) = (E^\varphi)^2$ and the metric (C1) can be given as

$$ds^2 = -\frac{R^2}{h_\mathfrak{g}(R)} dt^2 + \frac{h_\mathfrak{g}(R)}{R^2} dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad \text{(C2)}$$

We introduce the tortoise coordinate $x^*_\mathfrak{g}$ by the transformation

$$\frac{\partial R}{\partial x^*_\mathfrak{g}} = \frac{R^2}{h_\mathfrak{g}(R)}, \quad \text{(C3)}$$

with the tortoise coordinate $x^*_\mathfrak{g}$ satisfying

$$\lim_{R\to R(v)=R_\mathrm{h}} x^*_\mathfrak{g} \to -\infty, \quad \lim_{R\to R(v')=R_\mathrm{h}+\delta} x^*_\mathfrak{g} \to \delta^*_\mathfrak{g}, \quad 0 < \delta^*_\mathfrak{g} < +\infty. \quad \text{(C4)}$$

In the tortoise coordinate $x^*_\mathfrak{g}$, the metric in the region $R_\mathrm{h} < R < R_\mathrm{h} + \delta$ can be written as

$$ds^2 = -\frac{R^2}{h_\mathfrak{g}(R)} dt^2 + \frac{R^2}{h_\mathfrak{g}(R)} (dx^*_\mathfrak{g})^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad \text{(C5)}$$

Notice that the coordinate $(t, x^*_\mathfrak{g})$ can not cover $R = R_\mathrm{h}$. Thus, it is necessary to introduce the generalized Kruskal coordinates in the region $R_\mathrm{h} \le R < R_\mathrm{h} + \delta$. Let us define [87]

$$V_\mathfrak{g} \coloneqq t + x^*_\mathfrak{g}, \qquad U_\mathfrak{g} \coloneqq t - x^*_\mathfrak{g}, \quad \text{(C6)}$$

and

$$u_\mathfrak{g} \coloneqq \frac{1}{2}(e^{\xi_\mathfrak{g} V_\mathfrak{g}} + e^{-\xi_\mathfrak{g} U_\mathfrak{g}}), \quad v_\mathfrak{g} \coloneqq \frac{1}{2}(e^{\xi_\mathfrak{g} V_\mathfrak{g}} - e^{-\xi_\mathfrak{g} U_\mathfrak{g}}) \quad \text{(C7)}$$

with $\xi_\mathfrak{g} \coloneqq \frac{1}{2}\partial_R(\frac{R^2}{h_\mathfrak{g}(R)})|_{R=R_\mathrm{h}}$. Using these coordinates, the spacetime metric reads [87]

$$ds^2 = \frac{R^2}{\xi_\mathfrak{g}^2 h_\mathfrak{g}(R)} e^{-2\xi_\mathfrak{g} x^*_\mathfrak{g}}(-dv_\mathfrak{g}^2 + du_\mathfrak{g}^2) + R^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad \text{(C8)}$$

It is easy to see that this metric is regular at $R = R_\mathrm{h}$. Now, let us choose a point $u_\mathfrak{g} = v_\mathfrak{g} = 0$ on the horizon $R = R_\mathrm{h}$, and define a new set of coordinates by

$$T_\mathfrak{g} \coloneqq \lambda_\mathfrak{g} v_\mathfrak{g}, \qquad X_\mathfrak{g} \coloneqq \lambda_\mathfrak{g} u_\mathfrak{g} \quad \text{(C9)}$$

with $\lambda_\mathfrak{g} \coloneqq \lim_{R\to R_\mathrm{h}} \left(\frac{\sqrt{R^2/h_\mathfrak{g}(R)}}{|\xi_\mathfrak{g}| e^{-\xi_\mathfrak{g} x^*_\mathfrak{g}}}\right)$. Using these definitions, the spacetime metric on the given surface $u_\mathfrak{g} = v_\mathfrak{g} = 0$ takes the formulation [87,88]

$$ds^2 = -(dT_\mathfrak{g})^2 + (dX_\mathfrak{g})^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad \text{(C10)}$$

It is checked that a certain freely falling observer is momentarily at rest in this coordinate system $(T_\mathfrak{g}, X_\mathfrak{g})$ at the surface $u_\mathfrak{g} = v_\mathfrak{g} = 0$ [87].

Now, let us consider the Klein-Gordan equation for the massless scalar field in the tortoise coordinate and the generalized Kruskal coordinate, respectively. In the tortoise coordinate $x^*_\mathfrak{g}$, the Klein-Gordan equation for the massless scalar field $\phi(t, x^*_\mathfrak{g}) \eqqcolon \frac{F(t,x^*_\mathfrak{g})}{R}$ in the region $R_\mathrm{h} < R < R_\mathrm{h} + \delta$ is given by

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{(\partial x^*_\mathfrak{g})^2} + V(R)\right) F(t, x^*_\mathfrak{g}) = 0, \quad \text{(C11)}$$

where the "potential" $V(R) \coloneqq \frac{\left(\frac{R^2}{h_\mathfrak{g}(R)}\right)\partial_R\left(\frac{R^2}{h_\mathfrak{g}(x)}\right)}{R}$ satisfies

$$\lim_{R\to R_\mathrm{h}} V(R) = 0. \quad \text{(C12)}$$

By solving the Klein-Gordon equation in the region $R_\mathrm{h} < R < R_\mathrm{h} + \delta$, one obtains the solutions

$$S_k = C\frac{e^{-\mathbf{i}\omega t + \mathbf{i}k x^*_\mathfrak{g}}}{R}, \qquad \omega = |k|, \quad \text{(C13)}$$

which compose an orthonormal basis for ϕ in the region $R_\mathrm{h} < R < R_\mathrm{h} + \delta$. Moreover, we are interested in the behavior of the Klein-Gordan equation in the coordinate $\check{x}^*_\mathfrak{g}$ given by the coordinate transformation $\check{x}^*_\mathfrak{g} = x^*_\mathfrak{g} + L - \delta^*_\mathfrak{g}$. It is direct to give the corresponding solution of the Klein-Gordan equation in the coordinate $\check{x}^*_\mathfrak{g}$, which reads

$$\check{S}_k = C\frac{e^{-\mathbf{i}\omega t + \mathbf{i}k \check{x}^*_\mathfrak{g}}}{R} \quad \text{(C14)}$$

in the region $R_\mathrm{h} < R < R_\mathrm{h} + \delta$. Then, the quantum scalar field $\hat{\phi}(t, x^*_\mathfrak{g})$ in the region $R_\mathrm{h} < R < R_\mathrm{h} + \delta$ can be expanded as

$$\hat{\phi} = \int dk(a_k^{\mathfrak{g}}\check{S}_k + a_k^{\mathfrak{g},\dagger}\check{S}_k^*), \tag{C15}$$

where $\check{S}_k^*$ is the complex conjugate of $\check{S}_k$.

In the generalized Kruskal coordinate in the region $R_{\rm h} \leq R < R_{\rm h} + \delta$, we can write down the Klein-Gordan equation for the massless scalar field $\phi(T_{\mathfrak{g}}, X_{\mathfrak{g}}) =: \frac{\tilde{F}(T_{\mathfrak{g}}, X_{\mathfrak{g}})}{R}$ in a rest frame of the freely falling observer, which reads [87]

$$\left(\frac{\partial^2}{\partial T_{\mathfrak{g}}^2} - \frac{\partial^2}{\partial X_{\mathfrak{g}}^2} + \tilde{V}(R)\right)\tilde{F}(T_{\mathfrak{g}}, X_{\mathfrak{g}}) = 0, \tag{C16}$$

where

$$\tilde{V}(R) := \frac{1}{R}\frac{\left(\frac{R^2}{h_{\mathfrak{g}}(R)}\right)\partial_R\left(\frac{R^2}{h_{\mathfrak{g}}(R)}\right)}{\lambda_{\mathfrak{g}}^2\xi_{\mathfrak{g}}^2}e^{-2\xi_{\mathfrak{g}}x_{\mathfrak{g}}^*} \tag{C17}$$

with

$$\lim_{R\to R_{\rm h}}\tilde{V}(R) = \left.\frac{\partial_R\left(\frac{R^2}{h_{\mathfrak{g}}(R)}\right)}{R}\right|_{R=R_{\rm h}}. \tag{C18}$$

Consider the case that $R_{\rm h}$ is large so that $\tilde{V}(R)$ is small enough to be neglected. Then, by solving the Klein-Gordon equation (C16) in the region $R_{\rm h} \leq R < R_{\rm h} + \delta$, one obtains orthonormal solutions

$$\tilde{S}_{k'} = C'\frac{e^{-\mathbf{i}\omega' T_{\mathfrak{g}} + \mathbf{i}k' X_{\mathfrak{g}}}}{R}. \tag{C19}$$

In addition, the quantum scalar field $\hat{\phi}$ in the region $R_{\rm h} < R < R_{\rm h} + \delta$ can be expanded as

$$\hat{\phi} = \int dk'(A_{k'}^{\mathfrak{g}}\tilde{S}_{k'} + A_{k'}^{\mathfrak{g},\dagger}\tilde{S}_{k'}^*), \tag{C20}$$

where $\tilde{S}_{k'}^*$ is the complex conjugate of $\tilde{S}_{k'}$.

Let us then consider the Bogoliubov transformation between $(A_k^{\mathfrak{g}}, A_k^{\mathfrak{g},\dagger})$ and $(a_k^{\mathfrak{g}}, a_k^{\mathfrak{g},\dagger})$. Specifically, one has

$$A_{k'}^{\mathfrak{g}} = \int dk(\alpha_{kk'}a_k^{\mathfrak{g}} + \bar{\beta}_{kk'}a_k^{\mathfrak{g},\dagger}),$$
$$A_{k'}^{\mathfrak{g},\dagger} = \int dk(\bar{\alpha}_{kk'}a_k^{\mathfrak{g},\dagger} + \beta_{kk'}a_k^{\mathfrak{g}}). \tag{C21}$$

Recall the solution (C19) of the Klein-Gordan equation in the generalized Kruskal coordinate, one can know that the Bogoliubov coefficients $\alpha_{k'k}$ and $\beta_{k'k}$ are determined by the expansion

$$e^{-\mathbf{i}k\check{U}_{\mathfrak{g}}} = \int dk'(\alpha_{kk'}e^{-\mathbf{i}k'\tilde{U}_{\mathfrak{g}}} + \beta_{kk'}e^{\mathbf{i}k'\tilde{U}_{\mathfrak{g}}}), \tag{C22}$$

where $\tilde{U}_{\mathfrak{g}} := T_{\mathfrak{g}} - X_{\mathfrak{g}}$ and

$$\check{U}_{\mathfrak{g}} := t - \check{x}_{\mathfrak{g}}^* = -\xi_{\mathfrak{g}}^{-1}\ln(-\tilde{U}_{\mathfrak{g}}) + \xi_{\mathfrak{g}}^{-1}\ln\lambda_{\mathfrak{g}} + L - \delta_{\mathfrak{g}}^*. \tag{C23}$$

It is verified that the Bogoliubov coefficients $\alpha_{kk'}$ and $\beta_{kk'}$ satisfy [87]

$$|\alpha_{kk'}| = e^{\pi\xi_{\mathfrak{g}}^{-1}\omega_k}|\beta_{kk'}|. \tag{C24}$$

Then, one can define the local Hartle-Hawking vacuum $|0_{\rm HH}\rangle_{\mathfrak{g}}$ in the region $R_{\rm h} \leq R < R_{\rm h} + \delta$ by

$$A_k^{\mathfrak{g}}|0_{\rm HH}\rangle_{\mathfrak{g}} = 0, \quad \forall\ k. \tag{C25}$$

This leads to [88]

$$|0_{\rm HH}\rangle_{\mathfrak{g}} = \prod_k\Bigg((1 - e^{-2\pi\xi_{\mathfrak{g}}^{-1}\omega_k})^{1/2} \times \sum_{n_k=0}^{\infty}e^{-\pi n_k\omega_k/\xi_{\mathfrak{g}}}\frac{(a_k^{\mathfrak{g},\dagger})^{n_k}}{(\mathcal{N})^{n_k}\sqrt{n_k!}}\Bigg)\Bigg|0_{\rm B}\Bigg\rangle_{\mathfrak{g}}, \tag{C26}$$

where $n_k = 0, 1, 2, 3, \ldots$ is the particle number on the mode k, $\mathcal{N} := \frac{1}{\sqrt{2\pi\hbar\delta(k-k')|_{k=k'}}}$ is a normalization constant. Equivalently, $|0_{\rm HH}\rangle_{\mathfrak{g}}$ can be rewritten as

$$|0_{\rm HH}\rangle_{\mathfrak{g}} = \sum_{\vec{n}}\left(\prod_k(1 - e^{-2\pi\xi_{\mathfrak{g}}^{-1}\omega_k})^{1/2}e^{-\pi n_k\omega_k/\xi_{\mathfrak{g}}}\right)\left|\vec{k}_{\vec{n}}\right\rangle_{\mathfrak{g}}, \tag{C27}$$

where

$$|\vec{k}_{\vec{n}}\rangle_{\mathfrak{g}} := \left(\prod_k\frac{(a_k^{\mathfrak{g},\dagger})^{n_k}}{(\mathcal{N})^{n_k}\sqrt{n_k!}}\right)\Bigg|0_{\rm B}\Bigg\rangle_{\mathfrak{g}}, \tag{C28}$$

and

$$\vec{n} = (n_{k_1}, n_{k_2}, n_{k_3}, \ldots) \tag{C29}$$

with the sequence $k_1 < k_2 < k_3 < \ldots$ composed of all possible values in the spectrum of k.

[1] S. W. Hawking, Black hole explosions, Nature (London) **248**, 30 (1974).
[2] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975); **46**, 206(E) (1976).
[3] Maulik K. Parikh and Frank Wilczek, Hawking radiation as tunneling, Phys. Rev. Lett. **85**, 5042 (2000).
[4] Maulik K. Parikh, A secret tunnel through the horizon, Int. J. Mod. Phys. D **13**, 2351 (2004).
[5] S. W. Hawking, Breakdown of predictability in gravitational collapse, Phys. Rev. D **14**, 2460 (1976).
[6] William G. Unruh and Robert M. Wald, Information Loss, Rep. Prog. Phys. **80**, 092002 (2017).
[7] Samir D. Mathur, The Information paradox: A pedagogical introduction, Classical Quantum Gravity **26**, 224001 (2009).
[8] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini, The entropy of Hawking radiation, Rev. Mod. Phys. **93**, 035002 (2021).
[9] Don N. Page, Information in black hole radiation, Phys. Rev. Lett. **71**, 3743 (1993).
[10] Don N. Page, Time dependence of hawking radiation entropy, J. Cosmol. Astropart. Phys. 09 (2013) 028.
[11] Geoffrey Penington, Entanglement wedge reconstruction and the information paradox, J. High Energy Phys. 09 (2020) 002.
[12] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini, Replica wormholes and the entropy of hawking radiation, J. High Energy Phys. 05 (2020) 013.
[13] Ahmed Almheiri, Raghu Mahajan, Juan Maldacena, and Ying Zhao, The Page curve of Hawking radiation from semiclassical geometry, J. High Energy Phys. 03 (2020) 149.
[14] Zeng-Bing Chen, Universal entanglement and an information-complete quantum theory, Adv. Phys. Res. **5**, e00063 (2026).
[15] Zeng-Bing Chen, Quantum entanglement dynamics of spacetime and matter, Fundam. Res. **5**, 999 (2025).
[16] Stefan Hollands and Robert M. Wald, Quantum fields in curved spacetime, Phys. Rep. **574**, 1 (2015).
[17] Robert M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*, Chicago Lectures in Physics (University of Chicago Press, Chicago, IL, 1995).
[18] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1982).
[19] Jacob D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).
[20] Jacob D. Bekenstein, Generalized second law of thermodynamics in black hole physics, Phys. Rev. D **9**, 3292 (1974).
[21] Robert M. Wald, The thermodynamics of black holes, Living Rev. Relativity **4**, 6 (2001).
[22] A. Ashtekar, J. Baez, A. Corichi, and Kirill Krasnov, Quantum geometry and black hole entropy, Phys. Rev. Lett. **80**, 904 (1998).
[23] Steven Carlip, Black hole thermodynamics and statistical mechanics, Lect. Notes Phys. **769**, 89 (2009).
[24] Daniel Harlow, Jerusalem lectures on black holes and quantum information, Rev. Mod. Phys. **88**, 015002 (2016).
[25] Alejandro Perez and Daniel Sudarsky, Renormalization of the quantum stress tensor fluctuations and the limits of semiclassical gravity, Phys. Rev. Lett. **137**, 051501 (2026).
[26] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner, The dynamics of general relativity, Gen. Relativ. Gravit. **40**, 1997 (2008).
[27] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner, Canonical variables for general relativity, Phys. Rev. **117**, 1595 (1960).
[28] Abhay Ashtekar and Jerzy Lewandowski, Background independent quantum gravity: A status report, Classical Quantum Gravity **21**, R53 (2012).
[29] Abhay Ashtekar and Jorge Pullin, *Loop Quantum Gravity: The First 30 Years* (World Scientific Publishing Co. Pte Ltd, Singapore, 2017).
[30] Thomas Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2007).
[31] Carlo Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2007).
[32] Martin Bojowald, *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity* (Cambridge University Press, Cambridge, England, 2010).
[33] Abhay Ashtekar and Eugenio Bianchi, A short review of loop quantum gravity, Rep. Prog. Phys. **84**, 042001 (2021).
[34] A. Ashtekar, John C. Baez, and Kirill Krasnov, Quantum geometry of isolated horizons and black hole entropy, Adv. Theor. Math. Phys. **4**, 1 (2000).
[35] Romesh K. Kaul and Parthasarathi Majumdar, Logarithmic correction to the Bekenstein-Hawking entropy, Phys. Rev. Lett. **84**, 5255 (2000).
[36] Amit Ghosh and Alejandro Perez, Black hole entropy and isolated horizons thermodynamics, Phys. Rev. Lett. **107**, 241301 (2011); **108**, 169901(E) (2012).
[37] Rudranil Basu, Romesh K. Kaul, and Parthasarathi Majumdar, Entropy of isolated horizons revisited, Phys. Rev. D **82**, 024007 (2010).
[38] Jonathan Engle, Karim Noui, Alejandro Perez, and Daniele Pranzetti, Black hole entropy from an SU(2)-invariant formulation of Type I isolated horizons, Phys. Rev. D **82**, 044050 (2010).
[39] Alejandro Perez, Statistical and entanglement entropy for black holes in quantum geometry, Phys. Rev. D **90**, 084015 (2014); **90**, 089907(A) (2014).
[40] Gaoping Long, Qian Chen, and Jinsong Yang, Entanglement entropy of coherent intertwiner in loop quantum gravity, Phys. Rev. D **110**, 064017 (2024).
[41] Shupeng Song, Haida Li, Yongge Ma, and Cong Zhang, Entropy of black holes with arbitrary shapes in loop quantum gravity, Sci. China Phys. Mech. Astron. **64**, 120411 (2021).
[42] Shupeng Song, Gaoping Long, Cong Zhang, and Xiangdong Zhang, Thermodynamics of isolated horizons in loop quantum gravity, Phys. Rev. D **106**, 126007 (2022).
[43] Abhay Ashtekar and Parampreet Singh, Loop quantum cosmology: A status report, Classical Quantum Gravity **28**, 213001 (2011).

[44] Martin Bojowald, Absence of singularity in loop quantum cosmology, Phys. Rev. Lett. **86**, 5227 (2001).
[45] Abhay Ashtekar, Tomasz Pawlowski, and Parampreet Singh, Quantum nature of the big bang: Improved dynamics, Phys. Rev. D **74**, 084003 (2006).
[46] Gaoping Long, Yunlong Liu, and Xiangdong Zhang, Energy conditions in the new model of loop quantum cosmology, Chin. Phys. C **45**, 115102 (2021).
[47] Naresh Dadhich, Anton Joe, and Parampreet Singh, Emergence of the product of constant curvature spaces in loop quantum cosmology, Classical Quantum Gravity **32**, 185006 (2015).
[48] Xiangdong Zhang, Gaoping Long, and Yongge Ma, Loop quantum gravity and cosmological constant, Phys. Lett. B **823**, 136770 (2021).
[49] You Ding, Yongge Ma, and Jinsong Yang, Effective scenario of loop quantum cosmology, Phys. Rev. Lett. **102**, 051301 (2009).
[50] Abhay Ashtekar and Martin Bojowald, Quantum geometry and the Schwarzschild singularity, Classical Quantum Gravity **23**, 391 (2006).
[51] Rodolfo Gambini, Javier Olmedo, and Jorge Pullin, Quantum black holes in Loop Quantum Gravity, Classical Quantum Gravity **31**, 095009 (2014).
[52] Alejandro Corichi and Parampreet Singh, Loop quantization of the Schwarzschild interior revisited, Classical Quantum Gravity **33**, 055006 (2016).
[53] Abhay Ashtekar, Javier Olmedo, and Parampreet Singh, Quantum transfiguration of Kruskal black holes, Phys. Rev. Lett. **121**, 241301 (2018).
[54] Javier Olmedo, Sahil Saini, and Parampreet Singh, From black holes to white holes: A quantum gravitational, symmetric bounce, Classical Quantum Gravity **34**, 225011 (2017).
[55] Jarod George Kelly, Robert Santacruz, and Edward Wilson-Ewing, Black hole collapse and bounce in effective loop quantum gravity, Classical Quantum Gravity **38**, 04LT01 (2021).
[56] Kristina Giesel, Hongguang Liu, Parampreet Singh, and Stefan Andreas Weigl, Generalized analysis of a dust collapse in effective loop quantum gravity: Fate of shocks and covariance, Phys. Rev. D **110**, 104016 (2024).
[57] Muxin Han, Dongxue Qu, and Cong Zhang, Spin foam amplitude of the black-to-white hole transition, Phys. Rev. D **110**, 124055 (2024).
[58] T. Thiemann and O. Winkler, Gauge field theory coherent states (GCS): III. Ehrenfest theorems, Classical Quantum Gravity **18**, 4629 (2001).
[59] Andrea Calcinari, Laurent Freidel, Etera Livine, and Simone Speziale, Twisted geometries coherent states for loop quantum gravity, Classical Quantum Gravity **38**, 025004 (2020).
[60] Cong Zhang, Shicong Song, and Muxin Han, First-order quantum correction in coherent state expectation value of loop-quantum-gravity Hamiltonian, Phys. Rev. D **105**, 064008 (2022).
[61] Gaoping Long, Xiangdong Zhang, and Cong Zhang, Twisted geometry coherent states in all dimensional loop quantum gravity: Construction and peakedness properties, Phys. Rev. D **105**, 066021 (2022).
[62] Gaoping Long, Twisted geometry coherent states in all dimensional loop quantum gravity. II. Ehrenfest property, Phys. Rev. D **106**, 066021 (2022).
[63] David Shale, Linear symmetries of free boson fields, Trans. Am. Math. Soc. **103**, 149 (1962).
[64] Martin Bojowald and Rafal Swiderski, Spherically symmetric quantum geometry: Hamiltonian constraint, Classical Quantum Gravity **23**, 2129 (2006).
[65] Dah-Wei Chiou, Wei-Tou Ni, and Alf Tang, Loop quantization of spherically symmetric midisuperspaces and loop quantum geometry of the maximally extended Schwarzschild spacetime, p. 12, 2012.
[66] R. Gambini, J. Olmedo, and J. Pullin, Spherically symmetric loop quantum gravity: Analysis of improved dynamics, Classical Quantum Gravity **37**, 205012 (2020).
[67] Jarod George Kelly, Robert Santacruz, and Edward Wilson-Ewing, Effective loop quantum gravity framework for vacuum spherically symmetric spacetimes, Phys. Rev. D **102**, 106024 (2020).
[68] Rodolfo Gambini, Javier Olmedo, and Jorge Pullin, *Quantum Geometry and Black Holes* (Springer Nature, Singapore, 2023), pp. 1–34.
[69] Cong Zhang, Jerzy Lewandowski, Yongge Ma, and Jinsong Yang, Black holes and covariance in effective quantum gravity, Phys. Rev. D **111**, L081504 (2025).
[70] Cong Zhang, Jerzy Lewandowski, Yongge Ma, and Jinsong Yang, Black holes and covariance in effective quantum gravity: A solution without Cauchy horizons, Phys. Rev. D **112**, 044054 (2025).
[71] Charles W. Misner and David H. Sharp, Relativistic equations for adiabatic, spherically symmetric gravitational collapse, Phys. Rev. **136**, B571 (1964).
[72] Xiaotian Fei, Gaoping Long, Yongge Ma, and Cong Zhang, Expansion operators in spherically symmetric loop quantum gravity, Phys. Rev. D **113**, 084037 (2026).
[73] Robert M Wald, *General Relativity* (University of Chicago Press, Chicago, 2010).
[74] Charles R Evans, Lee Samuel Finn, and David W Hobill, *Frontiers in Numerical Relativity* (Cambridge University Press, Cambridge, England, 1989).
[75] Gaoping Long, Hongguang Liu, and Cong Zhang, Semiclassical propagator for coherent state on twisted geometry, Phys. Rev. D **111**, 046021 (2025).
[76] Gaoping Long, Cong Zhang, and Hongguang Liu, Quantum representation of reduced twisted geometry in loop quantum Gravity, Phys. Rev. D **112**, 024022 (2025).
[77] Gaoping Long and Yongge Ma, Effective dynamics of weak coupling loop quantum gravity, Phys. Rev. D **105**, 044043 (2022).
[78] Muxin Han and Hongguang Liu, Effective dynamics from coherent state path integral of full loop quantum gravity, Phys. Rev. D **101**, 046003 (2020).
[79] T. Thiemann and O. Winkler, Gauge field theory coherent states (GCS): II. Peakedness properties, Classical Quantum Gravity **18**, 2561 (2001).

[80] Gaoping Long and Norbert Bodendorfer, Perelomov-type coherent states of $\mathrm{SO}(D+1)$ in all-dimensional loop quantum gravity, Phys. Rev. D **102**, 126004 (2020).

[81] Gaoping Long, Cong Zhang, and Xiangdong Zhang, Superposition type coherent states in all dimensional loop quantum gravity, Phys. Rev. D **104**, 046014 (2021).

[82] Gaoping Long, Chun-Yen Lin, and Yongge Ma, Coherent intertwiner solution of simplicity constraint in all dimensional loop quantum gravity, Phys. Rev. D **100**, 064065 (2019).

[83] Gaoping Long and Hongguang Liu, Holonomy operator for spin connection in twisted geometry, Phys. Lett. B **866**, 139580 (2025).

[84] Gaoping Long and Yongge Ma, Polytopes in all dimensional loop quantum gravity, Eur. Phys. J. C **82**, 41 (2022).

[85] Muxin Han, M. A. Yongge, and Weiming Huang, Fundamental structure of loop quantum gravity, Int. J. Mod. Phys. D **16**, 1397 (2005).

[86] Eugenio Bianchi, Pietro Doná, and Simone Speziale, Polyhedra in loop quantum gravity, Phys. Rev. D **83**, 044035 (2011).

[87] Ari Peltola, Local approach to Hawking radiation, Classical Quantum Gravity **26**, 035014 (2009).

[88] T. Padmanabhan, Gravity and the thermodynamics of horizons, Phys. Rep. **406**, 49 (2005).