

False vacuum decay across the quantum-to-thermal crossover: A comparison of real-time observables

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We develop a real-time Wigner-functional lattice framework with positive Hartree-Gaussian initial sampling and introduce a connected-cluster survival criterion for extracting false vacuum decay rates across the crossover from quantum fluctuations to thermal nucleation. At high temperatures, the connected-cluster rate agrees well with the Hartree-resummed thermal nucleation benchmark, while the commonly used global-survival criterion can give substantially smaller rates because of multiseed dynamics and global averaging. At low temperatures, the connected-cluster and global-survival rates approach each other in the dilute-event regime, whereas the false vacuum fraction observable can be contaminated by transient spatial conversion and kink-antikink reflection. Our results clarify how different real-time observables encode distinct aspects of metastable decay.

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Introduction. False vacuum decay (FVD) can occur through quantum tunneling at zero temperature [1,2] and/or through thermal fluctuations in the early universe [3]. Quantum tunneling is crucial for understanding the decay rate of the electroweak vacuum or the age of our universe [4,5]. The foundational quantum field theory description of quantum tunneling was established in Refs. [1,2], and was later extended to finite-temperature scenarios to consider the thermal dynamical fluctuations [3,6] based on the functional integral treatment of classical metastability [7,8], which is closely related to Kramers's escape rate theory in the field theory [9]. Thermal fluctuations can generate localized field configurations, including oscillons, which may seed the false vacuum decay through vacuum bubble nucleation [10–15]. Vacuum bubble dynamics in the first-order cosmological thermal phase transition can generate primordial magnetic fields [16–23], detectable gravitational waves at space-based interferometers (such as LISA [24], TianQin [25] and Taiji [26,27]), and provide a possible mechanism for explaining the baryon asymmetry of the universe through electroweak baryogenesis [28–30].

It is known that the FVD rates (bubble nucleation rates) in both the quantum tunneling and thermal fluctuation

scenarios can be calculated based on the saddle point or the instanton approximation, which determines the semiclassical decay rate and the corresponding critical bubble configuration, leaving the prefactor to be determined through real-time simulations. Drawing an analogy from quantum mechanics, Refs. [31,32] developed a direct method for calculating the FVD at zero temperature. More recent progress has shown that FVD rates in the quantum tunneling regime can be computed using real-time semiclassical simulations, in which the truncated Wigner method incorporates quantum corrections through vacuum fluctuations in the initial state, while the subsequent classical evolution allows bubble nucleation to occur [33,34], which was found to be largely different from the prediction by the instanton method [35,36].

In this letter, we present a finite-temperature Wigner-functional lattice approach to FVD, which provides a real-time description of FVD across both quantum and thermal regimes, and complements conventional saddle point methods by directly evolving sampled field configurations and comparing different decay observables. The Wigner-functional formalism provides a phase-space quasiprobability description of field configurations, with both quantum and thermal fluctuations systematically incorporated via the Hartree approximation. In addition to the conventional global-survival criterion, we introduce a connected-cluster survival criterion for identifying persistent supercritical domains and a false vacuum fraction observable that tracks spatial conversion and postnucleation growth. We compare the corresponding rates across the quantum-to-thermal crossover and benchmark them against the Hartree-resummed thermal prediction.

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Wigner functional approach to the false vacuum probability. Since the Wigner function serves as a quasi-probability distribution, we extend this construction to scalar field theory [38], and define the probability of the false vacuum as

$$P_{\text{FV}}(t) = \frac{1}{Z} \int \mathcal{D}\phi \int \frac{\mathcal{D}\Pi}{2\pi} \Theta_{\text{FV}}[\phi] W[\phi, \Pi; t], \quad (1)$$

where $\Theta_{\text{FV}}[\phi]$ is an indicator functional that returns unity if the field configuration $\phi(x)$ is identified as belonging to the false vacuum basin, and vanishes otherwise. Here $W[\phi, \Pi; t]$ is the Wigner functional and Z is the normalization factor obtained from the same functional integral without the insertion of Θ_{FV} . By analogy with the Wigner function in quantum mechanics, its definition is given by

$$W[\phi(x), \Pi(x); t] = \int \mathcal{D}\varphi(x) \exp \left[-\frac{i}{\hbar} \int dx \Pi(x) \varphi(x) \right] \times \left\langle \phi(x) + \frac{\varphi(x)}{2} \left| \hat{\rho}(t) \right| \phi(x) - \frac{\varphi(x)}{2} \right\rangle. \quad (2)$$

Here $\phi(x)$ is the time-independent real scalar field in the Schrödinger picture. $\Pi(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)}$ is the conjugate momentum density.

The Wigner functional plays the role of a phase-space quasidistribution for the field and its conjugate momentum [38]. Starting from the von Neumann equation $i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}]$ and applying the Wigner transform, one obtains [34, 38, 39]

$$\frac{\partial}{\partial t} W[\phi, \Pi; t] = -2H \frac{1}{i\hbar} \sin \left(\frac{i\hbar}{2} \Lambda \right) W[\phi, \Pi; t], \quad (3)$$

where $\Lambda = \frac{\delta}{\delta \Pi} \frac{\delta}{\delta \phi} - \frac{\delta}{\delta \phi} \frac{\delta}{\delta \Pi}$ is the Poisson-bracket operator. Expanding the sine and neglecting terms of order $\hbar^2$ and higher gives the classical Liouville equation

$$\left[\frac{\partial}{\partial t} + \int dx \left(\frac{\delta H}{\delta \Pi} \frac{\delta}{\delta \phi} - \frac{\delta H}{\delta \phi} \frac{\delta}{\delta \Pi} \right) \right] W[\phi, \Pi; t] = 0. \quad (4)$$

Its solution along the classical characteristics is

$$W[\phi, \Pi; t] = W[\phi_0, \Pi_0; 0], \quad (5)$$

where (ϕ_0, Π_0) is the initial phase-space point whose classical Hamiltonian evolution reaches (ϕ, Π) at time t . Using Eq. (5) and Liouville's theorem, the false vacuum probability can therefore be written entirely as an average over the initial ensemble:

$$P_{\text{FV}}(t) = \frac{1}{Z} \int \mathcal{D}\phi_0 \frac{\mathcal{D}\Pi_0}{2\pi} W[\phi_0, \Pi_0; 0] \Theta_{\text{FV}}[\phi_t[\phi_0, \Pi_0]]. \quad (6)$$

We substitute the result of Eq. (5) and perform a transformation of the integral measure [40].

In $(1+1)$ spacetime dimensions, a true-vacuum domain embedded in the false vacuum is bounded by a kink-antikink pair. Such a critical configuration can be produced through thermal activation or quantum tunneling; once the kink-antikink separation exceeds the critical size, the domain expands under the pressure difference between the two vacua. Related kink-antikink production has been studied in double-well and ϕ^6 potentials [41–43]. Here we consider the quartic potential $V(\phi) = \frac{m^2}{2} \phi^2 + \frac{g}{3} \phi^3 + \frac{\lambda}{4} \phi^4$. The scalar field is assumed to be approximately in thermal equilibrium initially, with $\hat{\rho} = Z^{-1} e^{-\beta \hat{H}}$ and $\beta = T^{-1}$. This quantum thermal density operator should be distinguished from a fully classical thermal initialization, in which one samples a phase-space Gibbs's distribution $P_{\text{cl}}[\phi, \Pi] \propto e^{-\beta H_{\text{cl}}[\phi, \Pi]}$ in the metastable basin, as in classical lattice tests of Langer's rate [44]. In the present work, we do not sample such a classical Gibbs's ensemble. Instead, because the full interacting thermal Wigner functional is not analytically available, we approximate the initial density matrix by a self-consistent Hartree-Gaussian quasifree state.

We decompose the fields as $\phi(x) = \varphi + \eta(x)$ and $\Pi(x) = \bar{\Pi} + \xi(x)$, where $\langle \eta \rangle = \langle \xi \rangle = 0$ and $G = \langle \eta^2 \rangle$. The renormalized Hartree gap equations are then solved self-consistently for the background φ_* and effective mass M_H . Their lattice implementation and ultraviolet and infrared stability are documented in the Supplemental Material [45].

Within this Hartree-Gaussian approximation, the initial Wigner functional factorizes into independent momentum modes and is positive definite

$$W_k(\eta_k, \xi_k) = \exp \left[-\tanh \left(\frac{\beta \omega_k}{2} \right) \left(\frac{|\xi_k|^2}{\omega_k} + \omega_k |\eta_k|^2 \right) \right], \quad (7)$$

where $\omega_k = \sqrt{k^2 + M_H^2}$. The corresponding momentum space variances of the quantum-thermal distribution are

$$\begin{aligned} \langle |\eta_k|^2 \rangle &= \frac{1}{2\omega_k} \coth \frac{\beta \omega_k}{2} = \frac{1}{\omega_k} \left(\frac{1}{2} + \frac{1}{e^{\omega_k/T} - 1} \right), \\ \langle |\xi_k|^2 \rangle &= \frac{\omega_k}{2} \coth \frac{\beta \omega_k}{2} = \omega_k \left(\frac{1}{2} + \frac{1}{e^{\omega_k/T} - 1} \right). \end{aligned} \quad (8)$$

This positive definite initial Wigner functional allows us to sample the initial ensemble directly. We stress that our numerical procedure only requires positivity of $W[\phi, \Pi; 0]$ within the Hartree-Gaussian approximation; it is not necessary to assume that the full time dependent Wigner functional remains an ordinary probability density at all later times. Possible negativity generated by the exact Wigner evolution would correspond to corrections beyond the truncated-Wigner/classical-statistical approximation, rather than to a sign change of individual sampled trajectories.

Numerically, we sample independent Gaussian fluctuations mode by mode according to Eq. (8), perform the inverse Fourier transform to obtain $\eta(x)$ and $\xi(x)$ in real space, and construct the initial fields. Each realization is then evolved with the classical equations of motion, and the false vacuum probability Eq. (6) is estimated by the Monte Carlo average

$$P_{\text{FV}}(t) \approx \frac{1}{N_s} \sum_{i=1}^{N_s} \Theta_{\text{FV}}[\phi_t^{(i)}]. \quad (9)$$

Here, i is used to denote the different Monte Carlo samples of the ensemble, and N_s represents the total number of samples. The practical construction of the indicator functional Θ_{FV} is nontrivial because a field configuration is a spatially extended object. Different operational definitions lead to different observables. In the following sections, we will specify these criteria explicitly and use the corresponding observables as complementary cross-checks of the decay dynamics.

Decay observables and coarse-grained indicators. As discussed above, the sampled initial ensemble contains both thermal and zero-point fluctuations. Since the high- k zero-point sector produces substantial pointwise noise without controlling the long-wavelength decay pattern, all decay indicators are constructed from a coarse-grained field, which suppresses short-distance fluctuations while retaining the long-wavelength structures relevant for bubble nucleation. The coarse-graining scale is chosen as $\ell = c_1 \xi_{\text{FV}}$, where $\xi_{\text{FV}} \sim M_H^{-1}$ is the false vacuum correlation length. The microscopic field is nevertheless evolved with the original equations of motion; coarse graining is applied only when constructing the observables. Details of the smoothing prescription and its parameter dependence are given in the Supplemental Material.

We now turn to the implementation of the indicator functional Θ_{FV} . The threshold ϕ_{th} on the spatially averaged field has been used as a practical sample-level survival criterion [15,34,35,49], which is useful in regimes where a single bubble dominates the decay in a relatively small box. However, in the general case, especially at higher temperature and for modest vacuum energy difference, the simulation volume may contain multiple bubble seeds. Nearby seeds can collide rapidly, and the postcollision evolution may partially return to the false vacuum. Although the detailed collision dynamics is not the central focus of this work, such local collisions can substantially distort the global spatial average and therefore make a purely whole volume average criterion less reliable. A more physical sample-level criterion is to monitor whether a sufficiently large true vacuum cluster has emerged and persisted for a sufficiently long time. Accordingly, after coarse graining the field at each time step, we identify connected intervals in which the field exceeds a threshold

value ϕ_{th} . We then introduce a critical length scale L_c and a holding time τ_{hold} . A given realization is declared to have decayed when there first appears a connected interval satisfying

$$\phi_\ell(x, t) > \phi_{\text{th}} \quad \text{for all } x \text{ in the interval,} \quad (10)$$

whose length exceeds L_c and whose persistence time exceeds τ_{hold} . In this way, transient short-lived excursions above the threshold are not counted as genuine decay events. The corresponding indicator functional is denoted schematically by $\Theta_{\text{FV}}^{\text{clus}}[\phi_t^{(i)}]$: $\Theta_{\text{FV}}^{\text{clus}} = 1$ if no such persistent supercritical cluster has appeared up to time t , and $\Theta_{\text{FV}}^{\text{clus}} = 0$ otherwise. Eq. (9) then gives the associated false vacuum survival probability $P_{\text{surv}}(t)$. This differs from the local-box criterion of Ref. [50], which was introduced primarily to avoid double counting; here the connected domains diagnose decay in both single- and multibubble regimes.

As a complementary measure of spatial conversion, we also use the false vacuum fraction, following standard analyses of first-order transition dynamics [51–53]. Unlike the two sample-level survival criteria, this observable is sensitive to postnucleation domain growth. For each coarse-grained realization, we define

$$f_{\text{FV}}^{(i)}(t) = \frac{1}{L} \int dx \Theta(\phi_{\text{th}} - \phi_\ell^{(i)}(x, t)), \quad (11)$$

namely the fraction of space that lies on the false vacuum side of the threshold. Averaging over the ensemble yields

$$P_{\text{frac}}(t) = \frac{1}{N_s} \sum_{i=1}^{N_s} f_{\text{FV}}^{(i)}(t). \quad (12)$$

Numerical simulation. Unlike the analog potentials proposed to simulate vacuum decay in cold atom experiments [49,54–56], here we consider a scalar potential that is conventionally used to study gravitational waves from first-order phase transitions in the early Universe [57–59]. All real-time FVD lattice simulations are performed on a $(1+1)\text{D}$ lattice in natural units. We adopt a dimensionless rescaling scheme to simplify numerical implementation: $\tilde{t} = mt$, $\tilde{x} = mx$ and $\tilde{T} = T/m$, $\tilde{\phi} = \phi/\phi_0$, $\tilde{\Pi} = \frac{\Pi}{m\phi_0}$, where the scalar field is dimensionless, and we fix the scaling factor $\phi_0 = 1$ for simplicity. This rescaling fully eliminates all dimensionful parameters from the equations of motion, significantly improving the stability and efficiency of our lattice simulations. The Lagrangian density reads

$$\mathcal{L} = m^2 \left[\frac{1}{2} \dot{\tilde{\phi}}^2 - \frac{1}{2} (\nabla \tilde{\phi})^2 - \left(\frac{1}{2} \tilde{\phi}^2 + \frac{\tilde{g}}{3} \tilde{\phi}^3 + \frac{\tilde{\lambda}}{4} \tilde{\phi}^4 \right) \right]. \quad (13)$$

We fix the dimensionless coupling constants as $\tilde{g} = g/m^2 = -0.5888$ and $\tilde{\lambda} = \lambda/m^2 = 0.0703$. For brevity,

we omit the tilde accent on all dimensionless quantities in what follows. We set the dimensionless integration time step to $\Delta t = 0.01$, the lattice spacing to $a = 0.25$, and the number of lattice points to $N_x = 1024$. To ensure robust statistical convergence of our results, we generate $N_s = 10^5$ statistically independent initial (ϕ, Π) configurations. For each individual configuration, we solve the following equation of motion using the fourth-order Runge-Kutta method, which we employ to evolve the field configuration $\phi_i^{(i)}$ that enters the indicator functional Θ_{FV}

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} - (\phi + g\phi^2 + \lambda\phi^3). \quad (14)$$

After obtaining the real-time evolved field configurations, we coarse grain the raw field. We then adopt three distinct criteria with a threshold set as $\phi_{\text{th}} = \phi_b + c_2$, where ϕ_b denotes the field value at the potential barrier. For the connected-cluster criterion, we take the critical length scale $L_c = c_3 \xi_{\text{fv}}$ and the holding time $\tau_{\text{hold}} = c_4 \xi_{\text{fv}}$. This procedure yields three distinct real-time observables, namely $P_{\text{surv}}^{\text{glob}}(t)$, $P_{\text{surv}}^{\text{clus}}(t)$, and $P_{\text{frac}}(t)$. The three implementations of Θ_{FV} lead to qualitatively different time dependences of the false vacuum signal. The global-average criterion and the connected-cluster criterion define sample-level survival observables, while the fraction-based definition measures the fraction of space that remains on the false vacuum side.

For the sample-level survival observables, one can extract the decay rate per unit length from $\ln P_{\text{surv}}(t) = -\Gamma_{\text{surv}} L t + C$, within a suitable intermediate time window [15,34,35], where L is the lattice size and C is a constant. This method allows one to define an effective direct decay rate Γ , as emphasized in Refs. [31,32]. For survival observables, we fit the time intervals in which $\ln P_{\text{surv}}^{\text{glob}}(t)$ and $\ln P_{\text{surv}}^{\text{clus}}(t)$ are approximately linear in t , excluding the early transient from short-lived clusters and the late tail with poor survival statistics. The fitting uncertainty is estimated by varying the subwindow inside the admissible range. We also varied the coarse-graining scale, threshold offset, critical cluster length, and holding time. The extracted temperature dependence remains stable within the explored range. Increasing the connected-cluster length scale L_c delays the cluster decay assignment and makes the criterion more restrictive, but its relation to the global-survival result is temperature dependent: in the multibubble regime, larger L_c can move the cluster curve toward, or even slower than, the global criterion. Representative parameter scans and fitting details are shown in the Supplemental Material. For the false vacuum fraction based observable, we instead use the form of the Kolmogorov-Johnson-Mehl-Avrami (KJMA) formula in $(1+1)\text{D}$ [60,61]: $\ln P_{\text{frac}}(t) = -\Gamma_{\text{frac}} v t^2 + C$ within a suitable intermediate time window avoiding late-time contamination from bubble collisions, oscillations, and

partial returns toward the false vacuum side, where v is the bubble wall velocity estimated from the time evolution of several randomly selected coarse-grained sample profiles by tracking the growth of supercritical domains. The $P_{\text{frac}}(t)$ counts local conversion immediately through the false vacuum spatial fraction, while the global-average survival changes only when the converted fraction is large enough to shift the sample-averaged field across the threshold. Thus Γ_{frac} should be interpreted as a KJMA kinetic measure of spatial conversion and vacuum domain growth, complementary to the sample-level survival rates.

To compare our simulations with finite-temperature equilibrium field theory, we compute the Langer-Affleck semiclassical nucleation rate [3,6,8]. The critical bubble and fluctuation prefactor are obtained using FindBounce and BubbleDet [62,63], with the temperature-dependent Hartree effective potential constructed consistently with the initial-state calculation. Details of the effective-potential reconstruction, bounce equation, and prefactor are given in the Supplemental Material.

Here, the temperature should be interpreted as the effective temperature in our simulation. Since the initial ensemble contains both thermal and quantum fluctuations, the input temperature T_{ini} is not the most direct measure of the thermal energy stored in the long-wavelength modes that control the nucleation events. We therefore introduce an effective temperature T_{eff} as a diagnostic of the low- k modes. Concretely, we monitor the ensemble-averaged momentum spectrum during the early time evolution and extract T_{eff} from the low momentum window in which the spectrum approximately follows the equipartition theorem: $\mathcal{P}_{\Pi}(k, t) \simeq T_{\text{eff}}(t)$, $k < k_{\text{cut}}$. The cutoff k_{cut} is chosen to be of order the inverse critical bubble scale, with $R_c \simeq 2.44$ in the present potential. We then identify an early time plateau of $T_{\text{eff}}(t)$ before substantial false vacuum decay occurs and use this value as the effective temperature. The detailed extraction procedure and comparisons among alternative definitions of T_{eff} are given in the Supplemental Material.

Figure 1 shows the FVD rates extracted from three different methods as a function of $1/T_{\text{eff}}$, compared with the theoretical predictions from the Langer-Affleck formalism using the Hartree effective potential. In the high-temperature thermal nucleation regime ($1/T_{\text{eff}} < 1$), the KJMA description is a good effective description over the fitted intermediate time window, and the decay rates extracted from the fraction method and connected-cluster method are broadly consistent with the Hartree thermal benchmark, supporting the validity of our numerical approach and the Hartree-Gaussian initial ensemble. Here, we note that the semiclassical thermal-nucleation construction becomes less stable when the Hartree effective barrier becomes shallow at high temperature. This can lead to the turnover of the Hartree benchmark. However, in this regime, the global-average criterion introduces a spurious systematic delay in the assignment of decay events: by

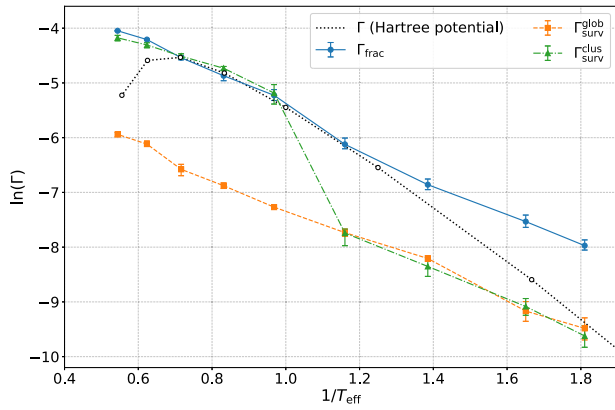


FIG. 1. Comparison between simulation results and theoretical predictions with $\eta = 0$. The quoted uncertainty is taken as the larger of the regression error and the fit-window systematic obtained from subwindow refits within the selected fitting range. See the fits to P_{frac} , $P_{\text{surv}}^{\text{glob}}$, and $P_{\text{surv}}^{\text{clus}}$ in the Supplemental Material for details.

spatially averaging over regions that have already decayed to the true vacuum and those remaining in the false vacuum, it fails to capture the onset of nucleation until the majority of the lattice has transitioned. This qualitative difference in decay dynamics explains the systematic discrepancy between the global survival method and our connected-cluster method and fraction method at high temperatures.

As the temperature decreases and the system enters the quantum tunneling regime, significant discrepancies emerge between the different methods. The false vacuum fraction observable can overestimate the sample-level decay rate, especially at the lowest temperatures studied. This behavior is associated with short-lived subcritical kink-antikinklike domains and kink-antikink reflection or partial-return dynamics, which can move transformed regions back toward the false vacuum side and thereby violate the irreversibility assumption underlying the KJMA interpretation. In contrast, the sample-level survival methods are less sensitive to these transient spatial-conversion effects. When nucleation events are sufficiently dilute, the global and connected-cluster criteria identify nearly the same decay events, with only a small difference in the assigned decay time.

The Hartree-potential prediction should be regarded as a thermal benchmark rather than an exact result. It is based on a spatially homogeneous, Gaussian self-consistent background and therefore does not include all fluctuation-determinant, dynamical-prefactor, and real-time nonequilibrium effects. In particular, it does not describe the detailed kink-antikink reflection or partial-return dynamics visible in the real-time simulations. For this reason, an overall normalization uncertainty remains in the theoretical curve. In Fig. 1, we include a constant normalization factor when comparing the Hartree benchmark with the simulation data. This factor affects the overall scale but not the temperature

dependence of the benchmark. The remaining discrepancies, especially at low temperatures, should therefore be interpreted as reflecting both the limitations of the benchmark and the observable dependence of the real-time rate extraction.

Additional uncertainties may arise because the metastable phase need not remain in full thermal equilibrium during real-time evolution [44,64–66]. Oscillon-like configurations that can arise from quantum fluctuations prior to bubble nucleation may contribute to residual systematic uncertainties in the extracted decay rate [14,59], resulting in the discrepancy between the numerically measured and the theoretical prediction of the FVD rate.

Conclusions and discussions. We developed a real-time Wigner-functional lattice framework for finite-temperature false vacuum decay using a positive Hartree-Gaussian initial ensemble. By applying global-survival, connected-cluster, and false vacuum fraction observables to the same evolved ensemble, we showed that their extracted rates reflect different aspects of nucleation and spatial conversion. The connected-cluster criterion provides a robust sample-level diagnostic: it agrees with the global criterion in the dilute regime while avoiding the global-averaging delay found in multiseed dynamics. The fraction observable instead probes spatial conversion and can be affected by transient subcritical domains and partial-return dynamics.

Our connected-cluster criterion identifies the local emergence of a persistent supercritical domain directly from evolved field configurations. Its practical advantage is that it remains tied to the local onset of a persistent supercritical domain, and is therefore less affected by global spatial averaging in multiseed regimes than the conventional global-survival criterion. The construction generalizes naturally to higher-dimensional scalar theories by replacing connected intervals with connected supercritical regions or volumes. It can thus complement saddle point and real-time approaches in numerical studies of metastable decay and cosmological first-order phase transitions, particularly when inhomogeneous nucleation, bubble growth, and domain conversion are important [1,2,8,67–69].

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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