

Horizon multipole moments of a Kerr black hole

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The horizon multipole moments of a Kerr black hole are computed from two distinct definitions that have been proposed in the literature. The first one [Ashtekar *et al.*, *Classical Quantum Gravity* **21**, 2549 (2004)] regards axisymmetric isolated horizons, while the second one [Ashtekar *et al.*, *J. High Energy Phys.* **01** (2022) 028] applies to generic (i.e., not necessarily axisymmetric) nonexpanding horizons. We review these definitions in a common frame and perform a detailed study of the resulting multipole moments for the Kerr event horizon. The horizon multipoles are found to share several properties with the (Hansen) field multipoles, including parity constraints and the leading scaling behavior with respect to the Kerr spin parameter a in the regime of small a . For the axisymmetry-based definition, we have obtained a closed-form expression of the multipole moments in terms of a and the spherical harmonic degree ℓ . For the generic definition, we have established closed-form expressions for the conformal unit round metric, the “electric” and “magnetic” potentials related to the multipoles, and the values of the multipoles in the small a limit. We show that the two definitions lead to different values of the Kerr horizon multipoles as soon as $\ell \geq 1$ (generic nonzero value of a) or $\ell \geq 2$ (small a limit).

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I. INTRODUCTION

A. Motivation

The notion of multipole moment plays a central role in classical electrodynamics [1] and Newtonian gravitational physics [2]. Indeed, whenever the details of the charge-current or mass distribution of the source can be ignored while addressing a specific problem, multipole expansions capture all of the physically relevant information about the structure of the electromagnetic or gravitational field. Two conceptually different notions of multipole moments need to be distinguished, however. On the one hand, the equations of motion for extended bodies can be formulated in terms of *source* multipoles, defined as volume integrals over the charge-current or the mass density distribution; on the other hand, *field* multipoles appear as coefficients in the asymptotic expansion of the (electromagnetic or gravitational) field itself, and are closely related to the fluxes of energy, angular momentum and linear momentum radiated by the source.

Because of such useful properties, there has been considerable interest in extending these notions to general relativity. In particular, for stationary, asymptotically flat spacetimes, the structure of the gravitational field in a neighborhood of spatial infinity has been shown to be fully characterized by two sets of field multipole moments $M_{\ell,m}$ and $S_{\ell,m}$, of mass-type and current-type, respectively [3–10]. For instance, for a spinning (Kerr) black hole of mass $M > 0$ and angular momentum $S \in [0, M^2]$, the non-vanishing field multipoles are captured by the elegant formula (due to Hansen [5])

$$M_{\ell,0} + iS_{\ell,0} = M(ia)^\ell, \quad (1.1)$$

with $a \equiv S/M$ the Kerr spin parameter and $\ell \in \mathbb{N}$ the spherical harmonic degree (thereafter we use the convention for which $0 \in \mathbb{N}$). Regarding the source multipoles, Dixon [11–15] gave a definition for extended bodies in terms of the energy-momentum tensor T_{ab} , which was later generalized by Harte [16,17] to account for self-interaction. The problem of defining the analog of source multipole moments for black holes, which correspond to *vacuum* solutions ($T_{ab} = 0$) of the Einstein equation, has only been tackled much more recently [18–22].

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A first proposal (2004) by Ashtekar *et al.* [18] regards axisymmetric isolated horizons, while a more recent proposal (2022) by Ashtekar *et al.* [22] deals with non-expanding horizons and does not require any axisymmetry assumption. In both proposals, the Coulomb-type Weyl scalar Ψ_2 at the horizon plays the role of a complex-valued *surface curvature density* and two sets of *horizon multipole moments*, $I_{\ell,m}$ and $L_{\ell,m}$, are defined according to

$$I_{\ell,m} + iL_{\ell,m} \equiv - \oint_S \Psi_2 \hat{Y}_{\ell,m} dS. \quad (1.2)$$

Here, the surface integral is taken over any (two-dimensional) cross section S of the horizon, dS is the area element of S induced by the spacetime metric g_{ab} , and $\hat{Y}_{\ell,m}$ are spherical harmonics with respect to a reference *unit round metric* $\hat{q}_{ab}$ on S , i.e., a Riemannian metric of constant scalar curvature $\hat{R} = 2$. The difference between the two proposals lies in the choice of this unit round metric: in [18], $\hat{q}_{ab}$ is uniquely defined from the assumed axisymmetry of the horizon, while in [22], $\hat{q}_{ab}$ is conformally related to the physical metric q_{ab} induced by g_{ab} on S . In the main text, we shall denote the first metric by $\hat{q}_{ab}^{\text{axi}}$ to distinguish it from the second one. In both cases, it has been shown that the whole horizon geometry can be reconstructed from the knowledge of the multipole sequences $(I_{\ell,m}, L_{\ell,m})$ [18,22].

Interestingly, the axisymmetry-based definition [18] has many applications in numerical relativity. Indeed, it has been used to assess the discrepancy between a black hole numerical spacetime and the Kerr solution [23]. Furthermore, it has been extended to axisymmetric *dynamical* horizons [19].¹ The resulting horizon multipole moments have proven a powerful coordinate-invariant diagnostic of the strongly curved geometry in binary black hole mergers, either in the early inspiral phase (where each horizon is axisymmetric to a high degree of precision) [24–27], or in the postmerger phase (when the common horizon becomes approximately axisymmetric) [24,26,28–34]; see Ref. [35] for a recent review.

In the context of Newtonian gravity (respectively classical electrodynamics), the linearity of the Poisson equation (respectively Maxwell equations) implies that field and source multipole moments coincide. In general relativity, however, this equality no longer holds: due to the nonlinear nature of the Einstein equation, the field multipoles are complicated functionals of the source multipoles [36]. In particular, the relationship between the horizon multipole moments (1.2) of a nonexpanding horizon and the corresponding field multipoles remain an open question. Even for the simplest case of an isolated, stationary rotating black hole, which is described by the

Kerr metric in four-dimensional general relativity by virtue of the no-hair theorem [37], it turns out that no closed-form formula expressing the horizon multipoles in terms of the spin parameter a and arbitrary spherical harmonic degree ℓ has been provided in the literature. It seems that the Kerr horizon multipoles have been computed only for the axisymmetry-based definition, and moreover only for $\ell \leq 3$ [18] or $\ell \leq 8$ [32]. Our aim here is to compute the Kerr horizon multipoles for all values of ℓ for both definitions, to compare them, and to study their relationship with the field multipoles (1.1). We achieve these goals as reviewed in the following subsection.

B. Main results

In this work, we review the definitions of horizon multipoles given in Refs. [18] and [22], and apply them to the event horizon of a Kerr black hole. This event horizon is a Killing horizon, and thus an isolated horizon, and is moreover axisymmetric, so that both definitions are applicable. For the axisymmetry-based definition [18], we derived formula (6.14) below, which generalizes to all $\ell \in \mathbb{N}$ the expression of the multipole moments obtained in Ref. [32] for $2 \leq \ell \leq 8$. For the generic definition [22], we have computed the conformal factor ψ between the unit round metric $\hat{q}_{ab}$ and the physical metric q_{ab} of the horizon cross sections [Eq. (6.32) below], thereby correcting a previous formula given in Ref. [22]. We have also obtained a rather simple expression of $\hat{q}_{ab}$ in terms of the standard Kerr angular coordinates (θ, φ) [Eq. (6.33)]. Moreover, we have derived a closed-form expression for both the “electric” and “magnetic” potentials involved in the construction of Ref. [22] [Eq. (6.48)]. We have expressed the resulting horizon multipole moments (1.2) in terms of a complicated integral [Eq. (6.42)]; while we could not express this integral in terms of standard functions for a generic (non small) value of a , we devised a numerical code to compute it to an arbitrary precision and have made it publicly available (cf. Appendix D).

Using either definition, the obtained horizon multipoles obey the same parity constraints as the Hansen field multipoles (1.1): the axisymmetry of the Kerr metric implies that the only nonvanishing multipoles have $m = 0$, and its discrete symmetry across the equatorial plane further implies that $I_{2n+1,0} = 0$ and $L_{2n,0} = 0$ for all $n \in \mathbb{N}$. In the regime $0 < a \ll M$ of small-spin values, we have shown that the nonvanishing horizon multipoles of a Kerr black hole behave as [Eqs. (6.18) and (6.45) below]

$$I_{\ell,0} + iL_{\ell,0} \sim \frac{\sqrt{(2\ell+1)\pi}}{M^\ell} (ia)^\ell \begin{cases} \frac{\ell!(\ell+2)!}{2(2\ell+1)!} & (\text{axisymmetric}) \\ 2^{-\ell} \alpha_\ell & (\text{generic}) \end{cases}, \quad (1.3)$$

where $(\alpha_\ell)_{\ell \in \mathbb{N}}$ is a sequence of rational numbers that increases “slowly” with ℓ [see Eq. (6.46) and Table I]. The

¹It has also been extended to nonaxisymmetric dynamical horizons that tend to an axisymmetric isolated horizon in the asymptotic future [21].

TABLE I. The first 15 numerical values of the sequence $(\alpha_\ell)_{\ell \in \mathbb{N}}$ of numbers (6.46) that appears in the small-spin limit (6.45) of the Kerr horizon multipole moments I_ℓ and L_ℓ .

ℓ	α_ℓ (exact)	α_ℓ (approximate)
0	1	1.00000000...
1	1	1.00000000...
2	$\frac{6}{5}$	1.20000000...
3	$\frac{44}{35}$	1.25714285...
4	$\frac{88}{63}$	1.39682539...
5	$\frac{16}{11}$	1.45454545...
6	$\frac{224}{143}$	1.56643356...
7	$\frac{10432}{6435}$	1.62113442...
8	$\frac{20864}{12155}$	1.71649536...
9	$\frac{7424}{4199}$	1.76804001...
10	$\frac{163328}{88179}$	1.85223239...
11	$\frac{1285120}{676039}$	1.90095541...
12	$\frac{514048}{260015}$	1.97699363...
13	$\frac{10145792}{5014575}$	2.02326059...
14	$\frac{20291584}{9694845}$	2.09302820...

scaling behavior (1.3) is clearly reminiscent of the corresponding field multipoles (1.1). As the Kerr black hole spin increases, however, that scaling breaks down and the horizon multipole moments deviate significantly from Hansen’s formula (see Figs. 5–7, 12, and 13). Moreover, even for small spin values, the coefficient in front of $(ia)^\ell$ differs between field and horizon multipoles, except for $\ell = 0$ and $\ell = 1$ for the (properly rescaled) axisymmetry-based multipoles. This coefficient also differs between the two families of horizon multipoles, except for $\ell = 0$ and $\ell = 1$, thereby showing that the generic definition [22] yields values of the Kerr horizon multipoles distinct from those arising from the axisymmetry-based definition [18]. For finite values of a , the difference occurs for any $\ell \geq 1$. In particular, the ratio between a multipole of the generic family and the axisymmetry-based one of the same ℓ diverges as $\ell \rightarrow +\infty$ [Eq. (6.54) and Figs. 15–16].

The remainder of this paper is organized as follows. The various geometrical objects involved in the two definitions of horizon multipole moments are introduced first at the level of a generic null hypersurface (Sec. II), and then at the level of a nonexpanding horizon (Sec. III). The axisymmetry-based definition of horizon multipoles is reviewed in Sec. IV, while that for generic nonexpanding horizons is reviewed in Sec. V. The application to the Kerr black hole is performed in Sec. VI, which presents the results summarized above. Section VII gives some concluding remarks and future prospects. Some technical aspects are relegated to Appendixes: Appendix A is devoted to the computation of the integral involved in the axisymmetry-based multipoles of the Kerr horizon; Appendix B provides an alternative derivation of the electric-type and magnetic-type potentials of the Kerr horizon; Appendix C regards the

small spin behavior of the Kerr horizon multipoles of the generic family; Appendix D provides the links to SageMath notebooks used for some symbolic or numerical computations.

Our conventions are those of Wald [38]. In particular, the metric signature is $(-, +, +, +)$ and the Riemann curvature tensor $R_{abc}{}^d$ is defined by $2\nabla_{[a}\nabla_{b]}\omega_c = R_{abc}{}^d\omega_d$ for any 1-form ω_a . The Latin letters $(a, b, c, \dots)$ over tensors denote abstract indices for tensor fields defined over a manifold, independently of the dimension of the latter, which can be 4 (spacetime), 3 (horizon) or 2 (horizon cross sections). In the calculations regarding Kerr spacetime, we use advanced Kerr coordinates (v, r, θ, ϕ) , which are regular on the future event horizon. Throughout the paper, an overbar denotes complex conjugation and we set $G = c = 1$.

II. BASIC GEOMETRY OF NULL HYPERSURFACES

The multipole moments introduced in Refs. [18] and [22] regard nonexpanding horizons, which are a specific type of null hypersurfaces adapted to the description of the event horizon of a black hole in equilibrium. In this section, we therefore review basic properties of null hypersurfaces and introduce geometrical objects—such as the rotation 1-form—which play a role in the definition of the horizon multipole moments.

A. First and second fundamental forms, expansion and shear

We consider a four-dimensional time-oriented spacetime $(\mathcal{M}, g_{ab})$ and a null hypersurface $\mathcal{H}$ of $\mathcal{M}$ with the topology

$$\mathcal{H} \sim \mathbb{R} \times \mathbb{S}^2. \quad (2.1)$$

For a stationary black hole with a connected event horizon, this is the only possible topology in four-dimensional general relativity (assuming vacuum or the null energy condition) [39,40].

The *first fundamental form* of $\mathcal{H}$ is the “metric” h_{ab} induced by g_{ab} , i.e., the pullback of g_{ab} on $\mathcal{H}$ by the inclusion map $\iota: \mathcal{H} \hookrightarrow \mathcal{M}$: $h \equiv \iota^*g$. In other words, for any vectors u^a and v^a tangent to $\mathcal{H}$, $h_{ab}u^a v^b \equiv g_{ab}u^a v^b$. By definition of a null hypersurface, at each point $p \in \mathcal{H}$, h_{ab} is a degenerate symmetric bilinear form, of signature $(0, +, +)$.

As a null hypersurface, $\mathcal{H}$ is ruled by a 2-parameter family $L_{(\theta_1, \theta_2)}$ of null geodesics, called the *generators of $\mathcal{H}$* (cf. Fig. 1). Given the topology (2.1), the parameters (θ_1, θ_2) span the sphere $\mathbb{S}^2$. For instance they can be polar coordinates, but any other coordinate system on $\mathbb{S}^2$ will do; what matters is that (θ_1, θ_2) stays constant along a given generator. For example, for the event horizon of a Kerr black hole described in terms of advanced Kerr coordinates

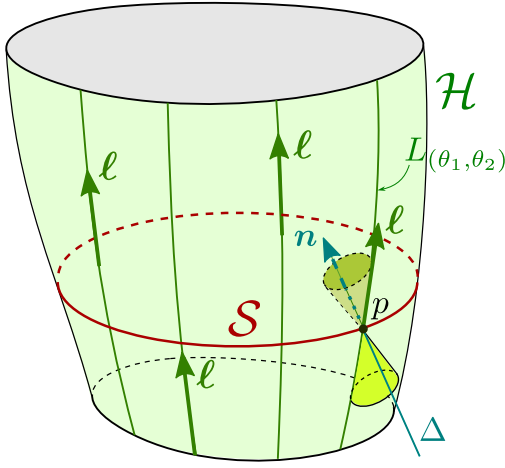


FIG. 1. Null hypersurface $\mathcal{H}$ of topology $\mathbb{R} \times \mathbb{S}^2$. Some of the null geodesic generators $L_{(\theta_1, \theta_2)}$ ruling $\mathcal{H}$ are depicted as green curves with tangent vectors ℓ , the latter being null normals to $\mathcal{H}$. $\mathcal{S}$ is a cross section of $\mathcal{H}$; it is drawn as a curve (of topology $\mathbb{S}^1$), instead of a surface (of topology $\mathbb{S}^2$), due to the dimensional reduction of the graphic. The metric null cone is depicted at some point $p \in \mathcal{S}$; this cone is tangent to $\mathcal{H}$ along the null generator $L_{(\theta_1, \theta_2)}$ through p , with its past (respectively future) nappe lying outside (respectively inside) $\mathcal{H}$. $\mathbf{n}$ is a future-directed null vector transverse to $\mathcal{H}$ and normal to $\mathcal{S}$ at p , while Δ is an ingoing null geodesic admitting $\mathbf{n}$ as a tangent vector.

(v, r, θ, ϕ) (cf. Sec. VI), one may choose $\theta_1 = \theta$ and $\theta_2 = \phi - \Omega_{\mathcal{H}} v$, where $\Omega_{\mathcal{H}}$ is the horizon's (constant) angular velocity.

Any vector field ℓ^a normal to $\mathcal{H}$ is null, tangent to $\mathcal{H}$ and lies in the kernel of h_{ab} , so that $h_{ab}\ell^b = 0$. In what follows, we consider only future-directed null normals ℓ^a . Since the kernel of h_{ab} is one-dimensional, given the signature $(0, +, +)$, ℓ^a is necessarily tangent to the null geodesic generators $L_{(\theta_1, \theta_2)}$ (cf. Fig. 1). It follows that ℓ^a is a *pregeodesic* vector field: there exists a scalar field $\kappa_{(\ell)}$ on $\mathcal{H}$ such that

$$\ell^b \nabla_b \ell^a = \kappa_{(\ell)} \ell^a. \quad (2.2)$$

Along a given generator $L_{(\theta_1, \theta_2)}$, ℓ^a can be viewed as the tangent vector associated to some parameter v of $L_{(\theta_1, \theta_2)}$,

$$\ell^a = \left. \frac{dx^a}{dv} \right|_{L_{(\theta_1, \theta_2)}}. \quad (2.3)$$

The quantity $\kappa_{(\ell)}$ measures the lack of affinity of v : v is an affine parameter of the geodesic $L_{(\theta_1, \theta_2)}$ if, and only if, $\kappa_{(\ell)} = 0$. If this happens, the right-hand side of (2.2) is zero, and ℓ^a is called a *geodesic* vector. By letting v vary smoothly from one generator to the other, one obtains a coordinate system (v, θ_1, θ_2) on $\mathcal{H}$ and, by construction, ℓ^a

is the first vector of the associated coordinate basis: $\ell^a = (\partial_v)^a$. It follows that $\ell^a \nabla_a v = 1$.

Since h_{ab} is degenerate, there is no canonical volume 3-form on $\mathcal{H}$ associated to it. However, there exists a unique (up to a sign) 2-form $\mathcal{H}\epsilon_{ab}$ such that $\mathcal{H}\epsilon_{ab}e_1^a e_2^b = \pm 1$ for any pair of unit spacelike vectors (e_1^a, e_2^a) tangent to $\mathcal{H}$ and orthogonal to each other. It is defined as follows: for any pair of vectors (u^a, v^a) in $T_p \mathcal{H}$,

$$\mathcal{H}\epsilon_{ab}u^a v^b \equiv \epsilon_{cdab}n^c \ell^d u^a v^b, \quad (2.4)$$

where ϵ_{abcd} is the Levi-Civita tensor (volume 4-form) associated to the spacetime metric g_{ab} , ℓ^a is any null normal to $\mathcal{H}$ and n^a is any null vector such that $\ell_a n^a = -1$ (which implies that n^a is transverse to $\mathcal{H}$; cf. Fig. 1). The definition (2.4) is independent of the choice of ℓ^a and n^a . Indeed, any other pair (ℓ'^a, n'^a) is necessarily related to (ℓ^a, n^a) by $\ell'^a = f\ell^a$ and $n'^a = f^{-1}n^a + \alpha^0 \ell^a + \alpha^1 w_1^a + \alpha^2 w_2^a$, where w_1^a and w_2^a are two vectors tangent to $\mathcal{H}$, such that (ℓ^a, w_1^a, w_2^a) is a basis of $T_p \mathcal{H}$. Accordingly,

$$\begin{aligned} \epsilon_{cdab}n'^c \ell'^d u^a v^b &= \epsilon_{cdab}n^c \ell^d u^a v^b + f\alpha^1 \underbrace{\epsilon_{cdab}w_1^c \ell^d u^a v^b}_0 \\ &\quad + f\alpha^2 \underbrace{\epsilon_{cdab}w_2^c \ell^d u^a v^b}_0 \\ &= \mathcal{H}\epsilon_{ab}u^a v^b, \end{aligned}$$

where the 0's occur because ϵ_{abcd} is a 4-form and the vectors $w_{1/2}^a$, ℓ^a , u^a and v^a are linearly dependent (four vectors in the 3-space $T_p \mathcal{H}$). The 2-form $\mathcal{H}\epsilon_{ab}$ is called the *area 2-form* of $\mathcal{H}$. Note that, by construction,

$$\mathcal{H}\epsilon_{ab}\ell^b = 0. \quad (2.5)$$

The extrinsic geometry of $\mathcal{H}$ is defined by the variation of the normal ℓ^a along $\mathcal{H}$ with respect to the ambient connection ∇_a . It is measured by the *Weingarten map* χ [41,42], which, at each point $p \in \mathcal{H}$, is an endomorphism of the tangent space $T_p \mathcal{H}$ defined by

$$\chi^a_b u^b \equiv u^b \nabla_b \ell^a, \quad (2.6)$$

for any vector $u^a \in T_p \mathcal{H}$. One has $\ell_a \chi^a_b u^b = u^b \ell_a \nabla_b \ell^a = 0$, which shows that $\chi^a_b u^b \in T_p \mathcal{H}$. Hence χ is well defined as an endomorphism of $T_p \mathcal{H}$. Moreover χ is self-adjoint with respect to h_{ab} : $h_{ab}u^a \chi^b_c v^c = h_{ab}\chi^a_c u^c v^b$. This follows from ℓ^a being normal to a hypersurface. In view of Eq. (2.2), ℓ^a is an eigenvector of the Weingarten map, of eigenvalue $\kappa_{(\ell)}$,

$$\chi^a_b \ell^b = \kappa_{(\ell)} \ell^a. \quad (2.7)$$

The *second fundamental form* of $\mathcal{H}$ is the field of symmetric bilinear forms Θ_{ab} on $\mathcal{H}$ defined by

$$\Theta_{ab} \equiv h_{ac} \chi^c{}_b. \quad (2.8)$$

It is symmetric because χ is self-adjoint. Given Eq. (2.6), one has $\Theta_{ab} u^a v^b = h_{ac} u^a v^b \nabla_b \ell^c = g_{ac} u^a v^b \nabla_b \ell^c = (\nabla_b \ell_a) u^a v^b$, so that Θ_{ab} can be viewed as the pullback of the tensor field $(\nabla \ell^b)_{ab} = \nabla_b \ell_a$ (defined on $\mathcal{M}$, given any extension of ℓ^a in some neighborhood of $\mathcal{H}$) by the inclusion map $\iota: \mathcal{H} \hookrightarrow \mathcal{M}$,

$$\Theta = \iota^* \nabla \ell^b. \quad (2.9)$$

Just like h_{ab} , Θ_{ab} is a degenerate symmetric bilinear form with ℓ^a in its kernel: $\Theta_{ab} \ell^b = 0$. The second fundamental form Θ_{ab} can be viewed as the deformation tensor of h_{ab} along the normal ℓ^a , since Θ_{ab} is (one half of) the Lie derivative of h_{ab} along ℓ^a ,

$$\Theta_{ab} = \frac{1}{2} \mathcal{L}_\ell h_{ab}. \quad (2.10)$$

This relationship easily follows from the definition of the Lie derivative. Indeed, $\mathcal{L}_\ell \mathbf{h} \equiv \lim_{\varepsilon \rightarrow 0} (\Phi_\varepsilon^* \mathbf{h} - \mathbf{h})/\varepsilon$, where $\Phi_\varepsilon^* \mathbf{h}$ is the pullback of $\mathbf{h}$ by the flow map Φ_ε of displacement ε along ℓ^a . Now, since $\mathbf{h} = \iota^* \mathbf{g}$, we have $\Phi_\varepsilon^* \mathbf{h} = \iota^* \Phi_\varepsilon^* \mathbf{g}$, so that $\mathcal{L}_\ell \mathbf{h} = \iota^* \mathcal{L}_\ell \mathbf{g}$. The identity $\mathcal{L}_\ell g_{ab} = \nabla_a \ell_b + \nabla_b \ell_a$ along with Eq. (2.9) and the symmetry of Θ_{ab} leads to Eq. (2.10).

The *expansion* $\theta_{(\ell)}$ of the null normal ℓ^a is the scalar field defined on $\mathcal{H}$ by

$$\theta_{(\ell)} \equiv \chi^a{}_a - \kappa_{(\ell)}. \quad (2.11)$$

Notice that $\chi^a{}_a$ is the trace of the endomorphism χ and is thus a well defined scalar field on $\mathcal{H}$. In terms of the coordinate system (v, θ_1, θ_2) introduced above, it follows from Eq. (2.7) that $\theta_{(\ell)} = \chi^{\theta_1}{}_{\theta_1} + \chi^{\theta_2}{}_{\theta_2}$. Moreover, it is easy to see that $\theta_{(\ell)}$ is related to the Lie derivative of the area 2-form $\mathcal{H}_{\varepsilon ab}$ along ℓ^a by

$$\mathcal{L}_\ell \mathcal{H}_{\varepsilon ab} = \theta_{(\ell)} \mathcal{H}_{\varepsilon ab}. \quad (2.12)$$

The *shear tensor* σ_{ab} of the null normal ℓ^a is the tensor field defined on $\mathcal{H}$ by

$$\sigma_{ab} \equiv \Theta_{ab} - \frac{1}{2} \theta_{(\ell)} h_{ab}. \quad (2.13)$$

Like h_{ab} and Θ_{ab} , σ_{ab} is a degenerate symmetric bilinear form, with ℓ^a in its kernel: $\sigma_{ab} \ell^b = 0$.

The above definitions of the first and second fundamental forms, $\mathbf{h}$ and Θ , and of the Weingarten map χ follow those for spacelike and timelike hypersurfaces. There is a major difference though: for a spacelike (respectively timelike) hypersurface, the normal vector ℓ^a can always be normalized by $\ell_a \ell^a = -1$ (respectively $\ell_a \ell^a = 1$). As a result, for non-null hypersurfaces, the tensor fields χ and Θ are unique (up to a sign). On the contrary, for the null

hypersurface $\mathcal{H}$, χ and Θ depend on ℓ^a . The latter can be arbitrarily rescaled as

$$\ell^a \rightarrow \ell'^a = f \ell^a, \quad (2.14)$$

for any smooth positive scalar field f on $\mathcal{H}$, with $f > 0$ to preserve the future orientation. It is easy to see that, under the rescaling (2.14), the geometric quantities introduced so far change as follows²:

$$\kappa_{(\ell)} \rightarrow \kappa_{(\ell')} = f \kappa_{(\ell)} + \ell^a (df)_a, \quad (2.15a)$$

$$\chi^a{}_b \rightarrow \chi'^a{}_b = f \chi^a{}_b + \ell^a (df)_b, \quad (2.15b)$$

$$\Theta_{ab} \rightarrow \Theta'_{ab} = f \Theta_{ab}, \quad (2.15c)$$

$$\theta_{(\ell)} \rightarrow \theta_{(\ell')} = f \theta_{(\ell)}, \quad (2.15d)$$

$$\sigma_{ab} \rightarrow \sigma'_{ab} = f \sigma_{ab}. \quad (2.15e)$$

B. Cross sections of a null hypersurface

A *cross section* of $\mathcal{H}$ is a two-dimensional submanifold $\mathcal{S}$ of $\mathcal{H}$ such that each null geodesic generator $L_{(\theta_1, \theta_2)}$ intersects $\mathcal{S}$ once, and only once, without being tangent to $\mathcal{S}$ (cf. Fig. 1). From (2.1) it is clear that all cross sections have the topology of $\mathbb{S}^2$. Moreover, given the signature $(0, +, +)$ of $\mathbf{h}$, any cross section $\mathcal{S}$ is necessarily a spacelike surface. Hence the pullback $\mathbf{q}$ of $\mathbf{h}$ by the inclusion map $j: \mathcal{S} \hookrightarrow \mathcal{H}$ is a Riemannian (i.e., positive definite) metric: for any pair of vectors (u^a, v^a) tangent to $\mathcal{S}$, $q_{ab} u^a v^b \equiv h_{ab} u^a v^b = g_{ab} u^a v^b$ and $q_{ab} u^a u^b > 0$ as long as $u^a \neq 0$. One can easily see that the Levi-Civita tensor (area 2-form) ε_{ab} of q_{ab} coincides with the pullback of the area 2-form of $\mathcal{H}$ [Eq. (2.4)] by the inclusion map,

$$\varepsilon_{ab} = j^* \mathcal{H}_{\varepsilon ab}. \quad (2.16)$$

A *cross section slicing* of $\mathcal{H}$ is a 1-parameter family $(\mathcal{S}_v)_{v \in \mathbb{R}}$ of nonintersecting cross sections such that $\mathcal{H} = \bigcup_{v \in \mathbb{R}} \mathcal{S}_v$. Combining the slicing parameter v with the parameters (θ_1, θ_2) labeling the generators of $\mathcal{H}$ yields a coordinate system (v, θ_1, θ_2) on $\mathcal{H}$. The slicing parameter v can also be considered as a regular parameter along each of the generators $L_{(\theta_1, \theta_2)}$. The tangent vector associated to this parametrization of $L_{(\theta_1, \theta_2)}$ by (2.3) provides a normal vector field ℓ^a to $\mathcal{H}$, which is called the *null normal adapted to the slicing* $(\mathcal{S}_v)_{v \in \mathbb{R}}$. By definition, the point where $\mathcal{S}_v$ intersects a given generator $L_{(\theta_1, \theta_2)}$ is connected to the nearby

²Notice a slight asymmetry in our notations: we have kept the dependency on ℓ^a explicit for the scalar quantities $\kappa_{(\ell)}$ and $\theta_{(\ell)}$, but have dropped it for the tensor quantities $\chi^a{}_b$, Θ_{ab} and σ_{ab} , to avoid cluttering with tensor indices.

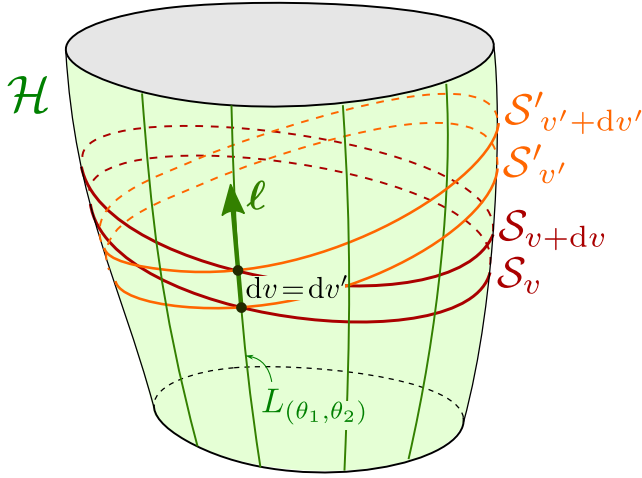


FIG. 2. Two cross section slicings $(S_v)_{v \in \mathbb{R}}$ and $(S'_{v'})_{v' \in \mathbb{R}}$ of a null hypersurface $\mathcal{H}$, both defining the same null normal ℓ to $\mathcal{H}$: $\ell^a = dx^a/dv = dx^a/dv'$ along the generators $L_{(\theta_1, \theta_2)}$ of $\mathcal{H}$.

point on S_{v+dv} along the same generator by the infinitesimal vector $dv\ell^a$ (cf. Fig. 2).

For a given cross section slicing, there is a unique null normal ℓ^a to $\mathcal{H}$ that is adapted to it. The converse is not true: given a normal ℓ^a to $\mathcal{H}$, there are many distinct slicings for which ℓ^a is an adapted normal (cf. Fig. 2). Two such slicings, $(S_v)_{v \in \mathbb{R}}$ and $(S'_{v'})_{v' \in \mathbb{R}}$, say, are connected by

$$v = v' + H(\theta_1, \theta_2), \quad (2.17)$$

where H is a (real-valued) smooth function. Indeed, Eq. (2.17) is a necessary and sufficient condition for having $dv' = dv$ along any geodesic generator $L_{(\theta_1, \theta_2)}$ of $\mathcal{H}$, which ensures that both v and v' parametrizations of $L_{(\theta_1, \theta_2)}$ lead to the same normal ℓ^a via (2.3).

Not only does a slicing $(S_v)_{v \in \mathbb{R}}$ of $\mathcal{H}$ define a unique null normal ℓ^a to $\mathcal{H}$, but it defines as well a unique vector field n^a on $\mathcal{H}$ that is null, normal to S_v , and obeys

$$\ell_a n^a = -1. \quad (2.18)$$

Notice that (2.18) implies that n^a is necessarily transverse to $\mathcal{H}$ (cf. Fig. 1). The uniqueness of n^a follows from the spacelike character of the cross sections. Indeed, the spacelike character of the tangent space $T_p S_v$ at any point $p \in S_v$ allows one to express $T_p \mathcal{M}$ as the direct sum $T_p \mathcal{M} = T_p S_v \oplus T_p^\perp S_v$, with the timelike 2-plane $T_p^\perp S_v$ intersecting the null cone of g_{ab} at p along two directions: one is necessarily along ℓ^a —for ℓ^a is normal to $\mathcal{H}$ and hence to S_v —and n^a , as defined above, lies along the second direction. The normalization relation (2.18) then determines n^a uniquely. Moreover the pullback of the 1-form n_a to $\mathcal{H}$ by the inclusion map $\iota: \mathcal{H} \hookrightarrow \mathcal{M}$ coincides with minus the differential of v considered as a scalar field on $\mathcal{H}$,

$$\iota^* n_a = -(dv)_a. \quad (2.19)$$

Indeed, $n_a \ell^a = -1 = -\ell^a \nabla_a v = -(dv)_a \ell^a$ by (2.18) and (2.3) and for any vector u^a tangent to S_v , $n_a u^a = 0 = -u^a \nabla_a v = -(dv)_a u^a$ (since n^a is normal to S_v), which proves (2.19); see also Eq. (4.34) in Ref. [43].

C. Rotation 1-form and Hájiček 1-form

Let $(S_v)_{v \in \mathbb{R}}$ be a cross section slicing of $\mathcal{H}$ and ℓ^a the associated normal to $\mathcal{H}$. The rotation 1-form ω_a associated to $(S_v)_{v \in \mathbb{R}}$ is the 1-form defined on $\mathcal{H}$ by

$$\omega_a \equiv (dv)_b \chi^b_a = -n_b \chi^b_a, \quad (2.20)$$

where χ^b_a is the Weingarten map associated to ℓ^a and n^a be the unique null normal to S_v such that $\ell_a n^a = -1$ [Eq. (2.18)]. The second equality in Eq. (2.20) stems from Eq. (2.19), with the understanding that n_a actually stands for $\iota^* n_a$. An immediate consequence of (2.7) and (2.18) is

$$\omega_a \ell^a = \kappa(\ell). \quad (2.21)$$

Under a rescaling $\ell^a \rightarrow f \ell^a$ of the null normal to $\mathcal{H}$, keeping the cross sections S_v fixed (albeit relabeling them to $S_{v'}$ with $dv' = f^{-1} dv$), so that n^a is merely rescaled as $n^a \rightarrow f^{-1} n^a$, one has $\chi^b_a \rightarrow f \chi^b_a + \ell^b (df)_a$ [Eq. (2.15b)], so that the rotation 1-form changes only by the addition of the differential of $\ln f$,

$$\omega_a \rightarrow \omega_a + (d \ln f)_a. \quad (2.22)$$

It is worth stressing that the 1-form ω_a not only depends on the scaling of ℓ^a , as expressed by (2.22), but it depends as well on the chosen cross section slicing of $\mathcal{H}$, contrary to h_{ab} , χ^a_b , Θ_{ab} , $\theta(\ell)$ and σ_{ab} . This is clear from the definition (2.20), which involves the $\mathcal{H}$ -transverse null normal n^a to the slices (cf. Sec. II B).

The Hájiček 1-form [42–44] associated to the cross section slicing $(S_v)_{v \in \mathbb{R}}$ is the 1-form defined on $\mathcal{H}$ by

$$\Omega_a \equiv \omega_b q^b_a, \quad (2.23)$$

where

$$q^b_a \equiv \delta^b_a + \ell^b n_a \quad (2.24)$$

is the orthogonal projector on S_v within $\mathcal{H}$. (The orthogonal projector on S_v within $\mathcal{M}$ is $q^b_a \equiv \delta^b_a + \ell^b n_a + n^b \ell_a$. Given that $\iota^* \ell_a = 0$, the two operators coincide when applied to vectors tangent to $\mathcal{H}$.) Since $q^b_a \ell^a = 0$, an immediate property is $\Omega_a \ell^a = 0$. We deduce from (2.23)–(2.24) and $\omega_b \ell^b = \kappa(\ell)$ [Eq. (2.21)] that

$$\Omega_a = \omega_a + \kappa(\ell) n_a = \omega_a - \kappa(\ell) (dv)_a, \quad (2.25)$$

where the second equality stems from (2.19). By combining Eqs. (2.23), (2.20) and (2.6), one gets an alternative expression of the Hájíček 1-form,

$$\Omega_a = -n_c \nabla_b \ell^c q^b_a. \quad (2.26)$$

Under a rescaling $\ell^a \rightarrow f \ell^a$ of the null normal, keeping the cross sections $\mathcal{S}_v$ fixed but relabeled to $\mathcal{S}_{v'}$ with $dv' = f^{-1} dv$, so that $n_a \rightarrow f^{-1} n_a$, we see from Eqs. (2.22) and (2.15) that the Hájíček 1-form changes as

$$\Omega_a \rightarrow \Omega_a + (d \ln f)_b q^b_a. \quad (2.27)$$

III. NONEXPANDING HORIZONS

The event horizon of an isolated, stationary black hole is a Killing horizon; see, e.g., [38]. Nonexpanding horizons generalize the concept of Killing horizon for “stationary” black holes embedded in a non necessarily stationary gravitational environment [35,43–47]. In this section we recall the definition of a nonexpanding horizon and review its main properties (Sec. III A), including the induced geometry and the characterization of its shape an angular momentum structure (Sec. III C), out of one of the five Weyl curvature scalars (Sec. III B).

A. Definition and main properties

A null hypersurface $\mathcal{H}$ is a *nonexpanding horizon* (NEH) if, and only if, (i) $\mathcal{H}$ has the topology (2.1): $\mathcal{H} \sim \mathbb{R} \times \mathbb{S}^2$, (ii) the expansion $\theta_{(\ell)}$ of any null normal ℓ^a to $\mathcal{H}$, as defined by Eq. (2.11), vanishes and (iii) the Ricci tensor of g_{ab} admits any null normal to $\mathcal{H}$ as an eigenvector: $R^a_b \ell^b = \alpha \ell^a$ for some (possibly zero) scalar field α on $\mathcal{H}$ [35,46–48]. Given the scaling law (2.15d), if $\theta_{(\ell)} = 0$ for some null normal, it remains zero for any other null normal. From now on, $\mathcal{H}$ will denote a NEH. In general relativity (i.e., assuming the Einstein equation), condition (iii) is implied by the null dominant energy condition. In our work, it is automatically fulfilled since we are considering vacuum spacetimes in the vicinity of $\mathcal{H}$. Thanks to (ii) and (iii), the Raychaudhuri equation implies the vanishing of the shear tensor σ_{ab} of ℓ^a [46,47]. It follows then from Eq. (2.13) that the second fundamental form Θ_{ab} associated to any null normal ℓ^a vanishes identically: $\Theta_{ab} = 0$. Equation (2.10) implies then the invariance of the first fundamental form by Lie transport along the null normal,

$$\mathcal{L}_\ell h_{ab} = 0. \quad (3.1)$$

If h_{ab} were a genuine (i.e., nondegenerate) metric, this would mean that ℓ^a is a Killing vector of h_{ab} . In addition, it follows from $\theta_{(\ell)} = 0$ and Eq. (2.12) that the area 2-form of a NEH is invariant as well by Lie transport along the null normal,

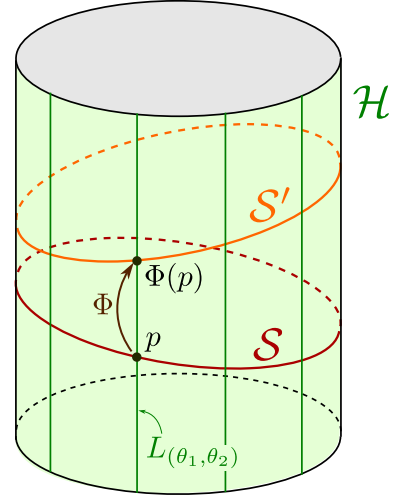


FIG. 3. Isometry Φ between two arbitrary cross sections $\mathcal{S}$ and $\mathcal{S}'$ of a NEH $\mathcal{H}$, each being endowed with the metric induced by the spacetime metric g_{ab} . The green vertical lines depict some null geodesic generators $L_{(\theta_1, \theta_2)}$ of $\mathcal{H}$.

$$\mathcal{L}_\ell \mathcal{H} \varepsilon_{ab} = 0. \quad (3.2)$$

A consequence of (3.1) is that all the cross sections $\mathcal{S}$ of $\mathcal{H}$, equipped with the Riemannian metric q_{ab} induced by h_{ab} (or equivalently by g_{ab}) (cf. Sec. II B), are isometric. The isometry between two arbitrary cross sections $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{H}$ is the diffeomorphism $\Phi: \mathcal{S} \rightarrow \mathcal{S}'$ defined by $\Phi(p)$ lying on the same null geodesic generator of $\mathcal{H}$ as p for every $p \in \mathcal{S}$ (cf. Fig. 3). Given that each generator of $\mathcal{H}$ intersects a given cross section exactly once, this clearly defines a bijective map. To show that Φ is an isometry, let us introduce a 1-parameter family $(\mathcal{S}_v)_{v \in [0,1]}$ of cross sections of $\mathcal{H}$ such that $\mathcal{S}_0 = \mathcal{S}$ and $\mathcal{S}_1 = \mathcal{S}'$. If $\mathcal{S} \cap \mathcal{S}' \neq \emptyset$ such a family does not constitute a slicing, as defined in Sec. II B, but this is not an issue here. Along the portion of each geodesic generator $L_{(\theta_1, \theta_2)}$ between $\mathcal{S}_0$ and $\mathcal{S}_1$, v can be considered as a parameter of $L_{(\theta_1, \theta_2)}$. Let ℓ^a be the tangent vector corresponding to this parametrization [Eq. (2.2)], with possibly $\ell^a = 0$ at points where $\mathcal{S}$ and $\mathcal{S}'$ intersect. Let us consider the family of flow maps $\Phi_v: \mathcal{S}_0 \rightarrow \mathcal{S}_v$, such that $\Phi_v(p)$ is the point of parameter v along the generator $L_{(\theta_1, \theta_2)}$ through p . By definition of a Lie derivative, we have $\mathcal{L}_\ell h_{ab} = \lim_{v \rightarrow 0} (\Phi_v^* h_{ab} - h_{ab})/v$, where $\Phi_v^* h_{ab}$ stands for the pullback of h_{ab} by Φ_v . The property (3.1) translates then to $\Phi_v^* h_{ab} = h_{ab}$. In particular, for $v = 1$, $\Phi_1 = \Phi$ and we get $\Phi^* h_{ab} = h_{ab}$. Given that the metric q_{ab} (respectively q'_{ab}) of $\mathcal{S}$ (respectively $\mathcal{S}'$) is that induced by h_{ab} , there comes $\Phi^* q'_{ab} = q_{ab}$, i.e., Φ is an isometry from $(\mathcal{S}, q_{ab})$ to $(\mathcal{S}', q'_{ab})$.

Since all the cross sections of a NEH are isometric, they share the same area,

$$A \equiv \oint_S \boldsymbol{\varepsilon} = \oint_S dS = \oint_S \sqrt{q} dx^1 dx^2, \quad (3.3)$$

where $\boldsymbol{\varepsilon}$ is the area 2-form of the metric q_{ab} on the cross section $\mathcal{S}$ (recall Sec. II B) and $dS = \sqrt{q} dx^1 dx^2$ is the area element of $\mathcal{S}$, expressed in terms of a coordinate system (x^1, x^2) covering $\mathcal{S}$ and the determinant q of q_{ab} with respect to (x^1, x^2) . Equivalently, $\boldsymbol{\varepsilon} = \sqrt{q} dx^1 \wedge dx^2$. Being independent of $\mathcal{S}$, the quantity A is called the *area of the NEH* $\mathcal{H}$. From A , one defines the *areal radius* of $\mathcal{H}$ by

$$R \equiv \sqrt{\frac{A}{4\pi}}. \quad (3.4)$$

Another important consequence of $\Theta_{ab} = 0$ is that the spacetime connection ∇_a induces a torsion-free affine connection $\mathcal{D}_a$ on $\mathcal{H}$, by setting $u^a \mathcal{D}_a v^b \equiv u^a \nabla_a v^b$ for any pair of vectors (u^a, v^a) tangent to $\mathcal{H}$. Indeed, since $\ell_b v^b = 0$ one has $\ell_b u^a \nabla_a v^b = -v_b u^a \nabla_a \ell^b = -v_b \chi^b_a u^a = -h_{ab} v^a \chi^b_c u^c = -\Theta_{ab} v^a u^b$, where use has been made of Eq. (2.8). Hence $\Theta_{ab} = 0$ implies $\ell_b u^a \nabla_a v^b = 0$, which shows that $u^a \nabla_a v^b$ is tangent to $\mathcal{H}$, i.e., that $\mathcal{D}_a$ is well defined. Moreover this connection is compatible with h_{ab} and $\mathcal{H}\varepsilon_{ab}$, as⁴

$$\mathcal{D}_c h_{ab} = 0 \quad \text{and} \quad \mathcal{D}_c \mathcal{H}\varepsilon_{ab} = 0. \quad (3.5)$$

For a NEH, the rotation 1-form ω_a introduced in Sec. II C depends only on the null normal ℓ^a and not on the cross section slicing $(\mathcal{S}_v)_{v \in \mathbb{R}}$ compatible with ℓ^a . Indeed, a change of slicing $(\mathcal{S}_v)_{v \in \mathbb{R}} \rightarrow (\mathcal{S}'_{v'})_{v' \in \mathbb{R}}$ of the type (2.17) leads to a change of the transverse null vector of the type $n^a \rightarrow n'^a = n^a + w^a$, with w^a tangent to $\mathcal{H}$ and the coefficient $+1$ in front of n^a ensuring $\ell_a n'^a = -1$. Then, according to the definition (2.20) of the rotation 1-form, for any vector $u^a \in T_p \mathcal{H}$, one has $\omega'_a u^a = -n'_b \chi^b_a u^a = \omega_a u^a - w_b \chi^b_a u^a$, with, thanks to Eqs. (2.6) and (2.9), $w_b \chi^b_a u^a = w_b u^a \nabla_a \ell^b = \nabla_a \ell_b u^a w^b = \Theta_{ab} u^a w^b = 0$ since $\Theta_{ab} = 0$. Hence ω_a depends only on the null normal ℓ^a , through the scaling law (2.22). The independence from the cross section slicing is also made clear by the following expression of the Weingarten map of $\mathcal{H}$ relative to ℓ^a ,

$$\chi^a_b = \mathcal{D}_b \ell^a = \omega_b \ell^a. \quad (3.6)$$

The first equality is a direct consequence of the definition (2.6) of χ^a_b , while the second one follows from setting $\Theta_{ab} = 0$ in Eq. (2.8): one gets $h_{ac} \mathcal{D}_b \ell^c = 0$, which means

that for any vector v^a tangent to $\mathcal{H}$, the vector $v^b \mathcal{D}_b \ell^a$ lies along the degenerate direction of h_{ab} , i.e., is collinear to ℓ^a . There exists then a 1-form α_a on $\mathcal{H}$ such that $v^b \mathcal{D}_b \ell^a = \alpha_b v^b \ell^a$, hence $\mathcal{D}_b \ell^a = \alpha_b \ell^a$. Equation (2.20) leads then to $\omega_a = (dv)_b \alpha_a \ell^b = \alpha_a$ since $\ell^b (dv)_b = 1$ [Eq. (2.3)], which completes the proof of (3.6).

By means of the Cartan identity and Eq. (2.21), we get $\mathcal{L}_\ell \omega_a = \ell^b (d\omega)_{ba} + (d\kappa(\ell))_a$. But thanks to (3.9b) below and the property $\mathcal{H}\varepsilon_{ab} \ell^b = 0$ [Eq. (2.5)], one readily obtains [46,47]

$$\mathcal{L}_\ell \omega_a = (d\kappa(\ell))_a. \quad (3.7)$$

Hence the rotation 1-form ω_a is Lie-transported along the null normal ℓ^a to a NEH $\mathcal{H}$ if, and only if, $\kappa(\ell)$ is constant. Applied to a Killing horizon, which is a special case of NEH, this property yields the zeroth law of black hole mechanics. Indeed, it suffices to choose ℓ^a to be the Killing vector ξ^a generating $\mathcal{H}$ and normalized by $\xi_a \xi^a = -1$ at the asymptotically flat end of $(\mathcal{M}, g_{ab})$; $\kappa(\ell)$ is then the so-called *surface gravity* and one has necessarily $\mathcal{L}_\ell \omega_a = 0$, so that Eq. (3.7) implies $\kappa(\ell) = \text{const}$ [42].

B. Weyl curvature scalars on a NEH

Let us consider a Newman-Penrose null tetrad $(\ell^a, n^a, m^a, \bar{m}^a)$ such that ℓ^a is normal to $\mathcal{H}$. By definition, n^a is a null vector fulfilling $\ell_a n^a = -1$ and m^a is a complex null vector orthogonal to ℓ^a and n^a fulfilling $m_a m^a = 0$ and $m_a \bar{m}^a = 1$, where $\bar{m}^a$ is the complex conjugate of m^a . Note that the two vectors ℓ^a and n^a discussed in Sec. II B can be considered as the first two legs of such a null tetrad; cf. Eq. (2.18) and Fig. 1. The ten independent components of the Weyl tensor C_{abcd} are neatly encoded into the five complex-valued curvature scalars

$$\Psi_0 \equiv C_{abcd} \ell^a m^b \ell^c m^d, \quad (3.8a)$$

$$\Psi_1 \equiv C_{abcd} \ell^a m^b \ell^c n^d, \quad (3.8b)$$

$$\Psi_2 \equiv C_{abcd} \ell^a m^b \bar{m}^c n^d, \quad (3.8c)$$

$$\Psi_3 \equiv C_{abcd} \ell^a n^b \bar{m}^c n^d, \quad (3.8d)$$

$$\Psi_4 \equiv C_{abcd} n^a \bar{m}^b n^c \bar{m}^d. \quad (3.8e)$$

Generally, Ψ_0 and Ψ_4 are associated with ingoing and outgoing transverse gravitational radiation, Ψ_1 and Ψ_3 with ingoing and outgoing longitudinal radiation, and Ψ_2 with a Coulomb field; see for instance Ref. [49].

It can be shown that the Weyl scalars Ψ_0 and Ψ_1 vanish identically on a NEH $\mathcal{H}$ [46]. It then follows from the transformation laws of the Weyl scalars under a change of null tetrad while keeping the direction of ℓ^a fixed (the

³For instance, one may choose (x^1, x^2) to coincide with the parameters (θ_1, θ_2) labeling the null generators $L_{(\theta_1, \theta_2)}$ of $\mathcal{H}$.

⁴Note, however, that $\mathcal{D}_a$ is not a Levi-Civita connection associated to the induced “metric” h_{ab} on $\mathcal{H}$, for the latter is degenerate. In particular, $\mathcal{D}_a$ cannot be reconstructed from the knowledge of h_{ab} alone.

so-called rotations of class I and III [50]) that Ψ_2 is *tetrad-invariant* on $\mathcal{H}$.

C. Shape and angular momentum structure

The frame-invariance on $\mathcal{H}$ of the Weyl scalar Ψ_2 implies that it “encodes” physically meaningful (Coulomb-type) curvature information about the NEH. More precisely, the geometry of $\mathcal{H}$ is encoded into the real and imaginary parts of Ψ_2 , according to [18,47,48,51]

$$\text{Re}\Psi_2 \stackrel{\mathcal{H}}{=} -\frac{1}{4}\mathcal{R}, \quad (3.9a)$$

$$(\text{Im}\Psi_2)^{\mathcal{H}}\varepsilon_{ab} \stackrel{\mathcal{H}}{=} \frac{1}{2}(\text{d}\omega)_{ab}, \quad (3.9b)$$

where $\mathcal{R}$ is the scalar curvature of an arbitrary 2-sphere cross section $\mathcal{S}$ of $\mathcal{H}$ (i.e., $\mathcal{R}$ is the Ricci scalar of the Riemannian metric q_{ab} induced by g_{ab} on $\mathcal{S}$) and ${}^{\mathcal{H}}\varepsilon_{ab}$ is the area 2-form of $\mathcal{H}$ [cf. Eq. (2.4)]. Since all cross sections are isometric (cf. Sec. III A), the value of $\mathcal{R}$ at a given point $p \in \mathcal{H}$ does not depend on the specific cross section $\mathcal{S}$ through p . Accordingly, $\mathcal{R}$ can be considered as a scalar field on $\mathcal{H}$, so that Eq. (3.9a) makes sense. As for the second equation, we note that the exterior derivative $(\text{d}\omega)_{ab}$ of the rotation 1-form ω_a is independent of the choice of the null normal ℓ^a , thanks to the behavior (2.22) under a generic rescaling $\ell^a \rightarrow f\ell^a$.⁵ This is in agreement with Ψ_2 and ${}^{\mathcal{H}}\varepsilon_{ab}$ in (3.9b) being independent of ℓ^a (recall that Ψ_2 on a NEH is tetrad independent). Moreover, for quasilocal horizons, among which NEHs, it is known that ω_a is involved in the definition of angular momentum (hence the name *rotation 1-form*); cf. Refs. [42,52–55] and Eq. (4.12) below. In view of Eqs. (3.9), we conclude that the real and imaginary parts of the Weyl curvature scalar Ψ_2 neatly encode the shape and angular momentum structure of $\mathcal{H}$, respectively.

Equation (3.9b) is an identity between two 2-forms on the three-dimensional manifold $\mathcal{H}$. We can recast it as an identity between two 2-forms on a two-dimensional manifold by pulling it back on a generic cross section $\mathcal{S}$ of $\mathcal{H}$ via the inclusion map $j: \mathcal{S} \hookrightarrow \mathcal{H}$. Thanks to Eq. (2.16), one has $j^*{}^{\mathcal{H}}\varepsilon_{ab} = \varepsilon_{ab}$. For evaluating $j^*(\text{d}\omega)_{ab}$, let us consider that $\mathcal{S}$ is the element $v = 0$ of a cross section slicing $(\mathcal{S}_v)_{v \in \mathbb{R}}$ of $\mathcal{H}$ and let Ω_a be the associated Hájíček 1-form (cf. Sec. II C). Thanks to relation (2.25) between ω_a and Ω_a , one has $\text{d}\omega = \text{d}\Omega + \text{d}\kappa_{(\ell)} \wedge \text{d}v$. Since $j^*\text{d}v = 0$ (for v is constant over $\mathcal{S}$), there comes $j^*\text{d}\omega = j^*\text{d}\Omega = \text{d}j^*\Omega$, where use has been made of the commuting property of pullbacks and exterior derivatives to write the second

equality. Accordingly, the pullback of Eq. (3.9b) onto $\mathcal{S}$ yields

$$(\text{Im}\Psi_2)\varepsilon_{ab} \stackrel{\mathcal{S}}{=} \frac{1}{2}(\text{d}\Omega)_{ab}, \quad (3.10)$$

where we use Ω_a to also denote the 1-form $j^*\Omega_a$ on $\mathcal{S}$. This slight abuse of notation is permissible since the Hájíček 1-form Ω_a “essentially lives” in the cross sections, as shown by Eq. (2.23).

Let ℓ^a be any null normal to $\mathcal{H}$. Because all cross sections of $\mathcal{H}$ are isometric by transport along the null generators of $\mathcal{H}$ (Sec. III A), we have $\mathcal{L}_\ell \mathcal{R} = 0$ and Eq. (3.9a) yields $\text{Re}\mathcal{L}_\ell \Psi_2 = 0$. On the other hand, (3.9b) and (3.2) lead to $2\mathcal{L}_\ell(\text{Im}\Psi_2)^{\mathcal{H}}\varepsilon = \mathcal{L}_\ell \text{d}\omega = \text{d}\mathcal{L}_\ell \omega = \text{d}\kappa_{(\ell)} = 0$, where we have used successively the commutation of Lie and exterior derivatives, Eq. (3.7) and the nilpotence of the exterior derivative. Hence, we get $\text{Im}\mathcal{L}_\ell \Psi_2 = 0$. Combining with $\text{Re}\mathcal{L}_\ell \Psi_2 = 0$, we conclude that

$$\mathcal{L}_\ell \Psi_2 \stackrel{\mathcal{H}}{=} 0. \quad (3.11)$$

Hence, on a NEH, not only the Weyl scalar Ψ_2 is frame-invariant, but it is also constant along the NEH’s generators.

IV. MULTIPOLE MOMENTS OF AN AXISYMMETRIC NEH

In Ref. [18], Ashtekar *et al.* have defined the geometrical multipole moments I_ℓ^{axi} and L_ℓ^{axi} of an axisymmetric isolated horizon. An *isolated horizon* is a pair $(\mathcal{H}, [\ell^a])$, where $\mathcal{H}$ is a NEH and $[\ell^a]$ is an equivalence class of null normals to $\mathcal{H}$, defined by $\ell'^a \sim \ell^a$ if, and only if, $\ell'^a = c\ell^a$ with c constant over $\mathcal{H}$, such that the affine connection $\mathcal{D}_a$ (cf. Sec. III A) is invariant by transport along the null normals belonging to $[\ell^a]$, i.e., $\mathcal{D}_a$ commutes with the Lie derivative along any representative ℓ^a of the equivalence class: $[\mathcal{L}_\ell, \mathcal{D}_a] = 0$ [46,47]. This condition supplements the NEH property $\mathcal{L}_\ell h_{ab} = 0$ [Eq. (3.1)] to make the full geometry $(h_{ab}, \mathcal{D}_a)$ of $\mathcal{H}$ be “time independent” (see Ref. [35] for a review). However it plays no role in the definition of the multipole moments I_ℓ^{axi} and L_ℓ^{axi} by formula (1.2), since the latter relies only on the frame-independent quantity Ψ_2 , the metric q_{ab} of an arbitrary cross section of $\mathcal{H}$ and the axisymmetric character of q_{ab} . The isolated horizon property is required only to reconstruct the whole horizon geometry from the knowledge of I_ℓ^{axi} and L_ℓ^{axi} [18], as well as to define the *source* multipole moments M_ℓ and S_ℓ to be discussed in Sec. IV C. We shall therefore provide the definition of the geometric multipole moments following the prescription of Ref. [18], but without requiring the isolated horizon structure atop of the NEH one. This has the advantage to put the definition on the same footing as that of multipole moments for a generic (nonaxisymmetric) NEH to be presented in Sec. V. We start by constructing a privileged

⁵This is analogous to the invariance of the Maxwell tensor $F_{ab} = (\text{d}A)_{ab}$ under a gauge transformation $A_a \rightarrow A_a + (\text{d}\Phi)_a$ of the electromagnetic potential A_a generated by a scalar potential Φ .

unit round metric on cross sections of an axisymmetric NEH (Sec. IV A); then we provide the definition of horizon multipole moments and discuss some of their properties (Sec. IV B), and the related notions of mass and current source multipoles (Sec. IV C).

A. Axisymmetric NEHs and their cross section geometry

Let us define a NEH $\mathcal{H}$ to be *axisymmetric* if, and only if, there exists a $\text{SO}(2)$ group action on $\mathcal{H}$, of generating vector field η^a , such that (i) η^a vanishes on exactly two null geodesic generators of $\mathcal{H}$, named the *North* and *South pole generators*, (ii) apart from these two generators, the orbits of η^a are closed spacelike curves and their parameter length with respect to η^a is 2π , (iii) there exists a null normal ℓ^a to $\mathcal{H}$ that commutes with η^a : $[\ell, \eta]^a = 0$, and (iv) $\mathcal{H}$'s intrinsic “metric” h_{ab} and extrinsic curvature, represented by the Weingarten map χ^a_b with respect to ℓ^a , are preserved by the group action: $\mathcal{L}_\eta h_{ab} = 0$ and $\mathcal{L}_\eta \chi^a_b = 0$.

The null normal ℓ^a obeying (iii) and (iv) (recall that χ^a_b depends on ℓ^a ; cf. Sec. II A) is far from unique: (iii) and (iv) are fulfilled by any other normal $\ell'^a = f\ell^a$, with f a positive scalar field on $\mathcal{H}$ respecting the axisymmetry, i.e., obeying $\eta^a(df)_a = 0$. Indeed, one has $[\ell', \eta]^a = f[\ell, \eta]^a - \eta^b(df)_b \ell'^a = 0 - 0 = 0$ and the Weingarten map χ'^a_b associated to ℓ'^a obeys $\mathcal{L}_\eta \chi'^a_b = 0$ thanks to the transformation law (2.15b) and $\eta^a(df)_a = 0$. We shall denote by $\{\ell^a\}_{\text{axi}}$ the set of all such normals. Note that the second requirement in (iv) is equivalent to $\mathcal{L}_\eta \omega_a = 0$, where ω_a is the rotation 1-form associated to ℓ^a . Indeed, from the NEH identity $\chi^a_b = \ell^a \omega_b$ [Eq. (3.6)], we have $\mathcal{L}_\eta \chi^a_b = \mathcal{L}_\eta \ell^a \omega_b + \ell^a \mathcal{L}_\eta \omega_b = \ell^a \mathcal{L}_\eta \omega_b$ since $\mathcal{L}_\eta \ell^a = [\eta, \ell]^a = 0$ by (iii). Accordingly, (iv) can be restated as

$$\mathcal{L}_\eta h_{ab} = 0 \quad \text{and} \quad \mathcal{L}_\eta \omega_a = 0. \quad (4.1)$$

Let $\mathcal{S}$ be a cross section of an axisymmetric NEH $\mathcal{H}$, such that η^a is tangent to $\mathcal{S}$. Since $\mathcal{L}_\eta h_{ab} = 0$, the induced metric $q_{ab} = j^* h_{ab}$ on $\mathcal{S}$ obeys $\mathcal{L}_\eta q_{ab} = 0$, i.e., η^a is a Killing vector of q_{ab} , or, in other words, $(\mathcal{S}, q_{ab})$ is an axisymmetric Riemannian manifold.⁶ Based on the axisymmetric structure, one may construct a fiducial unit round metric $\overset{\circ}{q}_{ab}^{\text{axi}}$ on $\mathcal{S}$ as follows [18]. First, let us consider

⁶If $\mathcal{S}'$ is a cross section of $\mathcal{H}$ to which η^a is not tangent everywhere, it is not globally invariant by the $\text{SO}(2)$ action on $\mathcal{H}$. It is nevertheless axisymmetric, given that all cross sections of a NEH are isometric (cf. Sec. III A). To show it explicitly, let us consider the $\text{SO}(2)$ group action on $\mathcal{S}'$ defined by $\Psi'_\phi \equiv \Phi \circ \Psi_\phi \circ \Phi^{-1}$, where Ψ_ϕ is the rotation of angle ϕ acting on the axisymmetric cross section $\mathcal{S}$ considered above and Φ is the isometry $\mathcal{S} \rightarrow \mathcal{S}'$ along the null generators of $\mathcal{H}$ considered in Sec. III A. Since Φ and Ψ_ϕ are isometries, Ψ'_ϕ is clearly an isometry of $(\mathcal{S}', q'_{ab})$. The corresponding Killing vector η'_a is then nothing but the pushforward of η^a (considered as a vector field tangent to $\mathcal{S}$) by Φ .

the Hodge dual $\star \eta_a$ of the 1-form $\eta_a \equiv q_{ab} \eta^b$ on $\mathcal{S}$, i.e., the 1-form defined by $\star \eta_a \equiv \eta^b \varepsilon_{ba}$, where ε_{ab} is the area 2-form of q_{ab} (cf. Sec. II B). One has necessarily $\mathcal{L}_\eta \varepsilon_{ab} = 0$, so that the Cartan identity $\mathcal{L}_\eta \varepsilon = \eta \cdot d\varepsilon + d\star \eta$ and $d\varepsilon = 0$ (vanishing of any 3-form on a 2-manifold) imply $d\star \eta = 0$, i.e., $\star \eta$ is closed. Now, on a simply connected manifold, such as $\mathcal{S} \sim \mathbb{S}^2$, any closed 1-form is exact: $\star \eta = d\mu$ for some scalar field μ on $\mathcal{S}$. *A priori* μ is determined up to a constant; one can uniquely fix the latter by demanding that the integral of μ over $\mathcal{S}$ vanishes. Rescaling μ by $\mathcal{H}$'s areal radius R [Eq. (3.4)], which is constant, we conclude that there exists a unique scalar field ζ on $\mathcal{S}$ such that

$$(d\zeta)_a = \frac{1}{R^2} \eta^b \varepsilon_{ba} \quad \text{and} \quad \oint_{\mathcal{S}} \zeta \varepsilon = 0. \quad (4.2)$$

Moreover, $\eta^a (d\zeta)_a = R^{-2} \varepsilon_{ba} \eta^b \eta^a = 0$, which implies that ζ is constant along the orbits of the Killing vector η^a . One can then introduce a parameter ϕ along these orbits such that $x^{A'} = (\zeta, \phi)$ is a coordinate system on $\mathcal{S}$ with $\zeta \in [-1, 1]$, $\zeta = -1$ (respectively $+1$) at the South (respectively North) pole, $\phi \in [0, 2\pi)$, $\eta^a = \partial_\phi^a$ and the metric q_{ab} takes the form (see Sec. 2.2 of Ref. [18] for details)

$$q_{A'B'} dx^{A'} dx^{B'} = R^2 (f^{-1} d\zeta^2 + f d\phi^2), \quad (4.3)$$

where $f = f(\zeta) \equiv q_{ab} \eta^a \eta^b / R^2$.

The coordinates $x^{A'} = (\zeta, \phi)$ defined above are unique, up to a constant shift in ϕ . Let us introduce the coordinates $x^A = (\theta, \phi)$ with $\theta \in [0, \pi]$ defined via $\cos \theta \equiv \zeta$ and define from them the metric $\overset{\circ}{q}_{ab}^{\text{axi}}$ on $\mathcal{S}$ by

$$\overset{\circ}{q}_{AB}^{\text{axi}} dx^A dx^B \equiv d\theta^2 + \sin^2 \theta d\phi^2. \quad (4.4)$$

Clearly, $\overset{\circ}{q}_{ab}^{\text{axi}}$ is a Riemannian metric of constant scalar curvature equal to 2, i.e., a unit round metric. Moreover, by construction, this metric is unique [since the coordinates (θ, ϕ) are uniquely defined—up to a shift in ϕ] and follows from the axisymmetry of $\mathcal{S}$, via Eq. (4.2), which defines ζ in terms of the rotational Killing vector η^a . The metric $\overset{\circ}{q}_{ab}^{\text{axi}}$ admits the same Killing vector $\eta^a = \partial_\phi^a$ as the physical metric q_{ab} . Expressed in terms of the coordinates (ζ, ϕ) instead of (θ, ϕ) , it reads $\overset{\circ}{q}_{A'B'}^{\text{axi}} dx^{A'} dx^{B'} = (1 - \zeta^2)^{-1} d\zeta^2 + (1 - \zeta^2) d\phi^2$, which takes the same form as in Eq. (4.3) with R replaced by 1 and $f(\zeta)$ replaced by $1 - \zeta^2$. Moreover, we note from Eq. (4.3) that $\det(q_{A'B'}) = R^4$, so that $\overset{\circ}{q}_{ab}^{\text{axi}}$ shares the same area 2-form as the physical metric q_{ab} , up to a constant factor R^2 ,

$$\varepsilon_{ab} = R^2 \overset{\circ}{\varepsilon}_{ab}^{\text{axi}}. \quad (4.5)$$

B. Horizon multipole moments

Given the unit round metric $\hat{q}_{ab}^{\text{axi}}$ introduced above on the cross section $\mathcal{S}$, we follow Ref. [18] to define the *shape* and *current* multipole moments I_ℓ^{axi} and L_ℓ^{axi} of the axisymmetric NEH $\mathcal{H}$ from the Weyl scalar Ψ_2 (cf. Sec. III C) by

$$I_\ell^{\text{axi}} + iL_\ell^{\text{axi}} \equiv - \oint_{\mathcal{S}} \Psi_2 \hat{Y}_{\ell,0}^{\text{axi}} dS = -R^2 \oint_{\mathcal{S}} \Psi_2 \hat{Y}_{\ell,0}^{\text{axi}} d\hat{S}^{\text{axi}}. \quad (4.6)$$

Here, dS (respectively $d\hat{S}^{\text{axi}}$) is the area element of $\mathcal{S}$ associated to the physical metric q_{ab} (respectively the unit round metric $\hat{q}_{ab}^{\text{axi}}$) [recall Eq. (3.3)], while $\hat{Y}_{\ell,0}^{\text{axi}}$ stands for the $m=0$ spherical harmonic of degree ℓ with respect to the unit round metric $\hat{q}_{ab}^{\text{axi}}$; hence $\hat{Y}_{\ell,0}^{\text{axi}}$ is the $m=0$ eigenfunction of the Laplace operator of $\hat{q}_{ab}^{\text{axi}}$ corresponding to the eigenvalue $-\ell(\ell+1)$. In terms of the coordinates (θ, ϕ) in which $\hat{q}_{ab}^{\text{axi}}$ takes the canonical form (4.4), we have

$$\hat{Y}_{\ell,0}^{\text{axi}} = Y_{\ell,0}(\theta, \phi), \quad (4.7)$$

where $Y_{\ell,0}$ are the ordinary spherical harmonic functions. The second equality in Eq. (4.6) holds because $dS = R^2 d\hat{S}^{\text{axi}}$, as a consequence of (4.5). Notice that the multipole moments I_ℓ^{axi} and L_ℓ^{axi} are dimensionless, given that Ψ_2 has the dimension of an inverse squared length and dS that of a squared length.

Since all cross sections of $\mathcal{H}$ are isometric (Sec. III A) and Ψ_2 is invariant by the isometry map along the null generators, thanks to Eq. (3.11), the definition (4.6) of the multipole moments is clearly independent of the choice of the cross section $\mathcal{S}$ of $\mathcal{H}$.

Thanks to expressions (3.9a) and (3.10) for, respectively, $\text{Re}\Psi_2$ and $\text{Im}\Psi_2$, the definition (4.6) is equivalent to

$$I_\ell^{\text{axi}} \equiv \frac{1}{4} \oint_{\mathcal{S}} \mathcal{R} \hat{Y}_{\ell,0}^{\text{axi}} dS \quad \text{and} \quad L_\ell^{\text{axi}} \equiv -\frac{1}{2} \oint_{\mathcal{S}} \hat{Y}_{\ell,0}^{\text{axi}} d\Omega. \quad (4.8)$$

For $\ell=0$, $\hat{Y}_{\ell,0}^{\text{axi}} = Y_{0,0}(\theta, \phi) = 1/(2\sqrt{\pi})$ and Eq. (4.8) yields

$$I_0^{\text{axi}} = \frac{1}{8\sqrt{\pi}} \underbrace{\oint_{\mathcal{S}} \mathcal{R} dS}_{8\pi} = \sqrt{\pi} \quad \text{and} \quad L_0^{\text{axi}} = -\frac{1}{4\sqrt{\pi}} \underbrace{\oint_{\mathcal{S}} d\Omega}_0 = 0, \quad (4.9)$$

where use has been made of the Gauss-Bonnet theorem for the first integral and of Stokes' theorem on a manifold without boundary for the second one. Hence we conclude that any axisymmetric NEH has a shape monopole I_0^{axi} equal to $\sqrt{\pi}$ and a vanishing current monopole L_0^{axi} . Moreover, thanks to the identity $\mathcal{R} = -f''(\zeta)/R^2$, one can show that the shape dipole I_1^{axi} always vanishes on an axisymmetric

NEH [18]. We thus have the identities

$$I_0^{\text{axi}} = \sqrt{\pi}, \quad L_0^{\text{axi}} = 0 \quad \text{and} \quad I_1^{\text{axi}} = 0. \quad (4.10)$$

C. Source multipole moments

The physical interpretation of the dimensionless, *geometrical* multipole moments (4.6) of a given axisymmetric NEH $\mathcal{H}$ is not straightforward. However, by introducing appropriate powers of the NEH mass $M_{\mathcal{H}}$ and areal radius R , one can define *physical source* multipole moments M_ℓ and S_ℓ of mass-type and current-type that have the appropriate physical dimensions [18]. The radius R is defined unambiguously from the NEH geometry by Eq. (3.4). On the other hand, defining the mass $M_{\mathcal{H}}$ of an axisymmetric NEH requires some Hamiltonian framework. The latter has been developed in Ref. [52] by adding some extra structure on $\mathcal{H}$, namely a privileged constant-rescaling equivalence class $[\ell^a]$ such that $[\mathcal{L}_\ell, \mathcal{D}_a] \ell^b = 0$ for any $\ell^a \in [\ell^a]$. The pair $(\mathcal{H}, [\ell^a])$ is called a *weakly isolated horizon (WIH)* [35,52]. Indeed the defining requirement is weaker than that of isolated horizons, which is $[\mathcal{L}_\ell, \mathcal{D}_a] v^b = 0$ for any v^b tangent to $\mathcal{H}$ (cf. the beginning of Sec. IV). Thanks to the NEH identity $\mathcal{D}_a \ell^b = \omega_a \ell^b$ [Eq. (3.6)], the WIH condition is equivalent to

$$\mathcal{L}_\ell \omega_a = 0, \quad (4.11)$$

for any $\ell^a \in [\ell^a]$. Note that the rotation 1-form ω_a is unique for a given WIH structure: the scaling law (2.22) with $f = c = \text{const}$ shows that ω_a does not depend on the representative ℓ^a in $[\ell^a]$. In view of Eq. (3.7), property (4.11) implies that $\kappa_{(\ell)}$ is uniform over $\mathcal{H}$, so that WIHs can be seen as objects for which a kind of “zeroth law” holds, keeping in mind that the value of $\kappa_{(\ell)}$ depends on the choice of the representative ℓ^a in the class $[\ell^a]$, since $\ell'^a = c\ell^a$ implies $\kappa_{(\ell')} = c\kappa_{(\ell)}$ [scaling law (2.15a) with $f = c = \text{const}$]. For any NEH $\mathcal{H}$, there exists infinitely many constant-rescaling classes $[\ell^a]$ such that $(\mathcal{H}, [\ell^a])$ is a WIH [47].⁷

Given an axisymmetric NEH $\mathcal{H}$, let us select a WIH structure $(\mathcal{H}, [\ell^a])$ such that $[\ell^a] \subset \{\ell^a\}_{\text{axi}}$, i.e., any $\ell^a \in [\ell^a]$ commutes with the axisymmetry generator η^a (cf. Sec. IV A). One may then build a Hamiltonian framework such that η^a is the value on $\mathcal{H}$ of a vector field φ^a on $\mathcal{M}$ that vanishes outside a compact neighborhood of $\mathcal{H}$ and that generates a Hamiltonian vector field on the phase space [52]. The value of the corresponding Hamiltonian reduces to a surface integral on $\mathcal{H}$, which one defines as the *angular momentum* of $\mathcal{H}$ [35,52],

⁷On the contrary, it is not always possible to endow a NEH with an isolated horizon structure [35,47].

$$J_{\mathcal{H}} \equiv -\frac{1}{8\pi} \oint_{\mathcal{S}} \omega_a \eta^a dS = -\frac{1}{8\pi} \oint_{\mathcal{S}} \Omega_a \eta^a dS, \quad (4.12)$$

where $\mathcal{S}$ is any cross section of $\mathcal{H}$ with η^a as a tangent vector. The second integral involves the Hájíček 1-form Ω_a of ℓ^a with respect to $\mathcal{S}$ and is equivalent to the first one thanks to Eq. (2.23) and η^a being tangent to $\mathcal{S}$. The value of $J_{\mathcal{H}}$ does not depend on the choice of the cross section $\mathcal{S}$. This follows from the Hamiltonian framework, but it is easy to show it directly by considering another cross section $\mathcal{S}'$ to which η^a is tangent. $\mathcal{S}'$ is necessarily connected to $\mathcal{S}$ by an isometry flow along the null generators of $\mathcal{H}$ (cf. Sec. III A) generated by a vector field $\ell'^a \in \{\ell^a\}_{\text{axi}}$. Writing $\ell'^a = f \ell^a$ with $\ell^a \in [\ell^a]$, we have then $\mathcal{L}_{\ell'}(\omega_a \eta^a) = f \mathcal{L}_{\ell}(\omega_a \eta^a) = f(\eta^a \mathcal{L}_{\ell} \omega_a + \omega_a \mathcal{L}_{\ell} \eta^a) = 0$ thanks to Eq. (4.11) and $[\ell, \eta]^a = 0$. Since the area element dS is preserved as well by the isometry flow, it follows that the integral (4.12) taken on $\mathcal{S}'$ equals that taken on $\mathcal{S}$. Furthermore, it can be shown [52] that if the whole spacetime $(\mathcal{M}, g_{ab})$ is axisymmetric, with the corresponding Killing vector coinciding with η^a on $\mathcal{H}$, then $J_{\mathcal{H}}$ is nothing but the standard Komar angular momentum.

The integral in the expression (4.12) of $J_{\mathcal{H}}$ can be written as the integral of the 2-form $(\Omega \cdot \eta) \epsilon$ on the 2-surface $\mathcal{S}$. It is easy to check that $(\Omega \cdot \eta) \epsilon = \Omega \wedge \star \eta$, where $\star \eta_a \equiv \eta^b \epsilon_{ba}$ (cf. Sec. IV A); in particular, as a 2-form, $\Omega \wedge \star \eta$ has to be proportional to ϵ . Now, Eq. (4.2) yields $\star \eta = R^2 d\zeta$. Hence, we may write

$$(\Omega \cdot \eta) \epsilon = -R^2 d\zeta \wedge \Omega = -R^2 [d(\zeta \Omega) - \zeta d\Omega]. \quad (4.13)$$

The integral over $\mathcal{S}$ of $d(\zeta \Omega)$ being zero by Stokes' theorem, we are left with

$$J_{\mathcal{H}} = -\frac{R^2}{8\pi} \oint_{\mathcal{S}} \zeta d\Omega. \quad (4.14)$$

By comparing with Eq. (4.8) for $\ell = 1$, taking into account that $Y_{1,0}(\theta, \phi) = \sqrt{3/(4\pi)} \cos \theta = \sqrt{3/(4\pi)} \zeta$, we see that the angular momentum is proportional to the current dipole L_1^{axi} ,

$$J_{\mathcal{H}} = \frac{R^2}{2\sqrt{3}\pi} L_1^{\text{axi}}. \quad (4.15)$$

The definition of the horizon mass $M_{\mathcal{H}}$ is more tricky than that of the horizon angular momentum $J_{\mathcal{H}}$ because the WIH structure does not allow one to single out a unique vector field ξ^a playing the role of a time translation generating a Hamiltonian on the phase space, which would define the energy, and hence the mass. On the contrary, the rotational vector η^a leading to $J_{\mathcal{H}}$ could be defined uniquely from the axisymmetry hypothesis. To be a symmetry generator of the WIH $(\mathcal{H}, [\ell^a])$, any candidate ξ^a must be such that its restriction to $\mathcal{H}$ takes the form

$\xi^a|_{\mathcal{H}} = c \ell^a - \Omega \eta^a$, where $\ell^a \in [\ell^a]$ and c and Ω are two constants: thanks to Eqs. (3.1), (4.1), and (4.11), this guarantees $\mathcal{L}_{\xi} h_{ab} = 0$ and $\mathcal{L}_{\xi} \omega_a = 0$. In Ref. [52] (see also the reviews [35,56]), it is shown that ξ^a generates a Hamiltonian on the phase space if, and only if, $\kappa \equiv c \kappa(\ell)$ and Ω are functions of $J_{\mathcal{H}}$ and the horizon area A only, and moreover obey $\partial \kappa / \partial J_{\mathcal{H}} = 8\pi \partial \Omega / \partial A$. The (on shell) value of the Hamiltonian is then the difference $H_{(\xi)} = E_{\infty}^{(\xi)} - E_{\mathcal{H}}^{(\xi)}$ between two surface integrals, one at spatial infinity, defining the ADM energy $E_{\infty}^{(\xi)}$, and the other one on $\mathcal{H}$, defining the horizon energy $E_{\mathcal{H}}^{(\xi)}$. The variations of $E_{\mathcal{H}}^{(\xi)}$, A and $J_{\mathcal{H}}$ in the phase space then obey $\delta E_{\mathcal{H}}^{(\xi)} = \kappa / (8\pi) \delta A + \Omega \delta J_{\mathcal{H}}$, which is known as the *first law of WIH mechanics*. The *horizon mass* $M_{\mathcal{H}}$ is defined as the value of $E_{\mathcal{H}}^{(\xi)}$ for a configuration where $\mathcal{H}$ is “at rest” with respect to ξ^a , i.e., when ξ^a is a global Killing vector of spacetime. It is argued in Ref. [52] that, in order to recover the Kerr solution, with ξ^a being the standard stationary Killing vector, this fixes the functions $\kappa(A, J_{\mathcal{H}})$ and $\Omega(A, J_{\mathcal{H}})$ to their Kerr values, thereby leading to the following expression of the horizon mass:

$$M_{\mathcal{H}} = \frac{R}{2} \sqrt{1 + 4J_{\mathcal{H}}^2/R^4}. \quad (4.16)$$

In view of Eq. (4.8) and by analogy with the electrostatics and magnetostatics of a charged, conducting sphere, one can assign to $\mathcal{H}$ a “surface mass density” $M_{\mathcal{H}} R / 8\pi$, a “surface momentum density” $\Omega_a / 8\pi$, as well as the associated mass and current multipole moments [18,35]

$$M_{\ell} \equiv \frac{M_{\mathcal{H}} R^{\ell}}{\sqrt{(2\ell+1)\pi}} I_{\ell}^{\text{axi}} \quad \text{and} \quad S_{\ell} \equiv \frac{R^{\ell+1}}{2\sqrt{(2\ell+1)\pi}} L_{\ell}^{\text{axi}}. \quad (4.17)$$

Incidentally, we note that interpreting $\Omega_a / 8\pi$ as a momentum density was first put forward in Refs. [41,42], while developing a “fluid bubble” analogy for the horizon dynamics. The scaling (4.17) ensures that $M_0 = M_{\mathcal{H}}$ [since $I_0^{\text{axi}} = \sqrt{\pi}$, cf. Eq. (4.10)] and $S_1 = J_{\mathcal{H}}$ [thanks to Eq. (4.15)]. Moreover, we shall see explicitly in Sec. VI B that for a Kerr black hole, M_0 and S_1 reproduce the Komar mass and angular momentum, respectively.

A natural question is how the *source* multipole moments (4.17) of a given axisymmetric NEH compare to the *field* multipole moments associated with that same NEH, defined for instance according to Hansen [5] [cf. Eq. (1.1) for a Kerr black hole] or Thorne [6], which are equivalent for stationary spacetimes [9]. In Newtonian gravity or electrodynamics, the linearity of the Poisson equation or the Maxwell equations imply that source and field multipoles necessarily coincide. In general relativity, however, the nonlinear nature of the Einstein equation means that

source and field multipoles generically differ [18,36]. As we shall see in Sec. VI B, in the case of Kerr black holes, the source multipoles (4.17) share several properties with the field multipoles, including their behavior for small spin values, but they do differ in their magnitude, except for $\ell = 0$ and $\ell = 1$.

V. MULTIPOLE MOMENTS OF A GENERIC NEH

In this section we review the definition of multipole moments of a generic NEH recently proposed by Ashtekar *et al.* [22]. Contrary to that discussed in Sec. IV, this definition does not require the NEH to be axisymmetric. Rather, it is based on a conformal decomposition of the cross section metric, devised by Korzyński [57] to define the horizon angular momentum (the $\ell = 1$ source multipole S_ℓ in the axisymmetric case). We successively discuss the conformal round metrics of horizon cross sections (Secs. V A and V B), the definition of the NEH multipole moments (Sec. V C), the uniqueness of this definition by imposing the condition of a vanishing area dipole moment (Sec. V D), and the expressions of the horizon multipoles in terms of an electric-type scalar potential and a magnetic-type pseudoscalar potential (Sec. V E).

A. Conformal unit round metrics on a cross section

Let us consider a cross section $\mathcal{S}$ of a NEH $\mathcal{H}$. As discussed in Sec. II B, $\mathcal{S}$ is endowed with the Riemannian metric q_{ab} induced by the spacetime metric g_{ab} . Let us denote by D_a the associated Levi-Civita connection and by $\mathcal{R}$ the associated scalar curvature, as in (3.9). Since $\mathcal{S}$ is topologically a 2-sphere, the uniformization theorem implies that there exists some unit round metrics $\hat{q}_{ab}$ on $\mathcal{S}$ (with Levi-Civita connection $\hat{D}_a$ and constant scalar curvature $\hat{\mathcal{R}} = 2$) that are conformally related to q_{ab} (see, e.g., Refs. [57–59]),

$$\hat{q}_{ab} = \psi^2 q_{ab}, \quad (5.1)$$

where the conformal factor ψ is a scalar field on $\mathcal{S}$ with the dimension of an inverse length, since $\hat{q}_{ab}$ is a round metric of a sphere of radius 1.⁸ Being a unit round metric means that there exists some coordinates $x^A = (\vartheta, \varphi)$ on $\mathcal{S}$ such that $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi]$ and

$$\hat{q}_{AB} dx^A dx^B = d\vartheta^2 + \sin^2 \vartheta d\varphi^2. \quad (5.2)$$

We will refer to (ϑ, φ) as *polar coordinates adapted to $\hat{q}_{ab}$* . Note that such coordinates are by no means unique: because of the SO(3) symmetry of $\hat{q}_{ab}$, there exists a 3-parameter family of coordinates (ϑ, φ) such that (5.2)

⁸One could make ψ dimensionless by introducing $\tilde{\psi} \equiv R\psi$, where R is the areal radius (3.4).

holds, two parameters defining the axis $\vartheta \in \{0, \pi\}$ and one parameter setting the origin of φ .

According to the analysis performed in Appendix D of Ref. [38],⁹ the conformal factor ψ is a solution of the nonlinear, second-order, elliptic partial differential equation

$$D^2 \ln \psi + \psi^2 = \frac{1}{2} \mathcal{R}, \quad (5.3)$$

where $D^2 \equiv q^{ab} D_a D_b$ is the Laplace operator associated with q_{ab} . Alternatively, using the conformal transformation law $D^2 \Phi = \psi^2 \hat{D}^2 \Phi$ valid for any scalar field Φ in dimension 2 [38], we may write the partial differential equation (5.3) as

$$\hat{D}^2 \ln \psi + 1 = \frac{1}{2} \psi^{-2} \mathcal{R}, \quad (5.4)$$

where $\hat{D}^2 \equiv \hat{q}^{ab} \hat{D}_a \hat{D}_b$ is the Laplace operator associated with $\hat{q}_{ab}$.

For a given q_{ab} , the round metric $\hat{q}_{ab}$ is by no means unique: any other unit round metric $\hat{q}'_{ab}$ on $\mathcal{S}$ that obeys $\hat{q}'_{ab} = \psi'^2 q_{ab}$ for some conformal factor ψ' is necessarily conformally related to $\hat{q}_{ab}$ via $\alpha \equiv \psi'/\psi$,

$$\hat{q}'_{ab} = \alpha^2 \hat{q}_{ab}, \quad \text{where } \alpha \text{ satisfies } \hat{D}^2 \ln \alpha + \alpha^2 = 1. \quad (5.5)$$

The above partial differential equation¹⁰ for α follows directly from (5.3) with q_{ab} (within D^2) substituted by $\hat{q}_{ab}$ and ψ substituted by α . The most general solution to (5.5) is given via a (suitably normalized) linear combination of the first four spherical harmonics of $\hat{q}_{ab}$ [51], namely

$$\alpha(\vartheta, \varphi)^{-1} = \alpha_0 + \alpha_1 \sin \vartheta \cos \varphi + \alpha_2 \sin \vartheta \sin \varphi + \alpha_3 \cos \vartheta, \quad (5.6)$$

where α_0 and $\vec{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ are real constants satisfying the constraint $\alpha_0^2 - |\vec{\alpha}|^2 = 1$. We conclude that the set of unit round metrics $\hat{q}_{ab}$ conformal to the physical metric q_{ab} on $\mathcal{S}$ forms a 3-parameter family. The set of conformal factors ψ does the same, since Eq. (5.5) is equivalent to $\psi' = \alpha\psi$.

B. Extension of the conformal decomposition

A priori, the conformal factor ψ introduced in Eq. (5.1) is defined only on the considered cross section $\mathcal{S}$. We now show that, on a NEH, where all cross sections are isometric, one can actually extend ψ to a scalar field defined on the whole of $\mathcal{H}$. To do so, let us consider another cross section

⁹See in particular Eq. (D9) in Ref. [38], with $n = 2$, $\Omega = \psi$, $\tilde{\mathcal{R}} = \mathcal{R}$ and $R = \mathcal{R}$.

¹⁰Note that there is a typo in the corresponding equation in Ref. [22], i.e., Eq. (2.6) there, which should be written as Eq. (5.5) above.

S' of $\mathcal{H}$, with metric q'_{ab} induced by g_{ab} , and the isometry $\Phi: S \rightarrow S'$ introduced in Sec. III A. By pulling back (5.1) to S' via Φ^{-1} , we get $(\Phi^{-1})^* \dot{q}_{ab} = (\psi \circ \Phi^{-1})^2 (\Phi^{-1})^* q_{ab}$. But $(\Phi^{-1})^* q_{ab} = q'_{ab}$ since Φ is an isometry for the induced metrics. Hence there comes

$$\dot{q}'_{ab} = \psi'^2 q'_{ab}, \quad (5.7)$$

with

$$\dot{q}'_{ab} \equiv (\Phi^{-1})^* \dot{q}_{ab} \quad \text{and} \quad \psi' \equiv \psi \circ \Phi^{-1}. \quad (5.8)$$

As it is defined, the metric $\dot{q}'_{ab}$ is isometric to $\dot{q}_{ab}$ —the isometry being Φ . This implies that $\dot{q}'_{ab}$ is a unit round metric on S' , so that (5.7) is exactly similar to (5.1). Let us then define ψ on $\mathcal{H}$ by $\psi(p) = \psi(p_0)$, where p_0 is the unique point on S lying on the same null geodesic generator $L_{(\theta_1, \theta_2)}$ as p . On S' , the function ψ hence defined coincides with ψ' . Accordingly, (5.1) can be extended to all cross sections of $\mathcal{H}$, thereby defining a unit round metric $\dot{q}_{ab}$ on each of them. By construction, the scalar field ψ is invariant along the generators of $\mathcal{H}$. It follows that, for any null normal ℓ^a to $\mathcal{H}$,

$$\mathcal{L}_\ell \psi = 0. \quad (5.9)$$

C. Horizon multipole moments

We are ready to define the multipole moments of a generic NEH $\mathcal{H}$, as introduced in [22]. For a given cross section S of $\mathcal{H}$ and a choice of the conformal unit round metric $\dot{q}_{ab}$ fulfilling Eq. (5.1), the *shape* and *current* multipole moments $I_{\ell, m}$ and $L_{\ell, m}$ are defined according to

$$\begin{aligned} I_{\ell, m} &\equiv - \oint_S (\text{Re} \Psi_2) \dot{Y}_{\ell, m} dS \quad \text{and} \\ L_{\ell, m} &\equiv - \oint_S (\text{Im} \Psi_2) \dot{Y}_{\ell, m} dS. \end{aligned} \quad (5.10)$$

Here, dS is the area element of S corresponding to the physical metric q_{ab} [Eq. (3.3)], while $\dot{Y}_{\ell, m}$ stands for the spherical harmonic with respect to the unit round metric $\dot{q}_{ab}$, i.e., the eigenfunction of order m corresponding to the eigenvalue $-\ell(\ell+1)$ of the Laplace operator $\dot{D}_a \dot{D}^a$ of $\dot{q}_{ab}$. In terms of polar coordinates (ϑ, φ) adapted to $\dot{q}_{ab}$ [cf. Eq. (5.2)], we have

$$\dot{Y}_{\ell, m} = Y_{\ell, m}(\vartheta, \varphi), \quad (5.11)$$

where $Y_{\ell, m}$ are the ordinary spherical harmonic functions. For $m = 0$, the above relationship is similar to Eq. (4.7), but one should keep in mind that, on an axisymmetric NEH, (ϑ, φ) and (θ, ϕ) are *a priori* distinct coordinates on S , as we shall see explicitly in the Kerr case in Sec. VI.

The multipole moments $I_{\ell, m}$ and $L_{\ell, m}$ are dimensionless, given that Ψ_2 has the dimension of an inverse squared length and dS that of a squared length. Besides, for all $|m| \leq \ell$, the multipole moments (5.10) obey the symmetry $I_{\ell, -m} = (-)^m \bar{I}_{\ell, m}$ and $L_{\ell, -m} = (-)^m \bar{L}_{\ell, m}$, inherited from standard properties of spherical harmonics $Y_{\ell, m}$. Therefore, it is possible to determine *separately* $I_{\ell, m}$ and $L_{\ell, m}$ from the knowledge of the convenient, complex-valued linear combination

$$\begin{aligned} K_{\ell, m} &\equiv I_{\ell, m} + i L_{\ell, m} = - \oint_S \Psi_2 \dot{Y}_{\ell, m} dS \\ &= - \oint_S \psi^{-2} \Psi_2 \dot{Y}_{\ell, m} d\dot{S}, \end{aligned} \quad (5.12)$$

where the second integral is expressed in terms of the area element $d\dot{S}$ of $\dot{q}_{ab}$, taking into account the relation $dS = \psi^{-2} d\dot{S}$, which follows from Eq. (5.1).

Recall that Ψ_2 is frame-independent on a NEH (cf. Sec. III B). Moreover, the multipoles $I_{\ell, m}$ and $L_{\ell, m}$ do not depend on the choice of the cross section S over which the integrals (5.10) are carried out, once a unit round metric has been chosen within the 3-parameter family given by (5.5)–(5.6) on a fiducial cross section and transported on $\mathcal{H}$ via (5.8). Indeed, let us consider a cross section S' of $\mathcal{H}$, distinct from the cross section S on which the unit round metric $\dot{q}_{ab}$ has been chosen. Let $K'_{\ell, m}$ be the integral (5.12) taken on S' . Since Ψ_2 is invariant along the null generators of $\mathcal{H}$ [cf. Eq. (3.11)], in the integral for $K'_{\ell, m}$, $\Psi_2 = \Psi_2|_{S'}$ is nothing but the pullback of $\Psi_2|_S$ by the inverse of the isometry $\Phi: S \rightarrow S'$ introduced in Sec. III A. Moreover, the unit round metric $\dot{q}'_{ab}$ on S' is also the pullback by Φ^{-1} of the unit round metric $\dot{q}_{ab}$ on S ; cf. Eq. (5.8). Therefore, the spherical harmonic $\dot{Y}_{\ell, m}$ in the integral for $K'_{\ell, m}$ is the pullback of that on S by Φ^{-1} . Finally, the area element dS' on S' is the pullback by Φ^{-1} of the area element of S since Φ is an isometry $(S, q_{ab}) \rightarrow (S', q'_{ab})$. Hence all the integrand of $K'_{\ell, m}$ is the pullback of the integrand of $K_{\ell, m}$ by Φ^{-1} . We conclude that $K'_{\ell, m} = K_{\ell, m}$. One may thus refer to (5.10) as *the* multipole moments of the NEH $\mathcal{H}$, irrespective of the choice of 2-sphere cross section S .

It should be noticed that the monopole moments $I_{0,0}$ and $L_{0,0}$ take the same values for all NEHs,

$$I_{0,0} = \sqrt{\pi} \quad \text{and} \quad L_{0,0} = 0. \quad (5.13)$$

The proof is the same as for the axisymmetry-based monopoles [Eq. (4.10)], given that $\dot{Y}_{0,0}$ is the constant $1/(2\sqrt{\pi})$, independently of the unit round metric $\dot{q}_{ab}$, so that Eq. (4.9) does not involve the choice of any $\dot{q}_{ab}$ and is valid outside axisymmetry. On the contrary the proof of $I_1^{\text{axi}} = 0$ in Sec. IV B is not applicable to $I_{1,0}$ here, even if q_{ab} is assumed to be axisymmetric.

D. Canonical round metric from vanishing area dipole moment

The definition (5.10) does not provide a unique set of horizon multipole moments, because the involved spherical harmonics $\hat{Y}_{\ell,m}$ are relative to a unit round metric $\hat{q}_{ab}$ on $\mathcal{S}$ that is arbitrary among the 3-parameter family described by Eqs. (5.5)–(5.6), all members of this family being conformal to the physical metric q_{ab} . To obtain a uniquely defined, “canonical” set of horizon multipoles, one must pick a “canonical” unit round metric $\hat{q}_{ab}$, and accordingly a “canonical” conformal factor ψ . One possible choice is to require the vanishing of the *area dipole moment* [22],

$$d^i \equiv \oint_{\mathcal{S}} n^i d\mathcal{S} = \oint_{\mathcal{S}} n^i \psi^{-2} d\hat{\mathcal{S}}, \quad (5.14)$$

where $n^i = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ can be viewed as a unit direction normal to $\mathbb{S}^2$ in terms of some polar coordinates (ϑ, φ) adapted to $\hat{q}_{ab}$, i.e., obeying Eq. (5.2). Hence $d^i = 0$ for $\hat{q}_{ab} = \hat{q}_{ab}$ and $\psi = \tilde{\psi}$. (This is analogous to the choice of mass-centered coordinates in other contexts). As shown in Appendix A of Ref. [22], the metric $\hat{q}_{ab}$ always exists and is *unique*.

Equivalently, using the bijection between the angular bases of symmetric trace-free unit tensors $n^{(i_1} \dots n^{i_{\ell})}$ and the spherical harmonics $Y_{\ell,m}$ [60], the condition $d^i = 0$ of a vanishing area dipole moment can be recast in terms of the $\ell = 1$ spherical harmonic modes of ψ^{-2} in adapted polar coordinates (ϑ, φ) , namely,

$$\begin{aligned} d_{1,m} &\equiv \oint_{\mathcal{S}} \hat{Y}_{1,m} d\mathcal{S} = \oint_{\mathcal{S}} \hat{Y}_{1,m} \psi^{-2} d\hat{\mathcal{S}} \\ &= \int_0^{2\pi} d\varphi \int_0^1 d(\cos \vartheta) Y_{1,m}(\vartheta, \varphi) \psi(\vartheta, \varphi)^{-2}. \end{aligned} \quad (5.15)$$

Setting $d_{1,m} = 0$ for $m \in \{-1, 0, 1\}$ provides three conditions obeyed by the four parameters α_0 and $\tilde{\alpha}$ in Eq. (5.6), in addition to the constraint $-\alpha_0^2 + |\tilde{\alpha}|^2 = -1$, and thus specifies *uniquely* the unit round metric $\hat{q}_{ab}$.

Alternatively, one could require the vanishing of the “mass dipole moment” $\propto \oint_{\mathcal{S}} \mathcal{R} n^i d\mathcal{S}$, instead of the area dipole moment (5.14). However, as discussed in Appendix A of Ref. [22], while there exist unit round metrics for which the mass dipole moment vanishes, in general that metric is not unique and additional conditions have to be imposed to guarantee uniqueness.

E. Electric and magnetic scalar potentials

Expression (3.10) of $\text{Im } \Psi_2$ involves the Hájíček 1-form Ω_a relative to any null normal ℓ^a with respect to the cross section $\mathcal{S}$. As noticed in Ref. [22], one can select null normals such that Ω_a is a divergence-free 1-form on $\mathcal{S}$;

we shall then denote it by $\hat{\Omega}_a$,

$$q^{ab} D_a \hat{\Omega}_b = 0. \quad (5.16)$$

By demanding further that the null normal are geodesic ($\kappa_{(\ell)} = 0$), condition (5.16) reduces the choice of null normals to $\mathcal{H}$ to a single constant-rescaling equivalence class $[\ell^a]$ [22].

Equation (5.16) is equivalent to $\hat{\Omega}_a$ being co-exact: there exists a (pseudo)-scalar field B on $\mathcal{S}$, which is referred to as the *magnetic potential* [22], such that

$$\hat{\Omega}_a = \varepsilon_a{}^b D_b B = \hat{\varepsilon}_a{}^b \hat{D}_b B. \quad (5.17)$$

The second equality involves the area 2-form $\hat{\varepsilon}_{ab}$ of the unit round metric $\hat{q}_{ab}$ and results from the conformal invariance of the Hodge dual of 1-forms on a two-dimensional manifold.¹¹ It follows from Eq. (5.17) that $(d\hat{\Omega})_{ab} = -(\hat{D}^2 B) \hat{\varepsilon}_{ab}$. Given that $\varepsilon_{ab} = \psi^{-2} \hat{\varepsilon}_{ab}$, Eq. (3.10) is then equivalent to

$$\psi^{-2} (\text{Im } \Psi_2) \stackrel{\mathcal{S}}{=} -\frac{1}{2} \hat{D}^2 B. \quad (5.18)$$

On the other hand, by combining (3.9a) and (5.4), we get $\psi^{-2} (\text{Re } \Psi_2) = -\frac{1}{2} (\hat{D}^2 \ln \psi + 1)$. In view of Eq. (5.18), this yields

$$\psi^{-2} \Psi_2 \stackrel{\mathcal{S}}{=} -\frac{1}{2} [(1 + \hat{D}^2 E) + i \hat{D}^2 B], \quad (5.19)$$

where

$$E \equiv \ln(R\psi) \quad (5.20)$$

is a scalar field on $\mathcal{H}$, which we shall refer to as the *electric potential*.¹² By virtue of (5.9) E is Lie-dragged along the null geodesic generators of $\mathcal{H}$. Notice the constant term in the electric sector in Eq. (5.19), absent from the magnetic sector, which originates from a topological constraint: indeed, by integrating over $\mathcal{S}$ the scalar curvature $\mathcal{R}$ (a topological invariant), we readily find

$$\begin{aligned} \frac{1}{2} \oint_{\mathcal{S}} \mathcal{R} d\mathcal{S} &= -2 \oint_{\mathcal{S}} (\text{Re } \Psi_2) d\mathcal{S} \\ &= -2 \oint_{\mathcal{S}} \psi^{-2} (\text{Re } \Psi_2) d\hat{\mathcal{S}} \\ &= \oint_{\mathcal{S}} d\hat{\mathcal{S}} = 4\pi, \end{aligned} \quad (5.21)$$

¹¹This is easy to establish: Eq. (5.1) implies $\hat{\varepsilon}_{ab} = \psi^2 \varepsilon_{ab}$, so that $\hat{\varepsilon}_a{}^b = \hat{\varepsilon}_{ac} \hat{q}^{cb} = \psi^2 \varepsilon_{ac} \psi^{-2} q^{cb} = \varepsilon_a{}^b$.

¹²The equivalent quantity introduced in Ref. [22] is $E = \ln \psi$; it differs from Eq. (5.20) by the additive constant $\ln R$, which plays no role in what follows. We simply have introduced the factor R in Eq. (5.20) to make the argument of the logarithm dimensionless.

where we successively used Eq. (3.9a), $dS = \psi^{-2} d\hat{S}$, the real part of Eq. (5.19), as well as $\oint_{\mathcal{S}} (\hat{D}^2 f) d\hat{S} = \int_{\partial\mathcal{S}} \hat{e}_{ab} \hat{D}^b f = 0$ for any scalar field f defined on $\mathcal{S} \sim \mathbb{S}^2$, as a consequence of Stokes' theorem and $\partial\mathcal{S} = \emptyset$. The formula (5.21) is in agreement with the Gauss-Bonnet theorem applied to a closed orientable 2-surface with *genus* $g = 0$.

Now, by substituting expression (5.19) for $\psi^{-2}\Psi_2$ into (5.12), we obtain, for any $\ell \geq 1$,

$$K_{\ell,m} = \frac{1}{2} \oint_{\mathcal{S}} \hat{D}^2 (E + iB) \hat{Y}_{\ell,m} d\hat{S}. \quad (5.22)$$

Integrating by parts twice while using Stokes' theorem and the fact that $\hat{Y}_{\ell,m}$ is an eigenfunction of the Laplace operator $\hat{D}^2$, with eigenvalue $-\ell(\ell+1)$, Eq. (5.22) can be expressed in the simpler form,

$$K_{\ell,m} = -\frac{1}{2} \ell(\ell+1) \oint_{\mathcal{S}} (E + iB) \hat{Y}_{\ell,m} d\hat{S}, \quad \ell \geq 1. \quad (5.23)$$

Consequently, the geometry of the NEH $\mathcal{H}$ is encoded into the electric potential E and the magnetic pseudopotential B , which can then be reconstructed from the multipole moments $I_{\ell,m}$ and $L_{\ell,m}$ according to¹³ [22]

$$E + iB = \langle E \rangle + i\langle B \rangle - \sum_{\ell=1}^{+\infty} \sum_{m=-\ell}^{\ell} \frac{2K_{\ell,m}}{\ell(\ell+1)} \hat{Y}_{\ell,m}, \quad (5.24)$$

where $\langle E \rangle$ and $\langle B \rangle$ stand for the mean values of E and B over $\mathcal{S}$, with respect to the measure $\hat{e}_{ab}$, respectively, and $\hat{Y}_{\ell,m}$ is the complex conjugate of $\hat{Y}_{\ell,m}$.

VI. APPLICATION TO KERR BLACK HOLES

In this section we apply the two distinct definitions of NEH multipole moments reviewed in Secs. IV and V to the event horizon of a Kerr black hole. We first discuss the main properties of Kerr horizon cross sections in Sec. VI A, before computing the multipole moments I_{ℓ}^{axi} and L_{ℓ}^{axi} resulting from the axisymmetry-based definition in Sec. VI B. Then, we move to the multipole family defined for generic NEHs, by determining first the conformal round metric with vanishing area dipole moment in Sec. VI C. We then evaluate in closed-form the magnetic-type potential on the Kerr horizon in Sec. VI D. Finally, we explore the resulting horizon multipoles in Sec. VI E, and compare them to the axisymmetry-based multipoles in Sec. VI F.

¹³We believe that there is a sign error in Eq. (2.14) of Ref. [22], which originates from a typo in the second line of Eq. (2.13) there. Also the mean values $\langle E \rangle$ and $\langle B \rangle$ are missing in Eq. (2.14) of Ref. [22].

A. Geometry of the horizon cross sections

Consider a Kerr black hole of mass M and spin angular momentum S . In advanced Kerr coordinates $(x^\alpha) = (v, r, \theta, \phi)$,¹⁴ which are adapted to the stationarity and axisymmetry of the Kerr metric, and are regular on the horizon (contrary to Boyer-Lindquist coordinates), the metric components $g_{\alpha\beta}$ read

$$\begin{aligned} g_{\alpha\beta} dx^\alpha dx^\beta = & -\left(1 - \frac{2Mr}{\Sigma}\right) dv^2 + 2dvdr \\ & - \frac{4Mr}{\Sigma} a \sin^2 \theta dv d\phi - 2a \sin^2 \theta dr d\phi \\ & + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Mr}{\Sigma} a^2 \sin^2 \theta\right) \sin^2 \theta d\phi^2, \end{aligned} \quad (6.1)$$

where $a \equiv S/M$ is the Kerr spin parameter and $\Sigma \equiv r^2 + a^2 \cos^2 \theta$. The black hole event horizon $\mathcal{H}$ is located at $r = r_+ \equiv M + \sqrt{M^2 - a^2}$; it is a *Killing horizon*, namely a null hypersurface admitting a Killing vector field as normal. This constitutes a special case of NEH. Therefore, according to the discussion in Sec. III B, the Weyl curvature scalar Ψ_2 is frame-invariant on $\mathcal{H}$, provided that ℓ^a coincides with a null normal to $\mathcal{H}$. In particular, it may be evaluated using either the Hartle-Hawking [64] or Hartle [65] tetrad. Using advanced Kerr coordinates, we explicitly have¹⁵

$$\Psi_2 = M\varrho^3, \quad \text{where } \varrho \equiv -\frac{1}{r - ia \cos \theta}. \quad (6.2)$$

Moreover, by using a null frame adapted to the two repeated principal null directions of the Kerr metric, such as the Hartle-Hawking [64] or Kinnersley [67] tetrad, the only nonvanishing Weyl scalar in Eqs. (3.8) is precisely (6.2) [50]. This is not the case for the Hartle [65] tetrad, for which Ψ_3 and Ψ_4 do not vanish [68].

Let $\mathcal{S}$ be a cross section of $\mathcal{H}$ defined by $v = \text{const.}$ ¹⁶ Since $\mathcal{H}$ is located at a constant value of r , namely $r = r_+$, $\mathcal{S}$ is spanned by the Kerr angular coordinates $x^A = (\theta, \phi)$, and the induced metric q_{ab} on $\mathcal{S}$ is obtained by setting $dv = 0$, $dr = 0$ and $r = r_+$ in Eq. (6.1); one obtains

¹⁴In the literature these coordinates are also referred to as *ingoing Kerr coordinates* [61], or more simply as *Kerr coordinates* [62]. They are related to the Boyer-Lindquist coordinates [63] $(t, r, \theta, \phi_{\text{BL}})$ by $dv = dt + (r^2 + a^2)dr/\Delta \equiv dt + dr_*$ and $d\phi = d\phi_{\text{BL}} + adr/\Delta$, where $\Delta \equiv r^2 - 2Mr + a^2$.

¹⁵See, e.g., Eqs. (2.30) and (2.33) in [66], which uses the opposite sign convention for the Weyl scalar Ψ_2 .

¹⁶Note that Hartle's tetrad [65] is adapted to such cross sections, since the vector n^a of that tetrad is normal to the slices $v = \text{const.}$ of $\mathcal{H}$. On the contrary, the Hartle-Hawking tetrad [64] is not, since its n^a vector is not normal to any slicing of $\mathcal{H}$ [the 2-planes $\text{Span}(\ell^a, n^a)^\perp$ are not integrable into 2-surfaces].

$$q_{AB}dx^A dx^B = R^2 \left[(1 - \beta^2 \sin^2 \theta) d\theta^2 + \frac{\sin^2 \theta}{1 - \beta^2 \sin^2 \theta} d\phi^2 \right], \quad (6.3)$$

where $R = \sqrt{r_+^2 + a^2} = \sqrt{2Mr_+}$ is the areal radius (3.4) and

$$\beta \equiv \frac{a}{R} = \frac{a}{\sqrt{r_+^2 + a^2}} = \frac{a}{\sqrt{2Mr_+}} = \frac{a}{\sqrt{2M(M + \sqrt{M^2 - a^2})}} \quad (6.4)$$

is the dimensionless distortion parameter introduced by Smarr [69], who pointed out that all cross sections of the Kerr horizon are isometric, in agreement with the general NEH result derived in Sec. III A. Notice that β ranges from 0 to $1/\sqrt{2}$ when a ranges from 0 to M and that $\beta^2 = a\Omega_{\mathcal{H}}$, with $\Omega_{\mathcal{H}} = a/(r_+^2 + a^2)$ the (constant) angular velocity of the horizon.

We read immediately from expression (6.3) that $\det(q_{AB}) = R^4 \sin^2 \theta$, so that the area 2-form ε_{ab} of $(\mathcal{S}, q_{ab})$ is simply

$$\varepsilon = R^2 \sin \theta \, \mathbf{d}\theta \wedge \mathbf{d}\phi. \quad (6.5)$$

One also deduces from Eq. (6.3) that the scalar curvature $\mathcal{R}$ of $(\mathcal{S}, q_{ab})$ is (cf. Ref. [69] or notebook 1 in Appendix D)

$$\mathcal{R} = \frac{2[1 - \beta^2(1 + 3\cos^2 \theta)]}{R^2(1 - \beta^2 \sin^2 \theta)^3}. \quad (6.6)$$

It is easy to check that $-\mathcal{R}/4$ coincides with the real part of Ψ_2 given by (6.2), in compliance with Eq. (3.9a).

B. Axisymmetry-based horizon multipoles

By introducing the coordinates $x^{A'} = (\zeta, \phi)$, with $\zeta \equiv \cos \theta$, we may recast the metric (6.3) and area 2-form (6.5) as, respectively,¹⁷

$$q_{A'B'} dx^{A'} dx^{B'} = R^2 \left[\left(\frac{1}{1 - \zeta^2} - \beta^2 \right) d\zeta^2 + \left(\frac{1}{1 - \zeta^2} - \beta^2 \right)^{-1} d\phi^2 \right], \quad (6.7)$$

$$\varepsilon = -R^2 d\zeta \wedge \mathbf{d}\phi. \quad (6.8)$$

We note that the metric (6.7) is of the canonical form (4.3) for an axisymmetric NEH, with $f = [(1 - \zeta^2)^{-1} - \beta^2]^{-1}$.

¹⁷As indicated by the minus sign in Eq. (6.8), which stems from $\mathbf{d}\zeta = -\sin \theta \mathbf{d}\theta$, $(\partial_\zeta, \partial_\phi)$ is a left-handed vector frame of $\mathcal{S}$ for the orientation set by Eqs. (2.16) and (2.4), given that $(\partial_v, \partial_r, \partial_\theta, \partial_\phi)$ is a right-handed vector frame of $\mathcal{M}$.

It follows that the coordinates (ζ, ϕ) coincide with those of the axisymmetric construction of Sec. IV A, and that the fiducial axisymmetry-based unit round metric (4.4) is given in terms of Kerr coordinates (θ, ϕ) by simply

$$\overset{\circ}{q}_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2. \quad (6.9)$$

Substituting Eqs. (6.2) and (4.7) into the definition (4.6) of the axisymmetry-based multipole moments, with $dS = -R^2 d\zeta d\phi$ from Eq. (6.8), we readily find

$$\begin{aligned} I_\ell^{\text{axi}} + iL_\ell^{\text{axi}} &= -M \oint_{\mathcal{S}} \varrho_+^3 \hat{Y}_{\ell,0}^{\text{axi}} dS \\ &= -MR^2 \int_0^{2\pi} d\phi \int_{-1}^1 d\zeta \varrho_+^3(\zeta) Y_{\ell,0}(\theta(\zeta), \phi), \end{aligned} \quad (6.10)$$

where $\varrho_+(\zeta)$ is the value of the coefficient $\varrho(r, \zeta)$ defined by (6.2) at $r = r_+$. Given expression (6.2) for ϱ and the identity $Y_{\ell,0}(\theta(\zeta), \phi) = \sqrt{(2\ell+1)/(4\pi)} P_\ell(\zeta)$, where P_ℓ is the Legendre polynomial of order ℓ , it follows that

$$I_\ell^{\text{axi}} + iL_\ell^{\text{axi}} = \frac{1}{2} (1 + \hat{a}^2)^2 \sqrt{(2\ell+1)\pi} \int_{-1}^1 d\zeta \frac{P_\ell(\zeta)}{(1 - i\hat{a}\zeta)^3}, \quad (6.11)$$

where we have introduced the dimensionless parameter

$$\hat{a} \equiv \frac{a}{r_+} = \frac{a}{M + \sqrt{M^2 - a^2}}, \quad (6.12)$$

and have made use of the identity $MR^2/r_+^3 = \frac{1}{2}(1 + \hat{a}^2)^2$. The relation between $\hat{a}$ and the distortion parameter β [Eq. (6.4)] is

$$\hat{a} = \frac{\beta}{\sqrt{1 - \beta^2}} \Leftrightarrow \beta = \frac{\hat{a}}{\sqrt{1 + \hat{a}^2}}. \quad (6.13)$$

Note that $\hat{a}$ ranges from 0 to 1 when a ranges from 0 to M and that, for small spin values, $\beta \sim \hat{a} \sim \chi/2$, with $\chi \equiv a/M = S/M^2$.

The integral in the right-hand side of Eq. (6.11) is evaluated in Appendix A, yielding

$$I_\ell^{\text{axi}} + iL_\ell^{\text{axi}} = 2^{\ell-1} \sqrt{(2\ell+1)\pi} \frac{\ell!(\ell+2)!}{(2\ell+1)!} (i\hat{a})^\ell G_\ell(\hat{a}^2), \quad (6.14)$$

where the real-valued function G_ℓ is defined in terms of the hypergeometric function ${}_2F_1$ by

$$G_\ell(\hat{a}^2) \equiv {}_2F_1\left(\frac{\ell}{2}, \frac{\ell-1}{2}, \ell + \frac{3}{2}; -\hat{a}^2\right), \quad (6.15)$$

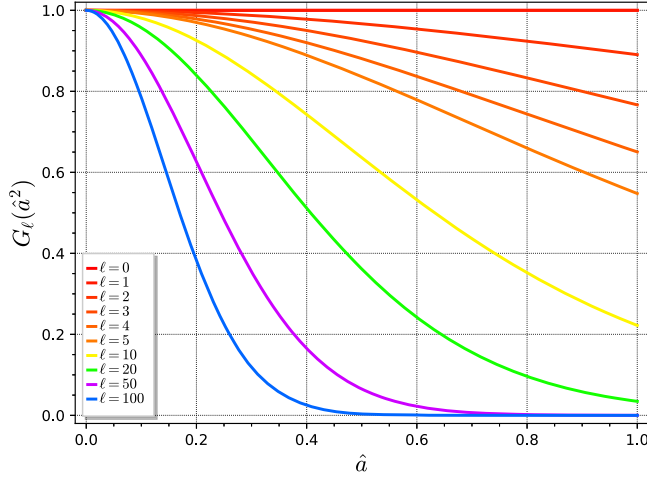


FIG. 4. Function $G_\ell(\hat{a}^2)$ defined by Eq. (6.15), for selected values of ℓ .

and is plotted in Fig. 4. It is shown in Appendix A that G_ℓ is expressible in terms of polynomials and the arctangent function as

$$G_0(\hat{a}^2) = G_1(\hat{a}^2) = 1, \quad (6.16a)$$

$$G_\ell(\hat{a}^2) = \frac{1}{\hat{a}^{2\ell}} \left[\mathcal{P}_{\lceil \ell/2 \rceil}(\hat{a}^2) + (1 + \hat{a}^2)^2 \mathcal{Q}_{\lfloor \ell/2 \rfloor - 1}(\hat{a}^2) \frac{\arctan \hat{a}}{\hat{a}} \right] \quad \ell \geq 2, \quad (6.16b)$$

where $\mathcal{P}_{\lceil \ell/2 \rceil}(\hat{a}^2)$ stands for a polynomial of degree $\lceil \ell/2 \rceil$ in $\hat{a}^2$ and $\mathcal{Q}_{\lfloor \ell/2 \rfloor - 1}(\hat{a}^2)$ for a polynomial of degree $\lfloor \ell/2 \rfloor - 1$ in $\hat{a}^2$. Explicit expressions for these polynomials are given in (A12) for $\ell \leq 5$ and in the notebook 3 of Appendix D for $\ell \leq 12$ (see also notebook 4). For any value of ℓ , such expressions can be computed from formulas (A23) and (A26). As apparent in Fig. 4, despite the singular factor of $\hat{a}^{-2\ell}$ in expression (6.16b), $G_\ell(\hat{a}^2)$ is bounded between 0 and 1 for all $\hat{a} \in [0, 1]$ and all $\ell \in \mathbb{N}$. Moreover, $G_\ell(0) = 1$ for all $\ell \in \mathbb{N}$, with the following small $\hat{a}$ behavior:

$$G_\ell(\hat{a}^2) = 1 - \frac{\ell(\ell-1)}{2(2\ell+3)} \hat{a}^2 + O(\hat{a}^4). \quad (6.17)$$

Inserting (6.17) into (6.14), we obtain the small spin behavior

$$I_\ell^{\text{axi}} + iL_\ell^{\text{axi}} = \sqrt{(2\ell+1)\pi} (i\hat{a})^\ell [\alpha_\ell^{\text{axi}} + O(\hat{a}^2)], \quad \alpha_\ell^{\text{axi}} \equiv 2^{\ell-1} \frac{\ell!(\ell+2)!}{(2\ell+1)!}. \quad (6.18)$$

For any spin, the factor i^ℓ in formula (6.14) implies the following parity properties:

$$\forall n \in \mathbb{N}, \quad I_{2n+1}^{\text{axi}} = 0 \quad \text{and} \quad L_{2n}^{\text{axi}} = 0. \quad (6.19)$$

Taking into account the expressions (A12) of $G_\ell(\hat{a}^2)$ for $0 \leq \ell \leq 3$, the values of the first nonvanishing multipole moments deduced from formula (6.14) are

$$I_0^{\text{axi}} = \sqrt{\pi} \quad \text{and}$$

$$I_2^{\text{axi}} = -\frac{\sqrt{5\pi}}{2\hat{a}^3} [3(1 + \hat{a}^2)^2 \arctan \hat{a} - 3\hat{a} - 5\hat{a}^3], \quad (6.20a)$$

$$L_1^{\text{axi}} = \sqrt{3\pi} \hat{a} \quad \text{and}$$

$$L_3^{\text{axi}} = \frac{\sqrt{7\pi}}{2\hat{a}^4} [15(1 + \hat{a}^2)^2 \arctan \hat{a} - 15\hat{a} - 25\hat{a}^3 - 8\hat{a}^5]. \quad (6.20b)$$

More values, up to $\ell = 12$, can be found in the notebook 3 of Appendix D. In particular, the monopole and dipole moments obey the general NEH relations (4.10). Moreover, injecting the value (6.20b) of L_1^{axi} in the expression (4.15) for the horizon angular momentum $J_{\mathcal{H}}$ and using the identities $\hat{a} = a/r_+$ and $R^2 = 2Mr_+$ leads to $J_{\mathcal{H}} = aM$, i.e., one recovers the Komar angular momentum of Kerr spacetime.

The multipoles I_ℓ^{axi} and L_ℓ^{axi} are plotted as functions of a/M in Fig. 5, for $0 \leq \ell \leq 9$. For large values of ℓ , Stirling's formula yields $\ell!(\ell+2)!/(2\ell+1)! \sim \sqrt{\pi} \ell^{3/2}/2^{2\ell+1}$ for $\ell \rightarrow +\infty$. Taking into account the asymptotic behavior (A13) of the hypergeometric function $G_\ell(\hat{a}^2)$, we arrive at

$$I_\ell^{\text{axi}} + iL_\ell^{\text{axi}} \sim \frac{\pi}{2} \frac{(1 + \hat{a}^2)^{3/4}}{(1 + \sqrt{1 + \hat{a}^2})^{1/2}} \ell^2 \left(\frac{i\hat{a}}{1 + \sqrt{1 + \hat{a}^2}} \right)^\ell \quad \text{for } \ell \rightarrow +\infty. \quad (6.21)$$

Since $0 \leq \hat{a}/(1 + \sqrt{1 + \hat{a}^2}) < 1$, we deduce from this formula that, for a fixed value of $\hat{a}$, the multipole moments decay to zero when ℓ increases,

$$\lim_{\ell \rightarrow +\infty} I_\ell^{\text{axi}} + iL_\ell^{\text{axi}} = 0, \quad (6.22)$$

which is consistent with the behavior observed in Fig. 5.

The source multipoles (cf. Sec. IV C) associated with the geometric multipoles (6.14) are obtained by setting $M_{\mathcal{H}} = M$ in Eq. (4.17), yielding

$$M_\ell + iS_\ell = \frac{MR^\ell}{\sqrt{(2\ell+1)\pi}} \left(I_\ell^{\text{axi}} + i \frac{RL_\ell^{\text{axi}}}{2M} \right). \quad (6.23)$$

Substituting I_ℓ^{axi} and L_ℓ^{axi} by their expressions from Eq. (6.14) and using the identities $R\hat{a} = a(1 + \hat{a}^2)^{1/2}$ and $R/(2M) = (1 + \hat{a}^2)^{-1/2}$, there comes

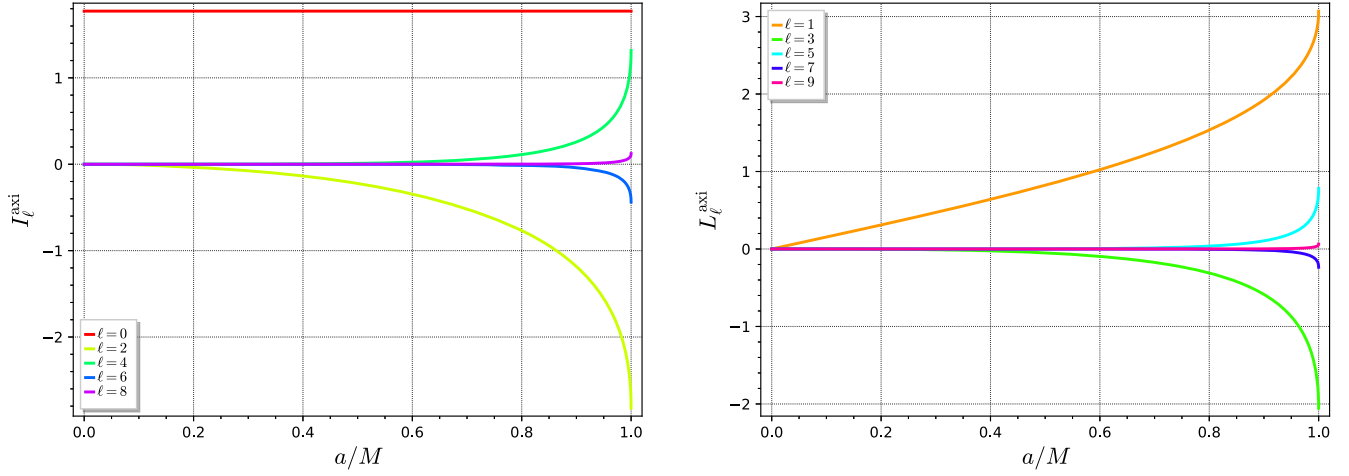


FIG. 5. Shape and current horizon multipole moments I_ℓ^{axi} and L_ℓ^{axi} of the Kerr black hole, resulting from the axisymmetry-based definition, as functions of the Kerr spin parameter a .

$$M_\ell + iS_\ell = 2^{\ell-1} \frac{\ell!(\ell+2)!}{(2\ell+1)!} (1 + \hat{a}^2)^{[\ell/2]} G_\ell(\hat{a}^2) M (ia)^\ell. \quad (6.24)$$

Since $G_0(\hat{a}^2) = G_1(\hat{a}^2) = 1$ [Eq. (6.16a)], we get $M_0 = M$ and $S_1 = aM$. In the small spin limit, we have, thanks to $G_\ell(\hat{a}^2) = 1 + O(\hat{a}^2)$ [recall Eq. (6.17)],

$$M_\ell + iS_\ell = M(ia)^\ell \left[2^{\ell-1} \frac{\ell!(\ell+2)!}{(2\ell+1)!} + O(\hat{a}^2) \right]. \quad (6.25)$$

Hence, to leading order in spin, the source multipole moments of a Kerr black hole share the same sign and scaling behavior as the corresponding field multipole moments, as given by the Hansen formula (1.1).

The source multipole moments M_ℓ and S_ℓ are plotted as functions of a/M in Figs. 6 and 7, for $0 \leq \ell \leq 9$. The right panels of these figures display the comparison to the Hansen field multipole moments M_ℓ^{field} and S_ℓ^{field} , as given by Eq. (1.1). Note that $M_0 = M_0^{\text{field}} = M$ (the mass of Kerr spacetime) and $S_1 = S_1^{\text{field}} = aM$ (the angular momentum of Kerr spacetime). But starting from $\ell = 2$, the relative discrepancy between the source and field multipoles gets quite large: for the mass quadrupole M_2 , it is 20% for a small¹⁸ and up to 40% for a large (cf. Fig. 6), while for the angular momentum octopole S_3 , it is 40% for a small and up to 15% for a large (cf. Fig. 7). Actually, the ratios $M_\ell/M_\ell^{\text{field}}$ and $S_\ell/S_\ell^{\text{field}}$ tend to zero for $\ell \rightarrow +\infty$. Indeed, from Eq. (6.24), the large- ℓ behaviors (A13) and $\ell!(\ell+2)!/(2\ell+1)! \sim \sqrt{\pi} \ell^{3/2}/2^{2\ell+1}$, we get

$$\frac{M_\ell}{M_\ell^{\text{field}}} \quad \text{or} \quad \frac{S_\ell}{S_\ell^{\text{field}}} \sim \sqrt{\pi} \left(\frac{\ell}{2} \right)^{3/2} \frac{(1 + \hat{a}^2)^{[\ell/2]+3/4}}{(1 + \sqrt{1 + \hat{a}^2})^{\ell+1/2}} \quad (6.26)$$

for $\ell \rightarrow +\infty$,

which tends to zero for $\ell \rightarrow +\infty$, whatever the value of $\hat{a}^2 \in [0, 1]$.

C. Unit round metric conformal to the physical one

Let us now move to the computation of the horizon multipole moments according to the generic definition discussed in Sec. V. The first step is to determine the unit round metric $\hat{q}_{ab}$ conformal to the metric q_{ab} induced by g_{ab} on the cross section $\mathcal{S}$ [Eq. (5.1)]. The physical metric q_{ab} is given in terms of the Kerr coordinates $x^A = (\theta, \phi)$ by Eq. (6.3). On the other hand, any round metric $\hat{q}_{ab}$ on $\mathcal{S}$ is given in terms of adapted coordinates $x^{\hat{A}} = (\vartheta, \varphi)$ by (5.2). Let us search for a metric $\hat{q}_{ab}$ that is compatible with the Kerr axisymmetry, i.e., that has the same rotational Killing vector ∂_ϕ ; this yields $\varphi = \phi + \text{const}$. Moreover, as we shall see below, this choice restricts the degrees of freedom on $\hat{q}_{ab}$ towards the vanishing area dipole metric advocated in Sec. V D. By introducing the coordinates $x^{\hat{A}'} = (z, \phi)$ with $z \equiv \cos \vartheta$, we may recast Eq. (5.2) (with $\varphi = \phi + \text{const}$) as

$$\hat{q}_{\hat{A}'\hat{B}'} dx^{\hat{A}'} dx^{\hat{B}'} = \frac{dz^2}{1-z^2} + (1-z^2) d\phi^2. \quad (6.27)$$

By comparing with expression (6.7) of q_{ab} in terms of $x^{A'} = (\zeta, \phi)$, we see that the conformal relation $\hat{q}_{ab} = \psi^2 q_{ab}$ [Eq. (5.1)] is equivalent to the following system of two equations:

$$\frac{dz^2}{1-z^2} = \psi^2 R^2 \left(\frac{1}{1-\zeta^2} - \beta^2 \right) d\zeta^2, \quad (6.28a)$$

¹⁸This somehow tempers the claim made in Ref. [18] that the “difference is insignificant.”

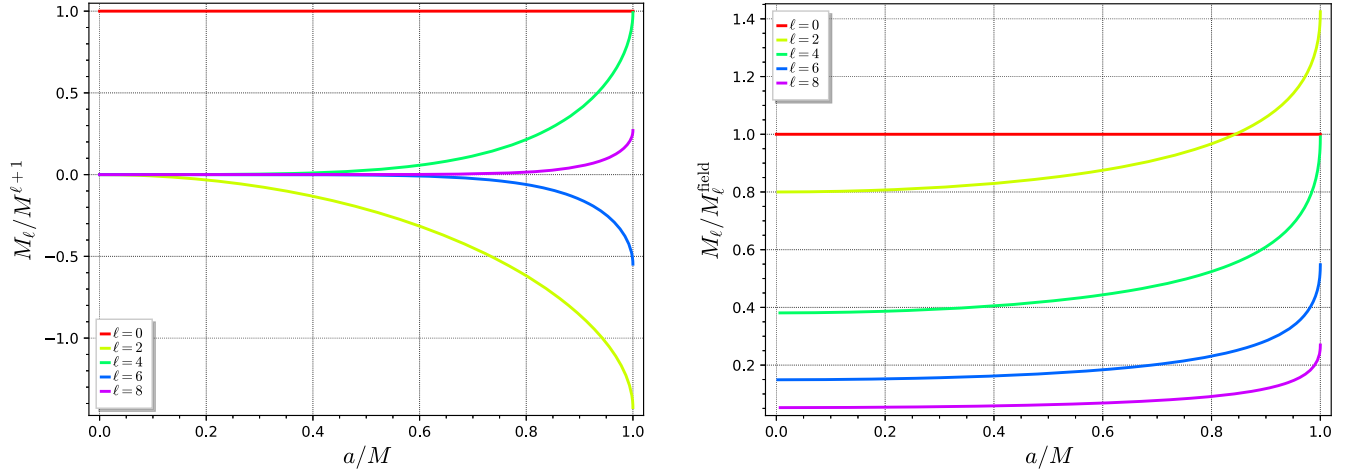


FIG. 6. Mass-type source multipole moments M_ℓ of the Kerr horizon as functions of the Kerr spin parameter a . The right panel shows M_ℓ rescaled by the mass-type field multipole moments given by Hansen's formula (1.1): $M_\ell^{\text{field}} = (-)^{\ell/2} M a^\ell$ for ℓ even.

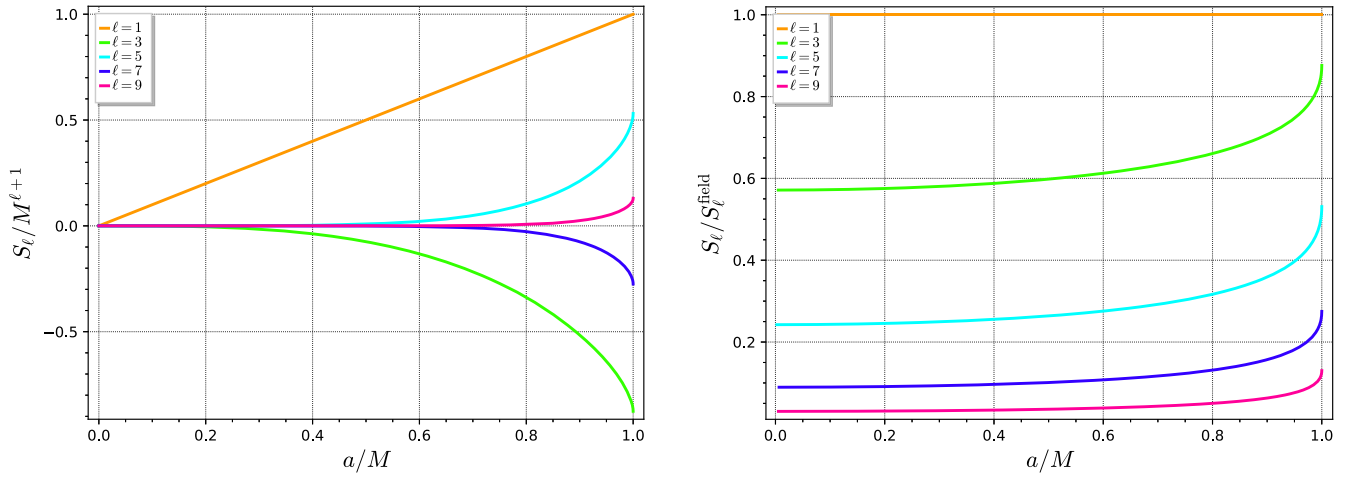


FIG. 7. Current-type source multipole moments S_ℓ of the Kerr horizon as functions of the Kerr spin parameter a . The right panel shows S_ℓ rescaled by the current-type field multipole moments given by Hansen's formula (1.1): $S_\ell^{\text{field}} = (-)^{(\ell-1)/2} M a^\ell$ for ℓ odd.

$$1 - z^2 = \psi^2 R^2 \left(\frac{1}{1 - \zeta^2} - \beta^2 \right)^{-1}. \quad (6.28b)$$

Extracting $\psi^2 R^2$ from Eq. (6.28b), substituting into Eq. (6.28a) and taking the square root while assuming that z is an increasing function of ζ , we get

$$\frac{dz}{1 - z^2} = \left(\frac{1}{1 - \zeta^2} - \beta^2 \right) d\zeta. \quad (6.29)$$

The integration leads to $\text{artanh } z = \text{artanh } \zeta - \beta^2 \zeta$, where the integration constant has been set to zero to make z an odd function of ζ . Hence, we get the equivalent closed-form expressions

$$\begin{aligned} z(\zeta) &= \tanh(\text{artanh } \zeta - \beta^2 \zeta) = \frac{\zeta - \tanh(\beta^2 \zeta)}{1 - \zeta \tanh(\beta^2 \zeta)} \\ &= \frac{1 + \zeta - (1 - \zeta)e^{2\beta^2 \zeta}}{1 + \zeta + (1 - \zeta)e^{2\beta^2 \zeta}}. \end{aligned} \quad (6.30)$$

Notice that (6.28) imply $z'(\zeta) = R^2 \psi^2$. The relation between $\vartheta = \arccos z$ and $\theta = \arccos \zeta$ resulting from (6.30) is depicted in Fig. 8. It shows a linear relation increasingly modulated as $a \rightarrow M$; for $a = M$, the discrepancy $|\vartheta - \theta|$ is at most 0.28, with a relative difference $|\vartheta - \theta|/\theta$ reaching 0.65 for small values of θ .

The expression for the conformal factor ψ is deduced from Eq. (6.28b). Using $\zeta = \cos \theta$, $R^2 = r_+^2 + a^2$ and $\beta^2 = a^2/(r_+^2 + a^2)$ [Eq. (6.4)], we may cast it as

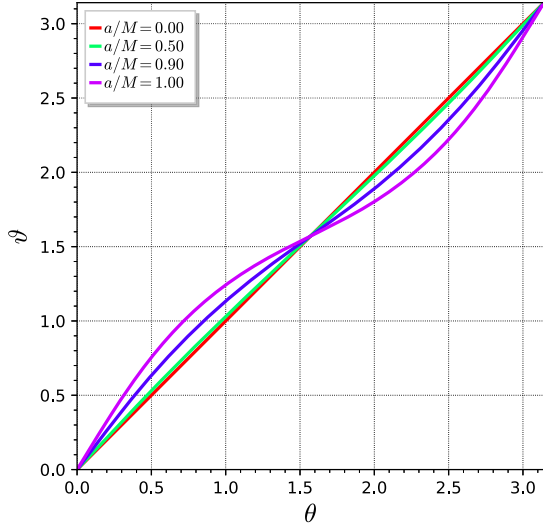


FIG. 8. Colatitude coordinate ϑ adapted to the unit round metric of cross sections of the Kerr horizon, as a function of the Kerr coordinate θ , for various values of the Kerr spin parameter a .

$$\psi = \frac{\sqrt{(r_+^2 + a^2 \cos^2 \theta)(1 - z^2)}}{(r_+^2 + a^2) \sin \theta}. \quad (6.31)$$

For a given z , expression (6.31) for ψ agrees with formula (A.8) in Ref. [22]. However, the function $z(\theta)$ that we found [Eq. (6.30) with $\zeta = \cos \theta$] is different (and simpler) from that given in Ref. [22] (which is denoted Z there). We checked that the conformal factor (6.31) is a solution of the partial differential equation (5.3), and that the scalar curvature of the metric $\psi^2 q_{ab}$ is equal to 2, as it should be (see below). Consequently we believe that there is a typo in the expression for $z(\theta)$ in Eq. (A.8) of Ref. [22].

We deduce from (6.30) that $\sin \theta / \sqrt{1 - z^2} = \cosh(\beta^2 \cos \theta) - \cos \theta \sinh(\beta^2 \cos \theta)$, so that Eq. (6.31) can be turned into an explicit expression for ψ in terms of θ ,

$$R\psi(\theta) = \frac{\sqrt{1 - \beta^2 \sin^2 \theta}}{\cosh(\beta^2 \cos \theta) - \cos \theta \sinh(\beta^2 \cos \theta)}. \quad (6.32)$$

By combining this result with Eq. (6.3), we get the components of the unit round metric $\hat{q}_{ab} = \psi^2 q_{ab}$ with respect to the Kerr angular coordinates $x^A = (\theta, \phi)$,

$$\hat{q}_{AB} dx^A dx^B = \frac{(1 - \beta^2 \sin^2 \theta)^2 d\theta^2 + \sin^2 \theta d\phi^2}{[\cosh(\beta^2 \cos \theta) - \cos \theta \sinh(\beta^2 \cos \theta)]^2}. \quad (6.33)$$

One can check that the above metric has a constant scalar curvature equal to 2, as expected (see notebook 1 in Appendix D). Notice that expression (6.33) is not the “canonical” one (5.2) for a unit round metric, except for

$\beta = 0$ (i.e., $a = 0$). In other words, the Kerr angular coordinates (θ, ϕ) are *not* polar coordinates adapted to $\hat{q}_{ab}$, contrary to (ϑ, φ) .

It turns out the obtained round metric (6.33) is the “canonical” one among the family of unit round metrics conformally related to q_{ab} , i.e., it fulfills the criterion of vanishing area dipole moment introduced in Sec. VD. Indeed, by definition, the modes (5.15) of the area dipole moment read

$$d_{1,m} = \oint_S \hat{Y}_{1,m} dS = R^2 \int_0^{2\pi} d\phi \int_{-1}^1 d\zeta Y_{1,m}(z(\zeta), \phi). \quad (6.34)$$

The nonaxisymmetric modes $m = \pm 1$ cancel out by averaging over ϕ of the periodic function $\exp(\pm i\phi)$. For the axisymmetric mode $m = 0$, we have instead $Y_{1,0}(z(\zeta), \phi) \propto z(\zeta)$. According to Eq. (6.30), the function $z(\zeta)$ is of odd parity [thanks to the choice of a zero integration constant while deriving Eq. (6.30)], which readily implies $d_{1,0} = 0$.

According to the analysis in Sec. VA, any other unit round metric $\hat{q}'_{ab}$ conformally related to q_{ab} must obey $\hat{q}'_{ab} = \alpha^2 \hat{q}_{ab}$, where α is given in terms of the three free parameters $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ by (5.6). The corresponding conformal factor is then $\psi' = \alpha\psi$. While $(\hat{q}_{ab}, \psi)$ in (6.33) and (6.32) manifestly respect the continuous and discrete symmetries of the Kerr horizon cross sections, namely the axisymmetry and symmetry with respect to the equatorial plane, a generic pair $(\hat{q}'_{ab}, \psi')$ does not, as $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ (respectively $\alpha_3 \neq 0$) explicitly breaks the continuous (respectively discrete) symmetry.

The logarithm of $R\psi(\theta)$, as given by Eq. (6.32), i.e., the function E defined by Eq. (5.20), is plotted in Fig. 9. Note that ψ is an even function of the Kerr parameter a , and consequently does not depend on the direction of the black

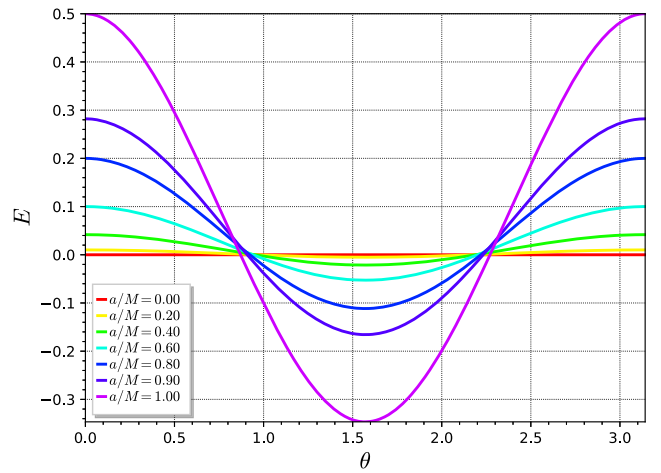


FIG. 9. Electric potential $E = \ln(R\psi)$ of the Kerr horizon, resulting from Eq. (6.32), for various values of the Kerr spin parameter a .

hole spin. It is also manifestly independent of ϕ , thus reflecting the axisymmetry of the Kerr metric (6.1), and is invariant under $\theta \rightarrow \pi - \theta$, corresponding to a reflection across the equatorial plane $\theta = \pi/2$. For a Schwarzschild black hole, $\beta = 0$ and $R = 2M$ so $\psi = 1/(2M)$ is constant (spherical symmetry). For a slowly spinning black hole, we have

$$R\psi(\theta) = 1 + \beta^2 P_2(\cos \theta) + O(\beta^4), \quad (6.35)$$

where $P_2(\cos \theta)$ is the Legendre polynomial of order $\ell = 2$, which captures an axisymmetric, quadrupolar deviation from spherical symmetry, with amplitude β^2 . For an extremal Kerr black hole ($a = M$) we have $r_+ = M$, and thus $R = \sqrt{2}M$ and $\beta^2 = \frac{1}{2}$. Then $M\psi(\theta) = f(\theta)$, where f is a strictly decreasing function varying in $[\sqrt{e}, 1/\sqrt{2}]$ over the interval $\theta \in [0, \pi/2]$.

The average $\langle E \rangle$ of the electric potential E with respect to the measure $\hat{\varepsilon}_{ab}$ is given by

$$\begin{aligned} \langle E \rangle &\equiv \frac{1}{4\pi} \oint_S E d\hat{S} = \frac{1}{4\pi} \oint_S E \psi^2 dS \\ &= \frac{1}{2} \int_{-1}^1 \ln(R\psi) (R\psi)^2 d(\cos \theta) = \frac{1}{5} \beta^4 + O(\beta^6), \end{aligned} \quad (6.36)$$

which vanishes “quickly” as $\beta \rightarrow 0$ (small-spin regime), as can be seen in Fig. 10. Moreover, the right panel shows that the dependence of $\langle E \rangle / \beta^4$ over the Kerr spin parameter a is weak. Notice that $\langle E \rangle$ is small, in the sense that $\langle E \rangle \simeq 0.07$ at most for $a = M$ while $\max E = 0.5$ for that value of a .

D. Magnetic-type potential

The Hájíček 1-form Ω_a associated to the slicing of the Kerr horizon by the $v = \text{const}$ cross sections (cf. Sec. II C) is expressed in terms of the coordinates $x^{A'} = (\zeta, \phi)$ by

$$\begin{aligned} \Omega_{A'} dx^{A'} &= \frac{\hat{a}}{1 + \hat{a}^2 \zeta^2} \left[\hat{a} \zeta d\zeta \right. \\ &\quad \left. - \frac{3 + \hat{a}^2 + \hat{a}^2(1 - \hat{a}^2)\zeta^2}{2(1 + \hat{a}^2 \zeta^2)} (1 - \zeta^2) d\phi \right], \end{aligned} \quad (6.37)$$

where we have let the parameter $\hat{a} \equiv a/r_+ = \beta/\sqrt{1 - \beta^2}$ [Eqs. (6.12)–(6.13)] appear instead of β . Formula (6.37) can be found by setting $\theta = \arccos \zeta$ in Eqs. (D33)–(D34) of Ref. [43]. It can also be derived directly from expression (2.26) of Ω_a in terms of the null vectors ℓ^a and n^a generating at each point the 2-plane orthogonal to $\mathcal{S}$, with $\ell^a \equiv (\partial_v)^a + \Omega_{\mathcal{H}}(\partial_\phi)^a$; cf. notebook 2 in Appendix D.

The 1-form Ω_a is not divergence-free. However, by defining $\hat{\Omega}_a \equiv \Omega_a + D_a \ln f$, with $f \equiv (1 + \hat{a}^2 \zeta^2)^{-1/2}$, we get a divergence-free 1-form: $D^a \hat{\Omega}_a = 0$ [Eq. (5.16)]. Note that $\hat{\Omega}_a$ is the Hájíček 1-form corresponding to the null normal $\hat{\ell}^a = f \ell^a$. Explicitly, we deduce from Eq. (6.37) and the above expression of f that

$$\hat{\Omega}_{A'} dx^{A'} = -\frac{\hat{a}(3 + \hat{a}^2 + \hat{a}^2(1 - \hat{a}^2)\zeta^2)}{2(1 + \hat{a}^2 \zeta^2)^2} (1 - \zeta^2) d\phi. \quad (6.38)$$

By definition, the magnetic potential B obeys Eq. (5.17): $q^{bc} \varepsilon_{ac} D_b B = \hat{\Omega}_a$, where ε_{ab} is the area 2-form of $(\mathcal{S}, q_{ab})$, as given by Eq. (6.8). Thanks to the diagonal form of the metric components $q_{A'B'}$ [recall Eq. (6.7)], Eq. (5.17) is equivalent to the system $q^{\phi\phi} \varepsilon_{\zeta\phi} \partial_\phi B = \hat{\Omega}_\zeta$ and $q^{\zeta\zeta} \varepsilon_{\phi\zeta} \partial_\zeta B = \hat{\Omega}_\phi$, with $\varepsilon_{\zeta\phi} = -\varepsilon_{\phi\zeta} = -R^2$. Given the components (6.7) and (6.38) of respectively q_{ab} and $\hat{\Omega}_a$, as well as the identity $\beta^2 = \hat{a}^2/(1 + \hat{a}^2)$, we get $\partial_\phi B = 0$ and

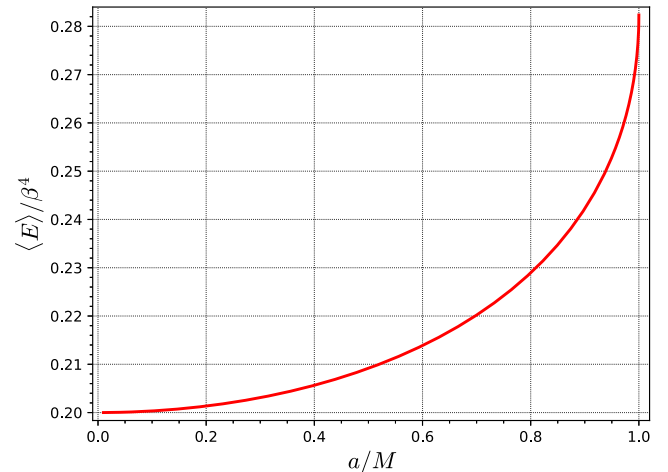
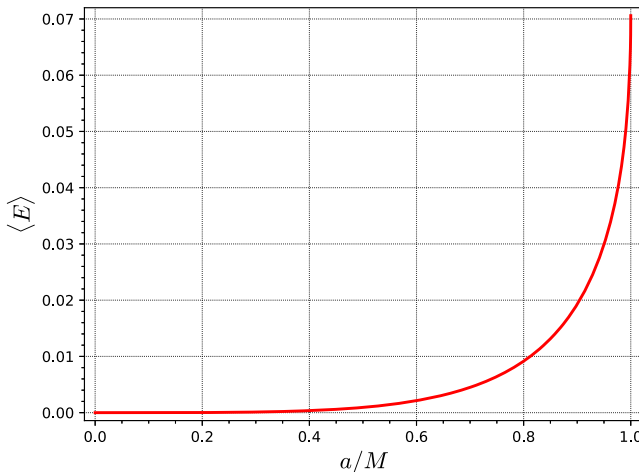


FIG. 10. Mean value $\langle E \rangle$ of E as a function of the Kerr spin parameter a ; the right panel shows the ratio $\langle E \rangle / \beta^4$.

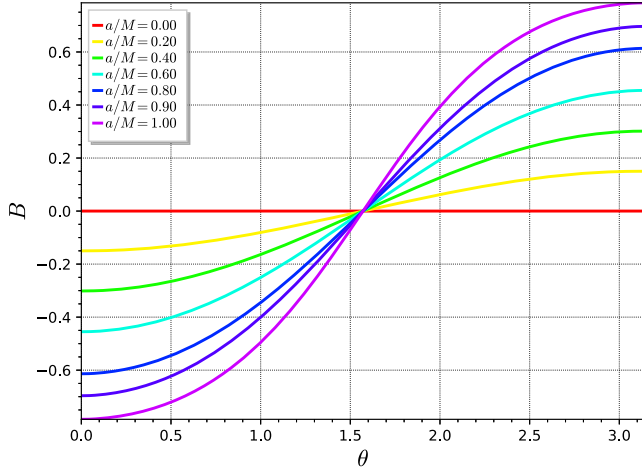


FIG. 11. Magnetic potential B of the Kerr horizon, as given by Eq. (6.40) with $\zeta = \cos \theta$, for various values of the Kerr spin parameter a .

$$\left(\frac{1}{1-\zeta^2} - \frac{\hat{a}^2}{1+\hat{a}^2} \right)^{-1} \frac{\partial B}{\partial \zeta} = - \frac{\hat{a}(3 + \hat{a}^2 + \hat{a}^2(1-\hat{a}^2)\zeta^2)}{2(1+\hat{a}^2\zeta^2)^2} (1-\zeta^2).$$

After simplification, there comes

$$\frac{\partial B}{\partial \zeta} = - \frac{\hat{a}}{1+\hat{a}^2\zeta^2} - \frac{\hat{a}(1-\hat{a}^2)}{2(1+\hat{a}^2)}. \quad (6.39)$$

In view of $\partial_\phi B = 0$, the integration is immediate,

$$B = -\arctan(\hat{a}\zeta) - \frac{1-\hat{a}^2}{2(1+\hat{a}^2)} \hat{a}\zeta. \quad (6.40)$$

The integration constant (which has no physical significance since B is a potential) has been set to zero. This makes B an odd function of ζ . Moreover, B is an odd function of $\hat{a}$ that scales as $\sim \hat{a}$ for small spin values. In Appendix B we provide an alternative proof of the key result (6.40). This equation yields B as a function of $\theta = \arccos \zeta$, which we plot in Fig. 11.

E. Horizon multipole moments

Substituting (6.2) and (5.11) into the generic-NEH definition (5.12) of the complex-valued multipole moments $K_{\ell,m}$, we readily find

$$\begin{aligned} K_{\ell,m} &= -M \oint_S \varrho_+^3 \hat{Y}_{\ell,m} dS \\ &= -MR^2 \int_0^{2\pi} d\phi \int_{-1}^1 d\zeta \varrho_+^3(\zeta) Y_{\ell,m}(\vartheta(\zeta), \phi), \end{aligned} \quad (6.41)$$

where we have used Eq. (6.8) to write $dS = -R^2 d\zeta d\phi$. Since ϱ_+ , as given by Eq. (6.2) at $r = r_+$, does not depend

on the angle ϕ , we deduce

$$\begin{aligned} K_{\ell,m} &= (I_\ell + iL_\ell) \delta_{m,0}, \quad \text{with} \\ I_\ell + iL_\ell &= \frac{1}{2} (1 + \hat{a}^2)^2 \sqrt{(2\ell+1)\pi} \int_{-1}^1 d\zeta \frac{P_\ell(z(\zeta))}{(1 - i\hat{a}\zeta)^3}, \end{aligned} \quad (6.42)$$

where the function $z(\zeta)$ is given by Eq. (6.30) and $\hat{a} \equiv a/r_+$ [recall Eq. (6.12)]. We have used the identities $MR^2/r_+^3 = \frac{1}{2}(1+\hat{a}^2)^2$ and $Y_{\ell,0}(\vartheta(\zeta), \phi) = \sqrt{(2\ell+1)/(4\pi)} P_\ell(z(\zeta))$. The quantities I_ℓ and L_ℓ introduced in (6.42) are nothing but the multipole moments $I_{\ell,0}$ and $L_{\ell,0}$ of the Kerr horizon [cf. Eq. (5.12)]. Notice that formula (6.42) is identical to formula (6.11) for the axisymmetry-based multipole moments, except for ζ replaced by the function $z(\zeta)$ in the argument of the Legendre polynomial P_ℓ . This function is too complicated to allow us to express the integral in (6.42) in terms of known functions like the hypergeometric ones used in Sec. VI B for I_ℓ^{axi} and L_ℓ^{axi} , except in the small $\hat{a}$ limit [Eqs. (6.45)–(6.46) below]. Instead, we have written a SageMath code capable of numerically evaluating I_ℓ and L_ℓ to an arbitrary precision¹⁹ and have made it publicly available via the notebook 3 listed in Appendix D. The values of I_ℓ and L_ℓ obtained with this code are plotted in terms of a/M in Figs. 12 and 13, for $0 \leq \ell \leq 9$.

Many properties of the multipoles moments $K_{\ell,m}$ can be inferred from the integral formula (6.42). First of all, the $K_{\ell,m}$'s are nonzero only for $m = 0$, with I_ℓ and L_ℓ real valued and functions of $\hat{a}$ only, or equivalently of a/M only (see Figs. 12 and 13). Secondly, the generic monopole properties (5.13) are easily recovered from (6.42): $I_0 = \sqrt{\pi}$ and $L_0 = 0$. Thirdly, the reflection symmetry across the equatorial plane $\zeta = 0$ and the parity properties of the Legendre polynomials $P_\ell(x)$ and the function $z(\zeta)$ given by Eq. (6.30) imply $I_\ell = 0$ when ℓ is odd and $L_\ell = 0$ when ℓ is even, similarly to the axisymmetry-based case [Eq. (6.19)]:

$$\forall n \in \mathbb{N}, \quad I_{2n+1} = 0 \quad \text{and} \quad L_{2n} = 0. \quad (6.43)$$

Moreover, the integral in (6.42) has a simple behavior under the change $a \rightarrow -a$ in the black hole spin direction. Indeed, using $z(-\zeta) = -z(\zeta)$ and $P_\ell(-x) = (-1)^\ell P_\ell(x)$ we readily find

$$I_\ell(-a) + iL_\ell(-a) = (-1)^\ell [I_\ell(a) + iL_\ell(a)]. \quad (6.44)$$

Combined with Eq. (6.43) we deduce that the shape multipoles I_ℓ only depends on the spin amplitude $|a|$, while the current multipoles L_ℓ change sign under $a \rightarrow -a$.

¹⁹Having a precision beyond the usual 16 digits (“double precision”) is required to numerically evaluate quantities like $I_\ell/(i\hat{a})^\ell$ (right panel of Fig. 12) for small $\hat{a}$ and ℓ larger than ~ 5 .

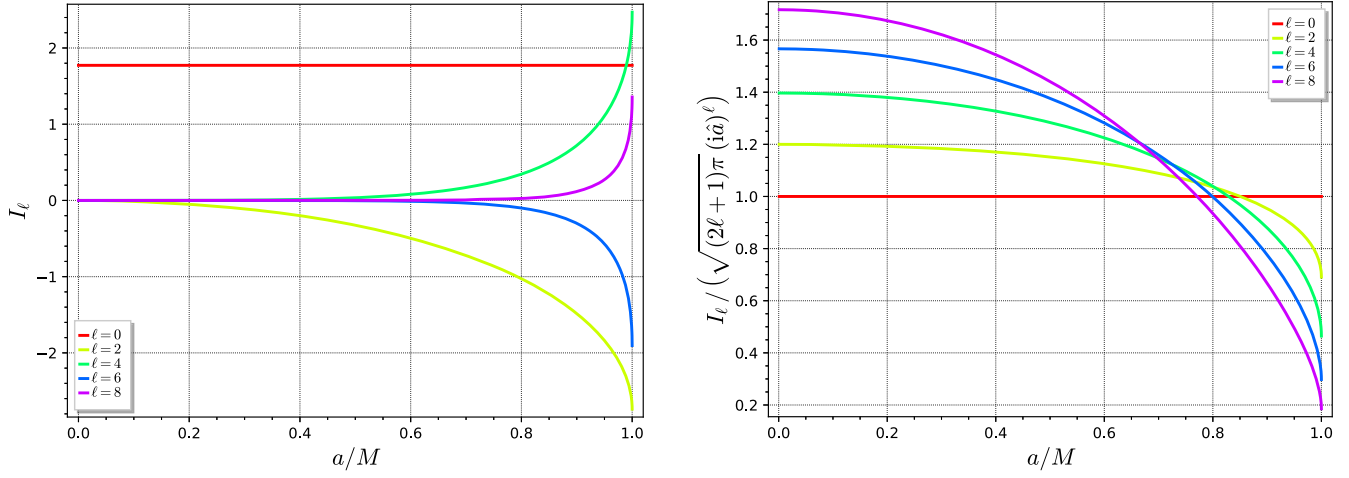


FIG. 12. Shape multipole moments I_ℓ of the Kerr horizon, as functions of the Kerr spin parameter a ; the right panel depicts I_ℓ rescaled by $\sqrt{(2\ell+1)\pi}(\hat{a})^\ell$.

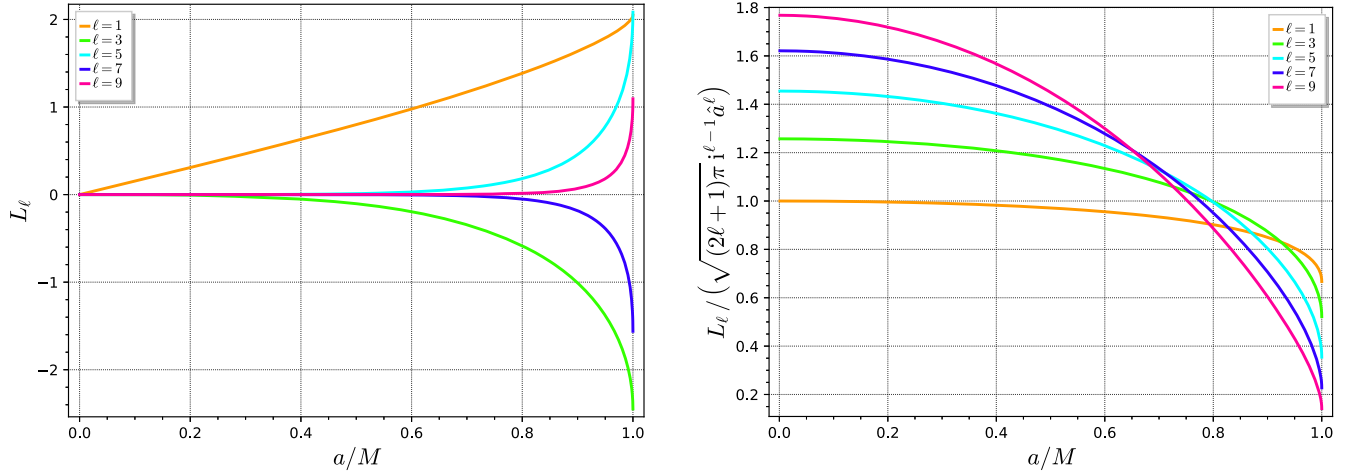


FIG. 13. Current multipole moments L_ℓ of the Kerr horizon, as functions of the Kerr spin parameter a ; the right panel depicts L_ℓ rescaled by $\sqrt{(2\ell+1)\pi}i^{\ell-1}\hat{a}^\ell$.

More precisely, the horizon multipoles I_ℓ (respectively L_ℓ) are even (respectively odd) in the Kerr spin parameter a , just like the Hansen (field) multipole moments (1.1) of the Kerr solution [5]. In the nonspinning limit $a \rightarrow 0$, we have $\hat{a} = 0$ and $\beta = 0$, so that according to Eq. (6.30) $z = \zeta$. Then $I_\ell = L_\ell = 0$ for all $\ell \geq 1$. In other words, the only nonvanishing horizon multipole moment of a Schwarzschild black hole is its shape monopole $I_0 = \sqrt{\pi}$.

For small spin values, the behavior of the multipoles can be determined in closed form, and we find (see Appendix C)

$$I_\ell + iL_\ell = \sqrt{(2\ell+1)\pi}(\hat{a})^\ell [\alpha_\ell + O(\hat{a}^2)], \quad (6.45)$$

where

$$\alpha_\ell = \frac{2^\ell \gamma_{[\ell/2]}}{\binom{2\ell+1}{\ell}} \begin{cases} 1 & (\ell \text{ even}) \\ \frac{1}{2} & (\ell \text{ odd}) \end{cases}$$

$$\text{with } \gamma_n \equiv \sum_{k=0}^n \binom{2n}{k} = \frac{2^{2n} + \binom{2n}{n}}{2}. \quad (6.46)$$

The first 15 numerical values of the sequence $(\alpha_\ell)_{\ell \in \mathbb{N}}$ are listed in Table I. In the limit $\ell \rightarrow +\infty$ we have $\alpha_\ell \sim \frac{1}{2}\sqrt{\pi\ell}$, so that $I_\ell + iL_\ell \sim (\pi/\sqrt{2})\ell(\hat{a})^\ell$. The scaling property (6.45) of the multipoles I_ℓ and L_ℓ as functions of $\hat{a}$ is confirmed graphically by the right panels of Figs. 12 and 13, with the value of α_ℓ from Table I being read at the intersection of the curves with the axis $a = 0$.

Finally, given the multipoles I_ℓ and L_ℓ , the mode decomposition (5.24) yields the electric and magnetic potentials E and B of a Kerr black hole horizon as series expansions over Legendre polynomials, according to

$$E - \langle E \rangle = -\frac{1}{\sqrt{\pi}} \sum_{\ell=1}^{+\infty} \frac{\sqrt{2\ell+1}}{\ell(\ell+1)} I_\ell(\hat{a}) P_\ell(\cos \vartheta), \quad (6.47a)$$

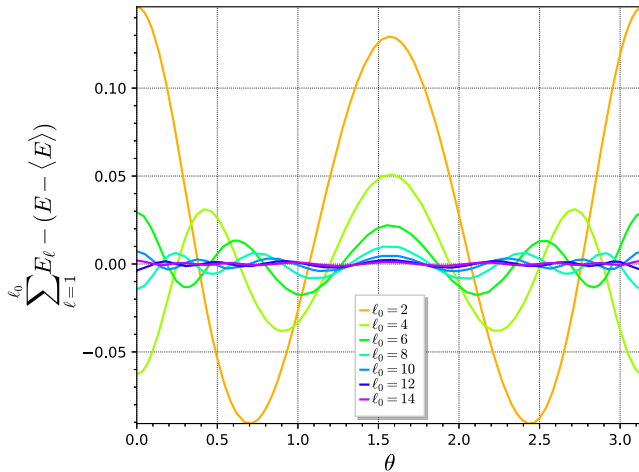
$$B = -\frac{1}{\sqrt{\pi}} \sum_{\ell=1}^{+\infty} \frac{\sqrt{2\ell+1}}{\ell(\ell+1)} L_\ell(\hat{a}) P_\ell(\cos \vartheta). \quad (6.47b)$$

Of course E and B are axisymmetric and have a reflection (anti)symmetry with respect to the equatorial plane, in agreement with the continuous and discrete spatial symmetries of the Kerr metric. Recall that the electric potential $E = \ln(R\psi)$ and the magnetic potential B are known in closed form from Eqs. (6.32) and (6.40), according to which

$$E = \frac{1}{2} \ln(1 - \beta^2 \sin^2 \theta) - \ln[\cosh(\beta^2 \cos \theta) - \cos \theta \sinh(\beta^2 \cos \theta)], \quad (6.48a)$$

$$B = -\arctan(\hat{a} \cos \theta) - \frac{1 - \hat{a}^2}{2(1 + \hat{a}^2)} \hat{a} \cos \theta. \quad (6.48b)$$

Note that in the above formulas, θ is the Kerr colatitude coordinate, while in (6.47), E and B are expressed in terms of ϑ —the colatitude adapted to the unit round metric $\hat{q}_{ab}$, the two angles being related by (6.30), where $\zeta = \cos \theta$ and $z = \cos \vartheta$ (see also Fig. 8). The functions E and B are depicted in Figs. 9 and 11 respectively, while Fig. 14 shows the convergence of the ℓ -sums (6.47) towards the closed-form formulas (6.48) as $\ell \rightarrow +\infty$.



F. Comparison of the two families of horizon multipoles

The multipole moments (I_ℓ, L_ℓ) are graphically compared to the axisymmetric-based multipole moments $(I_\ell^{\text{axi}}, L_\ell^{\text{axi}})$ computed in Sec. VIB in Figs. 15 and 16. In the small spin regime, we read from Eqs. (6.18), (6.45) and (6.46) that both families behave like a real-valued coefficient times $(i\hat{a})^\ell$, but with a generically different coefficient, as one can see from the ratios

$$\left. \frac{I_\ell}{I_\ell^{\text{axi}}} \right|_{\ell \text{ even}} \quad \text{or} \quad \left. \frac{L_\ell}{L_\ell^{\text{axi}}} \right|_{\ell \text{ odd}} = \frac{2^\ell + \binom{\ell}{2}}{\ell+2} + O(\hat{a}^2). \quad (6.49)$$

Since $z(\zeta) \sim \zeta$ for $a \rightarrow 0$ and the only difference between formulas (6.11) and (6.42) is $P_\ell(\zeta)$ replaced by $P_\ell(z(\zeta))$, one might have thought naively that in the limit of small spins, I_ℓ and L_ℓ coincide with I_ℓ^{axi} and L_ℓ^{axi} . However formula (6.49) shows that this is not the case, except for $\ell = 0$ and $\ell = 1$. The reason is that, in the limit $a \rightarrow 0$, the part $z = \zeta$ in the expansion of $z(\zeta)$ in powers of a contributes only to the multipoles $\ell = 0$ and $\ell = 1$, i.e., its integral in (6.42) vanishes for higher values of ℓ , so that the integrals depend on nonzero powers of a in the expansion of $z(\zeta)$, which do not appear in I_ℓ^{axi} and L_ℓ^{axi} .

Let us take the example of the shape quadrupole, I_2 . From Eq. (6.30) with $\beta = \hat{a}/\sqrt{1 + \hat{a}^2}$ [Eq. (6.13)], we have

$$z(\zeta) = \zeta - \hat{a}^2 \zeta(1 - \zeta^2) + O(\hat{a}^4). \quad (6.50)$$

Accordingly,

$$\text{Re} \frac{P_2(z(\zeta))}{(1 - i\hat{a}\zeta)^3} = \frac{1}{2}(3\zeta^2 - 1) - 6\hat{a}^2 \zeta^4 + O(\hat{a}^4), \quad (6.51)$$

while

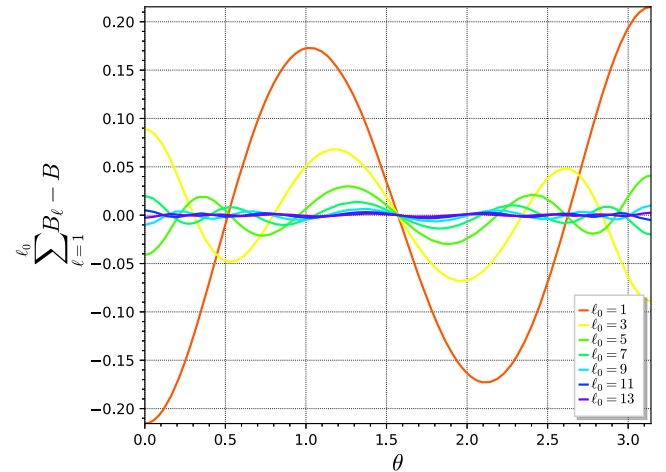


FIG. 14. Convergence of the multipole expansions (6.47) for the extremal Kerr horizon ($\hat{a} = 1$). Here, $\sum_{\ell=1}^{\ell_0} E_\ell$ and $\sum_{\ell=1}^{\ell_0} B_\ell$ stand for the sums appearing in the formulas (6.47), but truncated at a given order ℓ_0 .

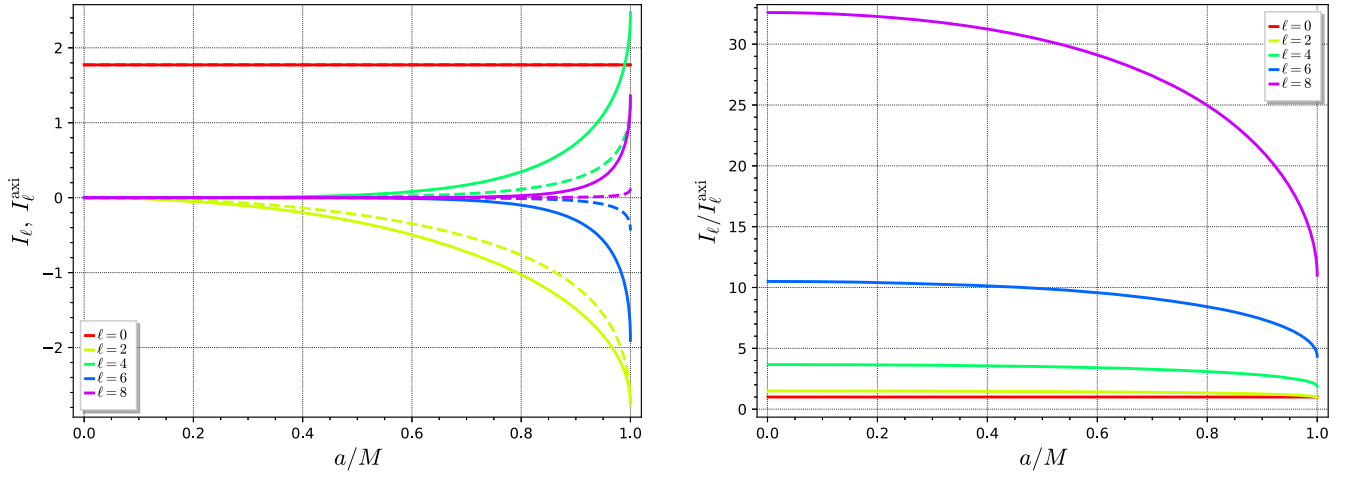


FIG. 15. Comparison, for the Kerr horizon, between the shape multipole moments I_ℓ resulting from the generic-NEH definition (solid curves in the left panel) and the axisymmetry-based moments I_ℓ^{axi} (dashed curves, same as solid ones in Fig. 5). The right panel depicts the ratio $I_\ell / I_\ell^{\text{axi}}$.

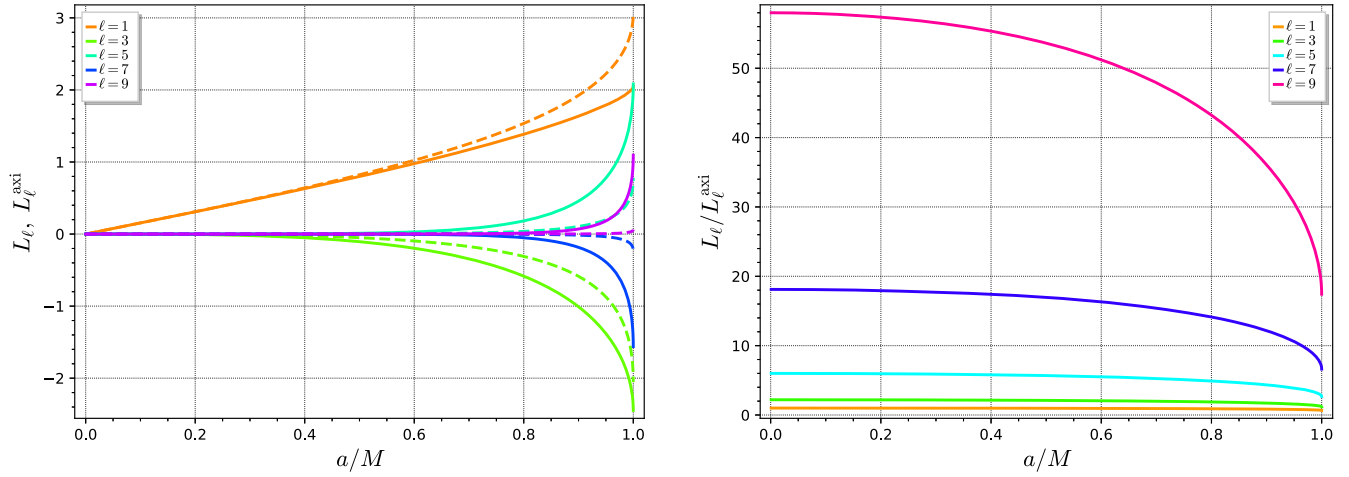


FIG. 16. Same as Fig. 15 but for the current multipole moments L_ℓ and L_ℓ^{axi} .

$$\text{Re} \frac{P_2(\zeta)}{(1 - i\hat{a}\zeta)^3} = \frac{1}{2}(3\zeta^2 - 1) + 3\hat{a}^2\zeta^2(1 - 3\zeta^2) + O(\hat{a}^4). \quad (6.52)$$

The integral of $\frac{1}{2}(3\zeta^2 - 1) = P_2(\zeta)$ yields 0 so that the value of I_2 is given at leading order by the term $\propto \hat{a}^2$ in Eq. (6.51), while the value of I_2^{axi} is given by the term $\propto \hat{a}^2$ in Eq. (6.52). These two terms are distinct quartic polynomials in ζ and hence, once integrated over $[-1, 1]$, they lead to distinct values for I_2 and I_2^{axi} ,

$$I_2 = -6\sqrt{\frac{\pi}{5}}\hat{a}^2 + O(\hat{a}^4) \quad \text{and} \quad I_2^{\text{axi}} = -4\sqrt{\frac{\pi}{5}}\hat{a}^2 + O(\hat{a}^4). \quad (6.53)$$

It follows that $I_2/I_2^{\text{axi}} \rightarrow 3/2$ in the limit $\hat{a} \rightarrow 0$, in agreement with formula (6.49) for $\ell = 2$.

In the small spin limit, Eq. (6.49) implies that the generic-NEH multipole moments are increasingly larger than the axisymmetry-based ones, as ℓ increases,

$$\lim_{\ell \rightarrow +\infty} \frac{I_\ell}{I_\ell^{\text{axi}}} = \lim_{\ell \rightarrow +\infty} \frac{L_\ell}{L_\ell^{\text{axi}}} = +\infty \quad \text{for } \hat{a} \rightarrow 0. \quad (6.54)$$

The numerical results displayed in Figs. 15 and 16 suggest that this behavior remains true for all values of a/M .

VII. CONCLUSIONS AND PROSPECTS

We have presented the first computation [see Figs. 12–13 and Eqs. (6.45)–(6.46)] of the horizon multipole moments of a Kerr black hole, following the generic definition for

nonexpanding horizons proposed by Ashtekar *et al.* in 2022 [22]. In particular, we have obtained a rather simple expression [Eq. (6.33)], in terms of the Kerr angular coordinates (θ, ϕ) , of the unique axisymmetric unit round metric $\hat{q}_{ab}$ conformally related to the physical metric q_{ab} of the horizon cross sections. Besides, we have extended to all spherical harmonic degrees $\ell \in \mathbb{N}$ the computation [Eqs. (6.14)–(6.16)] of the Kerr horizon multipole moments following the definition for axisymmetric isolated horizons proposed by Ashtekar *et al.* in 2004 [18], whereas previous results in the literature were limited to $\ell \leq 8$ [18,32]. We have compared the two families of horizon multipoles resulting from those definitions and shown that they differ, except for $\ell = 0$ and, in the small spin limit, also for $\ell = 1$. We have also compared these horizon multipoles to the (Hansen) field multipole moments. The results of these comparisons are detailed in Sec. VI and summarized in Sec. IB.

At first glance, it may be surprising that the two families of horizon multipole moments do not coincide for the Kerr black hole. This merely reflects the lack of a unique (canonical) construction of multipole moments, even in the particular case of axisymmetric horizons. It shall be kept in mind that the same situation occurs for the *field* multipole moments: for a nonspherically symmetric static body, the Geroch multipole moments [4] are different from the Hansen ones [5]. What matters is that the whole intrinsic and extrinsic geometry of the horizon can be reconstructed from the knowledge of any of the two multipole sets [18,22], similarly to the asymptotic space-time geometry being fully determined by the knowledge of the field multipole moments [7].

Astrophysical black holes are neither perfectly isolated, nor stationary. For instance, a supermassive black hole in a binary system of compact objects is subject to the tidal field of its companion. Just like for a Newtonian self-gravitating body, the tidal deformability of an astrophysical black hole can be characterized by means of two families of so-called *tidal Love numbers* (TLNs). In particular, the *field* TLNs characterize the linear response of the black hole at the level of the gravitational field itself. For stationary tidal perturbations, they can for instance be defined and computed from the linear perturbation in the (Geroch-Hansen) field multipole moments at spatial infinity [70,71].

By contrast, the *surfacial* TLNs characterize in a local manner the linear response of the black hole at the level of the event horizon, akin to the surface of a Newtonian self-gravitating body. The surfacial TLNs of nonspinning (Schwarzschild) black holes have been defined and computed by several authors [27,72,73]. For spinning (Kerr) black holes, a geometrically motivated definition of surfacial TLNs is still lacking, except in the small spin limit [27]. The horizon multipole moments of the second family explored here, namely those for generic nonexpanding horizons, pave the way to the definition and computation of

surfacial TLNs for Kerr black holes, since the definition of these multipoles [22] is valid beyond axisymmetry, contrary to the definition proposed previously in Ref. [18]. The surfacial TLNs should provide an invariant characterization of the shape and angular momentum structure of the horizon of a spinning black hole subject to a weak, but otherwise arbitrary tidal perturbation. This will be the topic of a forthcoming paper.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [74].

APPENDIX A: INTEGRAL FOR THE AXISYMMETRY-BASED MULTIPOLES OF THE KERR HORIZON

The integral appearing in the right-hand side of formula (6.11) for the axisymmetry-based multipole moments of the Kerr horizon is²⁰

$$J_{\ell,0} \equiv \int_{-1}^1 dx \frac{P_{\ell}(x)}{(1 - i\hat{a}x)^3}. \quad (\text{A1})$$

Explicit values of $J_{\ell,0}$ in terms of $\hat{a}$ can be found in Appendix A of Ref. [32] for $2 \leq \ell \leq 8$. In what follows, we derive a closed-form expression valid for any value of ℓ , first in terms of the hypergeometric function ${}_2F_1$, and then in terms of polynomials and the arctangent function.

1. Expression in terms of the hypergeometric function

Our starting point is the Rodrigues formula for the Legendre polynomials, namely,

$$P_{\ell}(x) = \frac{1}{2^{\ell} \ell!} \frac{d^{\ell}}{dx^{\ell}} [(x^2 - 1)^{\ell}]. \quad (\text{A2})$$

Substituting this formula into Eq. (A1) and integrating by parts ℓ times, while noticing that the boundary contributions at $x = \pm 1$ vanish thanks to the factor $(x^2 - 1)^{\ell}$ in Eq. (A2), we obtain

²⁰We denote it $J_{\ell,0}$ because this is the case $n = 0$ of a more general integral $J_{\ell,n}$ introduced in (C3) below.

$$J_{\ell,0} = \frac{(-)^\ell}{2^\ell \ell!} \int_{-1}^1 dx (x^2 - 1)^\ell \frac{d^\ell}{dx^\ell} [(1 - i\hat{a}x)^{-3}]. \quad (\text{A3})$$

Given that $d^\ell/dx^\ell [(1 - i\hat{a}x)^{-3}] = \frac{1}{2}(\ell + 2)!(i\hat{a})^\ell (1 - i\hat{a}x)^{-(\ell+3)}$, there comes

$$J_{\ell,0} = \frac{(\ell + 2)(\ell + 1)}{2^{\ell+1}} (i\hat{a})^\ell \int_{-1}^1 dx \frac{(1 - x^2)^\ell}{(1 - i\hat{a}x)^{\ell+3}}. \quad (\text{A4})$$

The change of variable $t \equiv \frac{1}{2}(1 + x)$ leads us to an integral between 0 and 1,

$$J_{\ell,0} = 2^\ell (\ell + 2)(\ell + 1) \frac{(i\hat{a})^\ell}{(1 + i\hat{a})^{\ell+3}} \int_0^1 dt \frac{t^\ell (1 - t)^\ell}{(1 - \frac{2i\hat{a}}{1+i\hat{a}}t)^{\ell+3}}, \quad (\text{A5})$$

and we may use the Euler formula [see DLMF-15.6.1 [75] with $a = \ell + 3$, $b = \ell + 1$ and $c = 2(\ell + 1)$] to express this integral in terms of the hypergeometric function ${}_2F_1$,

$$\begin{aligned} & \int_0^1 dt \frac{t^\ell (1 - t)^\ell}{(1 - \frac{2i\hat{a}}{1+i\hat{a}}t)^{\ell+3}} \\ &= \frac{(\ell!)^2}{(2\ell + 1)!} {}_2F_1\left(\ell + 3, \ell + 1, 2(\ell + 1); \frac{2i\hat{a}}{1 + i\hat{a}}\right). \end{aligned} \quad (\text{A6})$$

Then, since $(\ell + 2)(\ell + 1)\ell! = (\ell + 2)!$ and $(2\ell + 1)! = 2^\ell \ell!(2\ell + 1)!!$, Eq. (A5) becomes

$$\begin{aligned} J_{\ell,0} &= \frac{(\ell + 2)!}{(2\ell + 1)!!} \frac{(i\hat{a})^\ell}{(1 + i\hat{a})^{\ell+3}} \\ &\quad \times {}_2F_1\left(\ell + 3, \ell + 1, 2(\ell + 1); \frac{2i\hat{a}}{1 + i\hat{a}}\right). \end{aligned} \quad (\text{A7})$$

Next, let us introduce the variable $z \equiv 2i\hat{a}/(1 + i\hat{a})$, which appears as the argument of ${}_2F_1$. It fulfills $i\hat{a} = z/(2 - z)$ and $1 + i\hat{a} = (1 - z/2)^{-1}$, so that

$$\begin{aligned} J_{\ell,0} &= \frac{(\ell + 2)!}{(2\ell + 1)!!} (i\hat{a})^\ell \left(1 - \frac{z}{2}\right)^{\ell+3} \\ &\quad \times {}_2F_1(\ell + 3, \ell + 1, 2(\ell + 1); z). \end{aligned} \quad (\text{A8})$$

Now, by virtue of a standard quadratic transformation formula of the hypergeometric function (formula DLMF-15.8.13 [75] with $a = \ell + 3$ and $b = \ell + 1$), we have the identity,

$$\begin{aligned} & \left(1 - \frac{z}{2}\right)^{\ell+3} {}_2F_1(\ell + 3, \ell + 1, 2(\ell + 1); z) \\ &= {}_2F_1\left(\frac{\ell + 3}{2}, \frac{\ell + 4}{2}, \ell + \frac{3}{2}; \frac{z^2}{(2 - z)^2}\right). \end{aligned} \quad (\text{A9})$$

Given that $z^2/(2 - z)^2 = (i\hat{a})^2 = -\hat{a}^2$, we get the final expression of the integral $J_{\ell,0}$ in terms of the hypergeometric function ${}_2F_1$,

$$J_{\ell,0} = \frac{(\ell + 2)!}{(2\ell + 1)!!} (i\hat{a})^\ell {}_2F_1\left(\frac{\ell + 3}{2}, \frac{\ell + 4}{2}, \ell + \frac{3}{2}; -\hat{a}^2\right). \quad (\text{A10})$$

In expression (6.11) for the multipole moments, $J_{\ell,0}$ is multiplied by the factor $(1 + \hat{a}^2)^2$. Thanks to the Euler transformation formula ${}_2F_1(a, b, c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b, c; z)$ (cf. DLMF-15.8.1 [75]) with $a = \ell/2$, $b = (\ell - 1)/2$, $c = \ell + 3/2$ and $z = -\hat{a}^2$, the product happens to be the hypergeometric function with different arguments,

$$\begin{aligned} G_\ell(\hat{a}^2) &\equiv {}_2F_1\left(\frac{\ell}{2}, \frac{\ell - 1}{2}, \ell + \frac{3}{2}; -\hat{a}^2\right) \\ &= (1 + \hat{a}^2)^2 {}_2F_1\left(\frac{\ell + 3}{2}, \frac{\ell + 4}{2}, \ell + \frac{3}{2}; -\hat{a}^2\right). \end{aligned} \quad (\text{A11})$$

In conjunction with Eq. (A10), this yields (6.14). The function $G_\ell(\hat{a}^2)$ is depicted in Fig. 4. The low ℓ values are

$$G_0(\hat{a}^2) = G_1(\hat{a}^2) = 1, \quad (\text{A12a})$$

$$G_2(\hat{a}^2) = -\frac{5}{8\hat{a}^4} \left[5\hat{a}^2 + 3 - 3(1 + \hat{a}^2)^2 \frac{\arctan \hat{a}}{\hat{a}} \right], \quad (\text{A12b})$$

$$G_3(\hat{a}^2) = \frac{7}{8\hat{a}^6} \left[8\hat{a}^4 + 25\hat{a}^2 + 15 - 15(1 + \hat{a}^2)^2 \frac{\arctan \hat{a}}{\hat{a}} \right], \quad (\text{A12c})$$

$$G_4(\hat{a}^2) = -\frac{21}{32\hat{a}^8} \left[81\hat{a}^4 + 190\hat{a}^2 + 105 - 15(1 + \hat{a}^2)^2 (7 + \hat{a}^2) \frac{\arctan \hat{a}}{\hat{a}} \right], \quad (\text{A12d})$$

$$G_5(\hat{a}^2) = \frac{33}{32\hat{a}^{10}} \left[32\hat{a}^6 + 343\hat{a}^4 + 630\hat{a}^2 + 315 - 105(1 + \hat{a}^2)^2(3 + \hat{a}^2) \frac{\arctan \hat{a}}{\hat{a}} \right]. \quad (\text{A12e})$$

Other expressions of $G_\ell(\hat{a}^2)$, up to $\ell = 12$, are listed in the notebook 3 of Appendix D. For large values of ℓ , the following asymptotic behavior holds:

$$G_\ell(\hat{a}^2) \sim (1 + \hat{a}^2)^{3/4} \left(\frac{2}{1 + \sqrt{1 + \hat{a}^2}} \right)^{\ell+1/2} \text{ for } \ell \rightarrow +\infty. \quad (\text{A13})$$

To prove (A13), let us use Euler's integral representation of the hypergeometric function ${}_2F_1$ (formula DLMF-15.6.1 [75]) to rewrite Eq. (A11) as

$$G_\ell(\hat{a}^2) = \underbrace{\frac{\Gamma(\ell+3/2)}{\Gamma((\ell-1)/2)\Gamma(\ell/2+2)}}_{C_\ell} \underbrace{\int_0^1 dt h(t) e^{\frac{\ell}{2}f(t)}}_{E_\ell(\hat{a}^2)}, \quad (\text{A14})$$

with

$$h(t) \equiv \frac{1-t}{t^{3/2}} \quad \text{and} \quad f(t) \equiv \ln \left(\frac{t(1-t)}{1 + \hat{a}^2 t} \right). \quad (\text{A15})$$

Since $f(t)$ has a unique maximum over $]0, 1[$, at $t = t_0 \equiv (1 + \sqrt{1 + \hat{a}^2})^{-1}$, we may use Laplace's method of approximating the integrand in $E_\ell(\hat{a}^2)$ by a Gaussian function centered at t_0 , of standard deviation $\sigma = \sqrt{2/(\ell|f''(t_0)|)}$ and of height $h(t_0)e^{\frac{\ell}{2}f(t_0)}$, thereby getting an equivalent to the integral $E_\ell(\hat{a}^2)$ for large ℓ ,

$$E_\ell(\hat{a}^2) \sim \sqrt{\frac{4\pi}{\ell|f''(t_0)|}} h(t_0) e^{\frac{\ell}{2}f(t_0)} \text{ for } \ell \rightarrow +\infty. \quad (\text{A16})$$

Given that $f(t_0) = -2 \ln(1 + \sqrt{1 + \hat{a}^2})$, $f''(t_0) = -2(1 + \hat{a}^2)^{-1/2}(1 + \sqrt{1 + \hat{a}^2})^2$ and $h(t_0) = (1 + \hat{a}^2)^{1/2}(1 + \sqrt{1 + \hat{a}^2})^{1/2}$, there comes

$$E_\ell(\hat{a}^2) \sim \sqrt{\frac{2\pi}{\ell}} (1 + \hat{a}^2)^{3/4} (1 + \sqrt{1 + \hat{a}^2})^{-(\ell+1/2)} \text{ for } \ell \rightarrow +\infty. \quad (\text{A17})$$

On the other hand, thanks to Stirling's formula, it is easy to see that the prefactor C_ℓ in (A14) behaves as

$$C_\ell \sim 2^\ell \sqrt{\frac{\ell}{\pi}} \text{ for } \ell \rightarrow +\infty. \quad (\text{A18})$$

Combining (A17) and (A18) leads to (A13).

2. Hypergeometric function in terms of arctangent

Here we express the hypergeometric function $F_\ell(\hat{a}^2) \equiv {}_2F_1(\frac{\ell+3}{2}, \frac{\ell+4}{2}, \ell + \frac{3}{2}; -\hat{a}^2)$ appearing in (A10) in terms of $\arctan \hat{a}$ and rational functions of $\hat{a}$.

Let us first show the derivation for the case where ℓ is even. That is, we set $\ell = 2n$, with $n \in \mathbb{N}$, so that $F_\ell(\hat{a}^2) = {}_2F_1(n + \frac{3}{2}, n + 2, 2n + \frac{3}{2}; -\hat{a}^2)$. We first apply DLMF-15.5.2 [75] (with “ n ” there taking on the value of the “ $n + 1$ ” here), which yields

$$F_\ell(\hat{a}^2) = \frac{(n + 1/2)_{n+1}}{(1/2)_{n+1}(n + 1)!} \frac{d^{n+1}}{dz^{n+1}} {}_2F_1\left(\frac{1}{2}, 1, n + \frac{1}{2}; z\right), \quad (\text{A19})$$

where $z \equiv -\hat{a}^2 < 0$ and $(\alpha)_n \equiv \Gamma(\alpha + n)/\Gamma(\alpha)$ (with $\alpha \notin \mathbb{Z}_{\leq 0}$) is Pochhammer's symbol. We next apply DLMF-15.5.6 [75] (with “ n ” there taking on the value of the “ $n - 1$ ” here), yielding

$$F_\ell(\hat{a}^2) = A_\ell \frac{d^{n+1}}{dz^{n+1}} \left[(1 - z)^{n-1} \frac{d^{n-1}}{dz^{n-1}} {}_2F_1\left(\frac{1}{2}, 1, \frac{3}{2}; z\right) \right],$$

$$A_\ell \equiv \frac{2^\ell (2\ell + 1)!!}{(\ell + 2)!(\ell - 2)!}. \quad (\text{A20})$$

From DLMF-15.4.3 [75], we know that ${}_2F_1(\frac{1}{2}, 1, \frac{3}{2}; z) = \arctan(\hat{a})/\hat{a}$, which is how our sought-after $\arctan \hat{a}$ makes its appearance. We now apply the general Leibniz rule to obtain

$$F_\ell(\hat{a}^2) = A_\ell \sum_{m=0}^{n+1} \binom{n+1}{m} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{d^{n+1-m}}{dz^{n+1-m}} \left((1 - z)^{n-1} \frac{d^{n-1-k}}{dz^{n-1-k}} \right) \frac{d^{k+m} \arctan \hat{a}}{dz^{k+m}}. \quad (\text{A21})$$

It is easy to recognize that the only term with $\arctan \hat{a}$ in this double sum is the one with $k = m = 0$. Thus,

$$F_\ell(\hat{a}^2) = P_\ell^{(e)}(\hat{a}) + Q_\ell^{(e)}(\hat{a}) \arctan \hat{a} \quad (\ell \text{ even}), \quad (\text{A22})$$

where

$$P_\ell^{(e)}(\hat{a}) \equiv A_\ell \sum_{\substack{0 \leq m \leq n+1 \\ 0 \leq k \leq n-1 \\ (m,k) \neq (0,0)}} \binom{n+1}{m} \binom{n-1}{k} \frac{d^{n+1-m}}{dz^{n+1-m}} \left((1-z)^{n-1} \frac{d^{n-1-k} \hat{a}^{-1}}{dz^{n-1-k}} \right) \frac{d^{k+m} \arctan \hat{a}}{dz^{k+m}}, \quad (\text{A23a})$$

$$Q_\ell^{(e)}(\hat{a}) \equiv A_\ell \frac{d^{n+1}}{dz^{n+1}} \left((1-z)^{n-1} \frac{d^{n-1} \hat{a}^{-1}}{dz^{n-1}} \right). \quad (\text{A23b})$$

Note that in these expressions, $\hat{a}$ is to be considered as a function of z , according to $\hat{a} = \sqrt{-z}$. Is it easy to show that $Q_\ell^{(e)}(\hat{a})$ is equal to $\hat{a}^{-2\ell-1}$ times a polynomial of degree $\frac{\ell}{2} - 1$ in $\hat{a}^2$. Thus $Q_\ell^{(e)}(\hat{a})$ is odd in $\hat{a}$ [as expected, since it multiplies the odd function $\arctan \hat{a}$ in (A22) and $F_\ell(\hat{a}^2)$ is even]. Similarly, one can see²¹ that $P_\ell^{(e)}(\hat{a})$ is equal to $\hat{a}^{-2\ell}(1 + \hat{a}^2)^{-2}$ times a polynomial of degree $\ell/2$ in $\hat{a}^2$. Thus $P_\ell^{(e)}(\hat{a})$ is even in $\hat{a}$ [as expected, since $F_\ell(\hat{a}^2)$ in (A22) is even].

Let us now turn to the case where ℓ is odd, i.e., $\ell = 2n + 1$ with $n \in \mathbb{N}$. We first write $F_\ell(\hat{a}^2) = {}_2F_1(n + 2, n + 5/2, 2n + 5/2; -\hat{a}^2) = {}_2F_1(n + 5/2, n + 2, 2n + 5/2; -\hat{a}^2)$ and then apply DLMF-15.5.7 [75] (with ‘ n ’ there set to 1, $a = n + 3/2$, $b = n + 2$ and $c = 2n + 3/2$) to readily get

$$F_\ell(\hat{a}^2) = B_\ell \left[\left(n + \frac{3}{2} \right) F_{2n}(\hat{a}^2) - (1 - z) \frac{dF_{2n}(\hat{a}^2)}{dz} \right], \quad B_\ell \equiv \frac{2(2\ell + 1)}{(\ell + 2)(\ell - 2)}. \quad (\text{A24})$$

Using now Eq. (A22), it readily follows that

$$F_\ell(\hat{a}^2) = P_\ell^{(o)}(\hat{a}) + Q_\ell^{(o)}(\hat{a}) \arctan \hat{a} \quad (\ell \text{ odd}), \quad (\text{A25})$$

where

²¹In fact, we can show that $P_\ell^{(e)}(\hat{a})$ is equal to $\hat{a}^{-2\ell}(1 + \hat{a}^2)^{-\ell/2-1}$ times a polynomial of degree $\ell - 1$ in $\hat{a}^2$. However, on this instance, by giving various specific values to ℓ (cf. notebook 4 in Appendix D) we can then see that this latter polynomial can be factored out as $(1 + \hat{a}^2)^{\ell/2-1}$ times a polynomial of degree $\ell/2$ in $\hat{a}^2$, so that it is as we state in the main text, namely, that $P_\ell^{(e)}(\hat{a})$ is equal to $\hat{a}^{-2\ell}(1 + \hat{a}^2)^{-2}$ times a polynomial of degree $\ell/2$ in $\hat{a}^2$.

$$P_\ell^{(o)}(\hat{a}) \equiv B_\ell \left[\left(n + \frac{3}{2} \right) P_{2n}^{(e)}(\hat{a}) - (1 + \hat{a}^2) \frac{dP_{2n}^{(e)}(\hat{a})}{dz} + \frac{Q_{2n}^{(e)}(\hat{a})}{2\hat{a}} \right], \quad (\text{A26a})$$

$$Q_\ell^{(o)}(\hat{a}) \equiv B_\ell \left[\left(n + \frac{3}{2} \right) Q_{2n}^{(e)}(\hat{a}) - (1 + \hat{a}^2) \frac{dQ_{2n}^{(e)}(\hat{a})}{dz} \right]. \quad (\text{A26b})$$

We next use the facts, shown above, that $Q_\ell^{(e)}(\hat{a})$ is equal to $\hat{a}^{-2\ell-1}$ times a polynomial—call it $q_\ell^{(e)}(\hat{a})$ —of degree $\frac{\ell}{2} - 1$ in $\hat{a}^2$ and that $P_\ell^{(e)}(\hat{a})$ is equal to $\hat{a}^{-2\ell}(1 + \hat{a}^2)^{-2}$ times a polynomial of degree $\ell/2$ in $\hat{a}^2$, to derive the following properties for $P_\ell^{(o)}(\hat{a})$ and $Q_\ell^{(o)}(\hat{a})$. The term $P_\ell^{(o)}(\hat{a})$ is readily seen to be equal to $\hat{a}^{-4n-2}(1 + \hat{a}^2)^{-2}$ times a polynomial of degree $n + 1$ in $\hat{a}^2$. In its turn, the coefficient $Q_\ell^{(o)}(\hat{a})$ could be thought to be equal, naively and in principle, to $\hat{a}^{-4n-3}$ times a polynomial—call it $q_\ell^{(o)}(\hat{a})$ —of degree n in $\hat{a}^2$. However, a more detailed inspection shows that the coefficient of the highest-order term in the polynomial $q_\ell^{(o)}(\hat{a})$ of would-be degree n actually vanishes, so that it really is a polynomial of degree $n - 1$ in $\hat{a}^2$. Indeed, denoting by c_h the coefficient of the highest-order term in $q_\ell^{(e)}(\hat{a})$ [that is, $q_\ell^{(e)}(\hat{a})$ is equal to $c_h \hat{a}^{n-1}$ plus lower order terms], the contribution to the would-be term $\hat{a}^{2n}$ in $q_\ell^{(o)}(\hat{a})$ from the first and second terms in (A26) are equal to, respectively, $A c_h (n + 3/2) \hat{a}^{2n}$ and $-A c_h (n + 3/2) \hat{a}^{2n}$, so that they actually cancel and $q_\ell^{(o)}(\hat{a})$ is a polynomial of degree $n - 1$ in $\hat{a}^2$. Thus, $Q_\ell^{(o)}(\hat{a})$ is actually equal to $\hat{a}^{-4n-3}$ times a polynomial of degree $n - 1$ in $\hat{a}^2$.

By combining the above results, one obtains formula (6.16) for $G_\ell(\hat{a}^2) = (1 + \hat{a}^2)^2 F_\ell(\hat{a}^2)$, which is valid whatever the parity of ℓ .

APPENDIX B: ELECTRIC AND MAGNETIC POTENTIALS OF THE KERR HORIZON

In this appendix we give an alternative derivation of the electric and magnetic potentials E and B of a Kerr black hole horizon (Secs. VIC and VID), by solving for the linear partial differential equation (5.19) obeyed by the complex-valued linear combination $F \equiv E + iB$, namely,

$$\hat{D}^2 F = -2\psi^{-2}\Psi_2 - 1, \quad (\text{B1})$$

where $\hat{D}^2$ stands for the Laplace operator associated with the “canonical” unit round metric $\hat{q}_{ab}$. From Eqs. (6.2) and (6.32) we note that the conformal factor ψ and the Weyl curvature scalar Ψ_2 only depend on the Kerr polar coordinate θ . We thus look for a solution of the form $F(\theta)$. Actually, the polar coordinate adapted to $\hat{q}_{ab}$ is ϑ [Eq. (5.2)], and not θ . Hence,

$$\hat{D}^2 F = \frac{1}{\sin \vartheta} \frac{d}{d\vartheta} \left(\sin \vartheta \frac{dF}{d\vartheta} \right) = \frac{d}{dz} \left((1 - z^2) \frac{dF}{dz} \right), \quad (\text{B2})$$

where $z = \cos \vartheta$ is related to $\zeta = \cos \theta$ via Eq. (6.30). Then, using expression (6.2) for Ψ_2 , the relation $R^{-2}\psi^{-2} = d\zeta/dz$, which follows from the system (6.28), and $R^2 = r_+^2 + a^2$, Eq. (B1) becomes

$$\frac{d}{dz} \left((1 - z^2) \frac{dF}{dz} \right) = \frac{C}{(1 - i\hat{a}\zeta)^3} \frac{d\zeta}{dz} - 1, \quad (\text{B3})$$

where $C \equiv 2MR^2/r_+^3 = (1 + \hat{a}^2)^2$ is a dimensionless constant. Remarkably, this differential equation can be integrated to yield

$$\begin{aligned} (1 - z^2)F'(z) &= C \int^\zeta \frac{d\zeta'}{(1 - i\hat{a}\zeta')^3} - \int^z dz' \\ &= \frac{C}{2i\hat{a}} \frac{1}{(1 - i\hat{a}\zeta)^2} - z + z_0, \end{aligned} \quad (\text{B4})$$

where z_0 is a complex-valued constant of integration. Now by using Eq. (6.29) we readily get the following relationship between the differentials dF , $d\zeta$ and dz :

$$dF = \frac{C}{2i\hat{a}} \frac{1}{(1 - i\hat{a}\zeta)^2} \left(\frac{1}{1 - \zeta^2} - \beta^2 \right) d\zeta - \frac{z - z_0}{1 - z^2} dz. \quad (\text{B5})$$

Performing decompositions of the integrands in the right-hand side into simple elements and integrating, while using the identities $\beta^2 = \hat{a}^2/(1 + \hat{a}^2)$ and $C = (1 + \hat{a}^2)^2$, then gives the general solution for F in terms of ζ and $z(\zeta)$, in closed form,

$$\begin{aligned} F &= \ln(1 - i\hat{a}\zeta) - \frac{(1 + i\hat{a})^2}{4i\hat{a}} \ln|1 - \zeta| + \frac{(1 - i\hat{a})^2}{4i\hat{a}} \ln|1 + \zeta| \\ &\quad + \frac{1}{2} \ln(1 - z^2) + \frac{z_0}{2} \ln \left| \frac{1 - z}{1 + z} \right| + F_0, \end{aligned} \quad (\text{B6})$$

where F_0 is a second complex-valued constant of integration. This expression contains terms that diverge as $\zeta \rightarrow \pm 1$. In those limits, we have the asymptotic behaviors [recall Eq. (6.30)],

$$z(\zeta) = \begin{cases} 1 + e^{2\beta^2}(\zeta - 1) + O[(\zeta - 1)^2] & \text{if } \zeta \rightarrow +1 \\ -1 + e^{2\beta^2}(\zeta + 1) + O[(\zeta + 1)^2] & \text{if } \zeta \rightarrow -1 \end{cases}. \quad (\text{B7})$$

By requiring that the coefficients of the diverging terms vanish in those two limits, i.e., by imposing global regularity of the general solution (B6), we obtain

$$z_0 = \frac{1 - \hat{a}^2}{2i\hat{a}} \in i\mathbb{R}. \quad (\text{B8})$$

Finally, by substituting for this value of z_0 into the expression (B6), while using the identities $(1 - z)/(1 - \zeta) \times (1 + \zeta)/(1 + z) = e^{2\beta^2\zeta}$ and $\beta^2 = \hat{a}^2/(1 + \hat{a}^2)$, we find the remarkably compact expression²²

$$F = \ln(1 - i\hat{a}\zeta) + \frac{1}{2} \ln \left(\frac{1 - z^2}{1 - \zeta^2} \right) - \frac{i\hat{a}\zeta}{2} \frac{1 - \hat{a}^2}{1 + \hat{a}^2} + F_0. \quad (\text{B9})$$

The logarithms therein suggest to exponentiate Eq. (B9). According to the definition (5.20), the electric potential E is itself the logarithm of the rescaled conformal factor $R\psi > 0$. We thus have $e^F = (R\psi)e^{iB}$, which implies (with $\zeta = \cos \theta$)

$$R\psi = |e^F| = |e^{F_0}| \frac{\sqrt{(1 + \hat{a}^2 \cos^2 \theta)(1 - z^2(\theta))}}{\sin \theta}, \quad (\text{B10a})$$

$$B = \arg(e^F) = -\arctan(\hat{a} \cos \theta) - \frac{1 - \hat{a}^2}{2(1 + \hat{a}^2)} \hat{a} \cos \theta + B_0, \quad (\text{B10b})$$

where $B_0 \equiv \text{Im } F_0$ is a physically irrelevant constant since B is a (pseudoscalar) potential. These expressions are regular throughout $\theta \in (0, \pi)$, and are in perfect agreement with the results (6.31) [with $|e^{F_0}| = (1 + \hat{a}^2)^{-1/2}$] and (6.40) established in Secs. VIC and VID.

²²Note that the second term in Eq. (B9) can also be written explicitly in terms of ζ , according to

$$\frac{1}{2} \ln \left(\frac{1 - z^2}{1 - \zeta^2} \right) = -\ln[(1 - \zeta)e^{\beta^2\zeta} + (1 + \zeta)e^{-\beta^2\zeta}] + \ln 2.$$

APPENDIX C: ASYMPTOTIC BEHAVIOR IN THE SMALL-SPIN REGIME

In this appendix we establish that the horizon multipoles (6.42) of a Kerr black hole have the asymptotic behavior (6.45)–(6.46) in the small-spin regime. Let J_ℓ denote the definite integral appearing in Eq. (6.42), i.e.,

$$J_\ell(\hat{a}) \equiv \int_{-1}^1 dx \frac{P_\ell(z(x;\beta))}{(1-i\hat{a}x)^3}, \quad (\text{C1})$$

with $\hat{a} = a/r_+$ a given parameter in the range $[0, 1]$ and $\beta = \hat{a}/\sqrt{1+\hat{a}^2}$. Recall that $P_\ell(z)$ is the Legendre polynomial of order ℓ and the function $z(x;\beta)$ is given by formula (6.30).²³ We thus wish to prove that, in the regime $\hat{a} \ll 1$ of small spin values,

$$J_\ell \sim 2\alpha_\ell (i\hat{a})^\ell. \quad (\text{C2})$$

We shall first establish the scaling with spin, before determining in closed form the numerical prefactor α_ℓ .

1. Scaling with spin

Define $\epsilon(x;\beta) \equiv z(x;\beta) - x = O(\beta^2)$ and perform a Taylor series expansion of $P_\ell(z(x;\beta))$ about $z = x$. Then the integral (C1) becomes

$$J_\ell = \sum_{n=0}^{+\infty} J_{\ell,n} \quad \text{with} \quad J_{\ell,n} = \frac{1}{n!} \int_{-1}^1 dx \frac{\epsilon^n(x;\beta) P_\ell^{(n)}(x)}{(1-i\hat{a}x)^3}. \quad (\text{C3})$$

Next, using the fact that $\tanh'(x) = 1 - \tanh^2(x)$, we may perform a Taylor series expansion of $\epsilon(x;\beta)$ itself near $\beta = 0$, at fixed x , according to

$$\begin{aligned} \epsilon(x;\beta) &= \sum_{k=1}^{+\infty} \frac{(-\beta^2 x)^k}{k!} \tanh^{(k)}(\text{artanh} x) \\ &= \beta^2 (x^2 - 1) x \sum_{k=0}^{+\infty} \frac{\beta^{2k} (-x)^k}{(k+1)!} Q_k(x), \end{aligned} \quad (\text{C4})$$

with $Q_k(x)$ a polynomial of degree k that obeys the recurrence formula $Q_{k+1} = (1-x^2)Q'_k - 2xQ_k$. Its coefficients involve the (even) Bernoulli numbers and can be computed explicitly if necessary. Substituting expression (C4) into (C3) yields

$$J_{\ell,n} = \frac{\beta^{2n}}{n!} \int_{-1}^1 dx f_n(x) P_\ell^{(n)}(x), \quad (\text{C5})$$

with

²³In this appendix we make explicit the dependence of that function on the deformation parameter β and we denote ζ by x .

$$\begin{aligned} f_n(x) &= (x^2 - 1)^n x^n \left[\sum_{k=0}^{+\infty} \frac{\beta^{2k} (-x)^k}{(k+1)!} Q_k(x) \right]^n \\ &\times \left[\sum_{p=0}^{+\infty} \frac{(p+1)(p+2)}{2} (i\hat{a}x)^p \right]. \end{aligned} \quad (\text{C6})$$

For later use, it is convenient to rewrite (C6) in the following canonical form, which makes explicit both the minimal power of x and the associated scaling with $\hat{a}$ and β :

$$f_n(x) = (x^2 - 1)^n \sum_{m=0}^{+\infty} (i\hat{a})^m A_{n,m}(\beta^2) x^{n+m}, \quad (\text{C7})$$

where the coefficients $A_{n,m}$ are analytic in β^2 .

We may now integrate by parts n times the integral (C5), while noticing that the boundary contributions at $x = \pm 1$ vanish thanks to the factor $(x^2 - 1)^n$ in (C7). This gives

$$J_{\ell,n} = (-)^n \frac{\beta^{2n}}{n!} \int_{-1}^1 dx f_n^{(n)}(x) P_\ell(x). \quad (\text{C8})$$

We have thus reduced the problem of evaluating the integral (C1) to that of determining the projections over the Legendre polynomials of the (derivatives of the) functions (C7).

To extract the leading-order behavior of each $J_{\ell,n}$ as $\hat{a} \rightarrow 0$ (or equivalently as $\beta \rightarrow 0$), we now make use of the Rodrigues formula (A2) for the Legendre polynomials. By substituting for this formula into Eq. (C8) and integrating by parts ℓ times, we notice that the boundary contributions at $x = \pm 1$ vanish, once again, thanks to the factor $(x^2 - 1)^\ell$ in Eq. (A2). We thus obtain

$$J_{\ell,n} = (-)^{\ell+n} \frac{\beta^{2n}}{2^\ell \ell! n!} \int_{-1}^1 dx (x^2 - 1)^\ell f_n^{(\ell+n)}(x). \quad (\text{C9})$$

To evaluate the scaling of that integral as a function of (ℓ, n) in the regime where $\hat{a} \rightarrow 0$, we must control the derivative of order $\ell + n$ of the function (C7). Using the Leibniz rule, we readily find

$$\begin{aligned} f_n^{(\ell+n)}(x) &= \sum_{m=0}^{+\infty} (i\hat{a})^m A_{n,m}(\beta^2) \sum_{k=0}^{\ell+n} \binom{\ell+n}{k} \\ &\times \frac{d^k}{dx^k} [(x^2 - 1)^n] \frac{d^{n+\ell-k}}{dx^{n+\ell-k}} [x^{m+n}]. \end{aligned} \quad (\text{C10})$$

The derivative of order k of the polynomial $(x^2 - 1)^n$ vanishes if $k > 2n$, and the derivative of order $n + \ell - k$ of the monomial x^{m+n} vanishes if $\ell - k > m$. These conditions imply that the terms with $m < \ell - 2n$ do not contribute to (C10), so that whenever $2n \leq \ell$, Eq. (C9) with $\beta \sim \hat{a}$ implies the asymptotic behavior (coming from $m = \ell - 2n \geq 0$)

$$J_{\ell,n} \sim (-)^n \beta^{2n} c_{\ell,n} (i\hat{a})^{\ell-2n} \sim c_{\ell,n} (i\hat{a})^\ell, \quad (\text{C11})$$

for some real-valued coefficient $c_{\ell,n}$. If $2n > \ell$, however, then the series in Eq. (C10) does not truncate to a lower bound for m , but the factor of $\beta^{2n} \sim \hat{a}^{2n}$ in (C9) implies a subdominant contribution with respect to the leading-order scaling behavior (C11) of the integrals $J_{\ell,n}$ with $2n \leq \ell$. We thus conclude that the sum (C3) scales as $(i\hat{a})^\ell$, as claimed.

Notice that in the specific case where $n = 0$, the above analysis can be made more precise, and we readily find

$$\begin{aligned} J_{\ell,0} &= \frac{(-)^\ell}{2^\ell \ell!} \sum_{p=0}^{+\infty} \frac{(p+2)(p+1)}{2} (i\hat{a})^p \int_{-1}^1 dx (x^2 - 1)^\ell \frac{d^\ell x^p}{dx^\ell} \\ &\sim \frac{(\ell+2)!}{(2\ell+1)!!} (i\hat{a})^\ell. \end{aligned} \quad (\text{C12})$$

This coincides with the asymptotic behavior of the closed-form expression (A10) as $\hat{a} \rightarrow 0$.

$$\begin{aligned} (\beta_\ell)_{\ell \in \{0, \dots, 14\}} &= (1, 3, 12, 44, 176, 672, 2688, 10432, 41728, 163328, \\ &\quad 653312, 2570240, 10280960, 40583168, 162332672). \end{aligned} \quad (\text{C13})$$

Second, we observe a simple pattern relating even and odd members of the sequence (C13), namely $\beta_{2n} = 4\beta_{2n-1}$ for all $n \in \{1, \dots, 7\}$. Moreover, all even members of (β_ℓ) are divisible by 2^ℓ . We can thus write $\beta_{2n} = 2^{2n} \gamma_n$, and therefore $\beta_{2n-1} = 2^{2(n-1)} \gamma_n$, where

$$(\gamma_n)_{n \in \{0, \dots, 7\}} = (1, 3, 11, 42, 163, 638, 2510, 9908). \quad (\text{C14})$$

Third, we notice that this specific sequence of numbers matches perfectly the first eight elements of a particular sum of binomials, namely [76],

$$\gamma_n = \sum_{k=0}^n \binom{2n}{k} = \frac{2^{2n} + \binom{2n}{n}}{2}. \quad (\text{C15})$$

This leads us to infer the general formula (6.46) for α_ℓ . We then use this formula to *predict* the next few numerical values, e.g., for $\ell \in \{15, 16, 17\}$,

$$\alpha_{15} = \frac{642301952}{300540195}, \quad \alpha_{16} = \frac{1284603904}{583401555}, \quad \alpha_{17} = \frac{3080192}{1372525}. \quad (\text{C16})$$

As a check, one can repeat the Taylor-series expansion detailed above to determine J_{15} , J_{16} and J_{17} to leading order in $\hat{a}$; see notebook 3 in Appendix. D. The pattern (C2)

2. Numerical prefactor

Having established that $J_\ell \sim (i\hat{a})^\ell$ in the regime where $\hat{a} \ll 1$, we would like to determine in closed form the numerical prefactor, i.e., the coefficient α_ℓ in Eq. (C2). Given the intricate expression for the function $f_n(x)$ appearing in (C6), obtaining a closed-form formula for α_ℓ from the analysis above is challenging. Instead, we shall proceed by inductive reasoning as follows.

For a given $\ell \in \mathbb{N}$, we perform a Taylor-series expansion of the integrand in (C1) in powers of $\hat{a} \ll 1$ (e.g., by means of computer algebra; cf. notebook 3 in Appendix D), and proceed to compute the integral in closed form. Doing so, e.g., for all $\ell \in \{0, \dots, 14\}$, we find that (C2) holds, with the numerical values for α_ℓ listed in Table I above. Next, we look for patterns in that sequence of rational numbers and try to infer a general formula which would be valid for all $\ell \in \mathbb{N}$.

To do so, we first note that by multiplying $(\alpha_\ell)_{\ell \in \{0, \dots, 14\}}$ by the binomials $\binom{2\ell+1}{\ell}$, we obtain a sequence of *integers* $\beta_\ell \equiv \binom{2\ell+1}{\ell} \alpha_\ell$, namely,

is found to hold, with the corresponding numerical prefactors given by the values (C16), as predicted. Beware that the inductive reasoning that we followed here does not provide a rigorous (i.e., hypothetico-deductive) mathematical proof, but rather gives convincing arguments that the closed-form formula (6.46) is indeed correct for any $\ell \in \mathbb{N}$.

APPENDIX D: SAGEMATH NOTEBOOKS

Some exact and numerical computations in this article have been performed by means of the free Python-based mathematical software system SageMath [77]. The relevant Jupyter notebooks are publicly available, from the Zenodo repository [74] or directly from the links below. These notebooks have also been used to generate Figs. 4–16. Notebooks 1 and 2 are using SageMath differential geometry tools developed through the SageManifolds project [78,79], while notebook 3 is using SageMath's interface to FLINT [80] for arbitrary precision computations of complex-valued integrals with error bounds.

- (1) Unit round metric of a cross section of the Kerr horizon conformally related to the physical 2-metric: [81]
- (2) Hájíček 1-form and magnetic potential B on the Kerr horizon: [82]
- (3) Evaluation of the multipole moments of the Kerr horizon: [83]
- (4) Expression of the hypergeometric function $F_\ell(\hat{a}^2)$ (Appendix A) in terms of arctangent: [84]

- [1] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley, New York, 1998).
- [2] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic* (Cambridge University Press, Cambridge, England, 2014).
- [3] R. P. Geroch, Multipole moments. I. Flat space, *J. Math. Phys. (N.Y.)* **11**, 1955 (1970).
- [4] R. P. Geroch, Multipole moments. II. Curved space, *J. Math. Phys. (N.Y.)* **11**, 2580 (1970).
- [5] R. O. Hansen, Multipole moments of stationary space-times, *J. Math. Phys. (N.Y.)* **15**, 46 (1974).
- [6] K. S. Thorne, Multipole expansions of gravitational radiation, *Rev. Mod. Phys.* **52**, 299 (1980).
- [7] R. Beig and W. Simon, On the multipole expansion for stationary space-times, *Proc. R. Soc. A* **376**, 333 (1981).
- [8] W. Simon and R. Beig, The multipole structure of stationary space-times, *J. Math. Phys. (N.Y.)* **24**, 1163 (1983).
- [9] Y. Gürsel, Multipole moments for stationary systems: The equivalence of the Geroch-Hansen formulation and the Thorne formulation, *Gen. Relativ. Gravit.* **15**, 737 (1983).
- [10] G. Fodor, C. Hoenselaers, and Z. Perjés, Multipole moments of axisymmetric systems in relativity, *J. Math. Phys. (N.Y.)* **30**, 2252 (1989).
- [11] W. G. Dixon, A covariant multipole formalism for extended test bodies in general relativity, *Nuovo Cimento (1955–1965)* **34**, 317 (1964).
- [12] W. G. Dixon, Dynamics of extended bodies in general relativity I. Momentum and angular momentum, *Proc. R. Soc. A* **314**, 499 (1970).
- [13] W. G. Dixon, The definition of multipole moments for extended bodies, *Gen. Relativ. Gravit.* **4**, 199 (1973).
- [14] W. G. Dixon, Dynamics of extended bodies in general relativity III. Equations of motion, *Phil. Trans. R. Soc. A* **277**, 59 (1974).
- [15] W. Dixon, Extended bodies in general relativity: Their description and motion, in *Isolated Gravitating Systems in General Relativity*, Proceedings of the International School of Physics Enrico Fermi, Vol. 67, edited by J. Ehlers (North-Holland, Amsterdam, 1979), p. 156.
- [16] A. I. Harte, Mechanics of extended masses in general relativity, *Classical Quantum Gravity* **29**, 055012 (2012).
- [17] A. I. Harte, Motion in classical field theories and the foundations of the self-force problem, *Fund. Theor. Phys.* **179**, 327 (2015).
- [18] A. Ashtekar, J. Engle, T. Pawłowski, and C. Van Den Broeck, Multipole moments of isolated horizons, *Classical Quantum Gravity* **21**, 2549 (2004).
- [19] E. Schnetter, B. Krishnan, and F. Beyer, Introduction to dynamical horizons in numerical relativity, *Phys. Rev. D* **74**, 024028 (2006).
- [20] R. Owen, The final remnant of binary black hole mergers: Multipolar analysis, *Phys. Rev. D* **80**, 084012 (2009).
- [21] A. Ashtekar, M. Campiglia, and S. Shah, Dynamical black holes: Approach to the final state, *Phys. Rev. D* **88**, 064045 (2013).
- [22] A. Ashtekar, N. Khera, M. Kolanowski, and J. Lewandowski, Non-expanding horizons: Multipoles and the symmetry group, *J. High Energy Phys.* **01** (2022) 028.
- [23] N. Vasset, J. Novak, and J. L. Jaramillo, Excised black hole spacetimes: Quasilocal horizon formalism applied to the Kerr example, *Phys. Rev. D* **79**, 124010 (2009).
- [24] V. Prasad, Generalized source multipole moments of dynamical horizons in binary black hole mergers, [arXiv:2109.01193](https://arxiv.org/abs/2109.01193).
- [25] V. Prasad, A. Gupta, S. Bose, and B. Krishnan, Tidal deformation of dynamical horizons in binary black hole mergers, *Phys. Rev. D* **105**, 044019 (2022).
- [26] V. Prasad, Tidal deformation of dynamical horizons in binary black hole mergers and its imprint on gravitational radiation, *Phys. Rev. D* **109**, 044033 (2024).
- [27] A. Ribes Metidieri, B. Bonga, and B. Krishnan, Tidal deformations of slowly spinning isolated horizons, *Phys. Rev. D* **110**, 024069 (2024).
- [28] L. Rezzolla, R. P. Macedo, and J. L. Jaramillo, Understanding the “Antikick” in the Merger of Binary Black Holes, *Phys. Rev. Lett.* **104**, 221101 (2010).
- [29] J. L. Jaramillo, R. P. Macedo, P. Moesta, and L. Rezzolla, Black-hole horizons as probes of black-hole dynamics. I. Post-merger recoil in head-on collisions, *Phys. Rev. D* **85**, 084030 (2012), [arXiv:1108.0060](https://arxiv.org/abs/1108.0060) [gr-qc].
- [30] D. Pook-Kolb, O. Birnholtz, J. L. Jaramillo, B. Krishnan, and E. Schnetter, Horizons in a binary black hole merger II: Fluxes, multipole moments and stability, [arXiv:2006.03940](https://arxiv.org/abs/2006.03940).
- [31] P. Mourier, X. Jiménez Forteza, D. Pook-Kolb, B. Krishnan, and E. Schnetter, Quasinormal modes and their overtones at the common horizon in a binary black hole merger, *Phys. Rev. D* **103**, 044054 (2021).
- [32] A. Gupta, B. Krishnan, A. B. Nielsen, and E. Schnetter, Dynamics of marginally trapped surfaces in a binary black hole merger: Growth and approach to equilibrium, *Phys. Rev. D* **97**, 084028 (2018).
- [33] Y. Chen *et al.*, Multipole moments on the common horizon in a binary-black-hole simulation, *Phys. Rev. D* **106**, 124045 (2022).
- [34] A. Ribes Metidieri, B. Bonga, and B. Krishnan, Black hole tomography: Unveiling black hole horizon dynamics via ringdown observations, *Phys. Rev. D* **111**, 104075 (2025).
- [35] A. Ashtekar and B. Krishnan, Quasi-local black hole horizons: recent advances, *Living Rev. Relativity* **28**, 8 (2025).
- [36] L. Blanchet, Post-Newtonian theory for gravitational waves, *Living Rev. Relativity* **27**, 4 (2024).
- [37] P. T. Chruściel, J. L. Costa, and M. Heusler, Stationary black holes: Uniqueness and beyond, *Living Rev. Relativity* **15**, 7 (2012).
- [38] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [39] S. W. Hawking, Black holes in general relativity, *Commun. Math. Phys.* **25**, 152 (1972).
- [40] P. T. Chruściel and R. M. Wald, On the topology of stationary black holes, *Classical Quantum Gravity* **11**, L147 (1994).
- [41] T. Damour, Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs, Thèse de Doctorat d’État, Université Paris 6 (1979), available at <https://www.ihes.fr/damour/Articles/>.

- [42] T. Damour, Surface effects in black-hole physics, in *Proceedings of the Second Marcel Grossmann Meeting on General Relativity*, edited by R. Ruffini (North Holland, Amsterdam, 1982), p. 587.
- [43] E. Gourgoulhon and J. L. Jaramillo, A $3 + 1$ perspective on null hypersurfaces and isolated horizons, *Phys. Rep.* **423**, 159 (2006).
- [44] P. Hájíček, Exact models of charged black holes. I. Geometry of totally geodesic null hypersurface, *Commun. Math. Phys.* **34**, 37 (1973).
- [45] P. Hájíček, Can outside fields destroy black holes?, *J. Math. Phys. (N.Y.)* **15**, 1554 (1974).
- [46] A. Ashtekar, S. Fairhurst, and B. Krishnan, Isolated horizons: Hamiltonian evolution and the first law, *Phys. Rev. D* **62**, 104025 (2000).
- [47] A. Ashtekar, C. Beetle, and J. Lewandowski, Geometry of generic isolated horizons, *Classical Quantum Gravity* **19**, 1195 (2002).
- [48] P. Hájíček, Exact models of charged black holes: I. Geometry of totally geodesic null hypersurface, *Commun. Math. Phys.* **34**, 37 (1973).
- [49] P. Szekeres, The gravitational compass, *J. Math. Phys. (N.Y.)* **6**, 1387 (1965).
- [50] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).
- [51] A. Ashtekar and S. Bahrami, Asymptotics with a positive cosmological constant. IV. The no-incoming radiation condition, *Phys. Rev. D* **100**, 024042 (2019).
- [52] A. Ashtekar, C. Beetle, and J. Lewandowski, Mechanics of rotating isolated horizons, *Phys. Rev. D* **64**, 044016 (2001).
- [53] E. Gourgoulhon, Generalized Damour-Navier-Stokes equation applied to trapping horizons, *Phys. Rev. D* **72**, 104007 (2005).
- [54] S. A. Hayward, Angular momentum conservation for dynamical black holes, *Phys. Rev. D* **74**, 104013 (2006).
- [55] J. L. Jaramillo and E. Gourgoulhon, Mass and angular momentum in general relativity, in *Mass and Motion in General Relativity*, edited by L. Blanchet, A. Spallicci, and B. Whiting (Springer, Dordrecht, 2011), pp. 87–124.
- [56] A. Ashtekar and B. Krishnan, Isolated and dynamical horizons and their applications, *Living Rev. Relativity* **7**, 10 (2004).
- [57] M. Korzyński, Quasi-local angular momentum of non-symmetric isolated and dynamical horizons from the conformal decomposition of the metric, *Classical Quantum Gravity* **24**, 5935 (2007).
- [58] B. Chow, The Ricci flow on the 2-sphere, *J. Diff. Geom.* **33**, 325 (1991).
- [59] H.-P. Gittel, J. Jezierski, J. Kijowski, and S. Łeski, Rigid spheres in Riemannian spaces, *Classical Quantum Gravity* **30**, 175010 (2013).
- [60] L. Blanchet and T. Damour, Radiative gravitational fields in general relativity I. General structure of the field outside the source, *Phil. Trans. R. Soc. A* **320**, 379 (1986).
- [61] S. A. Teukolsky and W. H. Press, Perturbations of a rotating black hole. III. Interaction of the hole with gravitational and electromagnetic radiation, *Astrophys. J.* **193**, 443 (1974).
- [62] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).
- [63] R. H. Boyer and R. W. Lindquist, Maximal analytic extension of the Kerr metric, *J. Math. Phys. (N.Y.)* **8**, 265 (1967).
- [64] S. W. Hawking and J. B. Hartle, Energy and angular momentum flow into a black hole, *Commun. Math. Phys.* **27**, 283 (1972).
- [65] J. B. Hartle, Tidal shapes and shifts on rotating black holes, *Phys. Rev. D* **9**, 2749 (1974).
- [66] E. Poisson, Absorption of mass and angular momentum by a black hole: Time-domain formalisms for gravitational perturbations, and the small-hole or slow-motion approximation, *Phys. Rev. D* **70**, 084044 (2004).
- [67] W. Kinnersley, Type D vacuum metrics, *J. Math. Phys. (N.Y.)* **10**, 1195 (1969).
- [68] H. Fang and G. Lovelace, Tidal coupling of a Schwarzschild black hole and circularly orbiting moon, *Phys. Rev. D* **72**, 124016 (2005).
- [69] L. Smarr, Surface geometry of charged rotating black holes, *Phys. Rev. D* **7**, 289 (1973).
- [70] A. Le Tiec, M. Casals, and E. Franzin, Tidal Love numbers of Kerr black holes, *Phys. Rev. D* **103**, 084021 (2021).
- [71] A. Le Tiec and M. Casals, Spinning black holes fall in Love, *Phys. Rev. Lett.* **126**, 131102 (2021).
- [72] T. Damour and O. M. Lecian, Gravitational polarizability of black holes, *Phys. Rev. D* **80**, 044017 (2009).
- [73] P. Landry and E. Poisson, Relativistic theory of surficial Love numbers, *Phys. Rev. D* **89**, 124011 (2014).
- [74] <https://zenodo.org/records/18511585>.
- [75] DLMF, NIST Digital Library of Mathematical Functions, <https://dlmf.nist.gov>, Release 1.2.5 (2025), edited by F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain.
- [76] The On-Line Encyclopedia of Integer Sequences, published electronically at <https://oeis.org> (2026), sequence A032443.
- [77] The SageMath Developers, SageMath (2026), <https://www.sagemath.org>.
- [78] E. Gourgoulhon and M. Mancini, Symbolic tensor calculus on manifolds: A SageMath implementation, *Les cours du CIRM* **6**, 1 (2018).
- [79] The SageManifolds Developers, SageManifolds (2026), <https://sagemanifolds.obspm.fr>.
- [80] The FLINT team, FLINT: Fast Library for Number Theory (2025), <https://flintlib.org>.
- [81] https://nbviewer.org/url/zenodo.org/records/18511585/files/Kerr_cross_section.ipynb.
- [82] https://nbviewer.org/url/zenodo.org/records/18511585/files/Kerr_Hajicek_form_B.ipynb.
- [83] https://nbviewer.org/url/zenodo.org/records/18511585/files/Kerr_multipoles.ipynb.
- [84] https://nbviewer.org/url/zenodo.org/records/18511585/files/hypergeom_arctan.ipynb.