

Optimized quantum algorithms for simulating the Schwinger effect

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The Schwinger model, which describes lattice quantum electrodynamics in $1+1$ space-time dimensions, provides a valuable framework to investigate fundamental aspects of quantum field theory, and a stepping stone toward non-Abelian gauge theories. Specifically, it enables the study of physically relevant dynamical processes, such as the nonperturbative particle-antiparticle pair production, known as the Schwinger effect. In this work, we analyze the quantum computational resource requirements associated with simulating the Schwinger effect under two distinct scenarios: (1) a quench process, where the initial state is a simple product state of a non-interacting theory and then interactions are turned on at time $t = 0$, and (2) a splitting (or scattering) process where two Gaussian states, peaked at given initial momenta, are shot away from (or toward) each other. We explore different physical regimes in which the Schwinger effect is expected to be observable. These regimes are characterized by initial momenta and coupling strengths, as well as simulation parameters such as lattice size and electric-field cutoffs. Leveraging known rigorous bounds for electric-field cutoffs, we find that a reliable simulation of the Schwinger effect is provably possible at high cutoff scales. Furthermore, we provide optimized circuit implementations of both the second-order Trotter formula and an interaction-picture algorithm based on the Dyson series to implement the time evolution. Our detailed resource estimates show the regimes in which the interaction-picture approach outperforms the Trotter approach, and vice versa. The improved theoretical error bounds, optimized quantum circuit designs, and explicitly compiled subroutines developed in this study are broadly applicable to simulations of other lattice models in high-energy physics and beyond.

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I. INTRODUCTION

In Schwinger's seminal work on $(1+1)$ -dimensional quantum electrodynamics (QED) [1], known as the Schwinger model, it was shown that the vector boson could be massive, which is dubbed dynamical mass generation. Over the years, this model has been studied extensively with classical methods, both analytically [2–6] and numerically [7–17]. These studies have shown that it shares various properties with quantum chromodynamics (QCD), e.g. confinement and instantons.

Yet, nonperturbative behaviors, particularly dynamical behaviors, of the Schwinger model are hard to study using classical simulations, due to, e.g., the sign problem [18,19]. It is widely expected that quantum simulation will unlock dynamical studies of the Schwinger model, as evidenced by the numerous proposals for quantum simulation of this model [20–30]; some of them involve small-scale,

proof-of-concept simulations on error-prone, near-term quantum hardware, while a recent handful, such as Refs. [25–27,29,30], focus on algorithms with provable performance that could be implemented on a universal fault-tolerant quantum computer. Similar to these recent works, we devise algorithms to study the real-time dynamics of the Schwinger model and develop optimized fault-tolerant circuit implementations.

Quantum algorithms for real-time quantum dynamics have been studied since the late 1990s starting with the seminal work by Lloyd [31], which substantiated the proposal by Feynman [32]. This first method was based on a specific product formula called the Suzuki-Trotter formula [33]; product-formula algorithms have since been improved [34–36]. Alternative to this approach are methods based on various polynomial series approximations [37–43]. Each method has its own advantages and potential drawbacks.

While the best (time-independent) Hamiltonian simulation method achieves an additive gate complexity of $\mathcal{O}(\log(1/\epsilon))$ [40], quantum simulations based on a k th order product formula have a multiplicative gate complexity scaling as $\mathcal{O}((1/\epsilon)^{2k})$ [34]. With respect to simulation time t , both methods can, in principle, achieve linear scaling in complexity. For product formulas, however,

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achieving this requires a large k , which increases the complexity of the corresponding quantum circuit. This, in turn, motivates the use of a higher-order (rather than lower-order) product formula for large t or $1/\epsilon$. Other motivations include the fact that product formulas lend themselves more readily to additional physics-informed improvements, such as low-energy subspace [44] or symmetry-based [45] methods, and that error bounds based on the commutators of the Hamiltonian terms can be significantly lower than the worst-case analytical bounds [46,47].

From the family of series-approximation-based algorithms, another approach that exploits the physics of the problem is based on the Dyson series [42]. In particular, when a Hamiltonian is divided into a sum of two terms, such as $H = H_0 + V$ where $\|V\| \ll \|H_0\|$, H_0 is fast-forwardable [48,49], and the quantum simulation is performed in the interaction picture of H_0 ; it has gate complexity $\tilde{O}(\|Vt\|)$ rather than $\tilde{O}(\|Ht\|)$ (where $\tilde{O}$ is the complexity up to hidden logarithmic factors). In this work, we apply this interaction-picture-based quantum algorithm and the second-order Suzuki-Trotter formula to analyze the complexity of simulating the Schwinger effect.

Choosing the more gate-efficient algorithm for a given, finite problem instance, however, is hard because in addition to analyzing the asymptotic scaling, it also requires a detailed analysis of constant factors, which is typically accomplished by explicit compilations of the candidate algorithms. Reference [25] argued employing product formula rather than qubitization is likely to be advantageous due to the linear instead of quadratic scaling in terms of the electric field cutoff. Based on our analysis, we expect the electric field term to be much stronger than other terms when the Schwinger effect is to be observed. Therefore, we need a high electric field cutoff to accurately simulate the phenomenon [50].¹ Employing the interaction-picture-based algorithm, we can asymptotically achieve an exponential improvement,² i.e., a complexity scaling $O((\log \Lambda)^2)$, in terms of the electric field cutoff Λ .

Our main goal in this work is to detail this study. To this end, we provide optimized compilations of product-formula, which improves upon previous implementations in Refs. [25,26,29], and interaction-picture algorithms for various instances over a range of parameters, and compare their performances in terms of T-gate and qubit counts. The

considered range of parameters is determined such that the Schwinger effect, i.e., pair-production, can be reliably simulated. More specifically, if one simulates various instances of the Schwinger model over the range of parameters, an accurate simulation, in which the Schwinger effect are expected to be observed, is guaranteed. Furthermore, we optimize the cost of the interaction-picture algorithm mainly via improved and novel bounds on the discretization error of the Dyson series and improved compilation techniques, e.g., phase-gradient addition.

The manuscript is organized as follows. In Sec. II, we describe the Hamiltonian and the experiments to be simulated. We also define the conditions for observing the Schwinger effect arising from lattice approximations and accuracy considerations. Then, in Secs. III A and III B we summarize the quantum circuits and lay out their resource costs for product formula and interaction-picture based algorithms, respectively. In Sec. IV we give the detailed resource costs of each method. In Sec. IV C, in particular, we compare the asymptotic and numerical resource costs of two methods, and determine the method that has a lower resource cost for a wide range of parameter regimes. Finally, Sec. V concludes with a future outlook for improving the resource costs even further and extending the application of these methods to more complicated models in particle physics. Appendixes are reserved for deriving the final form of the lattice Hamiltonian that is used in Appendix A, and further technical and resource cost analysis of the two methods are in Appendixes B and C.

II. SCHWINGER MODEL: THE HAMILTONIAN, PARAMETERS, AND OBSERVABLES OF INTEREST

The Hamiltonian for the Schwinger model is given by [4]

$$H_{\text{nat}} = \frac{g}{2\sqrt{x}} \sum_{r=1}^N \left[(E_r + \alpha)^2 + (-1)^r \frac{2\sqrt{x}m}{g} \psi_r^\dagger \psi_r + x(\psi_r^\dagger U_r \psi_{r+1} + \text{H.c.}) \right], \quad (1)$$

in natural units ($\hbar = c = 1$). There are a few user specified parameters in this Hamiltonian: g , m , and α are respectively the bare coupling constant, bare mass, and the background electric field strength, independent of the specifications of the simulated lattice; a and N are the lattice spacing and the number of lattice sites, specified by the simulation lattice. We define the unitless quantities

$$x = \frac{1}{g^2 a^2} \quad \text{and} \quad \mu = \frac{2\sqrt{x}m}{g}, \quad (2)$$

which determine the strength of the mass and the kinetic term with respect to the electric field term. H_{nat} acts on fermionic (electron/positron) and spin (electric field)

¹Note that there are previous [51] and more recent [52] works that argue for lower field cutoffs. These works rely on energy based truncation and Hilbert space fragmentation arguments, respectively. While these heuristics may turn out to be approximately correct for certain observables, we base our assumption for the field cutoff on the rigorous bounds proved in Ref. [50].

²Complementary implementation-level resource estimates for the interaction-picture algorithms of Ref. [50] in higher-dimensional U(1), SU(2), and SU(3) lattice gauge theories can be found in Ref. [30].

degrees of freedom placed on the sites and links of the graph that defines the lattice. To be more precise, E_r and U_r are operators that act on the electric field degrees of freedom placed on the link labeled by index r that is placed right to the vertex labeled by r , and ψ_r and $\psi_r^\dagger$ are annihilation and creation operators on the fermion degree of freedom on the lattice site labeled by r . Below, we give definitions of these operators, after a slight massaging of the Hamiltonian. As is standard in numerical simulations, we render the Hamiltonian H and simulation time t both unitless, i.e.,

$$H = \sum_{r=1}^N [(E_r + \alpha)^2 + (-1)^r \mu \psi_r^\dagger \psi_r + x(\psi_r^\dagger U_r \psi_{r+1} + \text{H.c.})], \quad (3)$$

where going back to quantities with units is straightforward, i.e., $E^{\text{nat}} = \sqrt{\frac{g}{2\sqrt{x}}} E$ and $t^{\text{nat}} = \frac{2\sqrt{x}}{g} t$. For the sake of completeness, and for future sanity checks, we provide the units of scalars and operators:

$$[g] = [m] = [a]^{-1} = [E^{\text{nat}}]^2 = [\psi^{\text{nat}}]^2. \quad (4)$$

Note that, ultimately, the continuum limit needs to be taken, i.e., the limit $a \rightarrow 0$ or equivalently $N \rightarrow \infty$, which is done via extrapolation. As a is swept toward smaller values, x takes higher values and hence the model is driven toward the strong-coupling limit, and the physical result is obtained by extrapolating to $a \rightarrow 0$. More generally, the parameters, N , x , a , m/g , and the cutoff Λ of the electric field together with the constant external electric field α , are parameters that need to be chosen such that the simulation provides reliable results for the desired phenomena. During this process, the model is simulated under different parameter settings, and the more resource-efficient simulation algorithm may change depending on the parameter regime.

Note that the lattice spacing a does not appear directly in the Hamiltonian, yet appears in the definition of x . In fact, we can redefine the bare coupling constant g and the bare mass m such that they are rescaled by a , i.e., $\tilde{g} = ga$, $\tilde{m} = ma$. In this way, all quantities, including $\tilde{g}$ and $\tilde{m}$ are dimensionless, in addition to x . The Hamiltonian stays the same (for the sake of future convenience $\tilde{g}$ and $\tilde{m}$ are shown as g and m below) but with $x = 1/g^2$. Hence, for a set of end-to-end simulations, the parameters we need to specify are $N, x, m/g, \alpha, \Lambda$, the total evolution time t , and the parameters used to prepare the initial state (such as particle momenta, position, etc.; see Secs. II A and II B for details).

Following the naming convention made in Ref. [25], we partition the Hamiltonian into three terms as

$$H = H_E + H_M + H_I, \quad (5)$$

where

$$H_E = \sum_{r=1}^N (E_r + \alpha)^2, \quad (6)$$

$$H_M = \mu \sum_r (-1)^r \psi_r^\dagger \psi_r, \quad (7)$$

$$H_I = x \sum_r (U_r \psi_r^\dagger \psi_{r+1} - U_r^\dagger \psi_r \psi_{r+1}^\dagger) \quad (8)$$

are what we call the electric field, mass, and the interaction (or kinetic energy) term, respectively. A canonical choice of basis in the literature is as follows:

$$E_r = \sum_{\epsilon_r=-\Lambda}^{\Lambda-1} \epsilon_r |\epsilon_r\rangle \langle \epsilon_r|_r, \quad (9)$$

$$U_r = \sum_{\epsilon_r=-\Lambda}^{\Lambda-1} |\epsilon_r + 1 \bmod [-\Lambda, \Lambda - 1]\rangle \langle \epsilon_r|_r, \quad (10)$$

$$\psi_r^\dagger = \frac{X_r - iY_r}{2} \prod_{j=1}^{r-1} Z_j, \quad (11)$$

where a Jordan-Wigner transformation [53] is assumed to transform the fermionic operators. See Appendix A for details.

In what follows, we first determine bounds on α , Λ , N such that the Schwinger effect is expected to be observable. Then, we set up two experiments: a quench experiment and a scattering experiment, both of which aim to probe the dynamics of pair production. Next, we determine how to choose the lattice parameters so that spurious lattice effects are manageable. Finally, we obtain the constraints over the remaining simulation parameters such that the simulation is expected to describe the continuum accurately.

A. Conditions for observing the Schwinger effect

The Schwinger effect refers to the phenomenon of electron-positron pair creation. For this to be observable, there needs to be enough energy exchange between the electric field term and the mass term, i.e., the energy that can be stored in the electric field should be high enough to create many particles when that energy is transferred to mass. This corresponds to the following condition:

$$(\Lambda + \alpha)^2 - \alpha^2 \gg \mu, \quad (12)$$

where the left hand side is the maximum electric field energy that can be stored in one link on the lattice, whereas the right hand side is the minimum energy of an electron-positron pair that are only one link apart from each other on

the lattice (hence they are at the same position up to a lattice spacing a). This is equivalent to

$$\Lambda(\Lambda + 2\alpha) \gg \mu. \quad (13)$$

For our experiments, we can carry the effect of nonzero α to the electric field values in the initial state. Then, this is possible if Λ is chosen such that

$$\Lambda \gg \sqrt{\mu}. \quad (14)$$

Hence, we sweep through the parameter Λ in the interval given as

$$[100\mu \geq \Lambda^2 \geq 0.01\mu]. \quad (15)$$

This determines the range of Λ that we are interested in for the simulation of the Schwinger model. We remark that smaller values ($< \sqrt{\mu}$) of Λ may still give important information in order to determine the value of Λ at which the Schwinger effect starts to emerge.

We propose two experiments, where we probe the number of electron-positron pairs as a function of time. In both of these experiments the final observable and the time-evolution are the same, and they differ only in terms of the initial states. In the first experiment, we start from an initial state that contains no particles but nonzero electric field values. In the second experiment, we imagine a dressed electron-positron pair ($e^- - e^+$ pair) is created with particles distance l apart, and initiated as moving away from each other with momentum p_1 and p_2 , respectively (as depicted in Fig. 1). In both cases, we let the time evolve with the full Hamiltonian and monitor the dynamics of the particle density.

The first experiment: In the first experiment, we start with the following initial state:

$$|\psi(t=0)\rangle = |0\rangle_0 |\gamma\rangle_{0,1} |1\rangle_1 |\gamma\rangle_{1,2} |0\rangle_2 |\gamma\rangle_{2,3} |1\rangle_3 \\ |\gamma\rangle_{3,4} |0\rangle_4 \dots |1\rangle_{N-2} |\gamma\rangle_{N-2,N-1} |0\rangle_{N-1}, \quad (16)$$

where γ is a fixed electric field value that is uniform in the whole system, and a subscript $(\cdot)_r$ denotes the site r while a subscript $(\cdot)_{r,r+1}$ denotes the link that connects the sites r and $r+1$. By definition, $|\gamma| < \Lambda$. Note that from now on, we will also assume that the background field $\alpha = 0$. The

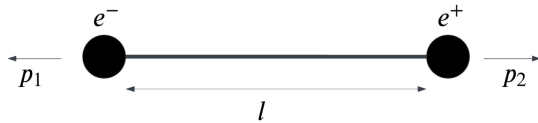


FIG. 1. Schematic representation of the initial state for the second experiment: an electron-positron pair ($e^- - e^+$) is created that are l apart, and moving away from each other with momentum p_1 and p_2 , respectively.

value of γ can be connected to the maximum particle density that is kinematically possible, which can be at most 1.

We wish to be able to have an initial state energy that can capture a pair production of about average (over the whole system) particle density, say $\rho < 1$. This would imply a choice of γ such that $\gamma^2 \geq \rho\mu$. Furthermore, by definition, γ can be at most as big as the cutoff Λ and hence we obtain

$$\rho \frac{\sqrt{xm}}{g} \leq \gamma^2 < \Lambda^2. \quad (17)$$

Note that this is only a rough estimate; we still sweep through the values of x and $\frac{m}{g}$ as independent parameters.

Then, we evolve the system with the full Hamiltonian for time t by applying the operator e^{-iHt} to the initial state $|\psi(t=0)\rangle$. The minimum time we want to carry out this time evolution is $t_{\min} = \rho/x$, which is the average minimum time (i.e., fastest possible) that $N\rho$ many ($e^- - e^+$) pairs can be generated. This is obtained by considering the time evolution with the term H_I . In the small time limit and when this term is perturbative it can be argued, similar to Fermi's golden rule, that the rate at which particles are generated is proportional to the transition amplitude of H_I between initial and intermediate states which generate a particle-antiparticle pair while decreasing the electric field. This overlap is given by the strength of the Hamiltonian term, x . Ideally, we observe the system for times $t \gg t_{\min}$, and settle with a maximum evolution time $t_{\max}$ when the observable profile seemingly starts to repeat itself in time. Note that this quantity, by itself, is an interesting observable and worth measuring as a result of the quantum simulation. To be more precise, we run the time-evolution for times

$$\frac{\rho}{x} = t_{\min} < t < t_{\max}, \quad (18)$$

where $t_{\max}$ is determined empirically as a result of the experiment. Furthermore, we need to choose a system large enough, i.e., N high enough, so that the boundary effects are negligible. This can be studied rigorously using Lieb-Robinson bounds, such as given in Refs. [54,55] (see Lemma 5). These results imply that for any observable (such as a defect, or any kind of operator that was not supposed to be there) on the boundary to affect considerably any other observable that is of distance l from the boundary, a certain time needs to pass. In other words, for a given time evolution t , we can bound a sufficient minimum system size, above which the boundary effects are only visible up to an error ϵ .

Following the notation from Lemma 5 of Ref. [55], where ζ_0 sums over all $H_I(Z)$ in H_I which acts nontrivially on lattice site r ,

$$\zeta_0 = \max_r \sum_{Z:r \in Z} \|H_I(Z)\| = 4|x|, \quad (19)$$

and given the extent of the initial state N_0 , we want to solve for l such that

$$N_0(8x|t|)^l/l! \leq \epsilon. \quad (20)$$

This results in

$$l \geq \max\{\ln(N_0/\epsilon), e8|x||t|\}. \quad (21)$$

Then for all

$$N \geq N_0 + 2l, \quad (22)$$

the simulation on N sites is accurate up to ϵ error. N_0 is usually given as the extent of a few well-separated quasiparticles that the system is expected to support. This can either be estimated via classical simulations or be calculated with various quantum simulations where N_0 is swept through a range of values. The first observable we measure is the particle density given as

$$O_{\text{dens}} := \frac{1}{N} \sum_{r=0}^{N-1} \frac{1}{2} (1 - (-1)^r \sigma_Z(r)). \quad (23)$$

This observable, measured at a given time, reveals the number of particles in the system, whether the time average of this observable reaches an equilibrium, and if so, the time in which it does. These observables are important to reveal the onset of *thermalization* and other properties of the system.

The second observable measures the *global* electric field polarization of the half-system, which is given as

$$O_{\text{pol}} := E_{[N/2]} - E_0, \quad (24)$$

where the subindices 0 and $[N/2]$ refer to the first and middle links on the lattice. This observable and its higher moments are interesting because they measure how many electrons versus positrons are on one half of the whole chain. This can be easily seen from Gauss' law, i.e., $(E_r - E_{r-1} - \rho_r)|\psi\rangle$ for any physical, i.e., *gauge-invariant* state $|\psi\rangle$ where

$$\rho_r = \frac{1 - (-1)^r}{2} - \psi_r^\dagger \psi_r \quad (25)$$

is the electric charge on site r .³ It is straightforward to see that the initial state fulfills Gauss' law and thus is

³For even sites, an occupied electron state gives rise to $\rho_r|1\rangle_r = +1|1\rangle_r$, for odd sites an occupied positron state gives rise to $\rho_r|0\rangle_r = -1|1\rangle_r$, and for unoccupied state at even/odd sites ρ_r results in 0 eigenvalue.

gauge-invariant. Note that by construction of the initial state and the Hamiltonian, the number of electrons is always equal to the number of positrons across the whole system. However, the positions of the electrons and positrons can fluctuate among the chain, which could lead to states with asymmetric electron/positron occupations with respect to the middle of the chain. This could be probed by $\langle O_{\text{pol}} \rangle$ or higher moments $\langle O_{\text{pol}}^n \rangle$ for $n \geq 2$, and gives information about the disparity of electron-positron distribution across the middle of the chain.

Note that we can also check the *k-local* field polarization at any region centered at k_0 on the lattice for size k ($k_0 > k/2$), which is given by

$$O_{\text{pol-k}} := E_{k_0 + \frac{k}{2}} - E_{k_0 - \frac{k}{2}}. \quad (26)$$

This would give information about the electron-positron disparity within *k-local* regions of the spatial lattice around the lattice point k_0 .

The second experiment: In the second experiment, we start from the following initial state:

$$|\psi(t=0)\rangle = O_{p_0} |\psi_{\text{gr}}\rangle, \quad (27)$$

where O_{p_0} is a unitary operator that initializes two gauge-invariant, quasiparticle wavepackets (a *real* positron-electron pair) moving away from each other with momenta p_0 , where $|\psi_{\text{gr}}\rangle$ is the ground state at the specified parameters of the given Hamiltonian. Note that we do not know, *a priori*, what this operator looks like, but we rely on classical methods such as matrix product states [12,15,56–61] and we assume an efficient quantum circuit that can prepare this state [61–63]. The choice of p_0 can be made depending on ρ , the particle-antiparticle density that we are hoping to produce within the experiment. Considering relativistic particles obeying the Dirac equation, as momentum is proportional to the energy, $p_0 \sim \rho \mu N$ which corresponds to the mass of ρN many particle-antiparticle pairs. Note that we should be careful that p_0 does not exceed a certain small fraction of N , because the lattice is only a good approximation of the continuum for $p \ll \pi/a \sim \pi N$ when the simulation cell is chosen to be $V \sim \mathcal{O}(1)$. The minimum time t_{min} for which we want to carry out the time evolution is determined by the minimum time it takes for all energy in the initial momenta to be transferred into the electric field and additional particle-antiparticle pairs. Using similar arguments to the first experiment, we can argue that the third term gives the rate of creating particle-antiparticle pairs, i.e., x . Hence we pick $t_{\text{min}} = \rho/x$. We then run the time-evolution for times

$$\frac{\rho}{x} = t_{\text{min}} < t. \quad (28)$$

The observables are the same as those in the first experiment; i.e., we measure the particle density and the global electric field polarization of the half-system.

We remark that more involved measurements following metrology could be done, such as those studied in Refs. [64–66], if efficiently employed within the overall quantum algorithm.

B. Conditions for accurate field truncation

The previous sections give a bound on the initial cutoff Λ_0 . However, as time evolution takes place, the support of the quantum state might leak out of the subspace defined as the image of the projector $\Pi_{[-\Lambda_0, \Lambda_0]}$, which projects onto the electric field values between $[-\Lambda_0, \Lambda_0]$ at each link. Ref. [50] demonstrates how one can bound the leakage outside of the subspace, and proves rigorous expressions for choosing sufficiently high electric field cutoffs $\Lambda = \Lambda_0 + \Delta$. Δ is chosen minimally such that the field cutoff error is bounded by ϵ_{cutoff} .

Following Ref. [50], we apply the long-time leakage bound (Theorem 3) to the Schwinger Hamiltonian. Our electric field term belongs to the $r = 0$ case, and we compute the quantity χ which appears in the norm of H_I^ν acting on ν th site, in the subspace $\Pi_{[-\Lambda_0, \Lambda_0]}$, i.e.,

$$\|H_I^{(\nu)} \Pi_{[-\Lambda, \Lambda]}\| \leq \chi(\Lambda + 1)^r \stackrel{(r=0)}{=} \chi. \quad (29)$$

In more detail,

$$\begin{aligned} \|H_I^{(\nu)} \Pi_{[-\Lambda, \Lambda]}\| &\leq \sum_{\nu} \|x(U_{\nu} \sigma_{\nu}^+ \sigma_{\nu+1}^- + U_{\nu}^{\dagger} \sigma_{\nu}^- \sigma_{\nu+1}^+)\| \\ &= \frac{x}{4} \sum_{\nu} \|(U_{\nu} + U_{\nu}^{\dagger})(X_{\nu} X_{\nu+1} + Y_{\nu} Y_{\nu+1})\| \end{aligned} \quad (30)$$

$$+i(U_{\nu} - U_{\nu}^{\dagger})(X_{\nu} Y_{\nu+1} - Y_{\nu} X_{\nu+1})\| \quad (31)$$

$$\leq 2x, \quad (32)$$

which implies that $\chi = 2x$. Using this together with Theorem 3 of Ref. [50], we can now bound the leakage of the electric field to higher values as a result of the full time evolution as follows:

$$\|\bar{\Pi}_{[-\Lambda(t), \Lambda(t)]} e^{-itH} \Pi_{[-\Lambda_0, \Lambda_0]}\| \leq \frac{[4xt]}{2^{\Delta-1} \Delta!} \quad (33)$$

$$< \frac{2[4xt]}{\sqrt{2\pi\Delta}} \left(\frac{e}{2\Delta}\right)^{\Delta}. \quad (34)$$

Assuming that $\Delta > e$, we have:

$$\|\bar{\Pi}_{[-\Lambda(t), \Lambda(t)]} e^{-itH} \Pi_{[-\Lambda_0, \Lambda_0]}\| \leq \frac{2[4xt]}{\sqrt{2\pi e}} \left(\frac{1}{2}\right)^{\Delta}. \quad (35)$$

To ensure that the above expression is smaller than ϵ_{cutoff} , we require

$$\Delta \geq \max\left(3, \left\lceil \log_2\left(2[4xt]/\epsilon_{\text{cutoff}}\sqrt{2\pi e}\right) \right\rceil\right). \quad (36)$$

This implies that for all

$$\Lambda(t) = \Lambda_0 + [2\chi t](\Delta - 1) = \Lambda_0 + [4xt](\Delta - 1) \quad (37)$$

$$\geq \Lambda_0 + [4xt] \max\left(2, \left\lceil \log_2\left(\frac{2[4xt]}{\epsilon_{\text{cutoff}}\sqrt{2\pi e}}\right) \right\rceil - 1\right) \quad (38)$$

the simulation of the truncated Hamiltonian is, within an ϵ_{cutoff} approximation, faithful to the original, infinite-dimensional Hamiltonian.

TABLE I. Parameter choices for the quantum simulation. The free parameters that determine the interaction strengths in the model, namely x and $\mu = 2\sqrt{x}m/g$, are swept from 0.1 up to 10. The parameter ρ , the density of electron-positron pairs expected to be produced at any time during the simulation, is kept free and implicitly depends on the evolution time t and the properties of the initial state. The parameter ϵ denotes the target precision. The parameter N_0 is the spatial extent of the initial state and depends on the type of experiment; it can be estimated as the extent of the support of a few quasi-particles. All other parameters, which are swept depending on x and μ , can be inferred from the above table. Notice that the provided expression for N acts as a lower bound for reliable simulation. In practice, we often take the number of links ($N - 1$ for nonperiodic boundary conditions and N for periodic boundary conditions) to be the next power of two for ease of implementation.

Symbol	Meaning	Expression
Λ_0	Electric field cutoff at time $t = 0$	$100\mu \geq \Lambda_0^2 \geq 0.01\mu$
$\Lambda(t)$	Electric field cutoff at time t	$\Lambda(t) \geq \Lambda_0 + [4xt] \max\left(2, \left\lceil \log_2\left(\frac{[4xt]}{\epsilon_{\text{cutoff}}\sqrt{2\pi e}}\right) \right\rceil - 1\right)$
γ	Initial electric field value in the 1st experiment	$\Lambda_0^2 > \gamma^2 \geq \rho\mu$
$ p_0 $	Initial momenta of wave packets in the 2nd experiment	$(\pi N \gg) p_0 \geq 0.1\rho\mu N$
$t_{\min}$	Minimum time for observing Schwinger effect	ρ/x
N	Number of sites on the lattice	$N_0 + 2 \max\{\ln(N_0/\epsilon), e8 x t \}$

In Table I, we provide the set of parameters that should be simulated to induce the Schwinger effect, and potentially access different phases of the Schwinger model. Note that the model has two essential parameters: the ratio $\mu = 2\sqrt{x}m/g$ and the interaction term $x(= 1/g^2)$, appearing as in Eq. (3). The simulation can be performed for the set of parameters that are swept through a range of values, such as $0.1 \leq \mu \leq 10$ and $0.1 \leq x \leq 10$, with increments of 0.1. One also makes a choice for a density $\rho \leq 1$ that corresponds to electron-positron pairs that can exist in the quantum state at any point during the simulation. Then, given a particular instance of μ , x , ρ , and target accuracy ϵ , other parameters, i.e., the cutoff Λ , the minimum time evolution $t_{\min}$, and the minimum system size required N , are deduced in terms of the interaction strengths μ and x , and the time evolution t . We assume that these parameters are already chosen high enough such that the error of making them finite is negligible. In fact, this is reflected in the choice of Λ explicitly by using the results of Ref. [50].

III. METHODS

The two quantum simulation methods considered in this work are the second-order Suzuki–Trotter product formula (PF₂) and interaction-picture (IP)-based algorithms. On the one hand, the PF₂ method allows one to exploit the Hamiltonian’s structure through the commutators of its individual terms, which has been shown to be particularly advantageous for local Hamiltonians [35,51]. On the other hand, the performance of the IP-based method scales linearly with the strength of the interaction term. Depending on the specific Hamiltonian, the scaling with respect to its parameters can favor either method.

There are two key differences between the methods that depend on the parameters t and $1/\epsilon$. Asymptotically, the IP-based method scales as $\tilde{\mathcal{O}}(t \log_2(t/\epsilon))$, whereas the second-order Suzuki–Trotter method scales as $\mathcal{O}(t^{3/2}/\epsilon^{1/2})$. Thus, for large t and small ϵ , the IP-based method is expected to outperform the PF₂ method. However, the overall picture remains nuanced once all parameters are considered. We examine the corresponding resource costs in detail in later sections.

In this section, we provide an overview of these two methods.

A. Quantum simulation of the Schwinger model with the Suzuki-Trotter method

The first fault-tolerant quantum algorithm for simulating the Schwinger model [25] is based on the second-order Suzuki-Trotter formula [33]. Subsequently, the second-order Suzuki-Trotter formula was applied to simulate higher-dimensional lattice QED [29] and non-Abelian lattice gauge theories including QCD [26]. In what follows, we briefly review the second-order Suzuki-Trotter formula, and then we compare the algorithms in [25,29].

The symmetric, second-order Suzuki-Trotter formula [33] admits an ordered decomposition of a given Hamiltonian H : $H = \sum_{l=1}^K H_l$, where every summand H_l is a Hermitian operator, and approximates the time-evolution operator according to

$$e^{-iHt} \approx S(t) \equiv U(t/r)^r \quad (39)$$

$$\equiv \left[\left(\prod_{l=1}^{K-1} e^{-iH_l \frac{t}{2r}} \right) e^{-iH_K \frac{t}{r}} \left(\prod_{l=K-1}^1 e^{-iH_l \frac{t}{2r}} \right) \right]^r, \quad (40)$$

where $\prod_{l=1}^N A_l = A_1 A_2 \dots A_N$, $\prod_{l=N}^1 A_l = A_N A_{N-1} \dots A_1$, and r is the number of Trotter steps. The approximation error ϵ , typically known as Trotter error, is given by [35]

$$\epsilon = \|e^{-iHt} - S(t)\| \quad (41)$$

$$\leq \frac{t^3}{12r^2} \sum_{i=1}^K \left\| \left[H_i, \sum_{j=i+1}^K H_j \right], \sum_{k=i+1}^K H_k \right\| + \frac{t^3}{24r^2} \sum_{i=1}^K \left\| \left[H_i, \sum_{j=i+1}^K H_j \right], H_i \right\| \equiv \frac{t^3 \rho}{r^2}, \quad (42)$$

where $\|\cdot\|$ is the spectral norm. Then, to reach a target accuracy ϵ , we choose the number of Trotter steps:

$$r = \left\lceil \sqrt{\frac{\rho t^3}{\epsilon}} \right\rceil. \quad (43)$$

In practice, we implement the operation $W(t, \epsilon_{\text{rot}})$, which is equivalent to $S(t)$ up to rotation synthesis error ϵ_{rot} . Furthermore, the gate cost of a simulation based on the Suzuki-Trotter formula is roughly the gate cost of $U(t)$ multiplied by r .

Note that the ordered decompositions adopted in [25,29] are different. We choose to use the one in [29] because it incurs lower gate costs per Trotter step and a smaller Trotter error. Specifically, we implement the commuting $e^{-iH_E t}$ and $e^{-iH_M t}$ first, then we divide H_I into four parts, i.e., $H_I = H_{1,e} + H_{1,o} + H_{2,e} + H_{2,o}$, before implementing their evolutions sequentially. The detailed resource cost of the resulting subroutines are explained Sec. IV A and given in Table II.

B. Quantum simulation of the Schwinger model with the interaction picture method

A Hamiltonian simulation method based on the interaction picture has been established in Ref. [42]. In this section, we give a brief overview of the algorithm with a more detailed account given in Appendix B and its detailed implementation in Appendix B 1.

Given $H = H_0 + V$, the basic idea of the method relies on operating in the rotating frame defined by $e^{-iH_0 t}$, i.e., we

TABLE II. Resource costs of each subroutine (in terms of the number of T-gates, the number of rotations, and the number of additional ancilla qubits) and the number of times they are used in the full quantum circuit. The unit Trotter evolution time $s = t/r$, where $r \in \mathbb{N}^+$, is predetermined such that $\|W_s(\epsilon/r) - e^{-iHs}\| \leq \epsilon/r$. Note that every operation is exact except the rotation syntheses, which are implemented precise enough to meet the total desired error [67].

Subroutine	No. T-gates	No. rotations	No. ancilla	No. total number of times
$e^{-iH_M s/2}, e^{-iH_M s}$	$4N - 4 + 4\lceil \log_2 N \rceil$	1	$N + \lceil \log_2 N \rceil + 1$	$2, r - 1$
$e^{-iH_E s/2}, e^{-iH_E s}$	$2(N - 1)(\eta^2 + \eta - 2)$	$(N - 1)\eta$	η	$2, r - 1$
$e^{-iH_{1,e/o} s/2}$	$6N - 4 + 4\lceil \log_2 N \rceil$	1	$3N/2 + \lceil \log_2 N \rceil$	$2r$
$e^{-iH_{2,e} s/2}, e^{-iH_{2,o} s}$	$6N - 4 + 4\lceil \log_2 N \rceil + 8N\eta - 8N$	1	$\max(3N/2 + \lceil \log_2 N \rceil, \eta)$	$2r, r$

make sure that at every stage, denoted by a time s , in the algorithm, the quantum state in the interaction picture is obtained by applying $e^{-iH_0 s}$ on the quantum state in the Schrödinger picture. To be more precise, as straightforwardly shown in Lemma B.1 or in a standard text [68]:

$$|\psi(t)\rangle = e^{-iHt}|\psi(t=0)\rangle \quad (44)$$

$$= e^{-iH_0 t} U_I(t) |\psi(t=0)\rangle, \quad (45)$$

where

$$U_I(t) = \mathcal{T}[e^{-i \int_0^t V(s) ds}] \quad (46)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \dots dt_n \mathcal{T} \left[\prod_{k=1}^n V(t_k) \right]. \quad (47)$$

Beyond product formulas, quantum algorithms for Hamiltonian simulation use block-encodings of the Hamiltonian, which is a unitary matrix that contains $H/\alpha_H - H$ rescaled by a factor $\alpha_H \in \mathbb{R}$ —in a chosen, accessible block; here, we follow what is typically done in the literature: the upper-left block, i.e., the subspace where the ancilla is in the 0 state. In general, the smaller the rescaling factor α_H , the more efficient the algorithm. Indeed, the algorithm based on the interaction picture, which we hereafter synonymously refer to as the IP-based algorithm, gives significant improvements when the block-encoding rescaling factors of H_0 and V satisfy $\alpha_{H_0} \gg \alpha_V$ and H_0 is fast-forwardable. Note that this is a common scenario, where V is a slight perturbation of an easily diagonalizable Hamiltonian H_0 . For the Schwinger model, this is true when the electric field is strong. In particular, we denote $H_0 = H_E + H_M$ and $V = H_I$, and the complexity of the algorithm will scale only mildly (polylogarithmically) with the rescaling factors of H_E and H_M .

The IP quantum algorithm approximates the operator $U_I(t)$ given in Eq. (46). In order to approximate this operator with a given precision ϵ in operator norm, we first expand the time-ordered integral via a Dyson series, and define two types of truncated discretized versions of it. These are given in Definitions B.2 and B.3, in which

collisions are disregarded and included, respectively; the collisions arise when the time integral is approximated as a discretized Dyson series [42]. K denotes the truncation degree of the Dyson series, while $1/M$ denotes the discretization coarseness in the time integrals. We analyze these approximations in Theorem B.4 and in Theorem B.9 and study the bounds for the truncation and discretization parameters K and M , in terms of $\|V\|$, $\|H_0\|$, t and ϵ . Corollary B.10 then combines the analysis for the final bounds for K and M that are used in the design of the IP-based algorithm. While our analysis for the algorithm without collisions leads to only minor improvements over previous analysis in [42], our analysis for the algorithm with collisions is novel and indicates a similar performance compared to the one without collisions. Indeed, our new error analysis reveals that including collisions in the block-encoding leads to negligibly bigger approximation error compared. In addition, it is more expensive to exclude collisions than to include them, as excluding collisions requires additional subroutines either to flag the collisions out of the success subspace of the block-encoding, or to prepare a state that ensures no collisions will occur. Therefore, we find including collisions to be the more practical and economical choice.

The IP-based quantum algorithm closely follows the linear combination of unitaries (LCU) approach combined with oblivious amplitude amplification (OAA), as described in Refs. [39,42,43]. We divide the total time evolution t into smaller pieces t_0/α_V , and implement each piece sequentially. The time step t_0 defines the rescaling factor of the block-encoding of the Dyson series [39], which dictates the number of rounds of OAA one needs to perform per time step. In practice, t_0 is optimized such that each time slice requires only a single round of OAA. We construct a quantum circuit $W_{(K,M)}(\frac{t_0}{\alpha_V}, \frac{\epsilon'}{r})$ that approximates $U_I(\frac{t_0}{\alpha_V})$ with an error that is at most $\frac{\epsilon'}{r}$, where $r = \lceil \frac{\alpha_V t}{t_0} \rceil$ is the number of time steps required to evolve for a total time t .

To enforce time ordering of the Dyson series, a natural choice is to implement coherently a sorting algorithm of K time registers of size $\lceil \log_2 M \rceil$. This introduces a qubit overhead that is polynomial in both K and $\lceil \log_2 M \rceil$. As an

TABLE III. Resource costs of each subroutine (in terms of the number of T-gates, the number of rotations and the number of additional ancilla qubits) and the number of times they are used for constructing the quantum circuit $W_{(K,M)}(t_0/\alpha_V, \epsilon'/r)$ ($\approx e^{-iHt_0/\alpha}$). To obtain $W(t, \epsilon')$ ($\approx e^{-iHt}$), one needs to repeat $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$, $r = \lceil t \frac{\alpha}{t_0} \rceil$ many times. Notice that we assumed an implementation of each Toffoli as described in [69], using 4 T-gates and one ancilla per Toffoli. Note that every operation is exact except the synthesis of rotation gates, which are implemented to our desired precision [70].

Subroutine	No. T-gates	No. rotations	No. ancilla	No. times
PREP $\sqrt{t_0/k!}$	0	$2K - 1$	0	6
Additional—PREP	$2K \lceil \log_2 M \rceil$	0	0	6
SORT	$4 \lfloor K/2 \rfloor (\lceil \log_2 K \rceil + 1) \lceil \log_2 M \rceil$	0	$\lceil \log_2 M \rceil + \lfloor \frac{K}{2} \rfloor (\lceil \log_2 K \rceil + 1) \frac{\lceil \log_2 K \rceil}{2}$	6
BE $_{V/\alpha}$	$8N + 4(N-1)(\eta-1) - 1$	0	$\eta - 1$	$3K$
$e^{-iH_M(\cdot)}$ (PGA)	$4N - 4 + 4 \lceil \log_2 M \rceil (\lceil \log_2 N \rceil + 1)$	$\lceil \log_2 M \rceil$	$N + \lceil \log_2 N \rceil + 1$	$3(K+1)$
$e^{-iH_M(\cdot)}$ (Mult)	$4[N + 2 \log_2 M \lceil \log_2 N \rceil + 7 \log_2 M + 5 \lceil \log_2 N \rceil + 4]$	1	$N + 2 \log_2 M + 2 \lceil \log_2 N \rceil + 1$	
$e^{-iH_E(\cdot)}$ (PGA)	$2(N-1) \lceil \log_2 M \rceil (\eta^2 + \eta - 2)$	$(N-1) \lceil \log_2 M \rceil \eta$	η	$3(K+1)$
$e^{-iH_E(\cdot)}$ (Mult)	$4N[4\eta^2 + 4\eta] + 4 \log_2 M [4\eta + 5 + 2 \lceil \log_2 N \rceil] + 20 \lceil \log_2 N \rceil - 8\eta^2 + 48\eta$	1	$8\eta + 3 \lceil \log_2 N \rceil + 2 \log_2 M$	
Additional—SEL	$8(\lceil \log_2 M \rceil - 1)(K-1) + 4K(N+1)$	0	$\max(N+1, \lceil \log_2 M \rceil - 1)$	3
Reflection	$8K + 4K \lceil \log_2 M \rceil - 4$	0	$2K + K \lceil \log_2 M \rceil - 1$	2

alternative, Ref. [42] proposes using only two time registers. This reduces the number of required qubits, but it increases the LCU rescaling factor, which in turn decreases the optimal choice of t_0 . Specifically, with K time registers and sorting, the optimal value $t_0 = \ln 2$, whereas the two-register implementation yields $t_0 = 0.5$. This corresponds to approximately a 39% increase in the total number of time steps r . We find that this increase leads to a higher overall T-count than the additional cost incurred by the sorting algorithm. Combined with the binary encoding of the truncation index, Ref. [42] reports a qubit scaling in $\lceil \log_2 K \rceil$ instead of K . In our case, since K time registers are already present, the marginal qubit cost of using a unary encoding for the truncation index register is justified by the resulting reduction in T-count.

This quantum circuit is given in Lemma B.11 which combines the block-encodings of V and fast-forwarded implementations of $e^{-iH_0 s}$. To be more precise, we execute the $W_{(K,M)}(t_0/\alpha_V, \epsilon'/r)$ subroutine $r-1 = \lfloor \alpha_V t/t_0 \rfloor$ times, and execute $W_{(K,M)}(t', \epsilon'/r)$ once for $t' = t - t_0 \lfloor \alpha_V t/t_0 \rfloor / \alpha_V$. We finally apply $e^{-iH_0 t}$. This implementation is given in detail in Corollary B.12.

We note that the piece-wise implementation of the LCU-based approach leads to a the multiplicative $\ln(1/\epsilon)$ scaling rather than an additive scaling. The quantum computational resource cost of implementing the algorithm is given in detail in Table III and explained in Sec. IV B.

IV. RESULTS

In this section, we summarize the resource cost analysis of implementing the PF₂ and IP-based algorithms. Both

algorithms divide the full time evolution into smaller time steps that are repeated sequentially. These smaller time evolution operators are further decompose into subroutines, for which their costs are listed in Tables II and III, respectively. These cost expressions are then run for specific parameter regimes to give explicit T-gate and qubit counts for the performances of the algorithms. We focus on the PF₂ approach in Sec. IV A, and on the interaction picture in Sec. IV B. In Sec. IV C, we give the asymptotic and numeric quantum resource estimates and compare the results. Further details on the resource estimates can be found in Appendixes B 2 and C.

A. Resource costs for the Suzuki-Trotter method

As shown in Eq. (39), the 2nd order Trotter formula approximates the time evolution operator as a product that involves the subevolutions $e^{-iH_E \tau}$, $e^{-iH_M \tau}$, and $e^{-iH_{1/2, \epsilon/\alpha} \tau}$. Whenever possible, we merge contiguous subevolutions. Thus, the value of τ is either $t/2r$ or t/r . Below, we briefly explain our quantum circuit implementation of the subevolutions and provide their associated cost, which we summarize in Table II. To simplify notation, let us define $\eta = \lceil \log_2 2\Lambda \rceil$. We provide further details in Appendix C.

- (1) $e^{-iH_E \tau}$: We start from the implementation in [25]: $e^{-iH_E \tau}$ is first expressed as a product of $e^{-iE^2 \tau}$ acting on every bosonic register. Then, each $e^{-iE^2 \tau}$ is decomposed into $\eta - 2$ layers—interleaved with layers of CNOT gates—of at most ηR_z rotations of the form $\otimes_i R_z(2^i \tau)$, as depicted in Fig. 4 of Ref. [25]. Instead of synthesizing every R_z gate individually as in [25], we effect each $\otimes_i R_z(2^i \tau)$

using the phase catalysis circuit from [71] shown in Fig. (168), which is more gate-efficient, mainly because one only needs to synthesize a reusable catalyst state $\bigotimes_{i=0}^{2\eta-3} R_z(2^i\tau)|+\rangle$ once for the entire time-evolution circuit.

- (2) $e^{-iH_M\tau}$: Following [29], we reduce $e^{-iH_M\tau}$ into N same-angle R_z rotations by applying 2 NOT gates each to half of the fermionic registers. Then, we implement the same-angle R_z gates using catalyzed Hamming-weight phasing [69], which improves upon the circuit in Fig. (168) from Ref. [71]. Note that [29] uses (uncatalyzed) Hamming-weight phasing to implement $e^{-iH_M\tau}$.
- (3) $e^{-iH_{1/2,e/o}\tau}$: Once again, we follow [29], differing mainly in choosing catalyzed Hamming-weight phasing over (uncatalyzed) Hamming-weight phasing. More specifically, $e^{-iH_{1,e/o}\tau}$ can be expressed as a product of operators $e^{-i\tau x\sigma^+\sigma^+\sigma^-+H.c.}$, where $\sigma^\pm$ are the Pauli raising and lowering operators, and $e^{-iH_{2,e/o}\tau}$ is a product of $Ue^{-i\tau x\sigma^+\sigma^+\sigma^-+H.c.}U^\dagger$, where U is an adder circuit. We then compile every $e^{-i\tau x\sigma^+\sigma^+\sigma^-+h.c.}$ into a layer of two same-angle R_z gates conjugated by temporary ANDs [72], Hadamards and NOT gates [70]. The resulting layer of N same-angle R_z gates in $e^{-iH_{1,e/o}\tau}$ is then implemented using catalyzed Hamming-weight phasing. $e^{-iH_{2,e/o}\tau}$ is implemented similarly, with the necessary, additional applications of adder circuits.

B. Resource costs for the interaction picture method

As explained in Sec. III B, the quantum circuit calls $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$, r times, where $r = \lceil t\alpha/t_0 \rceil$. As we find, $\alpha = 2Nx$ which is the coefficient 1-norm of the LCU expansion given in Eq. (A11). In this section we study the gate cost of implementing $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$, which is reported in Table III. The quantum circuit employed to realize the operator $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$ is given in Fig. 6. Its correctness is proven in Appendix B, more precisely referring to Lemma B.11 and Corollary B.12. More details of the circuit implementation can be found in Appendix B 2. The quantum circuit given in Fig. 6 plays a key role: it is used in OAA in order to implement $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$. We study the cost of the subroutines used in this quantum circuit, which are given Table III.

- (1) $\text{PREP}_{\sqrt{t_0^k/k!}}$: The first subroutine in PREP of the Dyson series is implemented by preparing a k -hot state:

$$\text{PREP}_{\sqrt{t_0^k/k!}}|0\rangle^{\otimes K} = \frac{1}{\beta} \sum_{k=0}^{K-1} \sqrt{\frac{t_0^k}{k!}} |1\rangle^{\otimes k} |0\rangle^{\otimes K-k}. \quad (48)$$

This can be obtained by the circuit in Fig. 7. The output register encodes the truncation index k , in

k -hot unary representation. It requires at most one rotation and $K - 1$ controlled-rotations (which can be implemented with two rotations each). It requires no additional ancilla and is called 6 times, three times due to oblivious amplitude amplification (OAA), and twice within each OAA, as PREP and $\text{PREP}^\dagger$.

- (2) *Additional PREP*: We now prepare the discretized time registers. There are K registers of size $\lceil \log_2 M \rceil$ each. Assuming M is an integer power of 2, one can use controlled-Hadamards to prepare a uniform superposition of states on these time discretization registers. In total, this costs $2K \log_2 M$ T-gates.
- (3) *SORT*: We then apply a bitonic sort [73,74] which arranges the integer values in the K time registers, each of size $\log_2 M$, in increasing order. This takes at most $\lfloor K/2 \rfloor (\lceil \log_2 K \rceil + 1) \lceil \log_2 K \rceil$ comparators (with equality if K is an integer power of 2), and the same amount of C-SWAPs. Both the comparator and C-SWAPs require $\lceil \log_2 M \rceil$ Toffolis each. This allocates at most $\lfloor K/2 \rfloor (\lceil \log_2 K \rceil + 1) \lceil \log_2 K \rceil / 2$ record qubits that will be deallocated at uncomputation. An extra $\lceil \log_2 M \rceil$ temporary ancillae are used for the comparator. The uncomputation can be done with a measurement-based procedure cheaper on average than the computation [75], whereas we assume calling this subroutine 6 times: three times due to OAA, and twice within each OAA, as SORT and $\text{SORT}^\dagger$, since the uncomputation is at worst the same cost as the computation.
- (4) $\text{BE}_{V/\alpha}$: The k th $\text{BE}_{V/\alpha}$ that appears in the circuit is singly controlled on the k th qubit of the state given in Eq. (48). Note that this k is related to the k th application of V in the truncated-discretized Dyson series. We implement each $\text{BE}_{V/\alpha}$ following the implementation of [76], which takes a linear combination of local terms that are of equal weight and implements $\text{PREP}_{\text{BE}} - \text{SEL}_{\text{BE}} - \text{PREP}_{\text{BE}}^\dagger$. Importantly, the control of the block-encoding only controls certain parts of PREP_{BE} , and some operations of SEL_{BE} (see Appendix B 2 for more details). PREP_{BE} prepares a W state of $N - 1$ qubits in a tensor product with $H^{\otimes 3}$. SEL_{BE} is implemented following Ref. [76] (see Sec. V.2, Fig. 4). By choosing the number of links ($N - 1$ for open boundary condition, N for periodic) as an integer power of two, the cost of $\text{BE}_{V/\alpha}$ is $8N + 4(N - 1)(\eta - 1) - 1$ T-gates and $\eta - 1$ ancillae. This subroutine is called $3K$ times in OAA.
- (5) $e^{-iH_M(\cdot)}$: This operation is conditioned on the discretized time registers, $m_1, m_2, \dots, m_K$. The implementation combines consecutive forward/backward time evolutions and first computes $m'_k = m_k - m_{k-1}$ for $k \in [2, K]$, which are counted as additional gates below. Then controlled on m'_k , we

implement $e^{-iH_M m'_k/(aM)}$. We compare two implementations that we call PGA and Mult. In the PGA approach, one first computes the Hamming weight of the fermions, n_f . Controlled on m'_k , we then perform a phase gradient addition with n_f . The cost of this is $N - 1 + \lceil \log_2 M \rceil (\lceil \log_2 N \rceil + 1)$ Toffolis and $\lceil \log_2 M \rceil$ rotations, $N + \lceil \log_2(N) \rceil + 1$ ancilla qubits. In the Mult approach, we perform a multiplication of m'_k and n_f and then perform a phase gradient addition to add the result to the phase. The cost of this approach is $N + 2 \log_2 M \lceil \log_2 N \rceil + 7 \log_2 M + 5 \lceil \log_2 N \rceil + 4$ Toffolis, 1 rotation and $N + 2 \log_2 M + 2 \lceil \log_2 N \rceil + 1$ ancillae. In both cases, the subroutines are called in total $3(K + 1)$ many times per application of $W_{(K,M)}$.

- (6) $e^{-iH_E(\cdot)}$: The phase applied in this operation also depends on the discretized time registers, $m_1, m'_2, \dots, m'_{K-1}, m_K$. Similar to the mass term, we also have two implementations, a PGA and a Mult approach. In the PGA case, each controlled time evolution is performed by a phase-gradient [77] version of the implementation in Ref. [25] (see their Sec. III.2). The cost of this subroutine is $(N - 1) \lceil \log_2 M \rceil (\eta^2 + \eta - 2)/2$ Toffolis, and $(N - 1) \lceil \log_2 M \rceil \eta$ rotations. An additional η ancillae are used (excluding the catalyst state). In the Mult approach, the squared electric fields are added in a binary tree fashion to reduce additional qubit costs. Then, the total electric term is multiplied by the time register. The result is then kicked back as a phase via phase-gradient addition. This costs $N + 2 \log_2 M \lceil \log_2 N \rceil + 7 \log_2 M + 5 \lceil \log_2 N \rceil + 4$ Toffolis, 1 rotation and $N + 2 \log_2 M + 2 \lceil \log_2 N \rceil + 1$ ancillae. In both cases, the subroutines are called $K + 1$ times in the circuit in Fig. 6, which is repeated 3 times via OAA, and hence in total called $3(K + 1)$ times.
- (7) *Additional-SEL*: We count the assisting elementary gates that appear in Fig. 6 for the in-place subtraction/addition and the compression gadget used to reduce the ancillary cost of multiplying block encodings. For the latter, we used a unary implementation of the method discussed in [78]. There are $2(K - 1)$ many subtractions/additions each of which costs $(\lceil \log_2 M \rceil - 1)$ Toffolis and $(\lceil \log_2 M \rceil - 1)$ temporary ancillae [72]. In parallel, the counter register is acted on via multi-controlled-NOTs, in total K times, and each of them costs $N + 1$ Toffolis. All of these are called 3 times in OAA.
- (8) *Reflection*: Furthermore, there is an additional set of Toffolis that is used in OAA for reflections on three registers: the one that holds the value of K ; the ones that hold the values of $m_1, m_2, \dots, m_K$; and the counter register that is used for multiplying the block-encodings $BE_{V/\alpha}$. In total, there are $K(2 +$

$\lceil \log_2 M \rceil$) qubits required for the reflection. This costs $(2K + K \lceil \log_2 M \rceil - 1)$ Toffolis and additional temporary ancillae. This subroutine is called twice for OAA.

C. Asymptotic and numerical QREs and comparison

In this section, we first give a comparison of the PF₂ method with the IP-based quantum algorithm in terms of the asymptotic scaling depending on the model and simulation parameters, i.e., x, μ, N, Λ, t , and ϵ . Then, we give instantiations of the quantum resource estimates for parameter regimes for which one expects to observe Schwinger effect; see Table I. Figure 4 shows the performance of the two methods, and can help with choosing the most efficient method to use.

PF₂ calls a unit Trotter evolution, $W_s(\epsilon/r)$,

$$r = \mathcal{O}(t^{3/2} \rho^{1/2} / \epsilon^{1/2}) \quad (49)$$

many times, where ρ is an upper bound on the norm of the third-order commutator. When we employ the asymptotic expression for ρ , given in Eq. (C14), and observing that each $W_{t/r}(\epsilon/r)$ costs $\tilde{\mathcal{O}}(N)$ (see Table II), we find the resulting cost given in Table IV for PF₂. On the other hand IP calls a unit time evolution $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$, $r = \lceil \frac{t\alpha}{t_0} \rceil$ many times, where the rescaling factors

$$\alpha = \mathcal{O}(Nx), \quad (50)$$

$$t_0 = \mathcal{O}(1), \quad (51)$$

$$\epsilon' = \mathcal{O}(\epsilon). \quad (52)$$

The most expensive part of implementing $W_{t/r}(\epsilon/r)$ is for implementing the time evolution of the electric term $e^{-iH_E(\cdot)}$ which scales as $\mathcal{O}(N(\log_2 M)\eta^2)$, as seen in Table III. M can be found in Eq. (B16) which for $t = t_0/\alpha$ gives $\mathcal{O}(\log_2 M) = \tilde{\mathcal{O}}(1)$. Finally, recalling that $\eta = \log_2(2\Lambda)$, we obtain the resulting cost given in Table IV for IP. This analysis shows that for larger time evolutions (large t), more precise simulation (large $1/\epsilon$), or high electric field cutoff Λ , IP should be the method of choice, assuming that one is not limited by qubits. On the other hand PF₂ performs better in terms of scaling in N , e.g.

TABLE IV. Asymptotic gate complexity for the Schwinger model: 2nd order product formula (PF₂) vs interaction-picture-based (IP) quantum algorithms in terms of $\Lambda, t, N, 1/\epsilon, \mu$ and x . Note that $\eta = \log_2 2\Lambda$.

Asymptotic gate complexity	
PF ₂	$\tilde{\mathcal{O}}(\Lambda^{(N^3 t^3 x)^{1/2}} \left(1 + \frac{\mu^2}{\Lambda^2} + \frac{x}{\Lambda^2} + \frac{\mu}{\Lambda} + \frac{x\mu}{\Lambda^2} + \frac{x}{\Lambda}\right)^{1/2})$
IP	$\tilde{\mathcal{O}}(N^2 x t (\log_2 \Lambda)^2 \log_2(1/\epsilon))$

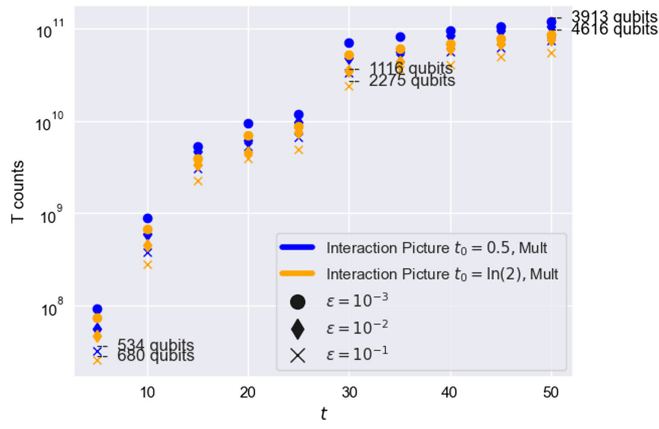


FIG. 2. Resource cost comparison of two different implementations of IP-based algorithms. The blue data points correspond to the implementation without SORT, using $t_0 = 0.5$, as described in Ref. [42]. The orange data points correspond to the implementation using the optimal $t_0 = \ln(2)$ with SORT, which requires more qubits but fewer T-gates. The comparison is shown in terms of T counts and qubit counts for time parameters $t \in (t_{\min}, 10t_{\min})$, and accuracies $\epsilon \in \{0.001, 0.01, 0.1\}$. We fix $N_0 = 8$, $x = 0.1$, $\mu = 1$, $\Lambda_0 = \sqrt{10}\mu = \sqrt{10}$, and $t_{\min} = 0.5/x$. The parameters N and Λ are chosen according to Table I.

$\mathcal{O}(N^{3/2})$ vs $\mathcal{O}(N^2)$, with minimal ancilla overhead. In terms of scaling with x , which can be a dominant factor in the continuum limit, PF_2 has a quadratic improvement.

Now, we compare the performance of different implementations of the IP-based algorithm, and the most-performant IP-based algorithm with the most performant PF_2 -based algorithm. Numerical results in Figs. 2 and 3 give the resource costs in terms of T count (y-axis) for varying time

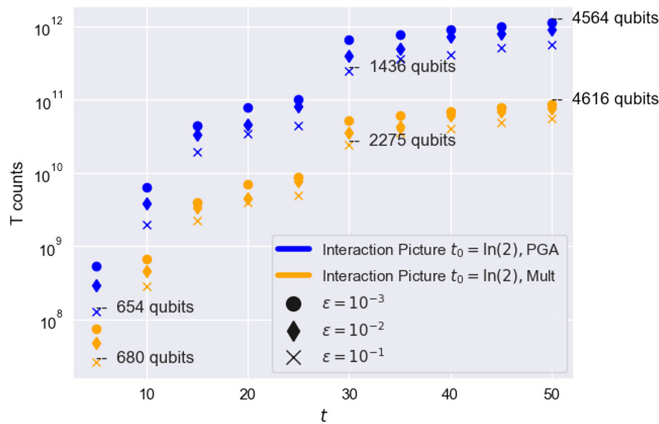


FIG. 3. Resource cost comparison of two different compilations of the IP-based algorithm with $t_0 = \ln(2)$. The blue data points correspond to the compilations that use the PGA, and the orange data points correspond to the Mult approach. The comparison is shown in terms of T counts and qubit counts for time parameters $t \in (t_{\min}, 10t_{\min})$, and accuracies $\epsilon \in \{0.001, 0.01, 0.1\}$. We fix $N_0 = 8$, $x = 0.1$, $\mu = 1$, $\Lambda_0 = \sqrt{10}\mu = \sqrt{10}$, and $t_{\min} = 0.5/x$. The parameters N and Λ are chosen according to Table I.

parameter t (x-axis), and for $\epsilon = 0.1, 0.01, 0.001$ precision in the implementation of the time evolution operator.

First, in Fig. 2 the choice of $t_0 = 0.5$ (which does not use SORT) vs $t_0 = \ln 2$ (which uses SORT) is compared. We find out that the choice of $t_0 = \ln 2$ is more efficient, as expected, and the fact that it has the additional subroutine SORT does not drastically deteriorate the expected gain of $\sim 1.4\times$. This is, however, at the expense of hundred(s) of additional qubits.

Second, in Fig. 3, we compare, for $t_0 = \ln 2$, the performance of two different implementations that we called PGA and Mult. Note that similar alternatives (PGA vs Mult) are also available for the PF_2 -based method. However, the best choice here is clearly the PGA approach, due to the fact that in the Trotter implementation, the time is a classically known value, and not a quantum one like in IP. In the IP-based algorithm, this control-structure on PGA leads to a $\log_2 M$ multiplicative factor. This turns out to be more expensive than the Mult approach which has additional arithmetic for computing the sum of the squares of electric fields. In the PF_2 approach, the absence of control of the PGA makes it far more efficient. In fact, our results, as seen in Fig. 3, show that the Mult approach is an order of magnitude better in terms of T count at the expense of less than a hundred of additional qubits.

Third, in Fig. 4, we compare the PF_2 vs IP-based algorithm with the optimal implementation choices, e.g., for $t_0 = \ln 2$ and the Mult approach. We perform this comparison for different coupling strengths $x = \{0.1, 1, 10, 100\}$ and for $\epsilon = \{10^{-3}, 10^{-2}, 10^{-1}\}$, over $\{t_{\min}, 2t_{\min}, \dots, 10t_{\min}\}$ where $t_{\min}$ is an optimistic estimate for the shortest time for which we expect to observe the Schwinger effect. Remark that the parameter regimes for different values of x , such as $t_{\min}$, are different, and tailored to optimistic estimates for observing the Schwinger effect. For example, for large x we expect a shorter time to observe the Schwinger effect given that is the strength of the particle-antiparticle creation term, hence a shorter $t_{\min}$ is chosen, e.g., as in Table I. It is however not guaranteed that $t_{\min}$, or even $10t_{\min}$, is sufficient to observe the phenomenon and one may need to carry out the simulation for longer times. With this remark in mind, we recover the expected performances from the asymptotic formula given in Table IV. The IP-based algorithm is more advantageous for smaller x , longer times, and high accuracy simulations, compared to the PF_2 approach.

It is clear that the PF_2 approach performs better for short simulation times, and when high accuracy is not required. While the simulations in this regime might not detect the Schwinger effect, they are the starting point for a set of numerical experiments. In fact, for an early fault-tolerant quantum computer which is limited in not only the number of operations but also the number of qubits, the PF_2 approach is clearly favorable. Thus far, we have considered rigorous cutoffs that scale with the simulation time t , as we

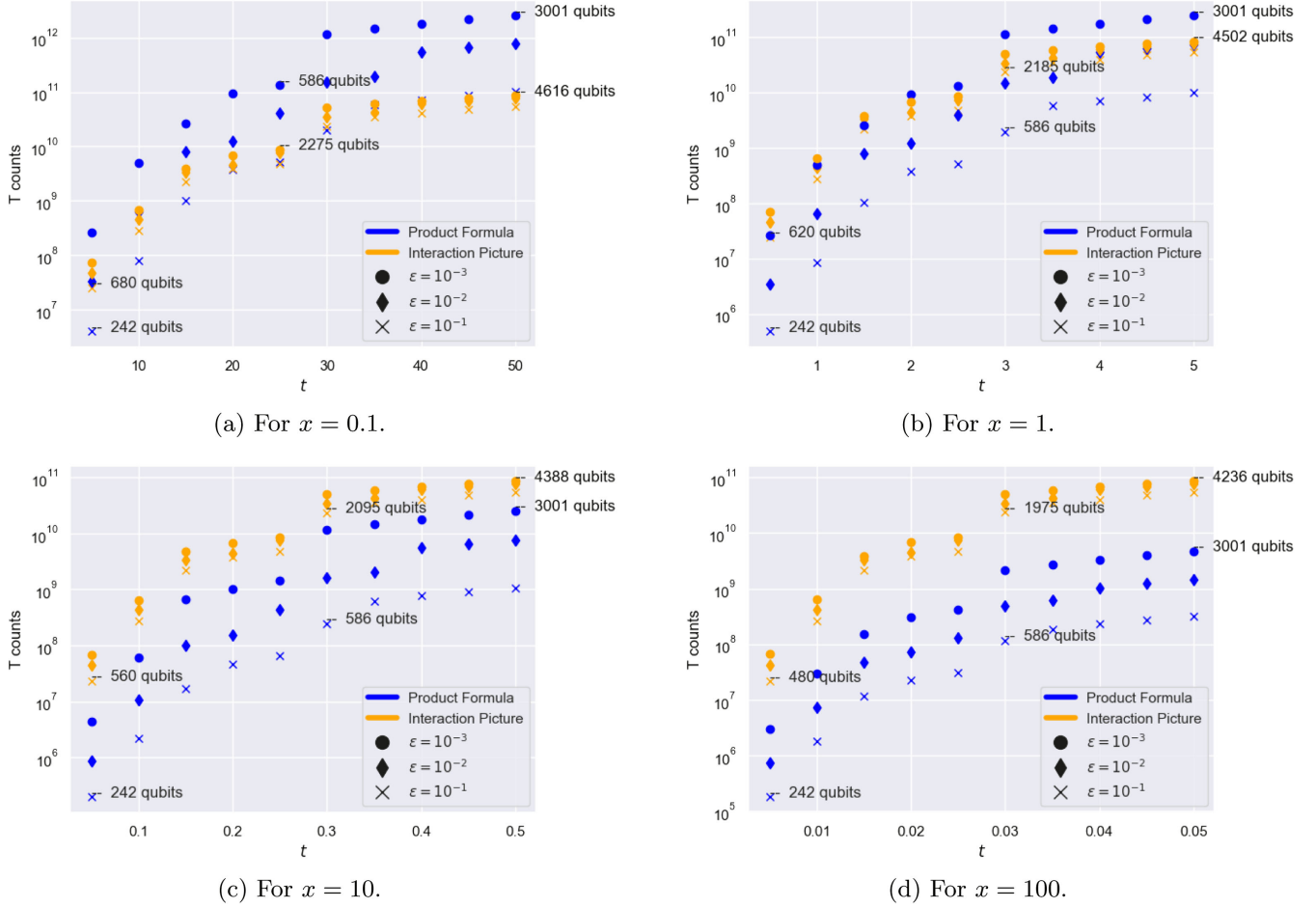


FIG. 4. Resource cost comparison between the best implementations of IP-based algorithm and PF_2 approach. The comparison is given in terms of T-counts and qubit counts, for different time parameters $t \in (t_{\min}, 10t_{\min})$, and for accuracy $\epsilon = \{0.001, 0.01, 0.1\}$. We fix $N_0 = 8$, $\mu = 1$, $\Lambda_0 = \sqrt{10}\mu = \sqrt{10}$, and $t_{\min} = 0.5/x$. The parameters N and Λ are chosen according to Table I.

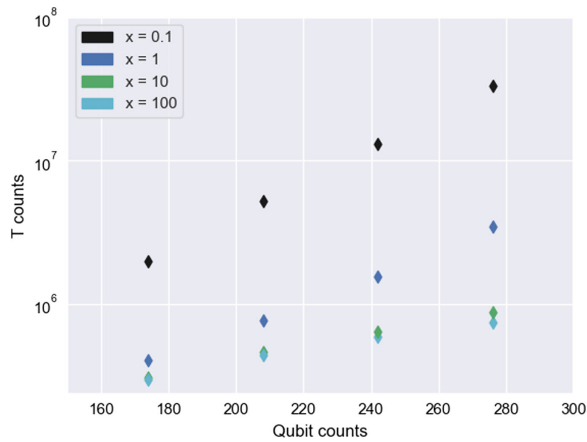


FIG. 5. Resource cost of simulating the time evolution of the Schwinger model with a second-order product-formula method, for $x \in \{0.1, 1, 10, 100\}$ (indicated by the colors) and time-independent $\eta \in \{2, 3, 4, 5\}$ (from the smallest to the largest number of qubits). We fix $\mu = 1$, $N_0 = 8$ and $t = t_{\min} = 0.5/x$. The number of fermions N is chosen according to Table I.

have explained in Sec. II. However, we do not preclude the emergence of evidence, e.g., Ref. [52], which indicates that small cutoffs, possibly independent of t , may suffice for measuring certain observables. Motivated by this, we plot the resource costs for small, time-independent cutoffs in Fig. 5. These optimistic quantum resource estimates assume a simulation time $t = t_{\min} \propto 1/x$ for which the Schwinger effect may be observed though is not guaranteed. This leads to quantum resource estimates that are higher in the weakly interacting regime than in the strongly interacting one, as shown in Fig. 5. Note that this will not be the case for a fixed time t , and will likely not be the case if one runs these simulations with varying time for probing the Schwinger effect or extracting another observable.

V. CONCLUSIONS AND OUTLOOK

In this work, we studied two algorithms and their various implementations for simulating the dynamics in the Schwinger model. Our findings demonstrate that while asymptotically the interaction picture based algorithm gives

rise to the best performance, product formulas perform better for parameter regimes that lead to smaller electric field cutoffs and less stringent accuracy requirements. We also studied the interdependency of the simulation parameters (such as N , Λ , $t_{\min}$, and optimal algorithm parameters) to the fundamental parameters of the model (such as the mass μ and the interaction strength x).

There are various directions to follow, both for increasing the performance of the algorithms and for extending the applications to more complicated models in the same or higher dimensions. Recent works proved that quantum time evolution can be simulated more efficiently with product formula [44,45,79,80] and quantum signal processing [81] when certain subspaces are considered, such as when the initial state is guaranteed to be in the low-energy subspace (with respect to the spectral norm of the Hamiltonian). Product formula approaches have been observed to perform better than the worst-case bounds [47,82]; hence, one may in fact hope for a better performance from those methods. Furthermore, one can consider transforming the Hamiltonian to an equivalent one where the gauge degrees of freedom are eliminated. This reduces the number of qubits, at the expense of more complicated, nonlocal Hamiltonian terms, yet it may still be more efficient to implement for early fault-tolerant quantum computers or quantum simulators [21,27,28,83,84]. However, this strategy is less extensible beyond one spatial dimensions, where the number of constraints, i.e., Gauss' law, is strictly less than the number of gauge degrees of freedom and thus the gauge fields cannot be completely eliminated. Similar to our work, an interaction-picture algorithm based on Dyson series was applied to $U(1)$, $SU(2)$ and $SU(3)$ lattice gauge theories in beyond $1+1$ space-time dimensions in Ref. [30]. Compared to our algorithm, the LCU-based algorithm for $U(1)$ lattice gauge theories from [30], if applied to the Schwinger model, will lead to different resource estimates mainly because: (i) H_M is not fast-forwarded, though it is ostensibly diagonal, and instead, it is block-encoded, thereby introducing an additional μ -dependence in the gate complexity. (ii) The fermions are encoded using mappings that introduce an additive $\mathcal{O}(N)$ ancilla overhead compared to our choice of Jordan-Wigner mapping. (iii) In [30], the authors optionally apply the HHKL algorithm [55] that provides an asymptotic advantage, which in practice, only manifests at large system sizes; the precise break-even point in system size for the Schwinger model, while interesting is beyond the scope of this work. Importantly, the improvements we made in the interaction-picture algorithm here, which are model-agnostic, e.g., better error bounds and more optimized circuit implementation, can be readily applied to improve the algorithms in Ref. [30]. In the future, it will be interesting to quantify such improvements in the context of non-Abelian theories, particularly QCD.

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Authors and contributions are listed alphabetically. A. K., J. L. and B. Ş. performed the technical analysis, conducted resource counts, and drafted the manuscript. J. L. and O. O. implemented the scripts that validate parts of the algorithm implementations. All authors contributed to discussions and code review.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available because they contain commercially sensitive information. The data are available from the authors upon reasonable request.

APPENDIX A: MAPPING THE HAMILTONIAN TO QUBITS: HAMILTONIAN AS LINEAR COMBINATIONS OF PAULI PRODUCTS

In this section, we introduce the Schwinger model again, and perform the explicit Jordan-Wigner mapping (fermion to qubit mapping), as could also be found in the literature [25] which we closely follow here. The Hamiltonian consists of three terms, the electric field term H_E , the gauge-matter interaction term H_I , and the mass (or matter) term H_M . The Hamiltonian is a sum of three terms, electric, interaction and mass terms, as

$$H = H_E + H_I + H_M, \quad (\text{A1})$$

where

$$H_E = \sum_{r=1}^N E_r^2, \quad (\text{A2})$$

$$H_I = x \sum_r (U_r \psi_r^\dagger \psi_{r+1} - U_r^\dagger \psi_r \psi_{r+1}^\dagger), \quad (\text{A3})$$

$$H_M = \mu \sum_r (-1)^r \psi_r^\dagger \psi_r. \quad (\text{A4})$$

Note that r denotes the lattice site, E_r and U_r act on the 2Λ dimensional qudit placed on the link $(r, r+1)$. The Kogut-Susskind construction [85] is such that electrons live on even numbered sites, whereas positrons live on odd numbered sites. On even sites r , $\psi_r, \psi_r^\dagger$ are annihilation/creation operators. On odd sites r , $\psi_r, \psi_r^\dagger$ are creation/annihilation operators. The electric term, H_E , acts like a

potential term that energetically penalizes higher electric field values. The interaction term, H_I , can create/annihilate electron-positron pairs on neighboring sites by incrementing/decrementing the electric field value. The mass term, H_M , penalizes states with higher numbers of fermions, proportional to the bare mass μ of each particle. x and μ are

the unitless coupling and mass parameters, given in terms of the physical parameters of the system as in Eq. (2).

After truncating the Hilbert space, the electric field value is considered to be an element of $\mathbb{Z}_{2\Lambda}$; i.e., it periodically wraps the electric field at a cutoff Λ , meaning the Hamiltonian term H_I has off-diagonal elements at $|\Lambda\rangle\langle\Lambda-1| + \text{H.c.}$:

$$H_E = \sum_{r=1}^N \sum_{\varepsilon=-\Lambda}^{\Lambda-1} \varepsilon^2 |\varepsilon\rangle\langle\varepsilon|_r, \quad (\text{A5})$$

$$H_I = x \sum_{r=1}^N \left\{ |-\Lambda\rangle\langle\Lambda-1|_r \otimes \psi_r^\dagger \psi_{r+1} - |\Lambda-2\rangle\langle\Lambda-1|_r \otimes \psi_r \psi_{r+1}^\dagger + |-\Lambda+1\rangle\langle-\Lambda|_r \otimes \psi_r^\dagger \psi_{r+1} \right. \\ \left. - |\Lambda-1\rangle\langle-\Lambda|_r \otimes \psi_r \psi_{r+1}^\dagger + \sum_{\varepsilon=-\Lambda+1}^{\Lambda-2} (|\varepsilon+1\rangle\langle\varepsilon|_r \otimes \psi_r^\dagger \psi_{r+1} - |\varepsilon-1\rangle\langle\varepsilon|_r \otimes \psi_r \psi_{r+1}^\dagger) \right\}, \quad (\text{A6})$$

$$H_M = \mu \sum_{r=1}^N (-1)^r \psi_r^\dagger \psi_r. \quad (\text{A7})$$

We then use the Jordan-Wigner transformation to represent the fermionic operators. For even sites, $|0\rangle$ denotes the vacuum state (i.e., the absence of an electron), while $|1\rangle$ denotes the occupied state (i.e., the presence of an electron), while for odd sites the convention is reversed. The Jordan-Wigner transform reads as

$$\psi_r^\dagger = \frac{X_r - iY_r}{2} \prod_{j=1}^{r-1} Z_j, \quad (\text{A8})$$

which leads to

$$\psi_r^\dagger \psi_{r+1} = \frac{1}{4} (-iY_r X_{r+1} + Y_r Y_{r+1} + X_r X_{r+1} + iX_r Y_{r+1}) \quad (\text{A9})$$

and

$$\psi_r \psi_{r+1}^\dagger = \frac{1}{4} (-iY_r X_{r+1} - Y_r Y_{r+1} - X_r X_{r+1} + iX_r Y_{r+1}). \quad (\text{A10})$$

Hence, combining Eqs. (A9) and (A10) we obtain

$$H_I = \frac{x}{4} \sum_{r=1}^N \{ (U_r + U_r^\dagger)(X_r X_{r+1} + Y_r Y_{r+1}) + i(U_r - U_r^\dagger)(X_r Y_{r+1} - Y_r X_{r+1}) \} \quad (\text{A11})$$

$$= \frac{x}{4} \sum_{r=1}^N \left\{ (|-\Lambda\rangle\langle\Lambda-1|_r + |\Lambda-2\rangle\langle\Lambda-1|_r + |-\Lambda+1\rangle\langle-\Lambda|_r + |\Lambda-1\rangle\langle-\Lambda|_r) \otimes (X_r X_{r+1} + Y_r Y_{r+1}) \right. \\ \left. + i(|-\Lambda\rangle\langle\Lambda-1|_r - |\Lambda-2\rangle\langle\Lambda-1|_r + |-\Lambda+1\rangle\langle-\Lambda|_r - |\Lambda-1\rangle\langle-\Lambda|_r) \otimes (X_r Y_{r+1} - Y_r X_{r+1}) \right. \\ \left. + \sum_{\varepsilon=-\Lambda+1}^{\Lambda-2} [(|\varepsilon+1\rangle\langle\varepsilon|_r + |\varepsilon-1\rangle\langle\varepsilon|_r) \otimes (X_r X_{r+1} + Y_r Y_{r+1}) + i(|\varepsilon+1\rangle\langle\varepsilon|_r + |\varepsilon-1\rangle\langle\varepsilon|_r) \otimes (X_r Y_{r+1} - Y_r X_{r+1})] \right\}. \quad (\text{A12})$$

For the mass term, we get

$$\psi_r^\dagger \psi_r = \frac{1}{2}(\mathbb{I} - Z_r). \quad (\text{A13})$$

Note that this differs from [25] by the identity term:

$$H_M = \frac{\mu}{2} \sum_{r=1}^N (-1)^r (\mathbb{I} - Z_r). \quad (\text{A14})$$

This completes the fermion/boson to qubit mappings that we use in order to simulate the system on a quantum computer.

APPENDIX B: OVERVIEW OF INTERACTION PICTURE SIMULATION

In this section, we first show the equivalence of time evolution operators in the Schrödinger and interaction pictures, given as in Lemma B.1. The nontrivial part of the time evolution operator, $U_I(t)$, is the time-ordered exponential operator as defined in Eq. (B4). We then give two operators $D_{(K,M)}(t)$ and $\tilde{D}_{(K,M)}(t)$ each of which approximates $U_I(t)$, by truncating and discretizing the series expansion of the time-ordered integral, i.e., the well-known Dyson series. These two approximations are given in Definitions B.2 and B.3, respectively. The former follows the operator in Ref. [42], which eliminates the overlapping time intervals in the discretization of the time-ordered integrals, while the latter does not. Then, Theorems B.4 and B.9 provide, respectively, the sufficient minimal values of K and M such that the truncated-discretized Dyson series $D_{(K,M)}(t)$ and $\tilde{D}_{(K,M)}(t)$ result in an ϵ -additive approximation of the time-ordered evolution operator $U_I(t)$. Combined with Lemma B.1, this implies Corollary B.10, which gives an approximation of the time evolution operator e^{-iHt} in terms of a product of e^{-iH_0t} and the truncated-discretized Dyson series. A similar result holds for $\tilde{D}_{(K,M)}(t)$. Using these general results, we set up a time-evolution circuit given by $W(t, \epsilon)$ that approximates e^{-iHt} up to an error ϵ . Lemma B.11 and its proof study the detailed quantum circuit for an implementation of $U_I(t_0/\alpha)$ and states the resource cost estimates. The final result is given in Corollary B.12.

The quantum algorithm includes a couple of approximations employed when implementing the subroutines.

While, in principle, we could distribute the overall error budget equally to the error sources, this is suboptimal compared to an optimized, uneven distribution of errors, which we will adopt in our implementation. Below we give a table of errors allocated to its sources. The results are stated in terms of these errors.

The total error is then given by

$$\epsilon = \epsilon_{\text{cutoff}} + \tilde{\epsilon}_1 + \tilde{\epsilon}_2 + \tilde{\epsilon}_3 = \epsilon_{\text{cutoff}} + \epsilon', \quad (\text{B1})$$

where

$$\tilde{\epsilon}_1 = r\epsilon_1, \quad \tilde{\epsilon}_2 = r\epsilon_2, \quad \tilde{\epsilon}_3 = n_{\text{rot}}\epsilon_3. \quad (\text{B2})$$

Recall that $r = \lceil t\alpha/t_0 \rceil$ is the number of times $U_I(t_0/\alpha)$ is called, and n_{rot} is the number of single-qubit rotations. Fixing these errors determines the parameters that feed into the description of the quantum algorithm, as summarized in Table V. Dyson was the first to explicitly formalize the transformations connecting the Schrödinger and interaction pictures [86]. The now-standard approach is summarized in Lemma B.1.

Lemma B.1. (Time evolution operator in the Schrödinger and in the interaction picture). Let $H = H_0 + V$ be a time-independent Hamiltonian with Hermitian H_0 and V . Then the time evolution operator in the Schrödinger picture, i.e., the operator that evolves $|\psi(0)\rangle$ to $|\psi(t)\rangle$, is

$$U(t) = e^{-iHt}, \quad (\text{B3})$$

whereas the time evolution in the interaction picture, i.e., the operator that evolves $|\psi(0)\rangle_I$ to $|\psi(t)\rangle_I$, is

$$U_I(t) = \mathcal{T}[e^{-i \int_0^t V(s) ds}], \quad (\text{B4})$$

where $|\psi(t)\rangle_I := e^{iH_0t}|\psi\rangle$ for any given $|\psi\rangle \in \mathcal{H}$ and $t \in \mathbb{R}$, and $V(s) := e^{iH_0s} V e^{-iH_0s}$ for any $s \in \mathbb{R}$. $\mathcal{T}$ denotes the time-ordering operator. The time evolution operator in the Schrödinger and interaction pictures are related as follows:

$$U(t) = e^{-iH_0t} U_I(t), \quad (\text{B5})$$

in the sense that

$$|\psi(t)\rangle = U(t)|\psi(t=0)\rangle = e^{-iH_0t} U_I(t)|\psi(t=0)\rangle_I. \quad (\text{B6})$$

TABLE V. Error symbols, their source, and what they determine.

Symbol	Source of the error	Determines
ϵ_1	Truncation error of Dyson series for time t_0/α	K
ϵ_2	Discretization error of Dyson series t_0/α	M
ϵ_3	Synthesizing single qubit rotations	b_{rot} (Bit precision of rotation angles)
ϵ_{cutoff}	Electric field cutoff	Λ

Proof. We start with the time-dependent Schrödinger equation:

$$\frac{d}{dt}|\psi(t)\rangle = -iH|\psi(t)\rangle. \quad (\text{B7})$$

Note that it is immediate that the time-independent Hamiltonian leads to a time-evolution operator $U(t) = e^{-iHt}$. For the rest of the claim, substitute $|\psi(t)\rangle = e^{-iH_0 t}|\psi(t)\rangle_I$. We obtain

$$\frac{d}{dt}|\psi(t)\rangle_I = -ie^{iH_0 t} V e^{-iH_0 t} |\psi(t)\rangle_I. \quad (\text{B8})$$

This implies that the time-ordered exponential $U_I(t) := \mathcal{T}[e^{-i \int_0^t V(s) ds}]$ is the time-evolution operator in the interaction picture. Namely,

$$e^{-iH_0 t} |\psi(0)\rangle = |\psi(t)\rangle = e^{-iH_0 t} U_I(t) |\psi(0)\rangle_I. \quad (\text{B9})$$

Next, we define two Dyson series that are truncated and discretized, both of which approximate the operator $U_I(t)$ given as in Lemma B.1. These are the operators that we use in the quantum algorithm. Following the notation introduced in [42], we defined the truncated-discretized Dyson series without collisions in Definition B.2 and with collisions in Definition B.3:

Definition B.2. ($D_{(K,M)}(t)$: (K, M) -truncated-discretized Dyson series). Let $H = H_0 + V$ be a Hamiltonian of a pair of Hermitian operators H_0, V . Let $t \in \mathbb{R}$, $K \in \mathbb{N}^+$ be the truncation parameter and $M \in \mathbb{N}^+$ be the time discretization parameter. The (K, M) -truncated-discretized Dyson series is defined as

$$D_{(K,M)}(t) := \sum_{k=0}^K (-i\Delta)^k B_{(k,M)}(t) \quad (\text{B10})$$

and

$$B_{(k,M)}(t) := \sum_{m_k=k-1}^{M-1} \dots \sum_{m_2=1}^{m_3-1} \sum_{m_1=0}^{m_2-1} V(m_k \Delta) \dots V(m_2 \Delta) V(m_1 \Delta), \quad (\text{B11})$$

where $\Delta := t/M$, and $V(s) := e^{iH_0 s} V e^{-iH_0 s}$ for $s \in \mathbb{R}$.

Note that the above definition excludes the *collisions*, i.e., the cases where there is at least one i such that $m_i = m_{i+1}$ in the expansion that defines the operators $B_{(k,M)}(t)$. Alternatively, one could include the collisions, and use an alternative discretization defined as follows. We will see that in terms of approximating the original operator, including/excluding the collision terms does not matter much. This is related to the fact that there are

overwhelmingly more cases where there is no collision compared to the number of cases with collision.

Definition B.3. ($\tilde{D}_{(K,M)}(t)$: (K, M) -truncated-discretized Dyson series with collisions). Let $H = H_0 + V$ be a Hamiltonian of a pair of Hermitian operators H_0, V . Let $t \in \mathbb{R}$, $K \in \mathbb{N}^+$ be the truncation parameter and $M \in \mathbb{N}^+$ be the time discretization parameter. The (K, M) -truncated-discretized Dyson series is defined as

$$\tilde{D}_{(K,M)}(t) := \sum_{k=0}^K (-i\Delta)^k \tilde{B}_{(k,M)}(t) \quad (\text{B12})$$

and

$$\tilde{B}_{(k,M)}(t) := \sum_{m_k=0}^{M-1} \dots \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} V(m_k \Delta) \dots V(m_2 \Delta) V(m_1 \Delta), \quad (\text{B13})$$

where $\Delta := t/M$, and $V(s) := e^{iH_0 s} V e^{-iH_0 s}$ for $s \in \mathbb{R}$.

We then employ the following Theorems B.4 and B.9 to find the minimal truncation and discretization parameters K, M , respectively, in order to guarantee an error ϵ in approximating the time evolution operator U_I in the interaction picture.

Theorem B.4. (Approximating time evolution in the interaction picture with truncated-discretized Dyson series). Let $H = H_0 + V$ be a Hamiltonian of a pair of Hermitian operators H_0, V . Let $D_{(K,M)}(t), \tilde{D}_{(K,M)}(t)$ be the (K, M) -truncated-discretized Dyson series without/with collisions defined as in Definitions B.2 and B.3 respectively, where $t \in \mathbb{R}$, $K \in \mathbb{N}^+$ and $M \in \mathbb{N}^+$, and let $\epsilon, \epsilon_1, \epsilon_2 \in \mathbb{R}^+$ be such that $\epsilon = \epsilon_1 + \epsilon_2$. Let $\mathcal{T}[e^{-i \int_0^t V(s) ds}]$ be the time evolution in the interaction picture defined in Lemma B.1. Then,

$$\begin{aligned} \|\mathcal{T}[e^{-i \int_0^t V(s) ds}] - D_{(K,M)}(t)\| &\leq \epsilon, \\ \|\mathcal{T}[e^{-i \int_0^t V(s) ds}] - \tilde{D}_{(K,M)}(t)\| &\leq \epsilon \end{aligned} \quad (\text{B14})$$

for all

$$K \geq -1 + \frac{\ln 1/\epsilon_1}{W(\frac{\ln 1/\epsilon_1}{t \max_s \|H(s)\|})},$$

or more explicitly, $K \geq \max\{2\|V\|t, e\|V\|t + \ln(1/\epsilon_1)\}$,

$$(\text{B15})$$

where $W(\cdot)$ is the Lambert W function, and for all

$$\begin{aligned} M &\geq \max\left\{2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{2t^2\|V\|e^{t\|V\|}(\|H_0\| + 2\|V\|)}{\epsilon_2}\right\}, \\ \tilde{M} &\geq \max\left\{2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{6t^2\|H_0\|\|V\|e^{t\|V\|}}{\epsilon_2}\right\} \end{aligned} \quad (\text{B16})$$

of the Dyson series without and with collisions, respectively.

Proof of Theorem B.4. We prove the theorem in two parts by first approximating the series expansion of $\mathcal{T}[e^{-i \int_0^t V(s) ds}]$ with a finite series (hence the truncation parameter K enters), and then discretizing the integrals in the series expansion with a finite sum (hence the discretization parameter M enters). More precisely, we bound the terms appearing in the rhs of

$$\left\| \mathcal{T}[e^{-i \int_0^t V(s) ds}] - D_{(K,M)}(t) \right\| \leq \left\| \mathcal{T}[e^{-i \int_0^t V(s) ds}] - D_K(t) \right\| + \|D_K(t) - D_{(K,M)}(t)\|, \quad (\text{B17})$$

where $D_K(t)$ is the truncated Dyson series defined as

$$D_K(t) := \sum_{k=0}^K (-i)^k B_k(t),$$

where $B_k(t) := \int_{t_k}^t dt_k \dots \int_{t_2}^{t_3} dt_2$

$$\times \int_0^{t_2} dt_1 V(t_k) \dots V(t_2) V(t_1). \quad (\text{B18})$$

To complete the proof, we first prove Lemma B.5 that bounds the first term in Eq. (B17), i.e., the truncation error.

Then we demonstrate Lemmas B.6 and B.7, without and with collisions, respectively, that bounds the second term in Eq. (B17), i.e., the discretization error.

Remark. The derivation of the truncation bound K in Lemma B.5 closely follows the approach described in [42]. However, we provide an improved bound for the discretization parameter M in the collision-free scenario in Lemma B.6. To the best of our knowledge, Lemma B.7 presents the first rigorous bound for the discretized Dyson series parameter $\tilde{M}$ that explicitly accounts for collisions.

Lemma B.5. (Approximation error of truncating the Dyson series). The truncation error of the Dyson series at order K satisfies the bound

$$\|\mathcal{T}[e^{-i \int_0^t V(s) ds}] - D_K(t)\| \leq \epsilon_1 \quad (\text{B19})$$

whenever

$$K \geq \max \left\{ 2\|V\|t, -1 + \frac{\ln 1/\epsilon_1}{W(\frac{\ln 1/\epsilon_1}{te\|V\|})} \right\},$$

$$\text{or more explicitly, } K \geq \max \{2\|V\|t, e\|V\|t + \ln(1/\epsilon_1)\}. \quad (\text{B20})$$

Proof of Lemma B.5. Following the definition,

$$\left\| \mathcal{T}[e^{-i \int_0^t V(s) ds}] - \sum_{k=0}^K (-i)^k B_k(t) \right\| \leq \sum_{k=K+1}^{\infty} \|B_k(t)\| \quad (\text{B21})$$

$$= \sum_{k=K+1}^{\infty} \frac{1}{k!} \left\| \int_0^t \dots \int_0^t \mathcal{T}[V(t_1) \dots V(t_k)] dt_1 \dots dt_k \right\| \quad (\text{B22})$$

$$\leq \sum_{k=K+1}^{\infty} \frac{1}{k!} \left\| \int_0^t \dots \int_0^t \prod_{j=1}^k \|V(t_j)\| dt_1 \dots dt_k \right\| \quad (\text{B23})$$

$$\leq \sum_{k=K+1}^{\infty} \frac{(t \max_s \|V(s)\|)^k}{k!} \quad (\text{B24})$$

$$= \frac{(t \max_s \|V(s)\|)^{K+1}}{(K+1)!} \sum_{k=K+1}^{\infty} \frac{(t \max_s \|V(s)\|)^{k-K-1}}{(K+2) \dots (k-1)k} \quad (\text{B25})$$

$$\leq \frac{(t \max_s \|V(s)\|)^{K+1}}{(K+1)!} \sum_{k=K+1}^{\infty} \left(\frac{t \max_s \|V(s)\|}{K+2} \right)^{k-K-1} \quad (\text{B26})$$

$$\leq \frac{(t \max_s \|V(s)\|)^{K+1}}{(K+1)!} \sum_{k=K+1}^{\infty} \left(\frac{K}{2(K+2)} \right)^{k-K-1} \quad (\text{B27})$$

$$= \frac{(t \max_s \|V(s)\|)^{K+1}}{(K+1)!} \sum_{k=0}^{\infty} \left(\frac{K}{2(K+2)} \right)^k \quad (\text{B28})$$

$$\leq \frac{(t \max_s \|V(s)\|)^{K+1}}{(K+1)!} \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k \quad (\text{B29})$$

$$= 2 \frac{(t \max_s \|V(s)\|)^{K+1}}{(K+1)!} \quad (\text{B30})$$

$$\leq \left(\frac{t \max_s \|V(s)\|}{K+1} \right)^{K+1} \frac{2}{\sqrt{2\pi(K+1)} e^{1/(12(K+1)+1)}} \quad (\text{B31})$$

$$\leq \left(\frac{t e \max_s \|V(s)\|}{K+1} \right)^{K+1}. \quad (\text{B32})$$

Note that $\|V(s)\| = \|e^{-iH_0 s} V e^{iH_0 s}\| = \|V\|$. The exact solution to

$$\left(\frac{e\|V\|t}{K+1} \right)^{K+1} \leq \epsilon_1 \quad (\text{B33})$$

is given by the Lambert-W function, i.e.,

$$K \geq \frac{\ln 1/\epsilon_1}{W\left(\frac{\ln 1/\epsilon_1}{te\|V\|}\right)} - 1. \quad (\text{B34})$$

We can derive a bound on K without using the Lambert W function as follows. Eq. (B33) is satisfied if (note that we assume $t > 0$ or otherwise use the absolute value of it, i.e., $|t|$)

$$K \ln \left(\frac{K}{e\|V\|t} \right) \geq \ln(1/\epsilon_1). \quad (\text{B35})$$

Assume $K \geq e\|V\|t$ (such as choose $K = e\|V\|t + a \ln(1/\epsilon_1)$ where $e\|V\|t > a \ln(1/\epsilon_1)$). Replace $K = e\|V\|t + a \ln(1/\epsilon_1)$ to obtain

$$(e\|V\|t + a \ln(1/\epsilon_1)) \ln \left(1 + \frac{a \ln(1/\epsilon_1)}{e\|V\|t} \right) \geq \ln(1/\epsilon_1). \quad (\text{B36})$$

The expansion $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$ indicates that $x > \log(1+x)$. Combined with the condition that $x < 1$ (equivalent to $e\|V\|t > a \ln(1/\epsilon_1)$), we observe that Eq. (B36) implies

$$(e\|V\|t + a \ln(1/\epsilon_1)) \frac{a \ln(1/\epsilon_1)}{e\|V\|t} \geq \ln(1/\epsilon_1), \quad (\text{B37})$$

which simplifies to

$$a \ln(1/\epsilon_1) + \frac{a^2 (\ln(1/\epsilon_1))^2}{e\|V\|t} \geq \ln(1/\epsilon_1). \quad (\text{B38})$$

This is satisfied for any $a \geq 1$; hence, $K = e\|V\|t + \ln(1/\epsilon_1)$ is one solution of the inequality with the condition that $e\|V\|t > \ln(1/\epsilon_1)$. ■

We now bound the second term of the rhs of Eq. (B17) for the case without collisions

Lemma B.6. (Approximation error of discretizing the truncated Dyson series). Let $H = H_0 + V$ be a given Hamiltonian. Let $D_K(t)$ and $D_{K,M}(t)$ be truncated and truncated-discretized Dyson series, with parameters K and (K, M) , respectively. Let $\epsilon_2 > 0$. Then,

$$\|D_K(t) - D_{(K,M)}(t)\| \leq \epsilon_2 \quad (\text{B39})$$

is satisfied for all choices of

$$M \geq \max \left\{ 2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{2t^2\|V\|(\|H_0\|e^{t\|V\|} + 2\|V\|e^{t\|V\|})}{\epsilon_2} \right\}. \quad (\text{B40})$$

Proof of Lemma B.6. We use the definitions for $D_K(t)$ [Eq. (B18)] and $D_{(K,M)}(t)$ (Definition B.2), and bound the approximation error term by term, i.e.,

$$\|D_K(t) - D_{(K,M)}(t)\| \leq \sum_{k=1}^K \|B_k(t) - \Delta^k B_{(k,M)}(t)\|. \quad (\text{B41})$$

Since we calculate errors on the rhs term by term, denote $\Xi_k := B_k(t) - \Delta^k B_{(k,M)}(t)$, for the sake of future notational ease. Below we bound $\|\Xi_k\|$. To get an idea of how it works, let us start from the easiest case, Ξ_1 , and proceed for higher values of k from there.

Bounding $\|\Xi_1\|$: Let us first write down the expressions explicitly:

$$\Xi_1 = \int_0^t dt_1 V(t_1) - \Delta \sum_{m_1=0}^{M-1} V(m_1 \Delta), \quad (\text{B42})$$

where $\Delta := t/M$, and $V(s) := e^{+iH_0 s} V e^{-iH_0 s}$ for $s \in \mathbb{R}$. We then partition the integral into M segments, each of which is of equal length and obtain

$$\Xi_1 = \sum_{m_1=0}^{M-1} \left[\int_{m_1 \Delta}^{(m_1+1)\Delta} dt_1 V(t_1) - V(m_1 \Delta) \Delta \right] \quad (\text{B43})$$

$$= \sum_{m_1=0}^{M-1} \left[\int_{m_1 \Delta}^{(m_1+1)\Delta} dt_1 (V(t_1) - V(m_1 \Delta)) \right] \quad (\text{B44})$$

$$= \sum_{m_1=0}^{M-1} e^{iH_0 m_1 \Delta} \left[\int_0^\Delta dt_1 (V(t_1) - V) \right] e^{-iH_0 m_1 \Delta}, \quad (\text{B45})$$

$$V(s) - V = \sum_{n=1}^{\infty} \frac{s^n}{n!} [iH_0, [iH_0, [\dots, [iH_0, V] \dots]]]. \quad (\text{B48})$$

where the second line follows straightforwardly, and the last line uses the definition of $V(s)$.

By the triangle inequality, we get the bound

$$\|\Xi_1\| \leq \sum_{m_1=0}^{M-1} \left\| \int_0^\Delta dt_1 (V(t_1) - V) \right\| \quad (\text{B46})$$

$$= M \left\| \int_0^\Delta dt_1 (V(t_1) - V) \right\|. \quad (\text{B47})$$

Using the Baker–Campbell–Hausdorff formula we show that

For general H_0 and V , i.e., no assumptions are made on H_0 or V , the integral can be bounded by

$$\left\| \int_{s=0}^\Delta ds (V(s) - V) \right\| \leq \sum_{n=1}^{\infty} \frac{|\Delta|^{n+1} 2^n}{(n+1)!} \|H_0\|^n \|V\|. \quad (\text{B49})$$

This bound could be improved considerably if we assumed local H_0 and V , by exploiting locality as in the Lieb–Robinson bounds, which can be interesting for a future work. Assuming $\Delta\|H_0\| \leq 1/2$ (hence $M \geq 2t\|H_0\|$), we arrive at the following upper bound:

$$\left\| \int_{s=0}^\Delta ds (V(s) - V) \right\| \leq \Delta^2 \|H_0\| \|V\| \sum_{n=1}^{\infty} (\Delta\|H_0\|)^{n-1} \leq 2\Delta^2 \|H_0\| \|V\|. \quad (\text{B50})$$

Bounding $\|\Xi_2\|$: This will be more complicated than the previous one, but it will be illuminating for bounding the higher order terms $\|\Xi_k\|$ for arbitrary $k > 2$. Let us proceed in a similar way as we did for Ξ_1 , and write down the expression for Ξ_2 explicitly:

$$\Xi_2 = \int_{t_2}^t dt_2 V(t_2) \int_{t_1=0}^{t_2} dt_1 V(t_1) - \Delta^2 \sum_{m_2=1}^{M-1} V(m_2 \Delta) \sum_{m_1=0}^{m_2-1} V(m_1 \Delta). \quad (\text{B51})$$

Notice that by partitioning the integral and then matching them with the corresponding discretized versions, we can rewrite Ξ_2 as follows:

$$\begin{aligned} \Xi_2 &= \sum_{m_2=1}^{M-1} \int_0^\Delta dt_2 V(m_2 \Delta + t_2) \sum_{m_1=0}^{m_2-1} \int_0^\Delta dt_1 V(m_1 \Delta + t_1) - \Delta^2 \sum_{m_2=1}^{M-1} V(m_2 \Delta) \sum_{m_1=0}^{m_2-1} V(m_1 \Delta) \\ &\quad + \frac{1}{2} \sum_{m=0}^{M-1} \int_0^\Delta dt_1 \int_0^\Delta dt_2 V(m \Delta + t_1) V(m \Delta + t_2). \end{aligned} \quad (\text{B52})$$

Observe that we can match the first and the second term on the rhs one by one, whereas the third term is not matched. More precisely,

$$\begin{aligned} \Xi_2 &= \sum_{m_2=1}^{M-1} \sum_{m_1=0}^{m_2-1} \left[\int_0^\Delta dt_2 \int_0^\Delta dt_1 (V(m_2 \Delta + t_2) V(m_1 \Delta + t_1) - V(m_2 \Delta) V(m_1 \Delta)) \right] \\ &\quad + \frac{1}{2} \sum_{m=0}^{M-1} \int_0^\Delta dt_1 \int_0^\Delta dt_2 V(m \Delta + t_1) V(m \Delta + t_2). \end{aligned} \quad (\text{B53})$$

Using the fact that $(A_1 A_2 - B_1 B_2) = ((A_1 - B_1) B_2 + A_1 (A_2 - B_2))$ we get

$$\begin{aligned}
\Xi_2 = & \sum_{m_2=1}^{M-1} \sum_{m_1=0}^{m_2-1} \left[\int_0^\Delta dt_2 (V(m_2\Delta + t_2) - V(m_2\Delta)) \int_0^\Delta dt_1 V(m_1\Delta) \right. \\
& + \left. \int_0^\Delta dt_2 V(m_2\Delta + t_2) \int_0^\Delta dt_1 (V(m_1\Delta + t_1) - V(m_1\Delta)) \right] \\
& + \frac{1}{2} \sum_{m=0}^{M-1} \int_0^\Delta dt_1 \int_0^\Delta dt_2 V(m\Delta + t_1) V(m\Delta + t_2).
\end{aligned} \tag{B54}$$

Taking the norm of the above expression, applying the triangle inequality and submultiplicative and unitary invariance property of the norm, we get

$$\|\Xi_2\| \leq 2 \binom{M}{2} \left\| \int_0^\Delta dt (V(t) - V) \right\| \Delta \|V\| + \frac{1}{2} \sum_{m=0}^{M-1} \int_0^\Delta dt_1 \int_0^\Delta dt_2 V(m\Delta + t_1) V(m\Delta + t_2) \tag{B55}$$

$$\leq 2 \binom{M}{2} \left\| \int_0^\Delta dt (V(t) - V) \right\| \Delta \|V\| + \frac{1}{2} M (\Delta \|V\|)^2. \tag{B56}$$

Bounding $\|\Xi_k\|$: Following the same lines of steps for Ξ_2 , we find the following bound for $\|\Xi_k\|$ for any arbitrary $k \in \mathbb{N}^+$:

$$\|\Xi_k\| \leq k \binom{M}{k} \left\| \int_0^\Delta dt (V(t) - V) \right\| (\Delta \|V\|)^{k-1} + \frac{2k(\Delta \|V\|)^k M^{k-1}}{(k-1)!}, \tag{B57}$$

where the second term comes from bounding the volume of the collision cases, i.e., the k -tuples where there is at least one collision such as $m_i = m_j$ for some $i \neq j$. The same analysis has been performed in Ref. [42] [see Eq. (A16)].

The bound on the total error will then be

$$\|D_K(t) - D_{(K,M)}(t)\| \leq \sum_{k=1}^K \|\Xi_k\| \tag{B58}$$

$$\leq \left\| \int_0^\Delta dt (V(t) - V) \right\| \sum_{k=1}^K k \binom{M}{k} (\Delta \|V\|)^{k-1} + \sum_{k=2}^K \frac{2k(\Delta \|V\|)^k M^{k-1}}{(k-1)!}. \tag{B59}$$

The sum in the first term can be bounded by

$$\sum_{k=1}^K k \binom{M}{k} (\Delta \|V\|)^k = \sum_{k=1}^K k \frac{M!}{k!(M-k)!} \left(\frac{t\|V\|}{M} \right)^k \tag{B60}$$

$$\leq \sum_{k=1}^K k \frac{t^k \|V\|^k}{k!} \tag{B61}$$

$$= t\|V\| \sum_{k=1}^K \frac{(t\|V\|)^{k-1}}{(k-1)!} \tag{B62}$$

$$\leq \|V\| \sum_{k=0}^{\infty} \frac{(t\|V\|)^k}{k!} \tag{B63}$$

$$= t\|V\| e^{t\|V\|}. \tag{B64}$$

The second term can be bounded by

$$\sum_{k=2}^K \frac{2k(\Delta\|V\|)^k M^{k-1}}{(k-1)!} = 2\Delta\|V\| \sum_{k=2}^K \frac{k(\Delta\|V\|)^{k-1} M^{k-1}}{(k-1)!} \quad (\text{B65})$$

$$\leq 4\Delta\|V\| \sum_{k=2}^K \frac{(\Delta\|V\|)^{k-1} M^{k-1}}{(k-2)!} \quad (\text{B66})$$

$$= 4\Delta^2\|V\|^2 M \sum_{k=2}^K \frac{(\Delta\|V\| M)^{k-2}}{(k-2)!} \quad (\text{B67})$$

$$\leq 4\Delta^2\|V\|^2 M e^{\Delta\|V\| M} \quad (\text{B68})$$

$$= \frac{4t^2\|V\|^2}{M} e^{t\|V\|}. \quad (\text{B69})$$

We put Eqs. (B60) and (B65) into Eq. (B58) and find

$$\|D_K(t) - D_{(K,M)}(t)\| \leq M e^{e t\|V\|+1} \left\| \int_0^\Delta dt (V(t) - V) \right\| + \frac{4t^2\|V\|^2}{M} e^{t\|V\|}. \quad (\text{B70})$$

Furthermore, assuming $\Delta\|H_0\| \leq 1/2$ implies

$$\left\| \int_0^\Delta dt (V(t) - V) \right\| \leq 2\Delta^2\|H_0\|\|V\|. \quad (\text{B71})$$

Using this, we obtain

$$\frac{2t^2\|H_0\|\|V\|e^{t\|V\|} + 4t^2\|V\|^2 e^{t\|V\|}}{M} \leq \epsilon_2, \quad (\text{B72})$$

and thus $\|D_K(t) - D_{(K,M)}(t)\| \leq \epsilon_2$ is satisfied for all

$$M \geq \frac{2t^2\|V\|(\|H_0\|e^{t\|V\|} + 2\|V\|e^{t\|V\|})}{\epsilon_2}. \quad (\text{B73})$$

Together with the condition $\Delta\|H_0\| \leq 1/2$, the proof is complete. $\blacksquare$

Lemma B.7. (Approximating time evolution in the interaction picture with the truncated-discretized Dyson series with collisions). Let $H = H_0 + V$ be a given Hamiltonian. Let $D_K(t)$ and $\tilde{D}_{K,\tilde{M}}(t)$ be the truncated Dyson series and the truncated-discretized Dyson series with collisions, with parameters K and $(K, \tilde{M})$, respectively. Let $\epsilon_2 > 0$. Then,

$$\|D_K(t) - \tilde{D}_{(K,\tilde{M})}(t)\| \leq \epsilon_2 \quad (\text{B74})$$

for all

$$\tilde{M} \geq \max \left\{ 2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{6t^2\|H_0\|\|V\|e^{t\|V\|}}{\epsilon_2} \right\}. \quad (\text{B75})$$

Proof. For notational ease, we use M instead of $\tilde{M}$ in this proof.

Following the same method as for the proof of Lemma B.6, we want to bound $\|\Xi_k\|$. The first term, $\|\Xi_1\|$, is identical, so let us consider Ξ_2 first:

$$\tilde{\Xi}_2 = \int_{t_2}^{t'} dt_2 V(t_2) \int_{t_1=0}^{t_2} dt_1 V(t_1) - \frac{\Delta^2}{2} \sum_{m_2=0}^{M-1} \sum_{m_1=0}^{M-1} \mathcal{T}(V(m_2\Delta)V(m_1\Delta)) \quad (\text{B76})$$

$$\begin{aligned} &= \sum_{m_2=1}^{M-1} \int_0^\Delta dt_2 V(m_2\Delta + t_2) \sum_{m_1=0}^{m_2-1} \int_0^\Delta dt_1 V(m_1\Delta + t_1) - \Delta^2 \sum_{m_2=1}^{M-1} V(m_2\Delta) \sum_{m_1=0}^{m_2-1} V(m_1\Delta) \\ &\quad + \frac{1}{2} \sum_{m=0}^{M-1} \int_0^\Delta dt_1 \int_0^\Delta dt_2 V(m\Delta + t_2) V(m\Delta + t_1) - \frac{\Delta^2}{2} \sum_{m=0}^{M-1} V(m\Delta)^2. \end{aligned} \quad (\text{B77})$$

Since the first two terms are exactly what we had previously, let us focus on the two last terms for now:

$$A_2 = \frac{1}{2} \sum_{m=0}^{M-1} \int_0^\Delta dt_1 \int_0^\Delta dt_2 (V(m\Delta + t_2)V(m\Delta + t_1) - V(m\Delta)^2) \quad (\text{B78})$$

$$= \frac{1}{2} \sum_{m=0}^{M-1} \int_0^\Delta dt_1 \int_0^\Delta dt_2 ((V(m\Delta + t_2) - V(m\Delta))V(m\Delta) + V(m\Delta + t_2)(V(m\Delta + t_1) - V(m\Delta))). \quad (\text{B79})$$

Taking the norm of the above expression, applying the triangle inequality and submultiplicative and unitary invariance property of the norm, we get

$$\|A_2\| \leq \binom{M}{1} \left\| \int_0^\Delta dt (V(t) - V) \right\| \Delta \|V\|. \quad (\text{B80})$$

In general, we have

$$\frac{\|A_k\|}{\left\| \int_0^\Delta dt (V(t) - V) \right\| (\Delta \|V\|)^{k-1}} \leq \sum_{q=1}^{k-1} \binom{k-1}{q} \binom{M}{k-q} \quad (\text{B81})$$

$$= \sum_{q=1}^{k-1} \frac{(k-1)!}{(k-1)!} \frac{(k-1)!}{q!(k-1-q)!} \binom{M}{k-q} \quad (\text{B82})$$

$$= \frac{1}{(k-1)!} \sum_{q=1}^{k-1} \frac{(k-1)!^2}{q!(k-1-q)!} \frac{M!}{(k-q)!(M-k+q)!} \quad (\text{B83})$$

$$\leq \frac{1}{(k-1)!} \sum_{q=1}^{k-1} \frac{(k-1)^{2q} M^{k-q}}{q!} \quad (\text{B84})$$

$$= \frac{M^k}{(k-1)!} \sum_{q=1}^{k-1} \frac{1}{q!} \left(\frac{(k-1)^2}{M} \right)^q \quad (\text{B85})$$

$$\leq \frac{M^k}{(k-1)!} \sum_{q=0}^{\infty} \frac{1}{q!} \left(\frac{(k-1)^2}{M} \right)^q \quad (\text{B86})$$

$$= \frac{M^k}{(k-1)!} e^{(k-1)^2/M}. \quad (\text{B87})$$

If we assume that $(K-1)^2/\ln 2 < M$, we obtain

$$\frac{\|A_k\|}{\left\| \int_0^\Delta dt (V(t) - V) \right\| (\Delta \|V\|)^{k-1}} \leq \frac{2M^k}{(k-1)!}. \quad (\text{B88})$$

When $\Delta \|H_0\| \leq 1/2$, we have

$$\|\tilde{\Xi}_k\| \leq \left(k \binom{M}{k} + \frac{2M^k}{(k-1)!} \right) \left\| \int_0^\Delta dt (V(t) - V) \right\| (\Delta \|V\|)^{k-1} \quad (\text{B89})$$

$$\leq 2 \left(k \binom{M}{k} + \frac{2M^k}{(k-1)!} \right) (\Delta \|V\|)^k \Delta \|H_0\|. \quad (\text{B90})$$

Now considering the summation over k , the first part can be bounded by

$$2\Delta \|H_0\| \sum_{k=1}^K k \binom{M}{k} (\Delta \|V\|)^k = 2\Delta \|H_0\| \sum_{k=1}^K k \frac{M!}{k!(M-k)!} \left(\frac{t\|V\|}{M} \right)^k \quad (\text{B91})$$

$$\leq 2\Delta \|H_0\| \sum_{k=1}^K k \frac{t^k \|V\|^k}{k!} \quad (\text{B92})$$

$$= 2\Delta t \|H_0\| \|V\| \sum_{k=1}^K \frac{(t\|V\|)^{k-1}}{(k-1)!} \quad (\text{B93})$$

$$\leq \frac{2t^2 \|H_0\| \|V\|}{M} \sum_{k=0}^{\infty} \frac{(t\|V\|)^k}{k!} \quad (\text{B94})$$

$$= \frac{2t^2 \|H_0\| \|V\|}{M} e^{t\|V\|}. \quad (\text{B95})$$

The second term can be bounded by

$$4\Delta \|H_0\| \sum_{k=1}^K \frac{M^k}{(k-1)!} (\Delta \|V\|)^k = 4\Delta \|H_0\| \sum_{k=1}^K \frac{(t\|V\|)^k}{(k-1)!} \quad (\text{B96})$$

$$= \frac{4t^2 \|H_0\| \|V\|}{M} \sum_{k=1}^K \frac{(t\|V\|)^{k-1}}{(k-1)!} \quad (\text{B97})$$

$$\leq \frac{4t^2 \|H_0\| \|V\|}{M} e^{t\|V\|}. \quad (\text{B98})$$

We then obtain that $\|D_K(t) - \tilde{D}_{(K,M)}(t)\| \leq \epsilon/2$ is satisfied for all

$$M \geq \frac{6t^2 \|H_0\| \|V\| e^{t\|V\|}}{\epsilon}. \quad (\text{B99})$$

Together with the condition $\Delta \|H_0\| \leq 1/2$, the proof is complete. $\blacksquare$

Theorem B.8. (Approximating time evolution in the interaction picture with truncated-discretized Dyson series without collisions). Let $H = H_0 + V$ be a Hamiltonian of a pair of Hermitian operators A, B . Let $D_{(K,M)}(t)$ be the (K, M) -truncated-discretized Dyson series defined as in Definition B.3, where $t \in \mathbb{R}$, $K \in \mathbb{N}^+$ and $M \in \mathbb{N}^+$, and let $\epsilon, \epsilon_1, \epsilon_2 \in \mathbb{R}^+$ be such that $\epsilon = \epsilon_1 + \epsilon_2$. Let $\mathcal{T}[e^{-i \int_0^t V(s) ds}]$ be the time evolution in interaction picture defined as inside of Lemma B.1. Then,

$$\|\mathcal{T}[e^{-i\int_0^t V(s)ds}] - D_{(K,M)}(t)\| \leq \epsilon \quad (\text{B100})$$

for all

$$K \geq \max \left\{ 2\|V\|t, -1 + \frac{\ln 1/\epsilon_1}{W(\frac{\ln 1/\epsilon_1}{te\|V\|})} \right\},$$

or more explicitly, $K \geq \max \{2\|V\|t, e\|V\|t + \ln(1/\epsilon_1)\},$

(B101)

and for all

$$M \geq \max \left\{ 2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{2t^2\|V\|e^{t\|V\|}(\|H_0\| + 2\|V\|)}{\epsilon_2} \right\}. \quad (\text{B102})$$

Proof. Follows immediately from Lemmas B.5 and B.6. ■

Theorem B.9. (Approximating time evolution in interaction picture with truncated-discretized Dyson series with collisions). Let $H = H_0 + V$ be a Hamiltonian of sum of Hermitian operators H_0, V . Let $\tilde{D}_{(K,\tilde{M})}(t)$ be the $(K, \tilde{M})$ truncated-discretized Dyson series defined as in Definition B.3, where $t \in \mathbb{R}$, $K \in \mathbb{N}^+$ and $\tilde{M} \in \mathbb{N}^+$, and let $\epsilon, \epsilon_1, \epsilon_2 \in \mathbb{R}^+$ be such that $\epsilon = \epsilon_1 + \epsilon_2$. Let $\mathcal{T}[e^{-i\int_0^t V(s)ds}]$ be the time evolution in interaction picture defined as inside of Lemma B.1. Then,

$$\|\mathcal{T}[e^{-i\int_0^t V(s)ds}] - \tilde{D}_{(K,\tilde{M})}(t)\| \leq \epsilon \quad (\text{B103})$$

for all

$$K \geq \max \left\{ 2\|V\|t, -1 + \frac{\ln 1/\epsilon_1}{W(\frac{\ln 1/\epsilon_1}{te\|V\|})} \right\},$$

or more explicitly, $K \geq \max \{2\|V\|t, e\|V\|t + \ln(1/\epsilon_1)\},$

(B104)

and for all

$$\tilde{M} \geq \max \left\{ 2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{6t^2\|H_0\|\|V\|e^{t\|V\|}}{\epsilon_2} \right\}. \quad (\text{B105})$$

Proof. Follows immediately from Lemmas B.5 and B.6. ■

We collect these results as a corollary:

Corollary B.10. (Approximating time evolution in Schrödinger picture with truncated-discretized Dyson series). Let $H = H_0 + V$ be a time-independent Hamiltonian with Hermitian H_0 and V . Let $D_{(K,M)}$ and $\tilde{D}_{(K,\tilde{M})}$ be given as in Definition B.2 and Definition B.3,

respectively. Then, in the case without collisions

$$\|e^{-iHt} - e^{-iH_0t}D_{(K,M)}(t)\| \leq \epsilon \quad (\text{B106})$$

for all

$$K \geq \max \left\{ 2\|V\|t, -1 + \frac{\ln 1/\epsilon_1}{W(\frac{\ln 1/\epsilon_1}{te\|V\|})} \right\} \quad \text{or more explicitly,}$$

$$K \geq \max \{2\|V\|t, e\|V\|t + \ln(1/\epsilon_1)\}, \quad (\text{B107})$$

and

$$M \geq \max \left\{ 2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{2t^2\|V\|e^{t\|V\|}(\|H_0\| + 2\|V\|)}{\epsilon_2} \right\}. \quad (\text{B108})$$

In the case with collisions,

$$\|e^{-iHt} - e^{-iH_0t}\tilde{D}_{(K,\tilde{M})}(t)\| \leq \epsilon \quad (\text{B109})$$

is satisfied for all

$$K \geq \max \left\{ 2\|V\|t, -1 + \frac{\ln 1/\epsilon_1}{W(\frac{\ln 1/\epsilon_1}{te\|V\|})} \right\} \quad \text{or more explicitly,}$$

$$K \geq \max \{2\|V\|t, e\|V\|t + \ln(1/\epsilon_1)\}, \quad (\text{B110})$$

and

$$\tilde{M} \geq \max \left\{ 2t\|H_0\|, \frac{(K-1)^2}{\ln 2}, \frac{6t^2\|H_0\|\|V\|e^{t\|V\|}}{\epsilon_2} \right\}. \quad (\text{B111})$$

Proof. Follows immediately from Lemma B.1 and Theorem B.4.

1. Quantum algorithm for simulating time-evolution in the interaction picture

Our Hamiltonian consists of various types of terms, such as those that are diagonal in the basis of electric fields and fermion operators. Furthermore, the electric field term (H_E) commutes with the mass term (H_M). These two terms are collected in the Hamiltonian H_0 , and the associated time evolution operators are fast-forwardable. In other words, for any time s , the Hamiltonian simulation

$$e^{-iH_0s} = e^{-i(H_E+H_M)s} = e^{-iH_Es}e^{-iH_Ms} \quad (\text{B112})$$

is implementable with polylogarithmic number of gates in $\{N, 1/\epsilon, \eta, \dots\}$. The quantum circuits for these fast-forwardable time evolutions are used while implementing $W_{(K,M)}$ and finally for e^{-iHt} . They consist of single-qubit

rotations and CNOTs. The error caused by these and other approximate single-qubit rotations (excluding Clifford and T-gates), in total n_{rot} -many with an error $\tilde{\epsilon}_3 = \epsilon_3/n_{\text{rot}}$, are considered altogether and give a total error at most ϵ_3 . Then the bit-precision for single qubit rotations are given by $b_{\text{rot}} = \lceil \log_2(1/\tilde{\epsilon}_3) \rceil = \lceil \log_2(n_{\text{rot}}/\epsilon_3) \rceil$. Given that each rotation synthesis costs $O(b_{\text{rot}})$ T-gates [67], and b_{rot} depends on $1/\epsilon_3$ only logarithmically, we typically choose $\epsilon_3 \approx \epsilon_1/10 = \epsilon_2/10$.

We first give the following intermediate result that lays out the complexity of implementing a unitary that approximates the interaction picture time evolution $U_I(t_0/\alpha)$ for a given time segment t_0/α . In the proof, we construct a quantum circuit that approximates $U_I(t_0/\alpha)$, and give a detailed quantum circuit compilation and its cost in Appendix B.2.

Lemma B.11. (Quantum circuit implementation of Dyson series $U_I(t_0/\alpha)$ for a time segment of t_0/α). Let $D_{(K,M)}(t)$ and $\tilde{D}_{(K,M)}(t)$ be given as in Definition B.2 and Definition B.3, respectively. Let K, M be the truncation and discretization parameters for time t_0/α and error ϵ' . Let α be the rescaling factor that arises as a result of block-encoding V , and let $t_0 \in \mathbb{R}$ be such that $\beta = \sum_{k=0}^K t_0^k/k! = 2$. Finally, let $\mathcal{M}$ be the number of local terms in H_0 . Then, there exist quantum circuits that

implement the unitary $W_{(K,M)}(t_0/\alpha, \epsilon)$ and $\tilde{W}_{(K,M)}(t_0/\alpha, \epsilon)$ such that

$$\begin{aligned} \|W_{(K,M)}(t_0/\alpha, \epsilon) - U_I(t_0/\alpha)\| &\leq \epsilon, \\ \text{and } \|\tilde{W}_{(K,M)}(t_0/\alpha, \epsilon) - U_I(t_0/\alpha)\| &\leq \epsilon. \end{aligned} \quad (\text{B113})$$

Both circuits use $\mathcal{O}(\mathcal{M} \log(k\mathcal{M}/\epsilon) + K)$ single qubit rotations and $\tilde{\mathcal{O}}(K \log M)$ additional T-gates, where $\tilde{\mathcal{O}}$ hides additional $\log$ factors in K . Furthermore, they call the block encodings $(\text{BE}_{V/\alpha}$ and $\text{BE}_{V/\alpha}^\dagger)$ $2K$ many times.

Proof. We implement the (K, M) -truncated-discretized Dyson series with quantum circuits using the method of LCU. Using either $D_{(K,M)}(t_0/\alpha)$ or $\tilde{D}_{(K,M)}(t_0/\alpha)$, we have a linear combination of products of Hermitian operators, i.e., V , interleaved with unitary time evolutions, i.e., $e^{-iH_0 s}$ for various times s that depend on the ancillae. The rest of the proof uses only the particular approximation of the Dyson series that is given in Definition B.2, except for one minor detail in the implementation of the bitonic sort, the proof and resource counts are identical.

The key circuit that implements an approximation to the Dyson series, conditioned on all the ancillae being in $|0\rangle$, is given in Fig. 6.

The cost of this circuit is

$$\begin{aligned} &2 \left[\mathcal{C}(\text{PREP}_{\sqrt{t_0^k/k!}}) + (K \log_2 M) \mathcal{C}(\text{C-Had}) + \mathcal{C}(\text{SORT}_{\log_2 M}^{(K)}) \right] \\ &+ (2K - 2) \mathcal{C}(\text{ADD}_{\log_2 M}) + K \mathcal{C}(\text{BE}_{V/\alpha}) + (K + 1) \mathcal{C}(e^{-iH_0 s}) + K(N + 1) \mathcal{C}(\text{Toffolis}). \end{aligned} \quad (\text{B114})$$

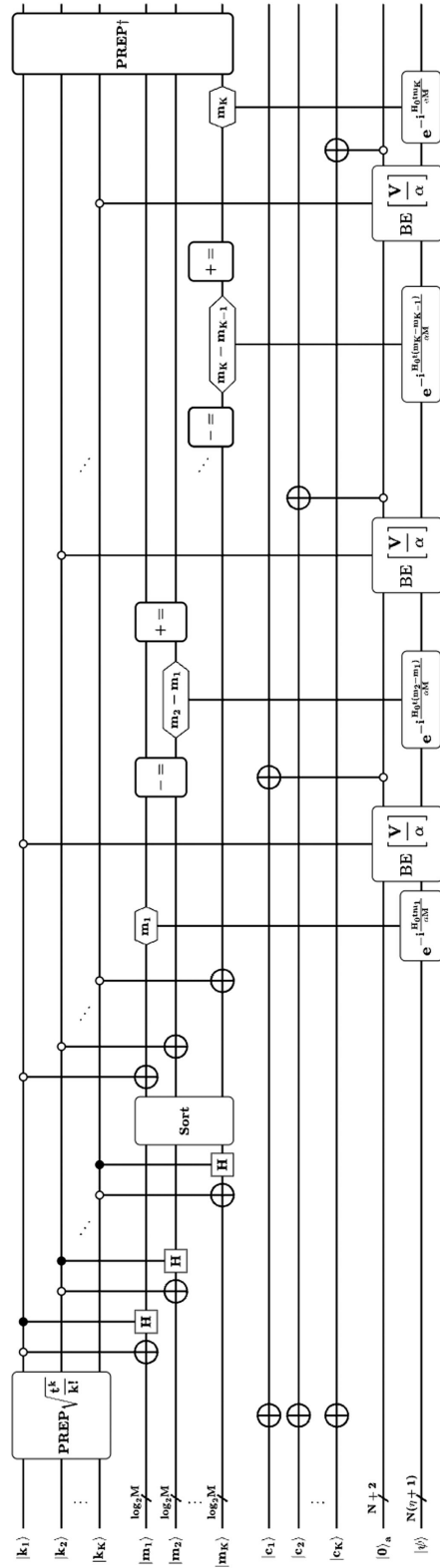
The first line is the cost of the PREP subroutine, whereas the second line is the cost of the SEL subroutine, and the construction is explained in Sec. IV B. Note that the prefactor 2 in the first line comes from the uncompute (or $\text{PREP}^\dagger$), and the rest can be straightforwardly derived from the circuit in Fig. 6. Moreover, M can always be chosen as the next power of two, such that we can create a uniform state preparation with Hadamards instead of a general method such as in Ref. [87]. Notice also that this is an upper bound, since some of the uncomputation can have a reduced cost. Up to infidelity from rotation synthesis, this circuit results in the following unitary U on any initial state $|0\rangle \otimes |\psi\rangle$:

$$U|0\rangle \otimes |\psi\rangle = \frac{1}{\beta} |0\rangle \otimes D_{(K,M)}(t_0/\alpha) |\psi\rangle + (\dots) |\Phi^\perp\rangle, \quad (\text{B115})$$

where $(|0\rangle\langle 0| \otimes \mathbb{1}) |\Phi^\perp\rangle = 0$, and β (or implicitly, t_0) is chosen such that one needs only (and exactly) one round of amplitude amplification, i.e., $\beta \geq 2$. We prove this by following the circuit given in Fig. 6. After applying $\text{PREP}_{\sqrt{t_0^k/k!}}$ and $\text{C-Had}^{\otimes K \log_2 M}$, we have

$$\frac{1}{\sqrt{\beta}} \sum_{k=0}^K \sqrt{\frac{t_0^k}{k!}} |1\rangle^{\otimes k} |0\rangle^{\otimes K-k} \otimes \frac{1}{\sqrt{M^k}} \sum_{m_1, m_2, \dots, m_k=0}^{M-1} |m_1\rangle |m_2\rangle \dots |m_k\rangle |0\rangle^{K-k} \otimes |\psi\rangle. \quad (\text{B116})$$

After SORT, we obtain

FIG. 6. Circuit that implements the unitary U that is given in Eq. (B115).

$$\begin{aligned} \frac{1}{\sqrt{\beta}} \sum_{k=0}^K \sqrt{\frac{t_0^k}{M^k k!}} |1\rangle^{\otimes k} |0\rangle^{\otimes K-k} \otimes \left[\sum_{0 \leq m_{\sigma(1)} < \dots < m_{\sigma(k)} \leq M-1} |m_{\sigma(1)}\rangle \dots |m_{\sigma(k)}\rangle |0\rangle^{K-k} \sum_{\sigma \in \mathcal{S}_k | m_{\sigma(1)} < \dots < m_{\sigma(k)}} |\sigma\rangle \right. \\ \left. + \sum_{\substack{0 \leq m_1, \dots, m_k \leq M-1 \\ \exists i \neq j, s, t, m_j = m_i}} \sum_{\sigma | m_{\sigma(1)} \leq \dots \leq m_{\sigma(k)}} |m_{\sigma(1)}\rangle \dots |m_{\sigma(k)}\rangle |0\rangle^{K-k} |\sigma\rangle \right] \otimes |\psi\rangle, \end{aligned} \quad (\text{B117})$$

where $\mathcal{S}_k$ is the permutation group of k objects and σ keeps track of the particular permutation that sorted the numbers $m_1, \dots, m_k$ to $m_{\sigma(1)}, \dots, m_{\sigma(k)}$ in increasing order. If one desires to implement $D_{(K,M)}$ rather than $\tilde{D}_{(K,M)}$, we will then need to flag the branch without collision [the second term in Eq. (B117)]:

$$\begin{aligned} \frac{1}{\sqrt{\beta}} \sum_{k=0}^K \sqrt{\frac{t_0^k}{M^k k!}} |1\rangle^{\otimes k} |0\rangle^{\otimes K-k} \otimes \left[\sum_{0 \leq m_{\sigma(1)} < \dots < m_{\sigma(k)} \leq M-1} |m_{\sigma(1)}\rangle \dots |m_{\sigma(k)}\rangle |0\rangle^{K-k} \sum_{\sigma \in \mathcal{S}_k | m_{\sigma(1)} < \dots < m_{\sigma(k)}} |\sigma\rangle |0\rangle \right. \\ \left. + \sum_{\substack{0 \leq m_1, \dots, m_k \leq M-1 \\ \exists i \neq j, s, t, m_j = m_i}} \sum_{\sigma | m_{\sigma(1)} \leq \dots \leq m_{\sigma(k)}} |m_{\sigma(1)}\rangle \dots |m_{\sigma(k)}\rangle |0\rangle^{K-k} |\sigma\rangle |1\rangle \right] \otimes |\psi\rangle. \end{aligned} \quad (\text{B118})$$

Note that one could amplify over the flag qubit being in state $|0\rangle$. However, the number of states in the no-collision branch is $\sum_{k=0}^K \binom{M}{k} k!$ which, for large M , makes the collision branch negligible. In other words, since the collisions contribute only to a slight increase of the induce one-norm, the amplification will be included in the overall OAA of the Dyson series.

Indeed, this is expected, because Theorem B.9 implies that $D_{(K,M)}(t)$ is ϵ -close to unitary. After applying the sequence of $e^{-iH_0 m_i t_0 / \alpha M}$ and block encodings of $-iV/\alpha$, we obtain the state

$$\begin{aligned} \frac{1}{\sqrt{\beta}} \sum_{k=0}^K \sqrt{\frac{t_0^k}{M^k k!}} |1\rangle^{\otimes k} |0\rangle^{\otimes K-k} \left[\sum_{0=m_1 < m_2 < \dots < m_k \leq M-1} |m_1\rangle |m_2\rangle \dots |m_k\rangle |0\rangle^{K-k} \sum_{\sigma \in \mathcal{S}_k} |\sigma\rangle |0\rangle \prod_{j=1}^k \frac{V(m_j \Delta)}{\alpha} |\psi\rangle |0\rangle_{BE} \right. \\ \left. + \sum_{\substack{0 \leq m_1, m_2, \dots, m_k \leq M-1 \\ \exists i \neq j, s, t, m_j = m_i}} \sum_{\sigma | m_{\sigma(1)} \leq \dots \leq m_{\sigma(k)}} |m_{\sigma(1)}\rangle |m_{\sigma(2)}\rangle \dots |m_{\sigma(k)}\rangle |0\rangle^{K-k} |0\rangle^{K-k} |\sigma\rangle |1\rangle \prod_{j=1}^k \frac{V(m_{\sigma(j)} \Delta)}{\alpha} |\psi\rangle |0\rangle_{BE} \right]. \end{aligned} \quad (\text{B119})$$

After uncomputing the SORT, controlled Hadamards, and applying $\text{PREP}^\dagger$, we obtain, in the $|0\rangle$ and no-collision branch of the state:

$$\begin{aligned} &= \frac{1}{\beta} |0\rangle^{K(\log_2 M + 1)} \otimes |0\rangle \sum_{k=0}^K \frac{t_0^k}{\alpha^k M^k} \sum_{0=m_1 < m_2 < \dots < m_k \leq M-1} \prod_{j=1}^k V(m_j \Delta) |\psi\rangle |0\rangle_{BE} \\ &+ \frac{1}{\beta} |0\rangle^{K(\log_2 M + 1)} \otimes |1\rangle \sum_{k=0}^K \frac{t_0^k}{\alpha^k M^k k!} \sum_{\substack{m_1, m_2, \dots, m_k \leq M-1 \\ \exists i \neq j, s, t, m_j = m_i}} \mathcal{T} \left(\prod_{j=1}^k V(m_j \Delta) \right) |\psi\rangle |0\rangle_{BE} + \dots \end{aligned} \quad (\text{B120})$$

$$= \frac{1}{\beta} |0\rangle^{K(\log_2 M + 1) + 1} |0\rangle_{BE} \otimes D_{(K,M)}(t_0/\alpha) |\psi\rangle + \dots \quad (\text{B121})$$

With similar arguments, and discarding the flag register for collision cases (by not creating it in the first place), we get

$$\begin{aligned} &= \frac{1}{\beta} |0\rangle^{K(\log_2 M + 1)} \sum_{k=0}^K \frac{t_0^k}{\alpha^k M^k} \sum_{0=m_1 < m_2 < \dots < m_k \leq M-1} \prod_{j=1}^k V(m_j \Delta) |\psi\rangle |0\rangle_{BE} \\ &+ \frac{1}{\beta} |0\rangle^{K(\log_2 M + 1)} \sum_{k=0}^K \frac{t_0^k}{\alpha^k M^k k!} \sum_{\substack{m_1, m_2, \dots, m_k \leq M-1 \\ \exists i \neq j, s, t, m_j = m_i}} \mathcal{T} \left(\prod_{j=1}^k V(m_j \Delta) \right) |\psi\rangle |0\rangle_{BE} + \dots \end{aligned} \quad (\text{B122})$$

$$= \frac{1}{\beta} |0\rangle^{K(\log_2 M+1)} |0\rangle_{BE} \otimes \tilde{D}_{(K,M)}(t_0/\alpha) |\psi\rangle + \dots \quad (\text{B123})$$

We further choose K and M such that $D_{(K,M)}(t_0/\alpha)$ approximates the Dyson series $U_I(t_0/\alpha)$ up to error $\epsilon/4 = (\epsilon_1 + \epsilon_2) \lceil \frac{t\alpha}{t_0} \rceil \frac{1}{4}$, then we know that $D_{(K,M)}(t_0/\alpha)$ is a unitary up to error $\epsilon/4$. After a single round of robust oblivious amplitude amplification [39], we ensure that we implement a unitary that is $3\epsilon/4$ close to $D_{(K,M)}(t_0/\alpha)$, which is also ϵ close to $U_I(t_0/\alpha)$ by the choice of K and M . This requires using two of the unitary circuits U and one $U^\dagger$. The resources estimate reported assume that the costs of U and $U^\dagger$ are identical. The remaining unbudgeted error is spared for other components in the rest of the circuit, e.g., error from cutoff and single-qubit rotations; see Table V. Hence the main claim follows.

The rest of the proof is devoted to the computational resource counts parts of the circuit except the cost of the block encoding $BE_{V/\alpha}$, since the circuit implementation of this highly depends on the structure of V and hence can vary case by case. For more details on these resource costs, see Appendix B 2. Expressing these costs in big-O notation completes the proof. ■

We finally put all the pieces together to find the total cost of implementing the quantum circuit $W(t, \epsilon)$ that approximates the time-evolution operator up to an error at most ϵ .

Corollary B.12. (Quantum circuit implementation of the time evolution based on the interaction picture). Let $H = H_0 + V$ be a Hamiltonian such that H_0 can be expressed as linear combination of $\mathcal{M}$ commuting k -local unitaries (or more generally exponentially fast-forwardable), and let $\alpha > 1$ be the minimal rescaling factor of V such that an efficient unitary block encoding $BE_{V/\alpha}$ of V/α is possible. Let $t \in \mathbb{R}$. Then, there exists a quantum circuit implementing the unitary $W(t, \epsilon')$ that approximates the time evolution e^{-iHt} such that

$$\|W(t, \epsilon') - e^{-iHt}\| \leq \epsilon'. \quad (\text{B124})$$

The circuit uses

$$\mathcal{O}(\alpha t \mathcal{M} \log(k \mathcal{M} \alpha t / \epsilon') + \log(\alpha t / \epsilon')) \quad (\text{B125})$$

single qubit rotations and

$$\mathcal{O}(\alpha t \log M + \alpha t \log M \log(\alpha t / \epsilon')) \quad (\text{B126})$$

additional Clifford + T gates. Furthermore, the circuit calls the block encoding $BE_{V/\alpha}$

$$\mathcal{O}(\alpha t \log(\alpha t / \epsilon')) \quad (\text{B127})$$

many times.

Proof. The circuit $W(t, \epsilon')$ we implement repeats the product of unitaries W_0 and $W_{(K,M)}(t_0/\alpha)$, $r = \lceil t\alpha/t_0 \rceil$ many times. More precisely,

$$W(t, \epsilon') = e^{-iH_0(t-(r-1)t_0/\alpha)} W_{(K,M)}(t - (r-1)t_0/\alpha, \epsilon'/r) \times \left[e^{-iH_0 t_0/\alpha} W_{(K,M)}(t_0/\alpha, \epsilon'/r) \right]^{r-1}. \quad (\text{B128})$$

Note that this circuit approximates the unitary operator e^{-iHt} up to an error at most ϵ' , which accounts for the truncation, discretization and rotation synthesis errors. The truncation and discretization parameters K and M are determined in Theorems B.4 and B.9, both for approximating the Dyson series without or with collisions. The complexity of $W(t, \epsilon')$ is then upper bounded by

$$r \left(\mathcal{C}(e^{-iH_0 t_0/\alpha}) + \mathcal{C}\left(W_{(K,M)}\left(\frac{t_0}{\alpha}, \frac{\epsilon'}{r}\right)\right) \right). \quad (\text{B129})$$

Then, the result follows by combining the costs $\mathcal{C}(e^{-iH_0 t_0/\alpha})$ and $\mathcal{C}(W_{(K,M)}(t_0/\alpha, \epsilon'/r))$, given in Appendix B 2. Note that the choice of t_0 affects the total number times one calls the block encoding of V in a few ways. Making the choice for $2 = \beta = \sum_{k=0}^{K-1} t_0^k/k!$, we find $t_0 = \ln 2 \approx 0.7$. Then, each circuit $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$ calls a number of $2K$ times the operator $BE_{V/\alpha}$ and K times its dagger $BE_{V/\alpha}^\dagger$. In total the number of calls to the block encodings is bounded by

$$\left\lceil \frac{3K\alpha t}{\ln 2} \right\rceil, \quad (\text{B130})$$

where $K = K(\frac{\ln 2}{\alpha}, \frac{\ln 2\epsilon'}{3t\alpha})$, whose value is

$$K = \mathcal{O}(\log(\alpha t / \epsilon')), \quad (\text{B131})$$

asymptotically. ■

2. Compilation and resource count for the Schwinger model

In this section, we give quantum circuit compilation details of the subroutines that appears in the implementation depicted in Fig. 6. In particular, these subroutines are elements that realize $W_{(K,M)}$, and the resource cost analysis is used for calculating the second term $\mathcal{C}(W_{(K,M)})$ in Eq. (B129). Note that the first term in Eq. (B129) is subdominant and just a special case of implementing $e^{-i(H_E + H_M)(\cdot)}$ for time t_0/α whose cost can be found below.

a. PREP

Our implementation uses a first register that encodes a weighted history state of the form:

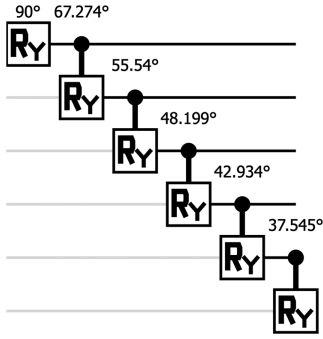


FIG. 7. Implementation of $\text{PREP}_{\sqrt{t_0^k/k!}}$ for $K = 6$ and $t_0 = \ln 2$.

$$\begin{aligned} & \text{PREP}_{\sqrt{t_0/k!}} |00\dots 0\rangle \\ &= \frac{1}{\sqrt{\beta}} \left(|00\dots 0\rangle + \sqrt{t_0} |10\dots 0\rangle + \frac{t_0}{\sqrt{2}} |110\dots 0\rangle + \dots \right. \\ & \quad \left. + \sqrt{\frac{t_0^K}{K!}} |11\dots 1\rangle \right), \end{aligned} \quad (\text{B132})$$

where β is the normalization factor

$$\beta = \sum_{k=0}^K \frac{t_0^k}{k!}. \quad (\text{B133})$$

As mentioned previously, in order to have exactly one step of AA, we choose t_0 such that β is arbitrarily close to 2. This step uses one rotation and $K - 1$ controlled rotations which can each be implemented with Clifford gates and 2 rotations, i.e., $2K - 1$ rotations of arbitrary angles. An example for $K = 6$ can be found in Fig. 7.

Then, we need K registers of size $\lceil \log_2 M \rceil$ that encodes the time steps. We choose M to be an integer power of 2, for the sake of ease in compilation/resource cost. Controlled on the k th qubit in the first register given in Eq. (B132), we create an equal weight superposition $\frac{1}{\sqrt{M}} \sum_{m_k=0}^{M-1} |m_k\rangle$ state in the k th register of this set of time parameters. This can be done with $K \log_2 M$ controlled-Hadamard gates. Each can be implemented with 2 T-gates, and hence it has a total cost of $2K \log_2 M$ T-gates. Controlled on the k th qubit being in the $|0\rangle$ state, the k th time register is set to $|M - 1\rangle$; performed via C-NOTs this operation only requires Cliffords. These last $K - k$ time registers will be set back to zero after the sort, importantly before calling BE_V . The result at this stage is

$$\propto \sum_{k=0}^K c_k |1\rangle^{\otimes k} |0\rangle^{\otimes K-k} \sum_{m_1, \dots, m_k=0}^{M-1} |m_1\rangle \dots |m_k\rangle |M-1\rangle \dots |M-1\rangle. \quad (\text{B134})$$

Then, the time registers are sorted in increasing order, i.e., they are put in order $m_1 \leq m_2 \leq \dots \leq m_k$. This is

achieved by a bitonic sort, as in Ref. [73]. The number of comparators in the bitonic sort is upper bounded by

$$\sum_{k=1}^{\lceil \log_2 K \rceil} k \lfloor K/2 \rfloor = \frac{\lceil \log_2 K \rceil + 1}{2} \left\lfloor \frac{K}{2} \right\rfloor \lceil \log_2 K \rceil. \quad (\text{B135})$$

Each comparator uses $\log_2 M$ Toffolis and an additional ancilla that stores the result of the comparator. The uncomputation of each Toffoli can be performed via phase-fixup measurement-based uncomputation similar to what is suggested in Ref. [72]. Additionally, $\log_2 M$ temporary ancillae are shared across all comparators. Controlled on the comparator's result, the two $\log_2 M$ qubit-size registers are swapped, with an additional $\log_2 M$ Toffolis. Hence, this stage costs

$$\log_2 M (\lceil \log_2 K \rceil + 1) \left\lfloor \frac{K}{2} \right\rfloor \lceil \log_2 K \rceil \quad (\text{B136})$$

Toffolis and

$$\frac{\lceil \log_2 K \rceil + 1}{2} \left\lfloor \frac{K}{2} \right\rfloor \lceil \log_2 K \rceil + \log_2 M \quad (\text{B137})$$

ancillae.

If we choose to not include the collisions, additional ancillae will be added to flag out the collisions. We do not need to apply AA since the probability of success of having zero collision is almost 1 for large M . To flag the no-collision branch, it requires $K - 1$ controlled-comparators, i.e., $(K - 1) \log_2 M$ Toffolis and $K - 1$ additional ancillae to record the results of these additional comparators. Finally, we AND the results of the individual comparators into a single flag qubit, which requires $(K - 2)$ Toffolis and $K - 2$ reusable/temporary additional ancillae.

This is the cost of one application of PREP. In the quantum circuit U , given in Fig. 6 and acting as in Eq. (B115), PREP and $\text{PREP}^\dagger$ are each called once. To construct $W_{(K,M)}(t_0/\alpha, \epsilon'/r)$, U itself is called twice and $U^\dagger$ is called once. Naively, the cost of a subroutine S has the same cost as $S^\dagger$ (= uncomputation of S), and hence Table III refers to the number of calls to the subroutine S as the total number of calls to S and $S^\dagger$. However, one can implement the uncomputations of certain subroutines more efficiently. For instance, in bitonic sort, the comparators can be uncomputed via phase fixup measurement-based uncomputation, which has been studied in detail in Ref. [75], which costs on average half of the Toffolis of the original subroutine itself.

b. SELECT

The select operators consist of the time evolution operator for the mass and electric terms and a block encoding of the interaction terms. The block encoding is

controlled on the history state register and the time evolution only needs to be controlled on the time steps registers. Note that since the operators are of the form $e^{-it_0 m_j / (\alpha M)} \text{BE}_{V/\alpha} e^{it_0 m_j / (\alpha M)}$, we will compute the difference between m_j and m_{j+1} and reduce by close to half the number of rotations required for the mass and electric terms. This costs $\log_2 M - 1$ Toffolis and $\log_2 M - 1$ reusable ancillae per addition or subtraction. In total, there are $2(K - 1)$ additions and subtractions. Moreover, in order to multiply the block encodings with minimal additional ancillae, a compression gadget is used. Specifically, we use the unary (k -hot to be specific) version of the compression gadget given in Ref. [78]. See Fig. 6. The compression gadget requires K many $(N + 2)$ -controlled Toffolis, where the controls are on the block-encoding ancillae (consisting of $N + 2$ qubits). This results in a total of $K(N + 1)$ Toffolis and $N + 1$ reusable ancillae in addition to the K ancillae used as the “counter” of the compression gadget. Hence, in addition to the subroutines we study below, $K(N + 1) + 2(K - 1)(\lceil \log_2 M \rceil - 1)$ additional Toffolis and $\max(\lceil \log_2 M \rceil - 1, N + 1)$ temporary ancillae are used.

The subroutine $\text{BE}_{V/\alpha}$ is implemented as a standard $\text{PREP}_{\text{BE}}^\dagger - \text{SEL}_{\text{BE}} - \text{PREP}_{\text{BE}}$ method. For the purpose of efficient implementation, we put the control on the PREP_W (and not on SEL_{BE}), due to the use of the unary encoding in the PREP_W registers. This register creates a W -state that indexes the link r on which V acts. This takes $N - 1$ qubits. For the number of links $(N - 1)$ that is an integer power of 2, the quantum circuit for the controlled- PREP_W is given in Fig. 8. This circuit costs $N - 2$ c-Had gates, which can be implemented with 2 T-gates each. There is an additional PREP' which creates an equal superposition over 3 qubits for block-encoding the interaction term, consisting of 8 terms, for a fixed link r . The complete subroutine $\text{PREP}_{\text{BE}} = \text{PREP}_W \otimes \text{PREP}'$ costs $2(N - 2)$ many T-gates. SEL_{BE} consists of gates that select between the linear combination of unitaries given in Eq. (A11). We implement the circuit given in Ref. [76]

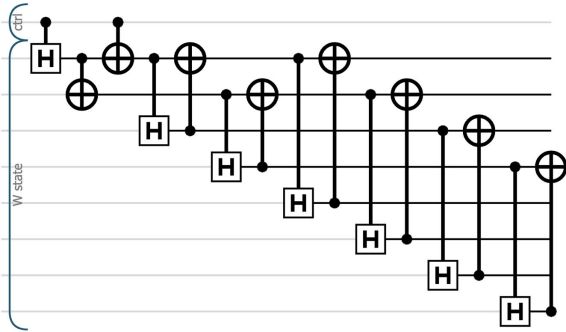


FIG. 8. Controlled- PREP_W when the number of links, i.e., $N - 1$, is a power of two. If it is not a power of 2, we modify this circuit by replacing the Hadamard gates with appropriate R_y rotations.

(see Sec. V.2, Fig. 4), with a slight modification: instead of controlling PREP' , we control the phase gates, i.e., S and CZ gates acting on the last qubit. The subroutines U are incrementers of size η , the subroutines Q are NOT gates, and we have an additional control- S and a CCZ gate. Then, SEL_{BE} costs $(N - 1)(\eta - 1) + 1$ Toffolis, $4N + 3$ T-gates, and $\eta - 1$ temporary ancillae. Hence, in total, $\text{BE}_{V/\alpha}$ costs $8N + 4(N - 1)(\eta - 1) - 1$ T-gates, derived as twice the cost of PREP_{BE} and once the cost of SEL , and an additional $\eta - 1$ ancillae.

We implement the fast-forwardable $e^{-i(H_E + H_M)s}$ for a given s , by implementing $e^{-i(H_M)s}$ and $e^{-i(H_E)s}$ in parallel, given that they act on different part of the system. Note that the duration of the time evolution s is stored in a quantum register, and hence both $e^{-i(H_M)s}$ and $e^{-i(H_E)s}$ are implemented controlled on the value of that register. The mass term as described in Eq. (A14) can be rewritten as follows:

$$e^{-iH_M s} = e^{-is\mu/2 \sum_{r=1}^N (-1)^r \mathbb{1}} e^{is\mu/2 \sum_{r=1}^N (-1)^r Z}. \quad (\text{B138})$$

Since the block-encoding is conjugated by the forward and backward time evolution operator, the terms proportional to the identity cancel, leaving us to implement only $e^{is\mu/2 \sum_{r=1}^N (-1)^r Z}$. To do so, we first compute the number of fermions, n_f . This can simply be done by first applying $\otimes_{r:\text{odd}} X_r$ on positron registers, computing the Hamming weight of the fermion registers [72], and then applying $\otimes_{r:\text{odd}} X_r$ on positron registers. The cost of computing n_f is $N - \omega(N)$ Toffolis, where $\omega(x)$ is the Hamming weight of x .

We then make use of phase gradient addition [77,88] in order to realize $e^{-im_k \mu n_f t_0 / (2\alpha M)}$ coherently depending on m_k and n_f . We present two different strategies to do so. The first one, which we will call the PGA approach, effectively multiplies m_k and n_f in the phase by performing $\log_2 M$ phase gradient additions with the catalyst state:

$$\left| \frac{\mu t_0}{2\alpha} \right\rangle = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} e^{-i\pi k \mu t_0 / (\alpha M)} |k\rangle. \quad (\text{B139})$$

The controls are on the discretized-time registers, and the targets are on the resulting Hamming weight of the fermions. The phase gradient addition is called $\log_2 M$ times, at a Toffolis cost of $\lceil \log_2(N) \rceil + 1$ each time, and a single qubit rotation each time as well. In total, considering the uncomputation of the temporary registers, the PGA approach has the following resource requirements: the number of Toffolis is at most $N - 1 + \lceil \log_2 M \rceil (\lceil \log_2(N) \rceil + 1)$, the number of rotations is $\lceil \log_2 M \rceil$, and the number of temporary ancillae is $N + \lceil \log_2(N) \rceil + 1$.

The second approach, which we call the Mult approach, performs the multiplication of m_k and n_f in the

computational basis, and then performs one big phase gradient addition with the catalyst state:

$$\left| \frac{\mu t_0}{2\alpha} \right\rangle = \frac{1}{\sqrt{M+N}} \sum_{k=0}^{N+M-1} e^{-i\pi k \mu t_0 / (\alpha M)} |k\rangle. \quad (\text{B140})$$

The cost of multiplying a register of size $\log_2 M$ with the register of size $\lceil \log_2 N \rceil + 1$ containing n_f is $\log_2 M(\lceil \log_2 N \rceil + 1) + 2(\log_2 M + \lceil \log_2 N \rceil + 1)$ Toffolis, and $\log_2 M + \lceil \log_2 N \rceil + 1 + \max(\log_2 M, \lceil \log_2 N \rceil + 1)$ ancillae [89]. The cost of the phase gradient addition is $\log_2 M + \lceil \log_2 N \rceil + 1$ Toffolis, 1 rotation and $\log_2 M + \lceil \log_2 N \rceil + 1$ ancillae. In total, considering the uncomputation of the temporary registers, the Mult approach has the resource cost as follows: $N + 2\log_2 M \lceil \log_2 N \rceil + 7\log_2 M + 5\lceil \log_2 N \rceil + 4$ Toffolis, 1 rotation and $N + 2\log_2 M + 2\lceil \log_2 N \rceil + 1$ ancillae.

Similarly, we can implement the evolution with the electric term in two ways. The first approach, which we call PGA again, follows the implementation in Ref. [25] (see Fig. 4 in Sec. III.1), modified with the use of Hamming weight phasing to implement the parallel rotations. This makes use of another catalyst state:

$$\left| \frac{t_0}{\alpha} \right\rangle = \frac{1}{\sqrt{\gamma}} \sum_{k=0}^{\gamma-1} e^{-2i\pi k t_0 / (\alpha \gamma)} |k\rangle, \quad (\text{B141})$$

where $\gamma = 2^{\log_2 M + \eta - 2} = 2M\Lambda/4$. There are in total $N - 1$ links each of $\eta (= \log_2(2\Lambda))$ qubits. This means, that we repeat the phase gradient addition, in a similar fashion to the mass term, for the $N - 1$ links and the η many parallel rotations. These η rotation “towers” are of decreasing size, from $j = \eta$ up to $j = 1$. Excluding the final tower ($j = 1$) consisting of only a single rotation, we perform a phase gradient addition of size j . Hence, the cost of the electric term is

$$(N - 1) \lceil \log_2 M \rceil \sum_{j=2}^{\eta} j = (N - 1) \lceil \log_2 M \rceil (\eta^2 + \eta - 2)/2 \quad (\text{B142})$$

Toffolis, $(N - 1) \lceil \log_2 M \rceil \eta$ single-qubit rotations, and η temporary ancillae.

The second approach, which we call Mult again, performs a square in the computational basis of the electric field, and then multiplies the result by the time register. Finally, a phase-gradient addition is performed. The cost of squaring is $\eta(\eta - 1)$ Toffolis [90] and η temporary ancillae and 2η qubits to store the result. We add the $(N - 1)$ -many (for each link) squared electric field values to find $\sum_r E_r^2$. This takes

$$\begin{aligned} & \sum_{i=0}^{\log_2 N} (2\eta + i)(N - 1)/(2^{i+1}) \\ &= \frac{((-1 + N)(-2\eta \ln(4) + N \ln(4) + 2\eta N \ln(16) - \ln(4N)))}{(N \ln(4))} \end{aligned} \quad (\text{B143})$$

$$\leq 4\eta N \quad (\text{B144})$$

Toffolis. The final register size is $2\eta + \lceil \log_2 N \rceil$. Then, we multiply this with the time register of size $\log_2 M$. This costs $\log_2 M(2\eta + \lceil \log_2 N \rceil) + 2(\log_2 M + 2\eta + \lceil \log_2 N \rceil)$ Toffolis and results in a register of size $2\eta + \lceil \log_2 N \rceil + \log_2 M$ and $\max(\log_2 M, 2\eta + \lceil \log_2 N \rceil)$ temporary ancillae. Finally, the phase gradient addition requires $2\eta + \lceil \log_2 N \rceil + \log_2 M$ Toffolis, 1 rotation and $2\eta + \lceil \log_2 N \rceil + \log_2 M$ ancillae. Hence, in total, considering the uncomputation of the temporary registers, the Mult approach has the resource cost as follows: $N[4\eta^2 + 4\eta] + \log_2 M[4\eta + 5 + 2\lceil \log_2 N \rceil] + 5\lceil \log_2 N \rceil - 2\eta^2 + 12\eta$ Toffolis, 1 rotation, and $8\eta + 3\lceil \log_2 N \rceil + 2\log_2 M$ ancillae.

In the implementation of *SELECT* of the Dyson series, the block-encoding of the interaction term is called K times, and the fast-forwardable term $e^{-i(H_E + H_M)s}$ is called $K + 1$ times. *SELECT* and *SELECT*[†] are called in total 3 times in oblivious amplitude amplification. Hence, $\text{BE}_{V/\alpha}$ is called $3K$ times, and $e^{-iH_E(\cdot)}$ and $e^{-iH_M(\cdot)}$ are each called $3(K + 1)$ times.

c. Reflection

In order to perform OAA, we must construct a reflection around the starting state and the success state:

$$W_{(K,M)} \left(\frac{t_0}{\alpha}, \frac{\epsilon'}{\mathbf{r}} \right) = UR_0 U^\dagger R_0 U, \quad (\text{B145})$$

where U is given in Eq. (B115) and the cost is described by the subroutine above. The reflection $R_0 = \mathbb{1} - 2P_0$ adds a phase when all the ancillae (for the truncation, discretization and the counter registers) are in the 0 state. This is performed by applying a multicontrolled Z . Hence, the cost of reflection is $K(2 + \lceil \log_2 M \rceil) - 1$ Toffolis and $K(2 + \log_2 M) - 1$ temporary ancillae, and it is called twice.

This concludes the resource estimation for the subroutines to implement the time evolution of the Schwinger model in the interaction picture.

APPENDIX C: TROTTER IMPLEMENTATION AND RESOURCE ESTIMATION

In this section, we report our implementation for the Trotter-based simulation of the Schwinger model, and the

associated resource estimation. We combine elements from [25,26] along with our own optimization.

First, we consider the interaction term, which, upon Jordan-Wigner transformation, can be written as [26]

$$H_I = x \sum_r (\sigma_r^+ U_r \sigma_{r+1}^- + \text{H.c.}), \quad (\text{C1})$$

where $\sigma^+ = |1\rangle\langle 0|$ and $\sigma^- = |0\rangle\langle 1|$ are the Pauli raising and lowering operators, respectively. Note that this decomposition does not lead to a linear combination of Pauli operators, which is more amenable to a block-encoding approach as given in Eq. (A11). However, this decomposition is favorable for a Trotter implementation, as it halves the number of terms that arise from H_I [26], compared to the scenario where we decompose into H_I Pauli operators that are evolved individually. Next, we decompose U as follows [26]:

$$U_r = \sigma_0^+ + U_r \sigma_0^+ U_r^\dagger, \quad (\text{C2})$$

where σ_0^+ is the raising operator acting on the least significant bit of the bosonic register. Intuitively, this decomposition is correct because the first term increments even numbers by flipping the least significant bit from 0 to 1; the second term decreases odd numbers by one with $U^\dagger$, increases the now even number by one with σ_0^+ , and then adds one again by U . Inserting this into the interaction term, we get

$$H_I = x \sum_r [\sigma_r^+ \sigma_0^+ \sigma_{r+1}^- + \sigma_r^+ (U \sigma_0^+ U^\dagger) \sigma_{r+1}^- + \text{H.c.}]. \quad (\text{C3})$$

This decomposition allows us to decompose H_I into two terms in the Trotter scheme, which we describe below. First, we recall that the simulation time is t and $\eta = \lceil \log_2(2\Lambda) \rceil$ is the size of a bosonic register with a cutoff Λ . We write the second-order Trotter formula as

$$S(t) = \left(\prod_{j=1}^l e^{-iH_j t/2r} \prod_{j=1}^1 e^{-iH_j t/2r} \right)^r, \quad (\text{C4})$$

where $H = \sum_{j=1}^l H_j$ is an ordered decomposition of the Hamiltonian into l terms.

We decompose the Hamiltonian into $l = 6$ terms, where

$$H_1 = H_E, \quad H_2 = H_M, \quad (\text{C5})$$

$$H_3 = H_{1,e} = x \sum_{\text{even } r} [\sigma_r^+ \sigma_0^+ \sigma_{r+1}^- + \text{H.c.}], \quad (\text{C6})$$

$$H_4 = H_{2,e} = x \sum_{\text{even } r} [\sigma_r^+ (U \sigma_0^+ U^\dagger) \sigma_{r+1}^- + \text{H.c.}], \quad (\text{C7})$$

$$H_5 = H_{1,o} = x \sum_{\text{odd } r} [\sigma_r^+ \sigma_0^+ \sigma_{r+1}^- + \text{H.c.}], \quad (\text{C8})$$

$$H_6 = H_{2,o} = x \sum_{\text{odd } r} [\sigma_r^+ (U \sigma_0^+ U^\dagger) \sigma_{r+1}^- + \text{H.c.}]. \quad (\text{C9})$$

For R Trotter steps, H_1 and H_2 are evolved $r+1$ times, H_{3-5} are evolved $2r$ times, and H_6 is evolved r times because contiguous evolutions of the same Hamiltonian can be combined into a single evolution.

The circuit implementation of the Trotter formula $W(t, \epsilon_{\text{rot}})$ implements an ϵ_{rot} approximation of $S(t)$, i.e.,

$$\|W(t, \epsilon_{\text{rot}}) - S(t)\| \leq \epsilon_{\text{rot}}, \quad (\text{C10})$$

where each rotation is synthesized with accuracy $\epsilon_{\text{rot}}/n_{\text{rot}}$. The Trotter error is given by

$$\|e^{-iHt} - S(t)\| \leq \epsilon_t = \frac{t^3 \rho}{r^2}, \quad (\text{C11})$$

where ρ is the Trotter commutator error bound [35]. The above adds up to a total error of at most $\epsilon = \epsilon_{\text{rot}} + \epsilon_t$.

We simply apply the bounds for U(1) lattice gauge theories at arbitrary spatial dimensions and with periodic boundary conditions from [26] to the Schwinger model with open boundary conditions (by removing irrelevant terms) to arrive at

$$\begin{aligned} \rho^{\text{nat}} = & \frac{1}{12} \left(4Nm^2 \sqrt{x}g + \frac{Ng^4(4\Lambda^2 - 1)}{4\sqrt{x}g} + 10(N-1)(\sqrt{x}g)^3 \right) \\ & + \frac{1}{24} \left(\frac{mg^2N(2\Lambda - 1)}{2} + 8mNxg^2 + 2Ng^3\sqrt{x}(2\Lambda + 1) + 9(N-1)(\sqrt{x}g)^3 \right). \end{aligned} \quad (\text{C12})$$

By removing the units, we equivalently have

$$\begin{aligned} \rho(x, \mu) = & \frac{1}{12} (8Nx\mu^2 + 2Nx(4\Lambda^2 - 1) + 80(N-1)x^3) \\ & + \frac{1}{24} (2x\mu N(2\Lambda - 1) + 32Nx^2\mu + 16Nx^2(2\Lambda + 1) + 72(N-1)x^3), \end{aligned} \quad (\text{C13})$$

when written in terms of only x and μ . Asymptotically, this is given as

$$\rho(x, \mu) = Nx\mathcal{O}(\mu^2 + \Lambda^2 + x + \mu\Lambda + x\mu + x\Lambda), \quad (\text{C14})$$

where N , m , x , and g are the number of lattice sites, base fermionic mass, lattice spacing, and bare gauge coupling. Then, the number of Trotter steps r is given by $r = \lceil t^{3/2} \rho^{1/2} / \epsilon_t^{1/2} \rceil$. Now we proceed to describe our circuit implementation of the evolutions of H_{1-6} .

1. Electric term

We begin with $H_1 = H_E$, the electric term. Following [25], by noting that

$$E^2 = \left(E + \frac{I}{2}\right)^2 - \left(E + \frac{I}{2}\right) + \frac{I}{4}, \quad (\text{C15})$$

e^{-itH_E} can be implemented, up to a global phase, as a product of $e^{-i(E+I/2)^2 t}$, where

$$(E + I/2)^2 = \frac{1}{12}(4^n - 1) + \sum_{j=0}^{\eta-2} \sum_{k>j}^{\eta-1} 2^{j+k-1} Z_j Z_k. \quad (\text{C16})$$

As a result, each $e^{-i(E+I/2)^2 t} e^{i(E+I/2)t}$, which is equivalent to $e^{-iE^2 t}$ up to a global phase, can be implemented using the circuit shown in Fig. 4 of Ref. [25]. Instead of synthesizing the R_z gates one by one, as done in [25], we implement them as one layer at a time using a phase catalysis circuit in Fig. 168 from Ref. [71]. Note the last layer is a single rotation gate and is implemented without using the phase catalysis circuit. Briefly, assuming access to a reusable catalyst state of the form $\bigotimes_{i=0}^{\eta-1} [R_z(2^i \theta)|+\rangle]$, this circuit applies $\bigotimes_{i=0}^{\eta-1} [R_z(2^i \theta)]$ using n Toffoli gates and one R_z gate. We synthesize the R_z gates using the mixed fallback method in [67], the costs of which are amortized over the number of times the catalyst is used which scales linearly in the number of Trotter steps here. The cost of this implementation is $(r+1)(N-1)(\eta-1)(2\eta+4)$ T-gates, $(2\eta-3) + (r+1)(N-1)\eta R_z$ gates, and $3\eta-3$ ancilla qubits, out of which $2\eta-3$ qubits are occupied by the catalyst state; we have used the fact that each Toffoli can be synthesized using four T-gates [91], and that η qubits are used (and reused whenever possible) for the applications of the phase catalysis circuit.

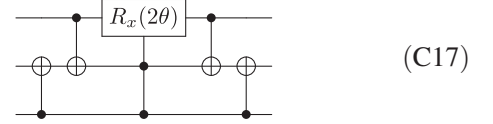
2. Mass term

Note $e^{-iH_M t}$ can be implemented as R_z gates, one per site, with angles of the same magnitude but of opposing signs depending on the parity of the site number. We conjugate every negative-angle R_z gate with a pair of NOT gates, transforming $e^{-iH_M t}$ into a layer of N same-angle R_z gates, which we implement using catalyzed Hamming-weight phasing [69,72]: (i) compute the Hamming weight of the to-be-rotated, N -qubit register; (ii) use the phase catalysis circuit in Fig. 168 from Ref. [71] to implement a phase

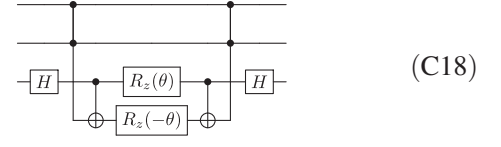
kick-back operation; (iii) uncompute (i). Catalyzed Hamming weight phasing costs $N - w(N) + \lfloor \log_2(N) \rfloor + 1$ Toffoli gates and clean ancilla qubits, where $w(N)$ is the Hamming weight of N , and 1 R_z gate per application; it further requires $\lfloor \log_2(N) \rfloor + 1$ ancilla qubits to store a catalyst state of the form $\bigotimes_{i=0}^{\lfloor \log_2(N) \rfloor + 1} [R_z(2^i \theta)|+\rangle]$.

3. Interaction terms

Finally we consider the interaction terms H_{3-6} . Each term consists of a layer of $N/2^4$ unitaries either of the form $e^{i\theta(\sigma^+ \sigma^+ \sigma^- + \text{H.c.})}$ or that but conjugated by a pair of adders. $e^{i\theta(\sigma^+ \sigma^+ \sigma^- + \text{H.c.})}$ can be constructed as [26,70]



and the doubly controlled $R_x(2\theta)$ can be compiled into



where the third bit from the top is the target. The circuit is largely taken from [70], except we uncompute the Toffoli using a Toffoli-free, measurement-feedforward circuit. Once we negate the negative angles via NOT gates, we get a layer of N same-angle R_z gates, which we implement using catalyzed Hamming weight phasing [69], as in the mass term. Catalyzed Hamming weight phasing costs $N - w(N) + \lfloor \log_2(N) \rfloor + 1$ Toffoli gates and 1 R_z gate per application.

In summary, evolving H_3 or H_5 costs $4(3N/2 - w(N) + \lfloor \log_2(N) \rfloor + 1)$ T-gates and 1 R_z gate, and evolving H_4 or H_6 costs an extra $4(2\eta - 2)$ T-gates for the adders. Furthermore, H_3 or H_5 requires each $3N/2 - w(N) + \lfloor \log_2(N) \rfloor + 1$ clean ancilla qubits and $\lfloor \log_2(N) \rfloor + 1$ to store the catalyst state. For H_4 or H_6 , the clean ancilla qubit count is $\max\{3N/2 - w(N) + \lfloor \log_2(N) \rfloor + 1, \eta - 1\}$ to account for the extra adders. Note that since H_6 is the final term and is evolved for twice as long, its catalyst states have twice the angle compared to those of H_{3-5} , i.e., $\bigotimes_{i=0}^{\lfloor \log_2(N) \rfloor} [R_z(2^{i+1} \theta)|+\rangle]$ vs $\bigotimes_{i=0}^{\lfloor \log_2(N) \rfloor} [R_z(2^i \theta)|+\rangle]$. Instead of preparing two $(\lfloor \log_2(N) \rfloor + 1)$ -qubit states, we can just prepare a $(\lfloor \log_2(N) \rfloor + 2)$ -qubit state, i.e., $\bigotimes_{i=0}^{\lfloor \log_2(N) \rfloor + 1} [R_z(2^i \theta)|+\rangle]$, and use the first and last $\lfloor \log_2(N) \rfloor + 1$ qubits for H_{3-5} and H_6 , respectively.

⁴We assume N is even for brevity, but our analysis can be simply adapted to the case of odd N by appropriate use of ceiling and floor functions.

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