

Forecasting constraints on cosmology and modified gravitational-wave propagation by combining strongly lensed gravitational waves and galaxy surveys

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(Received 12 February 2026; accepted 26 May 2026; published 17 June 2026)

Gravitational lensing of gravitational waves (GWs) will become the next frontier in studying cosmology and gravity. While time-delay cosmography using quadruply lensed GW events associated with optical images of the lens systems can provide precise measurement of the Hubble constant (H_0), they are considered to be much rarer than doubly lensed events. In this work, we analyze time-delay cosmography with doubly lensed GW events for the first time. We generate mock doubly lensed GW events with the designed sensitivity of the LIGO-Virgo-KAGRA (LVK) O5 network, with the LIGO post-O5 upgrade, and with the Einstein Telescope (ET) + Cosmic Explorer (CE) network, and we select the events that can be associated with future galaxy surveys. Over 1000 realizations, we find an average of $\sim 0.2(2.3)$ qualified events with the LVK O5 (post-O5) network, whereas with the ET + CE network, we find an average of 80.9 qualified events over 100 realizations. Using the singular isothermal sphere (SIS) lens model, we forecast the constraints on cosmological parameters and modified GW propagation by combining time-delay cosmography and the standard siren approach. The average posterior gives a constraint on H_0 with relative uncertainties of 14%, 10%, and 0.42% in the Λ CDM model for the LVK O5, LVK post-O5, and ET + CE networks, respectively. While the LVK network gives uninformative constraints on the (w_0, w_a) dynamical dark energy model, the ET + CE network yields moderate constraints of $w_0 = -1.02^{+0.31}_{-0.25}$ and $w_a = 0.39^{+1.01}_{-1.55}$. In addition, our method can provide precise constraints on modified GW propagation effects jointly with H_0 .

DOI: [10.1103/physrevd.113.123535](https://doi.org/10.1103/physrevd.113.123535)

I. INTRODUCTION

Despite the great success of the lambda cold dark matter (Λ CDM) standard cosmology model, it is facing many challenges today. For instance, the second data release of the Dark Energy Spectroscopic Instrument (DESI) indicates that a dynamical dark energy model is favored by the baryon acoustic oscillation (BAO) data, combined with data from the cosmic microwave background (CMB) and supernovae observations [1,2]. It challenges the assumption of the Λ CDM model, where dark energy is interpreted as the cosmological constant. In addition, the Hubble tension problem has long existed between the CMB data and the local Universe measurements—e.g., $H_0 = 67.49 \pm 0.50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ from the Planck mission [3] and $H_0 = 74.03 \pm 1.42 \text{ km s}^{-1} \text{ Mpc}^{-1}$ from the SH0ES Collaboration [4], which may hint for new physics beyond the Λ CDM model (see the review in Ref. [5]). To further examine

whether such observational anomalies are caused by systematics or unknown fundamental physics, more data from various aspects of observations are required in the near future.

On the other hand, gravitational wave (GW) detection has provided a unique laboratory in cosmological research. The latest Gravitational-Wave Transient Catalog 4.0 (GWTC-4) released by the LIGO-Virgo-KAGRA Collaboration (LVK) reported 128 new compact binary coalescence (CBC) events from the first part of the fourth LVK observing run (O4a) [6], which raises the total number of CBC events to 218. These events can provide an independent measurement of H_0 , as well as cosmological propagation effects, with the standard siren method [7–13]. For instance, the combination of the GW170817 bright siren event and the dark siren analysis for 142 of the GWTC-4 events yields $H_0 = 76.6^{+13.0}_{-9.5} \text{ km s}^{-1} \text{ Mpc}^{-1}$ [12]. As more CBC events will be detected by the next-generation (XG) detectors in the future, such as the Einstein Telescope (ET) [14] and Cosmic Explorer (CE) [15], the error of H_0 measurement by the dark siren method can be reduced to lower than 1% [16,17].

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With such precision in H_0 measurements, GWs are expected to provide a new perspective in solving the Hubble tension problem in the near future.

Apart from the standard siren method, H_0 can also be measured with GW detections in other aspects—in particular, the gravitational lensing of GWs [18–22]. The time-delay cosmography with strong lensing has been applied in electromagnetic (EM) observations in the past few decades [23–28]. For example, the TDCOSMO Collaboration obtained the latest constraints $H_0 = 71.6^{+3.9}_{-3.3} \text{ km s}^{-1} \text{ Mpc}^{-1}$ with eight strongly lensed quasars [29]. On the other hand, GW signals can also be lensed by foreground galaxies and result in multiple images for the same event. It has been proposed that these strongly lensed GW events can provide accurate measurements of H_0 , in association with strongly lensed images of their host galaxies. However, it requires precise estimation of the Fermat potential, which can be obtained either by strongly lensed EM counterparts of the events [30], or by more than two images of the lensed events [19,21,22]. These special events are considered to be rare in future detections, where 70% of the strongly lensed events are expected to have only two images [31]. Therefore, the chances to obtain cosmological constraints with these special strongly lensed events by LVK detections are very small.

In this work, we propose an alternative method to measure cosmological parameters with doubly lensed GW events, associating with images of the strong-lensing systems observed by large-scale galaxy surveys. The latest survey performed by the Euclid satellite is expected to discover about 75,000 strong-lensing systems [32], and the Legacy Survey of Space and Time (LSST) to be carried out by the Vera C. Rubin Observatory is also expected to detect about 70,000 strong-lensing systems [33]. The database of these strongly lensed galaxies can provide a precise map to identify the host galaxies of strongly lensed GW events. Using the singular isothermal sphere (SIS) lens model, one can infer the Einstein radius θ_E from the angles between the two images of the source galaxy and the lens galaxy in a strong-lensing system. In addition to θ_E , with the redshifts of the lens and the source galaxy, we can estimate H_0 using time-delay cosmography with strongly lensed GW signals, as well as dark energy or cosmological modified gravity effects described by the (Ξ_0, n) parametrization or the Horndeski-class parametrization [34–43]. Apart from this, the luminosity distance of the GW event and the redshift of the source galaxy can provide a standard siren measurement of the cosmological parameters jointly with time-delay cosmography. The main uncertainty of time-delay cosmography with the SIS model comes from the impact parameter y , which can be jointly estimated from the two images of lensed GW signals. However, there exists a caveat that degeneracies between lens parameters and cosmology could occur when not considering more complex lens models for possible missing images, which requires more careful consideration for precision cosmology.

The structure of this paper is listed as follows: Section II introduces the theoretical framework of strong lensing with the SIS model, cosmology models, and modified GW propagation. Section III describes our method to infer constraints on cosmological parameters. Section IV presents the mock data simulation we perform to obtain the cosmological constraints, and the inference results are shown in Sec. V. Finally, we summarize our work in Sec. VI.

II. THEORETICAL FRAMEWORK

A. Strong lensing with the SIS model

In this work, we use the SIS lens model to approximate galaxy lenses. It has been shown in literature that the eccentricity and detailed radial-mass distribution of the lens galaxy have a relatively small impact on lensing statistics [18,44–46]. However, the cosmological implications could be biased when using inaccurate lens models. In particular, for heavily elliptical lens galaxies, the cosmological implications would be biased when using the SIS model. Still, the accurate reconstruction of the elliptical lens model relies on more than two images, which is rare among all lensed events. Nonetheless, we use the SIS model to analyze doubly lensed events as an ideal condition in this work, in order to forecast the constraints on cosmology with a realistic lensed-event population. Detailed study on the biases by lens galaxy ellipticity would be needed in future works.

The SIS model describes the galaxy mass distribution by assuming that the behavior of stars and other mass components of the galaxy is similar to that of ideal gas particles in an adiabatic system [47]. The density profile of the SIS model is given by

$$\rho(r) = \frac{\sigma_v^2}{2\pi G} \frac{1}{r^2}, \quad (2.1)$$

where σ_v is the velocity dispersion. The Einstein radius of the SIS model is given by [48]

$$\theta_E = 4\pi \frac{\sigma_v^2}{c^2} \frac{D_{\text{LS}}}{D_S}, \quad (2.2)$$

where D_{LS} and D_S are the angular diameter distances between the lens and the source, and between the observer and the source, respectively. The lens mass for the SIS model is further given by [48]

$$M_L = 4\pi \frac{\sigma_v^4}{Gc^2} \frac{D_L D_{\text{LS}}}{D_S}, \quad (2.3)$$

where D_L is the angular diameter distance from the observer to the lens. Therefore, one can obtain the relation

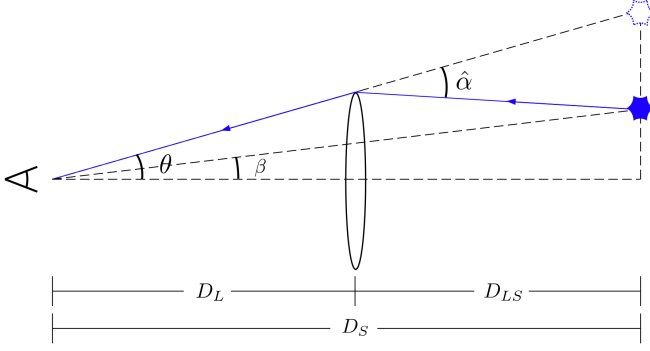


FIG. 1. Schematic figure for strong lensing with the thin-lens approximation.

$$\theta_E^2 = 4\pi \frac{GM_L}{c^2} \frac{D_{LS}}{D_L D_S}. \quad (2.4)$$

In addition, a special feature of the SIS model is that the reduced deflection angle¹ α equals θ_E , so that the lens equation becomes

$$\theta_{\pm} = \beta \pm \theta_E, \quad (2.5)$$

where $\theta_{\pm}$ stands for the angles between the two images and the lens on the line of sight, and β is the true source angular position (see Fig. 1). If $\beta < \theta_E$, there exist two images. Otherwise, there is only one image. Therefore, one can directly obtain the Einstein radius $\theta_E = (\theta_+ - \theta_-)/2$ for the doubly lensed system, which is half of the angle between the two images in observations.

On the other hand, the time delay of the lensed signals is given by

$$t_d(\vec{\theta}, \vec{\beta}) = \frac{1+z_L}{c} \frac{D_L D_S}{D_{LS}} \left[\frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right], \quad (2.6)$$

where $\psi(\vec{\theta})$ is the effective potential of the lens, and it causes the Shapiro time delay of the signals. The time delay can also be written in terms of dimensionless quantities:

$$t_d(\vec{x}, \vec{y}) = \frac{1+z_L}{c} \frac{D_L D_S}{D_{LS}} \theta_E^2 T(\vec{x}, \vec{y}), \quad (2.7)$$

where $\vec{x} = \vec{\theta}/\theta_E$ and $\vec{y} = \vec{\beta}/\theta_E$ are the dimensionless relative positions of the image and the source. In the SIS system, the dimensionless time delay for the two images is given by [49]

$$T_{\pm} = \mp y - \frac{1}{2}. \quad (2.8)$$

¹The reduced deflection angle α relates to the deflection angle $\hat{\alpha}$ in Fig. 1 by $\alpha = \hat{\alpha} D_{LS}/D_S$.

The uncertainty of time-delay measurement in EM strong lensing is a significant contribution to the uncertainty in the cosmological constraint [25]. Meanwhile, the arrival time of transient signals such as gamma-ray bursts and GWs can be precisely measured, so the observed time-delay measurement uncertainty in GW lensing is greatly reduced.

For gravitationally lensed GWs, the waveform is modified by $\tilde{h}_L(f) = F(\omega)\tilde{h}(f)$ in the frequency domain, where $F(\omega)$ is called the amplification factor ($\omega = 2\pi f$). In the region of CBC GWs lensed by galaxies, the wavelength of GWs is much lower than the lens size and satisfies $|\Delta t_d \cdot \omega| \gg 1$, so we can use the geometric optic approximation of $F(\omega)$, given by [48,50]

$$F(\omega) \simeq \sum_j \sqrt{|\mu_j|} \exp \left(i\omega t_{d,j} - i \text{sign}(\omega) n_j \frac{\pi}{2} \right), \quad (2.9)$$

where μ_j is the magnification factor for the j th image, and $n_j = 0, 1, 2$ is the Morse phase factor for the type-I, II, III images, respectively. In the SIS model, the magnification factor is given by [49]

$$\mu_{\pm} = 1 \pm \frac{1}{y}. \quad (2.10)$$

For lensed GWs, the waveform amplitude is multiplied by $|\mu_{\pm}|$, which introduces bias in the observed GW luminosity distance d_L^{GW} . Therefore, one can first obtain y by the ratio of the lensed signal,

$$y = \frac{\mu_+/\mu_- + 1}{\mu_+/\mu_- - 1}, \quad (2.11)$$

so that the lensing magnification can be removed from the lensed signals with Eq. (2.10), and the true d_L^{GW} can be obtained. In our analysis, we will jointly estimate d_L^{GW} and y with the two images of lensed GW signals (as shown in Sec. IV C).

For higher y within $y < 1$, the amplitude of the second image of lensed GWs is suppressed, so its signal-to-noise ratio (SNR) could be below the detection threshold, typically $\rho_{\text{th}} = 8$. Fortunately, with subthreshold signal searching techniques, lensed signals can be dug out with SNRs down to 6, and down to 4 in rare cases [51–55], which greatly improve the chance to find strongly lensed signal pairs. Therefore, in our work, subthreshold images of lensed GWs with network SNRs larger than 6 are included in our analysis.

B. Cosmology model and modified gravitational-wave propagation

With strongly lensed GWs, we can estimate cosmological parameters and modified gravity parameters that affect GW propagation. In a flat Universe described by the

Friedman-Lemaître-Robertson-Walker (FLRW) metric, the luminosity distance of an EM source at redshift z is given by [56]

$$d_L = (1+z) \int_0^z \frac{c \, dz'}{H_0 E(z')}. \quad (2.12)$$

For the Λ CDM model, $E(z) = \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$, where $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are the energy density of matter and dark energy today (the radiation energy density is negligible in the late-time Universe). The angular diameter distance is related to the luminosity distance by $d_A = d_L/(1+z)^2$.

However, dark energy can be dynamical over time, instead of a constant in the Λ CDM model. A simple parameterization of the equation of state of dynamical dark energy is the Chevallier-Polarski-Linder (CPL) parameterization [57,58], which depends on the scale factor a by

$$w(a) = w_0 + w_a(1+a). \quad (2.13)$$

The corresponding $E(z)$ is given by

$$E(z) = \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}(1+z)^{3(1+w_0+w_a)} e^{\frac{3w_a z}{(1+z)}}}. \quad (2.14)$$

The Λ CDM model is recovered by $w_0 = -1$ and $w_a = 0$. The first two BAO data releases by DESI show a preference for the CPL dark energy model over the Λ CDM model by over 2.8σ for the first time, combining with CMB data and supernova data [1,2].

The nature of dark energy remains a mystery nowadays, yet many explanations to it have been proposed. Modified gravity is one of the popular dark energy theories. In a general modified gravity framework, the propagation of GWs would deviate from general relativity (GR), making GWs a unique probe to dark energy. Meanwhile, the GW propagation speed has been tightly constrained by the multimessenger event GW170817 to be consistent² with GR [62–64]. Therefore, the remaining propagation effect without strong constraints is the damping effect,³ as described in the GW propagation equation [34,35,66–68]:

$$\tilde{h}_A'' + 2\mathcal{H}[1 - \delta(\eta)]\tilde{h}_A' + c^2 k^2 \tilde{h}_A = 0, \quad (2.15)$$

where $\tilde{h}_A(\eta, k)$ is the GW in Fourier space, the subscript A denotes the $+$ or $\times$ GW polarization, η is the conformal time, the prime ($'$) denotes a derivative with respect to η , and $\mathcal{H} = a'/a$ is the conformal Hubble parameter. The term

²Some theories predict a transition in GW speed at a lower energy scale than the LVK detection band, but it can be constrained by LISA in the future [59–61].

³However, see Ref. [65] for the correlation between GW speed and damping effect.

$\delta(\eta)$ is the extra damping effect induced by modified gravity, and GR is recovered at $\delta(\eta) = 0$. By solving Eq. (2.15), one can find the ratio between the modified GW luminosity distance and the EM luminosity distance as defined in Eq. (2.12):

$$\frac{d_L^{\text{GW}}(z)}{d_L^{\text{EM}}(z)} = \exp \left[- \int_0^z \frac{dz'}{1+z'} \delta(z') \right]. \quad (2.16)$$

The explicit form of $\delta(z)$ can be derived in specific modified gravity frameworks.⁴ But one widely used general parametrization originating from nonlocal modified gravity models is given by [71,72]

$$\frac{d_L^{\text{GW}}(z)}{d_L^{\text{EM}}(z)} = \Xi_0 + \frac{1 - \Xi_0}{(1+z)^n}, \quad (2.17)$$

where Ξ_0 and n are constants. At $z = 0$, the distance ratio is 1, and it approaches the constant Ξ_0 as z increases, which describes the deviation from GR consistent with the dark-energy-driven expansion of the Universe at low redshift. This parametrization can be mapped into different modified gravity models (see Ref. [35] for some examples). A first constraint on Ξ_0 has been given by the dark siren method with GWTC-3 and GWTC-4 data [12,38,40,41], though the uncertainty is large at this stage.

Another widely tested model is the alpha-basis parametrization built upon the Horndeski scalar-tensor theory framework, where deviation from GR at the linear perturbation level can be described by four parameters: α_M , α_K , α_B , and α_T [73,74]. While α_T is the deviation of GW speed from c , α_M describes the running rate of the effective Planck mass, which leads to the extra damping factor by $\delta(z) = -\alpha_M(z)/2$ [35]. The evolution of α_M is often parametrized to be proportional to the growth of dark energy density, $\alpha_M(z) = c_M \Omega_\Lambda(z)/\Omega_{\Lambda,0}$. Taking the dark energy density evolution in the Λ CDM model, the distance ratio is given by

$$\frac{d_L^{\text{GW}}(z)}{d_L^{\text{EM}}(z)} = \exp \left\{ \frac{c_M}{2\Omega_{\Lambda,0}} \ln \frac{1+z}{[\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}]^{1/3}} \right\}. \quad (2.18)$$

The constant c_M has been constrained by the CMB and large-scale structures [75–77], as well as the dark siren method [12,40,41,78]. The constraint of Ξ_0 and c_M parametrization has also been proposed using quadruply lensed GW events [42,79]. Meanwhile, it has been shown that the geometric optic time delay for modified GW propagation in scalar-tensor theories remains the same as in GR [80,81].

⁴See Refs. [69,70] for a more general GW damping term that includes frequency-dependent modification.

III. METHOD

In this work, we aim to obtain cosmological constraints with the doubly lensed GW events and the lensed images of their host galaxies from strong-lensing catalogues. Using the enhanced localization technique with the effective detector network consisting of ground-based GW detectors at different spacetime due to the time delay of strongly lensed signals, the sky localization area of strongly lensed GWs can be tightly constrained to $\Delta\Omega \sim 0.01 \text{ deg}^2$ with 3–4 images detected by three XG detectors [22], which greatly enhance our chances to find the corresponding strongly lensed host galaxies from galaxy surveys. On the other hand, the localization for LVK detectors has lower accuracy, which is expected to be $\Delta\Omega \sim 2 \text{ deg}^2$ for quadruply lensed events [82], and $\Delta\Omega \gtrsim 10 \text{ deg}^2$ for doubly lensed events with current LVK detectors [83]. In the case where there is more than one strongly lensed galaxy within the localization area, one can compare the Bayes factor between each system with the likelihood and identify the true lensed systems, which have been studied in previous works [19,84]. Here, we consider the optimistic case where true strongly lensed systems are identified for GW lensed events. Such systems should be within galaxy survey footprints, and the lensed-image angular positions can be resolved by the EM telescopes (see Sec. IV B for lensed event selection). Otherwise, lensed GW events with host galaxy EM images outside galaxy catalogues cannot be analyzed with our method.

Given the redshift of the source galaxy and the GW luminosity distance d_L^{GW} , we can constrain cosmological parameters via the standard siren method. However, the first analysis of lensed GWs will result in biases in d_L^{GW} because of magnification. Therefore, we need to reanalyze lensed GWs once the lensed signal pairs are confirmed. By joint reanalysis of the two images of lensed GWs, we can obtain the posterior samples of the true d_L^{GW} and the impact parameter y of lensed events. See Sec. IV C for details of waveform parameter reanalysis.

In addition, we can obtain cosmological constraints via time-delay cosmography of strongly lensed GWs. Given the observed time delay between GW signal images, $\Delta t = t_{d,2} - t_{d,1}$, as well as redshifts of the lens galaxy and the source galaxy, the Einstein radius θ_E , and the impact parameter y , we can estimate cosmological parameters with Eqs. (2.7) and (2.8) using the SIS model. For the resolved image of a strong-lensing system in galaxy surveys, the angular separation between the two images of the source can be observed. Then, one can obtain θ_E of the SIS lensing system by $\theta_E = (\theta_+ - \theta_-)/2 = \Delta\theta/2$, given the lens equation (2.5). Therefore, we construct the log-likelihood on cosmological parameters λ with χ^2 statistics over N lensed events, given by

$$\log \mathcal{L}(\lambda) = -\frac{1}{2} \sum_{i=1}^N \chi^2(\lambda, \Delta t_i^{\text{obs}}, \Delta\theta_i, z_{L,i}, z_{S,i}, y_i), \quad (3.1)$$

where

$$\begin{aligned} \chi^2(\lambda, \Delta t_i^{\text{obs}}, \Delta\theta_i, z_{L,i}, z_{S,i}, y_i) &= \left[\frac{d_{L,i}^{\text{obs}} - d_L(\lambda, z_{S,i})}{\delta d_{L,i}^{\text{obs}}} \right]^2 \\ &+ \left[\frac{\Delta t_i^{\text{obs}} - \Delta t(\lambda, \Delta\theta_i, z_{L,i}, z_{S,i}, y_i)}{\delta(\Delta t)} \right]^2. \end{aligned} \quad (3.2)$$

The uncertainty in time-delay reconstruction is dominated by δy from GW reanalysis, which typically gives a relative error of 10%–20%. The uncertainty in other factors—namely, $\Delta\theta, z_L, z_S$ —is on the order of at most a few percent from galaxy surveys.⁵ Moreover, uncertainty in the GW arrival time in milliseconds is negligible compared to the time delay in days, and we assume a perfectly smooth lens model so that time delay from small features in the galaxy lens model is neglected. Therefore, we have $\delta(\Delta t)/\Delta t \simeq \delta y/y$ in the likelihood. Using flat priors on λ , the posterior for λ can then be obtained by Bayesian inference with Markov chain Monte Carlo (MCMC) sampling.

The aim of this work is to forecast cosmological constraints given observed lensed GW pairs. In practice, however, one must also consider the detectability of lensed GW pairs, and weight the likelihood of observables by selection effects. This should be analyzed explicitly in future works.

IV. SIMULATION

To forecast the cosmological constraints with doubly lensed GW events, we first simulate mock data for strongly lensed GW events based on theoretical models for GW source population, merger rate, and lensing rate. Next, we filter out the events that can be associated with strong-lensing systems based on the resolution, the sky coverage, and the depth of galaxy surveys. We then perform joint parameter estimation for lensed GW signal pairs. Finally, we perform MCMC sampling to obtain constraints on cosmological parameters. Details of the simulation in each step are clarified in the following sections.

A. Mock GW events

Our first step of simulation is generating mock GW data for the LVK O5 stage, the post-O5 stage, and the next-generation detectors ET + CE, using the PYTHON tool GWSim [87]. We assume that the O5 detector network consists of LIGO Observatory at Livingston (L1) and Hanford (H1), Virgo (V1), and KAGRA (K1), with their

⁵The redshift uncertainties from spectroscopic and photometric surveys are expected to be on the order of 0.1% and 1%, respectively (e.g., Refs. [85,86]). Although photometric redshift of the source galaxy is affected by light bending due to gravitational lensing, a follow-up spectroscopic survey could provide more accurate redshift measurement.

designed sensitivity curves⁶ [88] that we implement in GWSim. For the post-O5 detector network, we assume that LIGO India (A1) will join detections with LIGO designed sensitivity, so that there are five detectors in total. We also assume that post-O5 upgrades will be installed in L1 and H1 with the A# sensitivity.⁷ For the ET + CE network, we use the orientation and the location of ET with the triangular configuration implemented in BILBY,⁸ with a sensitivity curve for the 10 km arm length design.⁹ We then assume two CE sites locating at L1 and H1, with a sensitivity curve for the 40 km arm length design.¹⁰ For each detector, we assume a duty cycle of 75% in an ideal operation, and it applies to the three ET detectors together as a whole. We generate the mock binary black hole (BBH) merger events using the “Powerlaw+Double Peak” black hole mass distribution model and the Madau-Dickinson redshift evolution model, which are used in a GWTC-4 cosmology paper [12]. We only consider BBHs here, since they comprise most of the GW events in LVK detections, while the event rates of binary neutron stars and neutron star–black hole mergers have large uncertainties. The description of these models and our injected values are shown in Appendix A. The luminosity distances of GW events are then generated from redshift with the fiducial Λ CDM model with $H_0 = 67.8 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $\Omega_{m,0} = 0.308$ from Planck 2018 TT, TE, and EE + lowE + lensing + BAO results [3]. We assume 5 years of observations for each detector network configuration, and that GW events distribute uniformly in the sky map. We select mock events with $\text{SNR} > 11$, and we proceed to lensed event generation.

B. Lensed event selection

We follow Ref. [18] to compute the optical depth for GW sources, which gives the probability for GW events being lensed and detected at different redshifts (see detailed derivation in Appendix B). The optical depth τ is given by

$$\tau = \frac{F_*}{30} [D_S(1 + z_S)]^3 y_{\max}^2 \left\{ 1 - \frac{1}{7} \frac{\Gamma[(8 + \alpha)/\beta]}{\Gamma[(4 + \alpha)/\beta]} \frac{\Delta t_*}{T_{\text{obs}}} \right\}, \quad (4.1)$$

⁶L1 and H1: <https://dcc.ligo.org/DocDB/0149/T1800042/005/AplusDesign.txt>.

V1: https://git.ligo.org/publications/detectors/obs-scenarios-2019/blob/master/Scripts/Figure1/data/avirgo_O5high_NEW.txt.

K1: https://git.ligo.org/publications/detectors/obs-scenarios-2019/blob/master/CurvesForSimulation/kagra_128Mpc.txt.

⁷https://dcc.ligo.org/LIGO-T2300041-v1/public/Asharp_strain.txt.

⁸<https://git.ligo.org/lscsoft/bilby/-/blob/d37609ccc9a878750c841b4b93da990e63acffb2/bilby/gw/detector/detectors/ET.interferometer>.

⁹<https://apps.et-gw.eu/tds/ql/?c=16492/ET10kmcolums.txt>.

¹⁰https://dcc.cosmicexplorer.org/CE-T2000017/public/cosmic_explorer_strain.txt.

where

$$F_* = 16\pi^3 n_{*,0} \left(\frac{\sigma_{*,0}}{c} \right)^4 \frac{\Gamma[(4 + \alpha)/\beta]}{\Gamma[\alpha/\beta]}, \quad (4.2)$$

$$\Delta t_* = 32\pi^2 \left(\frac{\sigma_{*,0}}{c} \right)^4 \frac{D_S}{c} (1 + z_S) y_{\max}. \quad (4.3)$$

D_S and z_S are the angular diameter distance and redshift of the source, Γ is the Euler gamma function, and T_{obs} is the observation time of GW detectors. We adopt the modified Schechter function parameters as $n_{*,0} = 8.0 \times 10^{-3} \text{ h}^3 \text{ Mpc}^{-3}$, $\sigma_{*,0} = 144 \text{ km s}^{-1}$, $\alpha = 2.49$, and $\beta = 2.29$, following previous study in Ref. [18] (also consistent with recent studies—e.g., in Refs. [89–91]).

The maximal impact parameter $y_{\max}$ depends on the network SNR detection threshold ρ_{th} and the network SNR of the unlensed GW signal ρ_{UL} by

$$y_{\max} = \left[\left(\frac{\rho_{\text{th}}}{\rho_{\text{UL}}} \right)^2 + 1 \right]^{-1}. \quad (4.4)$$

In our simulation, we compute $y_{\max}$ with ρ_{UL} for each event and $\rho_{\text{th}} = 6$ as the network SNR cut for the subthreshold search. We then randomly select lensed events from the mock GW catalogues, weighted by the optical depth with $y_{\max}$ for each event using Eq. (4.1). This criterion of $y_{\max}$ indicates that our simulated lensed events always have $y < y_{\max}$ as their second images pass the detection threshold.

For the selected lensed events, we create the foreground galaxy lenses based on the stellar mass distribution of galaxies. We adopt the double Schechter function to describe the galaxy density profile in stellar mass:

$$\Phi(M) = \exp\left(-\frac{M}{M^*}\right) \left[\Phi_1^* \left(\frac{M}{M^*} \right)^{\alpha_1+1} + \Phi_2^* \left(\frac{M}{M^*} \right)^{\alpha_2+1} \right]. \quad (4.5)$$

We adopt $\log(M^*/M_\odot) = 10.79$, $\log(\Phi_1^*/h^3 \text{ Mpc}^{-3}) = -3.31$, $\log(\Phi_2^*/h^3 \text{ Mpc}^{-3}) = -2.01$, $\alpha_1 = -1.69$, and $\alpha_2 = -0.79$ from Ref. [92], and we randomly draw the lens galaxy stellar mass with probability weighted by Eq. (4.5). Since the ratio between baryonic matter and dark matter is 1:5 in the Universe, and baryonic matter in galaxies includes stars and interstellar media, we adopt a representative value of 10% for the stellar mass contribution to the galaxy’s total mass including a dark matter halo. Hence, we obtain a lens galaxy total mass of 10 times that of the drawn stellar mass. In addition, we draw lens redshifts z_L with probability weighted by the differential comoving volume dV_c/dz , and we draw the impact parameter y with probability weighted by y in $(0, y_{\max})$.

Since our method relies on the resolved images of lensed host galaxies, we need to filter out the lensed events whose

EM counterparts are observable and resolvable in galaxy surveys. We target the LSST survey, which will cover redshift up to 3 and a sky area of $18,000 \text{ deg}^2$ [86,93], which is about 43% of the sky, so we only keep the same portion of events out of all mock events. In addition, the resolvable angle for the LSST camera is 0.2 arcseconds [94]. For the selected lensed events, we compute θ_E with Eq. (2.4) from M_L and D_L, D_S, D_{LS} converted from z_L and z_S with the fiducial Λ CDM model. Then, we select resolvable lensed events with $\theta_{\pm} = (y \pm 1)\theta_E > 0.2''$. The observed time delay between images of each event is computed with lens model parameters using Eqs. (2.4), (2.7), and (2.8).

We simulate 1000 realizations for LVK O5 and post-O5 observations. A realization means that we generate random mock lensed events based on the optical depth of GW events in our mock catalogue given a different random seed, and we collect all events with resolvable EM images in LSST's footprint. For the O5 run, we obtain at least one lensed event in 331 realizations. Considering events in the LSST's footprint, the average lensed event number is 0.40 over all 1000 realizations. However, we only obtain resolvable lensed events by LSST in 152 realizations after filtering, with an average of 0.17 resolvable events across all realizations. For the post-O5 run, we obtain resolvable lensed events in 895 realizations. Compared to an average of 5.4 total lensed events within LSST coverage, there are only 2.3 resolvable events across all realizations. For the ET + CE network, we obtain an average of 191.8 total lensed events, and 80.9 resolvable events in all of the 100 realizations. In our simulation, the fraction of resolvable lensed events by LSST is 24%–43%, which is consistent with the lensed event localization study in Ref. [84].

C. Waveform parameter estimation

For each resolvable lensed event detected by LVK, we perform waveform parameter estimation (PE) jointly for the two signal images. We assume that both signals from a lensed event have been analyzed individually prior to our analysis, so that parameters like sky location, spin, time, and phase at merger have been measured, which can be fixed in the reanalysis. In reality, there could be large uncertainty in individual PE and degeneracy between parameters, so a more robust approach is to jointly reanalyze all parameters. However, to reduce computational cost in our analysis, we run joint PE on five parameters while fixing the rest as a proof of principle for our method, assuming that the true location is well recovered for individual images. We choose to jointly estimate the chirp mass $\mathcal{M}_c$, mass ratio q , GW luminosity distance d_L^{GW} , inclination angle ι , and the impact parameter y . PE of $\mathcal{M}_c$ and d_L^{GW} from single images could be biased by the magnification factor due to lensing, so we need to find their true values from reanalyses with the SIS lens model. Meanwhile, we include the mass ratio because it

degenerates with $\mathcal{M}_c$, and the inclination angle because it degenerates with d_L^{GW} .

We first generate the unlensed waveform signal in frequency domain using the IMRPhenomXPHM waveform [95], and then create the two lensed signal images magnified by $\mu_{\pm}(y)$ with Eq. (2.10) and a Morse phase difference of $\pi/2$ between images. In reality, the Morse phase depends on the type of observed images, but we assume that the true Morse phase is found by the phase difference between images from individual PE, and we use $\pi/2$ for all events for simplicity. For simplicity, we generate spin-free waveforms. The posterior of waveform parameters $\{\phi\}$ given image data $x_{j=1,2}$ in the joint analysis is thus given by

$$P(\phi|x_1, x_2) = \int P(x_1, x_2|\phi)\Pi(\phi)d\phi, \quad (4.6)$$

where $\Pi(\phi)$ is the prior, and the joint log likelihood is given by

$$\log P(x_1, x_2|\phi) = -\frac{1}{2} \sum_{j=1}^2 \langle \tilde{h}(x_j) - \tilde{h}(\phi) | \tilde{h}(x_j) - \tilde{h}(\phi) \rangle. \quad (4.7)$$

Here, f is the frequency, and $\tilde{h}(f)$ is the detector response to GW signals, given by

$$\tilde{h}(f) = \sum_k [F_+^k \tilde{h}_+(f) + F_{\times}^k \tilde{h}_{\times}(f)], \quad (4.8)$$

where F_+^k and $F_{\times}^k$ are the antenna pattern functions of the k detector for the plus and cross GW polarizations. The scalar product $\langle \tilde{h}(f) | \tilde{g}(f) \rangle$ is defined as

$$\langle \tilde{h}(f) | \tilde{g}(f) \rangle \equiv 2\text{Re} \left[\int df \frac{\tilde{h}(f)\tilde{g}^*(f) + \tilde{h}^*(f)\tilde{g}(f)}{\text{Sn}(f)} \right], \quad (4.9)$$

where $\text{Sn}(f)$ is the power spectral density (PSD) of the detector noise. The priors on d_L^{GW} and y are $\Pi(d_L^{\text{GW}}) \propto (d_L^{\text{GW}})^2$ and $\Pi(y) \propto y$, respectively, while the priors for other parameters are uniform. Finally, the posterior samples are inferred with NESSAI [96,97], which incorporates normalizing flows in nested sampling within the BILBY package [98].

Figure 2 shows an example of the posterior samples from joint PE for lensed GW signals. We can see that, in spite of moderate degeneracy, the true d_L^{GW} and y are well recovered, and their marginalized posteriors generally follow Gaussian distribution. We can therefore obtain the observables and their uncertainties from the mean and the standard deviation of the posterior samples, respectively.

For the ET + CE network, however, the waveform parameter estimation is computationally expensive given a large event number. We thus perform Fisher matrix analysis to forecast the uncertainty of the waveform

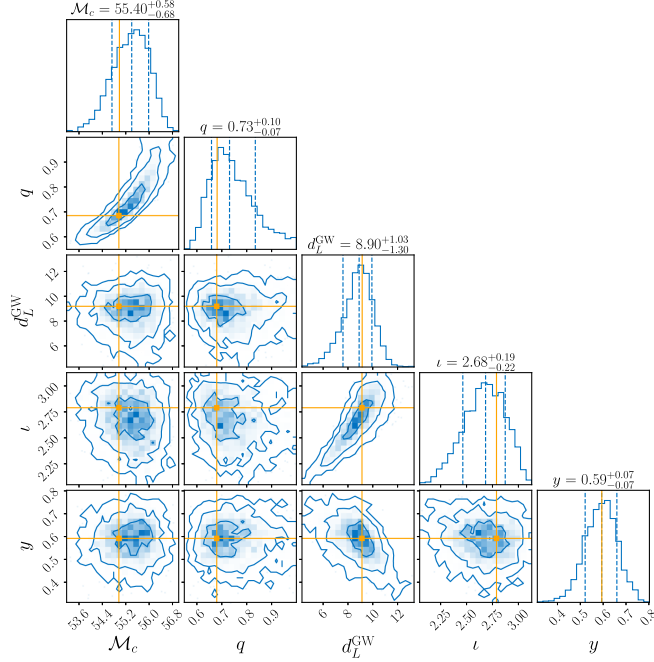


FIG. 2. Posterior samples from joint reanalysis of one of the mock doubly lensed GW events detected by the LVK O5 network using NESSAI nested sampling in BILBY. The estimated parameters include chirp mass $\mathcal{M}_c$ [$M_\odot$], mass ratio q , GW luminosity distance d_L^{GW} [Gpc], inclination angle i , and impact parameter y . The orange lines show the injected values for parameters.

parameters for the ET + CE events, which is given by the square root of the diagonal elements of the covariance matrix Σ^{ii} [99,100]. The Fisher matrix is the inverse of the covariance matrix, constructed by $(\Sigma^{ij})^{-1} = \langle \partial \tilde{h} / \partial \phi_i | \partial \tilde{h} / \partial \phi_j \rangle$. We numerically compute the partial derivatives of the waveform by finite differencing. The joint Fisher matrix for the two lensed GW signals is the sum of the Fisher matrices for the individual signals. Note that our ET + CE forecasts cannot be compared to our LVK forecasts directly, since we use the Fisher matrix to approximate parameter errors for ET + CE instead of Bayesian inference.

V. RESULTS

We infer the posterior for cosmological parameters with Eqs. (3.1) and (3.2) from all qualified lensed GW events in each realization using MCMC sampling with the EMCEE package [101]. For each realization, we estimate the posterior probability density function (PDF) from marginalized posterior samples with the Gaussian kernel density estimation (KDE) method using `scipy.gaussian_kde` under Scott's rule. Then we compute the average posterior PDF over all N_{real} realizations by¹¹

¹¹Realizations with no qualified lensed events are not included in averaging.

$$P_{\text{avg}}(\lambda|x) = \frac{1}{N_{\text{real}}} \sum_{i=1}^{N_{\text{real}}} P_i(\lambda|x). \quad (5.1)$$

In our analysis, we assume that we have identified the true lens system associated with the lensed GW events, and therefore use a Gaussian likelihood distribution for the lens parameter y . In reality, it would be more rigorous to marginalize over lens parameters given all possible lens systems within the localization area, which would allow for more weight to be assigned to more confident GW-EM pairs. Such analysis will require tests with a mock large-scale galaxy catalogue, which is beyond the scope of this paper.

We perform the inference in four different cosmology models—namely, the Λ CDM model, the $w_0 w_a$ CDM model, the (Ξ_0, n) modified gravity model, and the α -basis scalar-tensor gravity model. The inference results are presented in the following subsections.

A. Λ CDM model

For the Λ CDM model, we estimate H_0 and $\Omega_{m,0}$, while assuming $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$ in the late Universe. The averaged posteriors over all realizations are shown in Fig. 3. We see that for the LVK O5 network and the post-O5 network, the constraints on H_0 and $\Omega_{m,0}$ are weak, which is due to the limited number of lensed events. The relative 1σ uncertainty of H_0 is around 14% and 10% for the O5 and the post-O5 networks, respectively (see the exact number in Table I). For $\Omega_{m,0}$, the relative 1σ uncertainty reaches 100% and 73%, respectively. However, for the ET-CE network, the cosmological constraints are greatly improved, because the lensed event number is over 1 order

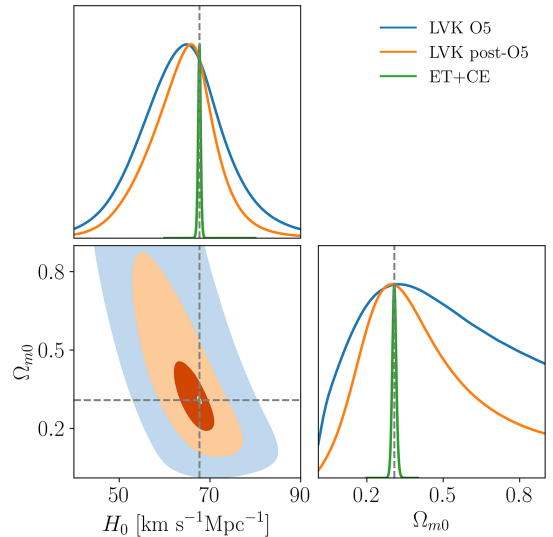


FIG. 3. The figure shows the forecast posterior on cosmological parameters in the Λ CDM model by jointly analyzing all lensed GW events in each realization and averaging over all realizations with different detector networks. The gray dashed lines indicate the parameter values in the fiducial Λ CDM model.

of magnitude larger than the LVK network, and the event SNRs are much larger. The relative 1σ uncertainties on H_0 and $\Omega_{m,0}$ for the ET-CE network reduce to around 0.42% and 2.6%, respectively. Such precision is comparable to Planck's measurement and other local-Universe measurements, making doubly lensed GWs + the strong-lensing galaxy survey an accurate independent probe of H_0 that could provide a new insight to the Hubble tension.

For comparison, the bright siren + dark siren constraints on H_0 with 142 of the GWTC-4 events yields an uncertainty of $\sim 15\%$ [12], which can be achieved with 1–2 strongly lensed events with EM images. In addition, constraints on H_0 and $\Omega_{m,0}$ can also reach an uncertainty of 2.2% and 5.5%, respectively, by spectral siren + golden dark siren analysis with the ET + CE network in 1 year [16], and the bright siren constraint on H_0 alone can reach an uncertainty below 1% with ~ 300 events [16]. Meanwhile, with a much smaller detection number, strongly lensed GW-EM pairs can provide a constraint at the same level as unlensed events. However, there exist different systematics in the two approaches. While the precision of the standard siren approach is affected by GW source population models and galaxy catalogue completeness, the strongly lensed GW approach relies on accurate lens signal identification, lens system localization, and lens model reconstruction. Therefore, these two approaches can complement each other in GW cosmology.

B. w_0w_a CDM model

In the dynamical dark energy model with the CPL parametrization, we estimate H_0 , $\Omega_{m,0}$, w_0 , and w_a , based on modification in d_L^{GW} with Eq. (2.14). As shown by averaged posteriors over all realizations in Fig. 4, w_0 and w_a are basically unconstrained with the LVK O5 and post-O5 networks. On one hand, there are too few lensed events that can be associated with galaxy surveys for LVK sensitivity. On the other hand, the redshift range of LVK events is mostly at $z \lesssim 2$, while precise constraints on w_0 and w_a rely on high-redshift events. For the ET + CE network, we can see moderate constraints on w_0 and w_a , with uncertainties of ~ 0.3 for w_0 , and around the order of unity for w_a . It demonstrates the potential of doubly lensed GW events to constrain dynamical dark energy in the near future, though such a constraint could be weakened if the true lenses deviate heavily from the SIS model. If the observation period of ET + CE can last longer, the uncertainty on w_0 and w_a can be further reduced, making our method competitive to other EM probes.

In addition, constraints on H_0 and $\Omega_{m,0}$ in the w_0w_a CDM model are also weakened compared to the Λ CDM model. The relative 1σ uncertainty on H_0 becomes 21% for both the LVK O5 and post-O5 networks. Due to high degeneracy between H_0 and the dark energy parameters, as well as poor constraints on w_0 and w_a , the difference in H_0 constraints between the O5 and post-O5 networks becomes insignificant compared to the decrease in accuracy. For the ET + CE

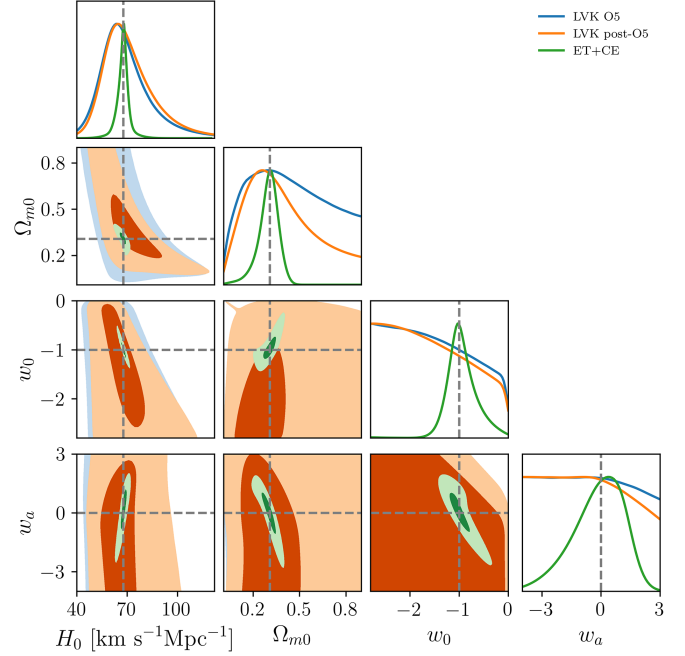


FIG. 4. The figure shows the forecast posterior on cosmological parameters in the w_0w_a CDM model by jointly analyzing all lensed GW events in each realization and averaging over all realizations with different detector networks. The gray dashed lines indicate the parameter values in the fiducial Λ CDM model.

network, the uncertainty on H_0 becomes 4.6%, which is increased by ~ 10 times. The evolution of the precision on the H_0 measurement by different detector networks is shown in Fig. 5. However, the uncertainty on $\Omega_{m,0}$ only increases slightly for the LVK O5 and post-O5 networks, which may be due to the fact that $\Omega_{m,0}$ is already poorly constrained by LVK. But it has climbed to 19% for the ET + CE network,

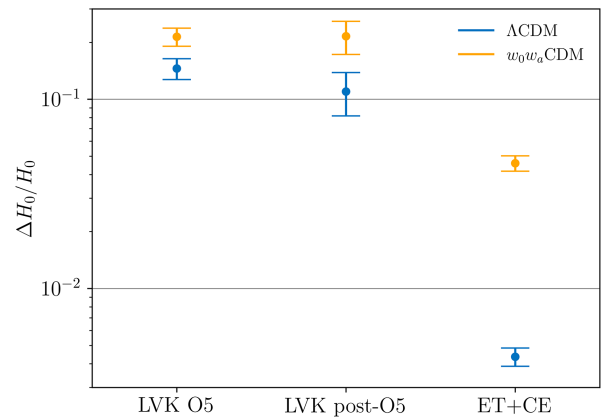


FIG. 5. Fractional 1σ uncertainty of H_0 in the Λ CDM model (blue) and w_0w_a CDM model (orange) with different detector networks. The upper and lower bars show the upper and lower 1σ uncertainty bounds, respectively, and the circles represent their means.

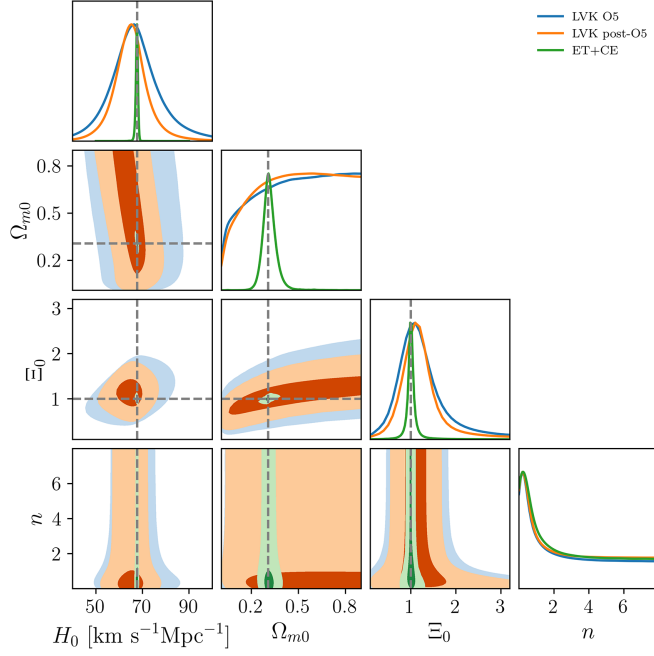


FIG. 6. The figure shows the forecast posterior on cosmological parameters in the (Ξ_0, n) modified gravity model by jointly analyzing all lensed GW events in each realization and averaging over all realizations with different detector networks. The gray dashed lines indicate the parameter values in the fiducial Λ CDM model.

which is ~ 7.4 times the $\Omega_{m,0}$ uncertainty in the Λ CDM model.

C. (Ξ_0, n) modified gravity model

We also analyze the constraints on modified gravity models with our method. For the (Ξ_0, n) parametrization, we jointly estimate H_0 , $\Omega_{m,0}$, Ξ_0 , and n . The averaged posteriors over all realizations are presented in Fig. 6. The averaged parameter constraints over all realizations are listed in Table II. For the LVK network, we obtain moderate constraints on H_0 and Ξ_0 , with a relative uncertainty in H_0 of 16% and 11% for the O5 network and the post-O5 network, respectively, which is comparable to the Λ CDM case. But the relative uncertainty in Ξ_0 is 49% and 36% for the O5 network and the post-O5 network, respectively. The constraint on Ξ_0 is not as tight as that on H_0 , because Ξ_0 only modifies d_L^{GW} via Eq. (2.17) in the GW waveform, but not the angular diameter distances in lens systems from EM probes, while H_0 affects all these distances. However, we obtain poor constraints on $\Omega_{m,0}$ for the LVK network, probably due to its degeneracy with H_0 and Ξ_0 . On the other hand, with the ET + CE network, we obtain strong constraints on H_0 and Ξ_0 , and a moderate constraint on $\Omega_{m,0}$. The relative uncertainties are 0.7% and 8.0% for H_0 and Ξ_0 , and 12% for $\Omega_{m,0}$. Finally, we find that the constraints on n favor a value near the lower bound, but they are not converged at the higher bound for all of the detector networks.

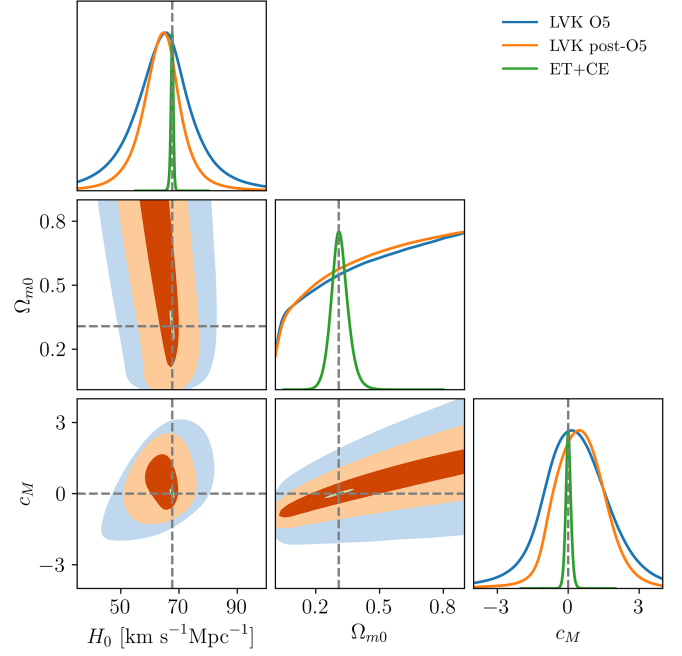


FIG. 7. The figure shows the forecast posterior on cosmological parameters in the scalar-tensor gravity model by jointly analyzing all lensed GW events in each realization and averaging over all realizations with different detector networks. The gray dashed lines indicate the parameter values in the fiducial Λ CDM model.

This could be explained by the fact that the fiducial Λ CDM case corresponds to either $n = 0$ or any value of n for $\Xi_0 = 1$.

For comparison, the constraint on Ξ_0 has a relative uncertainty of $\sim 50\%$ from the GWTC-4 dark siren analysis [12]. Forecasts on the future dark siren constraint on Ξ_0 show an uncertainty of $\sim 10\%$ with LIGO/Virgo design sensitivity [39], and $\sim 0.4\%$ with the ET + CE network with precessing BBHs [43]. Although the expected constraint for our method is slightly weaker than the dark siren constraint in the LVK O5 or the ET + CE era, the GW + EM strong-lensing constraint can still contribute to achieving a tighter combined constraint on modified gravity when used alongside other methods.

D. α -basis modified gravity model

Apart from the general (Ξ_0, n) modified gravity parametrization, we estimate constraints on the α -basis parametrization as well. Here, d_L^{GW} is modified by Eq. (2.18), so we jointly estimate H_0 , $\Omega_{m,0}$, and c_M . As shown by the averaged posteriors over all realizations in Fig. 7, the constraints on H_0 , $\Omega_{m,0}$, and c_M are similar to those on H_0 , $\Omega_{m,0}$, and Ξ_0 in the (Ξ_0, n) parametrization. The relative uncertainty on H_0 is 11%, 7%, and 0.7% for the LVK O5, post-O5, and ET + CE networks, respectively. $\Omega_{m,0}$ is again unconstrained with the LVK network, but it is moderately constrained with a relative error of 13% for the ET + CE network. For c_M , the fiducial value in the Λ CDM model is $c_M = 0$. The mean uncertainty of the

TABLE I. Cosmological parameter constraints with 1σ uncertainty in the Λ CDM and the w_0w_a CDM models from marginalized posteriors averaged over all realizations.

	Λ CDM		w_0w_a CDM			
	$H_0[\text{km s}^{-1} \text{Mpc}^{-1}]$	$\Omega_{m,0}$	$H_0[\text{km s}^{-1} \text{Mpc}^{-1}]$	$\Omega_{m,0}$	w_0	w_a
LVK O5	$64.77^{+8.24}_{-10.60}$	$0.325^{+0.382}_{-0.196}$	$63.83^{+15.17}_{-12.16}$	$0.301^{+0.373}_{-0.211}$	...	...
LVK post-O5	$65.88^{+5.38}_{-9.12}$	$0.297^{+0.277}_{-0.158}$	$64.61^{+16.67}_{-11.14}$	$0.260^{+0.285}_{-0.171}$	...	...
ET + CE	$67.70^{+0.33}_{-0.26}$	$0.308^{+0.009}_{-0.007}$	$67.94^{+2.83}_{-3.41}$	$0.312^{+0.057}_{-0.064}$	$-1.02^{+0.31}_{-0.25}$	$0.39^{+1.01}_{-1.55}$

TABLE II. Cosmological parameter constraints with 1σ uncertainty in modified gravity models from marginalized posteriors averaged over all realizations.

	(Ξ_0, n)				α -basis		
	$H_0[\text{km s}^{-1} \text{Mpc}^{-1}]$	$\Omega_{m,0}$	Ξ_0	n	$H_0[\text{km s}^{-1} \text{Mpc}^{-1}]$	$\Omega_{m,0}$	c_M
LVK O5	$66.38^{+10.35}_{-10.80}$	...	$1.07^{+0.58}_{-0.47}$	$0.21^{+6.21}_{-0.21}$	$65.44^{+10.04}_{-11.79}$	...	$0.15^{+1.57}_{-1.30}$
LVK post-O5	$65.30^{+6.56}_{-7.37}$	$0.587^{+0.354}_{-0.260}$	$1.13^{+0.39}_{-0.43}$	$0.21^{+6.44}_{-0.21}$	$64.77^{+6.48}_{-7.54}$	...	$0.48^{+0.98}_{-1.12}$
ET + CE	$67.68^{+0.41}_{-0.56}$	$0.310^{+0.040}_{-0.038}$	$1.00^{+0.09}_{-0.07}$	$0.23^{+6.37}_{-0.23}$	$67.69^{+0.38}_{-0.51}$	$0.309^{+0.043}_{-0.037}$	$0.01^{+0.11}_{-0.12}$

upper bound and the lower bound of c_M is 1.44, 1.05, and 0.12, for the LVK O5, post-O5, and the ET + CE networks, respectively. Similarly to the degeneracy between $\Omega_{m,0}$ and Ξ_0 in the (Ξ_0, n) model, we can also see degeneracy between $\Omega_{m,0}$ and c_M , but it is a bit stronger. This is likely because $d_L^{\text{GW}}/d_L^{\text{EM}}$ depends on both $\Omega_{m,0}$ and c_M in the α -basis model, while it is independent of $\Omega_{m,0}$ in the (Ξ_0, n) model.

VI. CONCLUSION

In this work, we generalize GW lensing cosmography by associating doubly lensed GWs with strong-lensing systems in galaxy surveys. By jointly reanalyzing the two lensed GW images using the SIS lens model, we can measure d_L^{GW} for the unlensed waveform and the impact parameter y . Under precise lensed GW localization with the effective detector network, the lens galaxy and the host galaxy of lensed GWs are very likely to be identified from strong-lensing catalogues, which could provide the redshift of the lens and the source, as well as Einstein's radius in the SIS model from the two image positions. With these observables, we can constrain cosmological parameters and modified GW propagation effects by combining time-delay cosmography and the standard siren approach. We simulate lensed GWs from BBH mergers detected by the LVK O5/post-O5 network and the ET + CE network, and select events within the range and resolvable by the LSST given its design, without using real or mock LSST data.

By simulating 1000 realizations of five-year observations, we find that the chance to observe lensed GWs by the LVK O5 network that can be associated with LSST is low, with an average detection number of 0.17 over all

realizations. But once such events are detected, they are expected to provide moderate constraints on H_0 with 14% uncertainty and modified GW propagation effects. With the LVK post-O5 network, the average detection number of lensed GWs associated with LSST increases to 2.3 over all realizations, which slightly enhances the cosmological constraints, but dynamical dark energy remains unconstrained. However, using the ET + CE network, we can detect an average of 80.9 lensed GWs associated with LSST over 100 realizations. They can provide a strong constraint on H_0 with 0.42% uncertainty and on $\Omega_{m,0}$ with 2.6% uncertainty in the Λ CDM model. It can also provide a moderate constraint on w_0 and w_a in the w_0w_a CDM model for dynamical dark energy, yielding $w_0 = -1.02^{+0.31}_{-0.25}$ and $w_a = 0.39^{+1.01}_{-1.55}$. It demonstrates the potential of our method to give an independent measurement of H_0 and shed light on the Hubble tension problem using the SIS lens model. Furthermore, it could also jointly constrain modified GW propagation effects with a high precision.

As discussed in literature, a small portion of strongly lensed GWs can produce more than two images. For triply lensed or quadruply lensed GWs, one can reconstruct the lens system with a more comprehensive lens model, such as the singular isothermal ellipsoid (SIE) model, using software like LENSTRONOMY [102]. With 10 quadruply lensed GWs detected by ET + CE and the images of their lensing systems, one can also constrain H_0 with an uncertainty $< 1\%$ [22]. Although the constraint by a single doubly lensed event is less accurate than a quadruply lensed event due to the difficulty in reconstructing the Fermat potential, the combination of a larger number of doubly lensed events can still provide a competitive constraint.

However, there exist some limitations in our method as well. First, the localization of doubly lensed GW events is more challenging than quadruply lensed events, so that the identification of the true lens systems with EM images would be more difficult, especially for the LVK network when the GW sky localization is not sufficiently precise. The detectability of lensed events also depends on accurate matching between lensed signals, careful lens model selection [103], and accounting for the selection effects of lensed signals (e.g., in Ref. [104]). Second, in reality, reconstruction of the lens model would be less accurate when the EM images have irregular shapes such as arcs, so that a clear image position angle is difficult to obtain. Third, using the SIS model for elliptical lens galaxies would lead to significant biases in cosmography. Therefore, despite the larger difficulty in lens reconstruction with only two images, a more careful choice is to use a more complex lens model than the SIS model for strong lensing by galaxies even for doubly lensed events, which however would lead to higher degeneracies between lens model parameters and cosmology, and thus weaker constraints on cosmology. Nevertheless, a combination of doubly lensed GWs and quadruply lensed GWs will further enhance the cosmological constraints.

Apart from direct constraints from strongly lensed GWs and their EM counterparts in galaxy catalogues, there are other proposals to constrain cosmology and modified gravity using lensed GWs as well—for example, using lensed GW population and time-delay distribution [105–108], or cross-correlation between weak lensing of GWs and galaxy surveys [109,110]. These independent methods could further provide great complements to the cosmological constraints with GW lensing. In conclusion, with the promising future of next-generation GW detectors and large-scale galaxy surveys, GW lensing cosmography will be a powerful tool in the era of precision cosmology.

ACKNOWLEDGMENTS

We thank Eungwang Seo for insightful comments during the internal review of our manuscript. A. C. is supported by the China Postdoctoral Science Foundation under Grant No. 2025M773325, and the National Natural Science Foundation of China (NSFC) under Grants No. E414660101 and No. 12147103. J. Z. is supported by the NSFC under Grants No. E414660101 and No. 12147103, and the Fundamental Research Funds for the Central Universities under Grants No. E4EQ6604X2 and No. E3ER6601A2. This material is based upon work supported by NSF’s LIGO Laboratory, which is a major facility fully funded by the National Science Foundation. We are grateful to the High Performance Computing Center (HPCC) of ICTP-AP for performing the numerical computations in this paper.

DATA AVAILABILITY

The data that support the findings of this article are openly available [111].

APPENDIX A: GW POPULATION MODEL

In our work, we simulate mock BBH events with a population model in the space of mass and redshift given by

$$p_{\text{pop}}(m_1^s, m_2^s, z | \Lambda_m, \Lambda_r) \propto p(m_1^s, m_2^s | \Lambda_m) p_{\text{rate}}(z | \Lambda_r) \frac{dV_c}{dz}, \quad (\text{A1})$$

where m_1^s, m_2^s are the source-frame masses. dV_c/dz is the differential comoving volume. We adopt the power-law + double peak black hole mass distribution model, which is a phenomenological model well fitted by GWTC-4 data [12]. It is described by ten parameters: minimal mass $M_{\min}$, maximal mass $M_{\max}$, slope of the power-law component of the primary mass distribution α , fraction of both Gaussian components in the primary mass distribution λ_g , fraction of the lower Gaussian component in the two Gaussian components $\lambda_{g,\text{low}}$, means of the lower and higher Gaussian components of the primary mass distribution $\mu_{g,\text{low}}, \mu_{g,\text{high}}$, widths of the lower and higher Gaussian components of the primary mass distribution $\sigma_{g,\text{low}}, \sigma_{g,\text{high}}$, and the range of mass tapering at the lower end of the mass distribution δ_m . The primary mass prior for this population model is given by

$$p(m_1^s | \Lambda_m) = (1 - \lambda_g) \mathcal{B}(m_1^s | M_{\min}, M_{\max}, \alpha) + \lambda_g \lambda_{g,\text{low}} \mathcal{G}(m_1^s | \mu_{g,\text{low}}, \sigma_{g,\text{low}}) + \lambda_g (1 - \lambda_{g,\text{low}}) \mathcal{G}(m_1^s | \mu_{g,\text{high}}, \sigma_{g,\text{high}}), \quad (\text{A2})$$

where $\mathcal{B}$ represents the broken power-law distribution, and $\mathcal{G}$ is the Gaussian distribution. The secondary mass is then drawn with a power-law mass ratio distribution with a slope of β .

We also apply the Madau-Dickinson merger rate redshift evolution model inspired from the Madau-Dickinson star formation rate [112] to our simulation, which is used by the GWTC-4 cosmology paper [12]. The merger rate in the function of redshift is given by

$$p_{\text{rate}}(z | \Lambda_r) = R_0 [1 + (1 + z_p)^{-\gamma-k}] \frac{(1 + z)^\gamma}{1 + \left(\frac{1+z}{1+z_p}\right)^{\gamma+k}}, \quad (\text{A3})$$

where R_0 is the merger rate at $z = 0$, while γ and k are the slopes of the power-law rate evolution at redshifts lower and higher than the turning point z_p , respectively. In our simulation, we adopt the parameter values from GWTC-4 cosmology analysis [12], which are $M_{\min} = 4.72 M_\odot$,

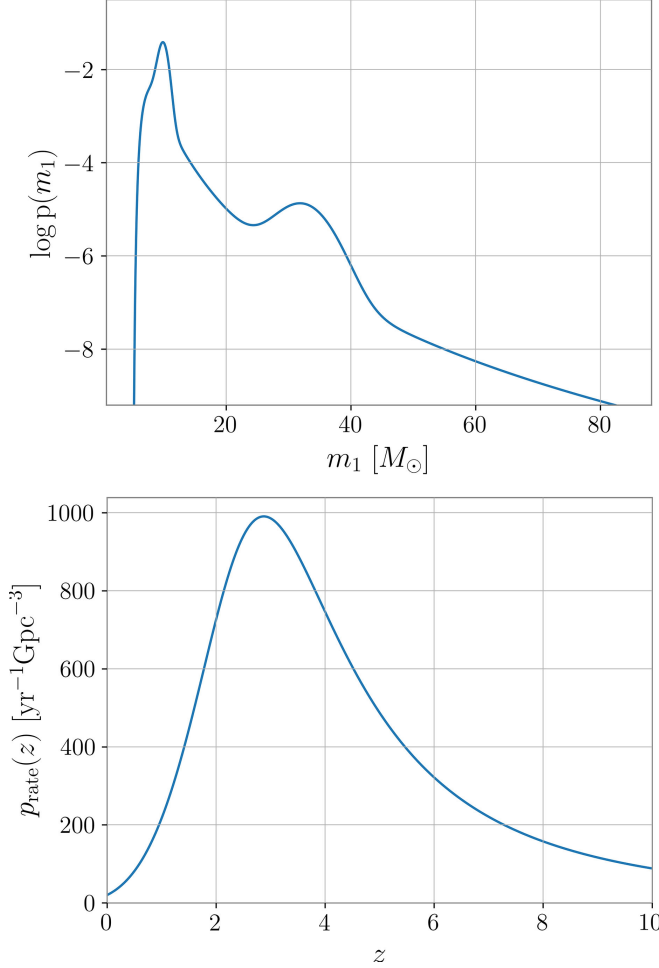


FIG. 8. Upper panel: Power-law + double peak black hole mass distribution model. Lower panel: Madau-Dickinson merger rate redshift evolution model. Both figures are plotted with the parameter values from GWTC-4 cosmology analysis [12].

$M_{\text{max}} = 84.28 M_\odot$, $\alpha = 2.97$, $\beta = 0.83$, $\mu_{\text{g,low}} = 9.91 M_\odot$, $\sigma_{\text{g,low}} = 0.83 M_\odot$, $\lambda_{\text{g,low}} = 0.86$, $\mu_{\text{g,high}} = 32.31 M_\odot$, $\sigma_{\text{g,high}} = 4.24 M_\odot$, $\lambda_{\text{g,high}} = 0.28$, $\delta_m = 4.71 M_\odot$, $\gamma = 3.46$, $k = 2.90$, and $z_p = 2.77$, as listed in Table III. For the local merger rate density, we adopt a typical value $R_0 = 20 \text{ yr}^{-1} \text{Gpc}^{-3}$. The corresponding mass distribution and redshift evolution are plotted in Fig. 8.

APPENDIX B: OPTICAL DEPTH OF SOURCES

The probability of a background source to be lensed by a foreground galaxy is also known as the optical depth. Its

differentiation with respect to the redshift and the velocity dispersion of the lens galaxy is given by [18]

$$\frac{d^2 \tau}{dz_L d\sigma} = \frac{dn}{d\sigma}(z_L, \sigma) s_{\text{cr}}(z_L, \sigma) \frac{cdt}{dz_L}(z_L), \quad (\text{B1})$$

where s_{cr} is the cross section of the signal deflection in the lens system. The lens number density distribution with respect to the galaxy velocity dispersion σ can be modeled by a modified Schechter function [113] as

$$\frac{dn}{d\sigma} = n_* \left(\frac{\sigma}{\sigma_*} \right)^\alpha \exp \left[- \left(\frac{\sigma}{\sigma_*} \right)^\beta \right] \frac{\beta}{\Gamma[\alpha/\beta]} \frac{1}{\sigma}, \quad (\text{B2})$$

where α is the faint-end slope, β is the high-velocity cutoff, and n_* and σ_* are the characteristic number density and velocity dispersion, which could be parametrized in a form evolving with redshift as $n_*(z) = n_{*,0}(1+z)^{3-\nu_{n^*}}$ and $\sigma_*(z) = \sigma_{*,0}(1+z)^{\nu_{\sigma^*}}$. In our work, we adopt the constant comoving lens number density, so that $\nu_{n^*} = \nu_{\sigma^*} = 0$.

The lensing statistic in a finite transient observation needs to account for the missing events due to the time delay of lensed signals [114]. The SIS cross section in a transient survey is given by

$$s_{\text{cr}}(z_L, \sigma) = \pi D_L^2 \theta_E^2 \int_{y_{\text{min}}}^{y_{\text{max}}} f(\Delta t(y; \sigma, z_L)) y dy, \quad (\text{B3})$$

where $f(\Delta t)$ is the fraction of lensed events with time delay Δt that can be observed. Assuming a uniform distribution of lensed signal arrival time during the continuous transient observation time T_{obs} , we have

$$f(\Delta t) = 1 - \frac{\Delta t}{T_{\text{obs}}} \quad (\text{B4})$$

for $\Delta t < T_{\text{obs}}$, and $f(\Delta t) = 0$ otherwise. Then, the cross section in Eq. (B3) results in

$$s_{\text{cr}} = \pi D_L^2 \theta_E^2 \left[(y_{\text{max}}^2 - y_{\text{min}}^2) - \frac{2}{3} \frac{\Delta t_z}{T_{\text{obs}}} (y_{\text{max}}^3 - y_{\text{min}}^3) \right]. \quad (\text{B5})$$

For the SIS model, we have $y_{\text{min}} = 0$. By inserting the above equation into Eq. (B1) and integrating it over σ in

TABLE III. Parameters for the power-law + double peak black hole mass distribution model and the Madau-Dickinson merger rate redshift evolution model used in our mock event simulation.

$M_{\text{min}} [M_\odot]$	$M_{\text{max}} [M_\odot]$	α	β	$\mu_{\text{g,low}} [M_\odot]$	$\sigma_{\text{g,low}} [M_\odot]$	$\lambda_{\text{g,low}}$	$\mu_{\text{g,high}} [M_\odot]$	$\sigma_{\text{g,high}} [M_\odot]$	$\lambda_{\text{g,high}}$	$\delta_m [M_\odot]$	γ	k	z_p
4.72	84.28	2.97	0.83	9.91	0.83	0.86	32.31	4.24	0.28	4.71	3.46	2.90	2.77

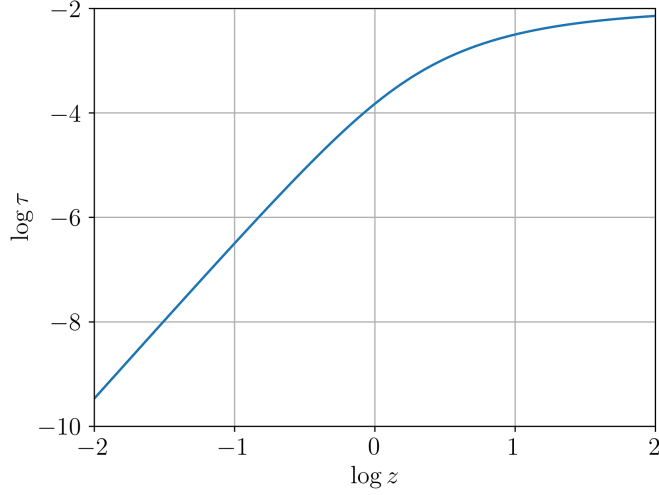


FIG. 9. Optical depth of background sources computed with Eq. (B6) using $n_{*,0} = 8.0 \times 10^{-3} h^3 \text{Mpc}^{-3}$, $\sigma_{*,0} = 144 \text{ km s}^{-1}$, $\alpha = 2.49$, and $\beta = 2.29$.

$(0, \infty)$, we arrive at

$$\tau = \frac{F_*}{30} [D_S(1+z_S)]^3 y_{\text{max}}^2 \left\{ 1 - \frac{1}{7} \frac{\Gamma[(8+\alpha)/\beta]}{\Gamma[(4+\alpha)/\beta]} \frac{\Delta t_*}{T_{\text{obs}}} \right\}, \quad (\text{B6})$$

where

$$F_* = 16\pi^3 n_{*,0} \left(\frac{\sigma_{*,0}}{c} \right)^4 \frac{\Gamma[(4+\alpha)/\beta]}{\Gamma[\alpha/\beta]}, \quad (\text{B7})$$

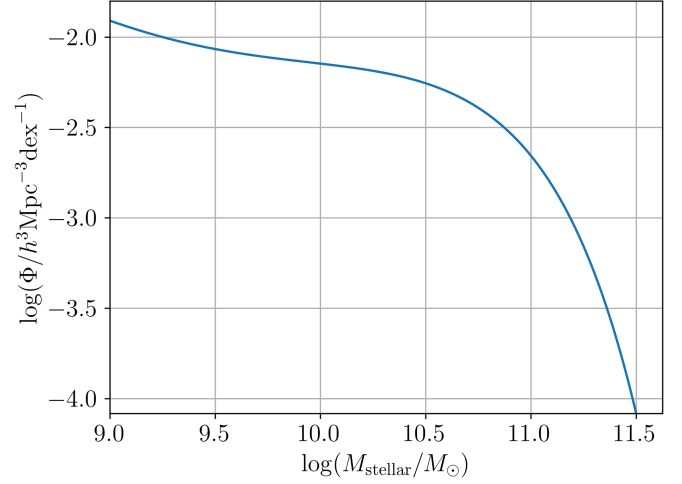


FIG. 10. The density profile of lens galaxies in stellar masses computed with Eq. (4.5).

$$\Delta t_* = 32\pi^2 \left(\frac{\sigma_{*,0}}{c} \right)^4 \frac{D_S}{c} (1+z_S) y_{\text{max}}. \quad (\text{B8})$$

Figure 9 shows the optical depth computed with parameters used in Sec. IV B. In addition, Fig. 10 shows the density profile of lens galaxies in stellar masses described by the double Schechter function in Eq. (4.5).

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