Toward a causal effective thermodynamics of scalar-tensor gravity

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The thermal analogy between the effective fluid of scalar-tensor gravity and Eckart's irreversible thermodynamics is extended to the causal Israel-Stewart model, adopting the minimal ansatz of promoting the heat flux density to a timelike vector. This choice yields analytically manageable constitutive equations, allowing for the first consistent decoupling of the effective temperature $\mathcal{T}$ and the effective thermal conductivity $\mathcal{K}$ of scalar-tensor gravity. Crucially, this new framework preserves the interpretation of general relativity as the equilibrium state approached via a dynamical relaxation process in the vanishing-$\mathcal{K}\mathcal{T}$ limit. This new causal formalism is applied to cosmology.

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I. INTRODUCTION

There are strong indications that general relativity (GR), might not be the ultimate theory of gravity. Indeed, despite its success and its powerful predictions, it still suffers from the problem of spacetime singularities. Furthermore, at cosmological scales, the standard ΛCDM model relies heavily on a dark sector [1], which is affected by worrisome tensions [2,3]. To address these issues, one option is that GR could be the low-energy limit of another more general theory. The simplest GR extension is scalar-tensor gravity, which introduces a scalar field ϕ in the Einstein-Hilbert action, nonminimally coupled with the gravitational sector [4–8]. The progenitor of scalar-tensor gravity is Brans-Dicke gravity [4]. Their idea was that a proper gravitational theory should incorporate Mach's principle, which is implemented by the relation $\phi \simeq 1/G_{\text{eff}}$, i.e., the scalar field as the inverse of the effective gravitational coupling strength. This new ingredient entails a dynamical Newton constant G, as first proposed by Dirac [9]. In parallel, one of the most popular alternatives to GR is $f(\mathcal{R})$ gravity, where $\mathcal{R}$ is the Ricci scalar and f a nonlinear function [10–12]. Notably $f(\mathcal{R})$ gravity turns out to be a subclass of scalar-tensor gravity, which makes this class of theories even more appealing.

Scalar-tensor gravity has recently been investigated from a thermal perspective, originating an analogy called first order thermodynamics of scalar-tensor gravity [13–18] (see Ref. [19] for an exhaustive review). Within this framework, the authors formalized a map between the constitutive equations of Eckart's irreversible thermodynamics [20] and the dissipative components of the effective energy-momentum tensor resulting from the modified field equations. Remarkably, this procedure provides an expression for the quantity $\mathcal{K}\mathcal{T}$, where $\mathcal{K}$ is an effective thermal conductivity and $\mathcal{T}$ the so-called effective temperature of gravity whose value determines the relaxation of scalar-tensor gravity to (or departure from) GR. This thermal analogy interprets scalar-tensor gravity as an out-of-equilibrium state that can dynamically relax toward GR in the limit $\mathcal{K}\mathcal{T} \to 0$. This feature provides an elegant interpretation of the attractor-to-GR mechanism [21,22], a topic extensively analyzed by Damour and Nordtvedt in [23,24], generating a large literature and debate.

The main limitation of the thermal view of scalar-tensor gravity is shared with Eckart's irreversible thermodynamics, which is known to be noncausal, despite having been

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successfully and widely used for many years. The goal of this work is to extend the thermal view of scalar-tensor gravity to the Israel-Stewart model [25], which entails hyperbolic, causal constitutive relations. In the remainder of this section we briefly review scalar-tensor gravity, while the thermal analogy is recalled in Sec. II, followed by its causal extension in Sec. III. We then apply this new formalism to Bianchi I and Friedmann-Lemaître-Robertson-Walker (FLRW) universes in Secs. IV and V and conclude with an analysis of exact solutions in Sec. VI.

The action of scalar-tensor gravity reads[1]

$$S_{\rm ST} = \int d^4x \frac{\sqrt{-g}}{16\pi}\left[\phi\mathcal{R} - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(\rm m)}, \tag{1.1}$$

where $\mathcal{R} \equiv \mathcal{R}^c{}_c$ is the Ricci scalar, the Brans-Dicke scalar $\phi > 0$ is approximately the inverse of the effective gravitational coupling, $\omega(\phi)$ is the "Brans-Dicke coupling," $V(\phi)$ is a potential, g is the determinant of the metric and $S^{(\rm m)} = \int d^4x\sqrt{-g}\mathcal{L}^{(\rm m)}$ is the matter action. The field equations [4,5,7,8], written as effective Einstein equations, are

$$\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R} = \frac{8\pi}{\phi}T^{(\rm m)}_{ab} + \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}, \tag{1.2}$$

$$\Box\phi = \frac{1}{2\omega+3}\left(8\pi T^{(\rm m)} + \phi\frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi}\nabla^c\phi\nabla_c\phi\right), \tag{1.3}$$

where $\mathcal{R}_{ab}$ is the Ricci tensor and $T^{(\rm m)}$ is the trace of the matter stress-energy tensor

$$T^{(\rm m)}_{ab} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S^{(\rm m)}}{\delta g^{ab}}.$$

The core idea of the thermal formulation is to group the contributions of the field ϕ and its derivatives into the effective stress-energy tensor

$$T^{(\phi)}_{ab} = \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}, \tag{1.4}$$

whose structure resembles the one of an imperfect fluid.

[1]We shall follow the notation of Ref. [26] in which $G = c = 1$, with G and c Newton's constant and speed of light respectively.

II. THERMAL VIEW OF SCALAR-TENSOR GRAVITY *À LA* ECKART

Let us introduce the effective dissipative fluid corresponding to the Brans-Dicke-like scalar field ϕ which originates the thermal analogy, beginning with the kinematic quantities [27]. Assuming that the gradient $\nabla^a\phi$ is timelike and future-oriented, it defines the effective fluid 4-velocity

$$u^a = \frac{\nabla^a\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}}. \tag{2.1}$$

Then spacetime splits into the time direction u^c plus the 3-dimensional space seen by the observers comoving with the fluid and orthogonal to u^a. The Riemannian metric in this 3-space is $h_{ab} \equiv g_{ab} + u_au_b$, while $h_a{}^b$ is the projection operator onto it. The effective fluid 4-acceleration $\dot{u}^a \equiv u^b\nabla_bu^a$ is orthogonal to the 4-velocity. By projecting the velocity gradient one obtains the tensor

$$V_{ab} \equiv h_a{}^ch_b{}^d\nabla_du_c = \Theta_{ab} = \sigma_{ab} + \frac{\Theta}{3}h_{ab} \tag{2.2}$$

(where $\Theta \equiv \Theta^c{}_c = \nabla_cu^c$ is the expansion scalar), which coincides with the symmetric expansion tensor Θ_{ab}. The vorticity $\omega_{ab} = V_{[ab]}$ vanishes because u^a derives from a gradient, then u^a is irrotational and hypersurface-orthogonal [26,27]. $\sigma_{ab} \equiv \Theta_{ab} - \Theta h_{ab}/3$ is the symmetric, trace-free shear tensor and the velocity gradient reads [27]

$$\nabla_bu_a = \sigma_{ab} + \frac{\Theta}{3}h_{ab} - \dot{u}_au_b. \tag{2.3}$$

Specializing these general definitions [26,27] to the effective ϕ-fluid, we compute [13,28]

$$\dot{u}_a = (-\nabla^e\phi\nabla_e\phi)^{-2}\nabla^b\phi[(-\nabla^e\phi\nabla_e\phi)\nabla_a\nabla_b\phi + \nabla^c\phi\nabla_b\nabla_c\phi\nabla_a\phi], \tag{2.4}$$

$$\Theta = \frac{\Box\phi}{(-\nabla^e\phi\nabla_e\phi)^{1/2}} + \frac{\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi}{(-\nabla^e\phi\nabla_e\phi)^{3/2}}, \tag{2.5}$$

$$\sigma_{ab} = (-\nabla^e\phi\nabla_e\phi)^{-3/2}\Big[-(\nabla^e\phi\nabla_e\phi)\nabla_a\nabla_b\phi - \frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi)\Box\phi - \frac{1}{3}\left(g_{ab} + \frac{2\nabla_a\phi\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right)\nabla_c\nabla_d\phi\nabla^d\phi\nabla^c\phi + (\nabla_a\phi\nabla_c\nabla_b\phi + \nabla_b\phi\nabla_c\nabla_a\phi)\nabla^c\phi\Big]. \tag{2.6}$$

The Brans-Dicke field then generates the effective stress-energy tensor (1.4) with imperfect fluid form

$$T_{ab} = \rho u_a u_b + q_a u_b + q_b u_a + \Pi_{ab}, \qquad \Pi_{ab} = P h_{ab} + \pi_{ab} \tag{2.7}$$

and effective energy density, heat flux density, stress tensor, isotropic pressure, and anisotropic stresses [15,28,29]

$$8\pi\rho^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi} + \frac{1}{\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{2.8}$$

$$8\pi q_a^{(\phi)} = \frac{\nabla^c\phi\nabla^d\phi}{\phi(-\nabla^e\phi\nabla_e\phi)^{3/2}}(\nabla_d\phi\nabla_c\nabla_a\phi - \nabla_a\phi\nabla_c\nabla_d\phi), \tag{2.9}$$

$$8\pi\Pi_{ab}^{(\phi)} = \left(-\frac{\omega}{2\phi^2}\nabla^c\phi\nabla_c\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi}\right)h_{ab} + \frac{1}{\phi}h_a{}^c h_b{}^d\nabla_c\nabla_d\phi, \tag{2.10}$$

$$8\pi P^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} - \frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{2.11}$$

$$\begin{aligned}8\pi\pi_{ab}^{(\phi)} = \frac{1}{\phi\nabla^e\phi\nabla_e\phi}\Bigg[&\frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi)\left(\Box\phi - \frac{\nabla^c\phi\nabla^d\phi\nabla_d\nabla_c\phi}{\nabla^e\phi\nabla_e\phi}\right)\\ &+ \nabla^d\phi\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi - \nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi}\right)\Bigg],\end{aligned} \tag{2.12}$$

while the isotropic pressure is the sum of perfect fluid and viscous contributions

$$P^{(\phi)} = P_{\rm pf}^{(\phi)} + P_{\rm v}^{(\phi)}. \tag{2.13}$$

Let us briefly review Eckart's thermal view of scalar-tensor gravity. Eckart's first order thermodynamics for *real* fluids ([20,30]) assumes three constitutive relations relating viscous pressure $P_{\rm v}$, heat current density q^c, and anisotropic stresses π_{ab} to the expansion Θ, temperature $\mathcal{T}$, and shear tensor σ_{ab}:

$$P_{\rm v} = -\zeta\Theta, \tag{2.14}$$

$$q_a = -\mathcal{K}h_{ab}(\nabla^b\mathcal{T} + \mathcal{T}\dot{u}^b), \tag{2.15}$$

$$\pi_{ab} = -2\eta\sigma_{ab}, \tag{2.16}$$

where ζ is the effective bulk viscosity coefficient, $\mathcal{K}$ is the effective thermal conductivity, and η is the effective shear viscosity coefficient. q^a is purely spatial in the frame of the effective fluid. The key point of the thermal analogy is that, by comparing Eqs. (2.9) and (2.4), one miraculously obtains [28]

$$q_a^{(\phi)} = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}\dot{u}_a. \tag{2.17}$$

In the frame comoving with the ϕ-fluid, the spatial temperature gradient vanishes identically and the heat flow is only due to the inertia of energy. The temperature of the ϕ-fluid, referred to as the "effective temperature of scalar-tensor gravity," pops out [28] as

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}. \tag{2.18}$$

This quantity is positive-definite and vanishes when the scalar ϕ is constant, which reproduces GR. Furthermore, the comparison of Eqs. (2.12) and (2.6) for $\pi_{ab}^{(\phi)}$ and $\sigma_{ab}^{(\phi)}$ and the use Eq. (2.16) give

$$\eta = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{16\pi\phi} = -\frac{\mathcal{K}\mathcal{T}}{2} < 0. \tag{2.19}$$

(Negative viscosities are common in fluid mechanics and in nonisolated systems [31], and our ϕ-fluid is not isolated due to the explicit coupling of ϕ to $\mathcal{R}$ in the action.)

The thermal view of scalar-tensor gravity *à la* Eckart includes an equation describing the approach to the GR equilibrium state, or its departure from GR [13–15]:

$$\frac{\mathrm{d}(\mathcal{K}\mathcal{T})}{\mathrm{d}\tau} = 8\pi(\mathcal{K}\mathcal{T})^2 - \Theta\mathcal{K}\mathcal{T} + \frac{\Box\phi}{8\pi\phi}. \tag{2.20}$$

In the simplified situations in which $\Box\phi = 0$ one deduces [13–15] that near spacetime singularities, where the world-lines of the ϕ field converge ($\Theta < 0$), the deviations of scalar-tensor gravity from GR are extreme. By contrast, the expansion of 3-space ($\Theta > 0$) "cools" gravity [13–15].

III. THERMAL VIEW OF SCALAR-TENSOR GRAVITY IN THE CAUSAL FORMALISM

A dissipative fluid in causal thermodynamics [30,32] is described as follows.[2] In causal thermodynamics, the Eckart constitutive relations [20] are promoted to the differential equations (e.g., [30,32])

$$\tau_0 \dot{P}_{\rm v} + P_{\rm v} = -\zeta\Theta - \frac{\zeta\mathcal{T}}{2}\left[\nabla_c\left(\frac{\tau_0 u^c}{\zeta\mathcal{T}}\right)\right]P_{\rm v}, \quad (3.1)$$

$$\tau_1 h_a{}^b \dot{q}_b + q_a = -\mathcal{K}(h_{ab}\nabla^b\mathcal{T} + \mathcal{T}\dot{u}_a) - \frac{\mathcal{K}\mathcal{T}^2}{2}\left[\nabla_b\left(\frac{\tau_1 u^b}{\mathcal{K}\mathcal{T}^2}\right)\right]q_a, \quad (3.2)$$

$$\tau_2 h_a{}^c h_b{}^d \dot{\pi}_{cd} + \pi_{ab} = -2\eta\sigma_{ab} - \frac{\eta\mathcal{T}}{2}\left[\nabla_c\left(\frac{\tau_2 u^c}{\eta\mathcal{T}}\right)\right]\pi_{ab}, \quad (3.3)$$

where τ_0, τ_1, τ_2 (which, in general, are not constant) are relaxation timescales and where an overdot denotes differentiation with respect to the proper time τ of the ϕ-fluid, i.e., $\dot{} \equiv u^c\nabla_c$.

In general, Eqs. (3.1)–(3.3) are difficult to solve analytically and the purpose of this work is to find a special sector of the theory of causal thermodynamics that satisfies judicious, although restrictive, choices of constitutive relations. The result will then be applied to the effective ϕ-fluid of scalar-tensor gravity. It is possible that the thermal view ultimately makes sense for the effective fluid (our main subject of interest here) but only marginally for real fluids—this possibility will be assessed in future work. Specifically, we look for the special thermal scenarios satisfying

$$P_{\rm v} = -\zeta\Theta, \quad (3.4)$$

$$q_a = -\mathcal{K}(\nabla_a\mathcal{T} + \mathcal{T}\dot{u}_a), \quad (3.5)$$

$$\pi_{ab} = -2\eta\sigma_{ab}, \quad (3.6)$$

where now the heat flux density 4-vector is timelike, $q_c q^c < 0$. This causal vector field decomposes as

$$q_a = (-q_c u^c)u_a + h_a{}^b q_b \equiv q_a^{(\parallel)} + q_a^{(\perp)} \quad (3.7)$$

with

$$h_a{}^b q_b^{(\parallel)} = 0, \qquad q_a^{(\perp)} u^a = 0. \quad (3.8)$$

[2]This is the description of a real fluid and, at this stage, we do not yet apply it to the effective ϕ-fluid of scalar-tensor gravity.

The difference with the noncausal thermodynamics of Eckart [20] is that, while for Eckart q^a is a purely spatial noncausal vector field satisfying the generalized Fourier law $q_a = -\mathcal{K}h_{ab}(\nabla^b\mathcal{T} + \mathcal{T}\dot{u}^b)$ and implying superluminal heat flow [20], now this vector field is promoted to a timelike one, with its spatial projection $q_a^{(\perp)}$ still satisfying the Eckart-Fourier law. By promoting q^a to a timelike vector, Eq. (3.5) seems the most natural and straightforward generalization of the Eckart-Fourier law and constitutes the most important and original assumption of this work. The scenario proposed here is a minimal generalization of Eckart's thermodynamics to make it causal. The fact that Eqs. (3.4) and (3.6) are still satisfied restricts the picture to Newtonian fluids, i.e., those in which viscous pressure and anisotropic stresses are linear in the gradient of the four-velocity, which is a statement about the mechanical behavior of the fluid. Even in prerelativistic physics, non-Newtonian fluids can be found among common materials (e.g., starch or custard) but Newtonian fluids are even more common and constitute the standard textbook situation (see, e.g., [33]). Eventually, we want to apply the causal generalization of Eckart's theory to the effective fluid of first-generation scalar-tensor gravity, which is well known to behave as a Newtonian fluid. More general theories, even in the well-explored "viable" Horndeski class, give rise to non-Newtonian effective (dissipative) fluids [34–36].

Using the decomposition (3.7), the dissipative fluid stress energy tensor

$$T_{ab} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a u_b + q_b u_a \quad (3.9)$$

with causal heat flux density q_a retains the Eckart form and is rewritten as

$$T_{ab} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a^{(\perp)} u_b + q_b^{(\perp)} u_a, \quad (3.10)$$

which contains explicitly only the spatial part $q_a^{(\perp)}$ of q_a. The total energy density now reads

$$\rho = \rho_{\rm pf} + \rho_{\rm v} = \rho_{\rm pf} - 2q_c u^c \quad (3.11)$$

and, similar to the total pressure P, is the sum of the usual perfect fluid energy density plus a "viscous energy density" contribution $\rho_{\rm v} = -2q_c u^c$, which is non-negative since q^a is now timelike and $q_a u^a < 0$ to keep q^a future-oriented. To this regard, the effective stress energy tensor $T_{ab}^{(\phi)}$ contains two terms: the perfect fluid contribution

$$\frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right) - \frac{V}{2\phi}g_{ab} \quad (3.12)$$

similar to the stress-energy tensor of a minimally coupled matter scalar field, which is quadratic in the first order derivatives of ϕ; and the dissipative term

$$\frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) \quad (3.13)$$

linear in the second derivatives of ϕ, which is the only term describing dissipation. Accordingly, the effective fluid energy density and pressure $\rho^{(\phi)}$ and $P^{(\phi)}$ given by Eqs. (2.8) and (2.11) clearly split into nonviscous and viscous contribution, while $q_a^{(\phi)}$ and $\pi_{ab}^{(\phi)}$ in Eqs. (2.9) and (2.12) contain second derivatives and are purely dissipative.

Let us comment on the choices (3.4)–(3.6). Equations (3.4) and (3.6) are Eckart's constitutive relations which mimic the three-dimensional nonrelativistic relations of a Newtonian fluid and describe its mechanical properties, detailing the response to expansion, contraction, and shear deformations. The ansatz (3.5) is the natural generalization of Eckart's law to the case of a timelike heat flux density (Eckart's law, in turn, generalizes Fourier's law of heat conduction). Since

$$q_a = -\mathcal{K}(\nabla_a \mathcal{T} + \mathcal{T}\dot{u}_a) = q_a^{(\|)} + q_a^{(\perp)} \equiv q^{(\|)} u_a + q_a^{(\perp)}, \tag{3.14}$$

we have

$$q^{(\|)} = -q_c u^c = \mathcal{K}\dot{\mathcal{T}}. \tag{3.15}$$

For scenarios satisfying Eqs. (3.4)–(3.6), the causal thermodynamics equations (3.1)–(3.3) reduce to

$$\tau_0 \dot{P}_{\rm v} + \frac{\zeta \mathcal{T}}{2}\left[\nabla_c\left(\frac{\tau_0 u^c}{\zeta \mathcal{T}}\right)\right] P_{\rm v} = 0, \tag{3.16}$$

$$\tau_1 h_a{}^b \dot{q}_b + \frac{\mathcal{K}\mathcal{T}^2}{2}\left[\nabla_b\left(\frac{\tau_1 u^b}{\mathcal{K}\mathcal{T}^2}\right)\right] q_a - (q_c u^c) u_a = 0, \tag{3.17}$$

$$\tau_2 h_a{}^c h_b{}^d \dot{\pi}_{cd} + \frac{\eta \mathcal{T}}{2}\left[\nabla_c\left(\frac{\tau_2 u^c}{\eta \mathcal{T}}\right)\right] \pi_{ab} = 0. \tag{3.18}$$

Consider now the effective ϕ-fluid of first-generation scalar-tensor gravity, for which the effective stress-energy tensor has the Eckart form (3.10) [15,28,29]. By comparing the calculated heat flux density and 4-acceleration of ϕ [13–15],

$$8\pi q_a^{(\phi)} = \frac{\nabla^c\phi\nabla^d\phi}{(-\nabla^e\phi\nabla_e\phi)^{3/2}\phi}(\nabla_d\phi\nabla_a\nabla_c\phi - \nabla_a\phi\nabla_c\nabla_d\phi) \tag{3.19}$$

and

$$\dot{u}_a = \frac{\nabla^b\phi[(-\nabla^e\phi\nabla_e\phi)\nabla_a\nabla_b\phi + \nabla^c\phi\nabla_b\nabla_c\phi\nabla_a\phi]}{(-\nabla^e\phi\nabla_e\phi)^2} \tag{3.20}$$

one obtains, similarly to noncausal thermodynamics[3] [13–15],

$$q_a^{(\perp)} = -\mathcal{K}\mathcal{T}\dot{u}_a \tag{3.21}$$

(with the difference that in noncausal thermodynamics q_a coincides with its purely spatial projection $q_a^{(\perp)}$, while $q_a^{(\|)} = 0$) and identifies again

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi} \tag{3.22}$$

for $\nabla^a\phi$ timelike and future-oriented. Furthermore, in the frame comoving with the ϕ-fluid it is $h_{ab}\nabla^b\mathcal{T} = 0$, that is,

$$\mathcal{T} = \mathcal{T}(\tau), \tag{3.23}$$

where τ is the proper time of the observer comoving with the ϕ-fluid. This property does not imply that the choice (3.4)–(3.6) is limited to spatially homogeneous manifolds (e.g., FLRW or Bianchi cosmologies) because the effective thermal conductivity $\mathcal{K}$ can still depend on the spatial position.

Continuing, we have $q_a = q^{(\|)}u_a + q_a^{(\perp)}$ with

$$q^{(\|)} = -q_c u^c = \mathcal{K}u^c(\nabla_c\mathcal{T} + \mathcal{T}\dot{u}_c) = \mathcal{K}\dot{\mathcal{T}}, \tag{3.24}$$

$$q_a^{(\perp)} = h_a{}^b q_b = -\mathcal{K}h_a{}^b(\nabla_b\mathcal{T} + \mathcal{T}\dot{u}_b) = -\mathcal{K}\mathcal{T}\dot{u}_a, \tag{3.25}$$

$$q_a = \mathcal{K}\dot{\mathcal{T}}u_a - \mathcal{K}\mathcal{T}\dot{u}_a \tag{3.26}$$

and (given that u^a is timelike and future-oriented) q^a is future-oriented if $q^a u_a < 0$, i.e., $\mathcal{K}\dot{\mathcal{T}} > 0$, then

$$\rho_{\rm v} = 2q^{(\|)} = 2\mathcal{K}\dot{\mathcal{T}} > 0. \tag{3.27}$$

The relations between $\mathcal{K}\mathcal{T}$ and the viscosity coefficients derived in [13–15] for noncausal thermodynamics still hold in the new causal thermal view since they only involve purely spatial tensors:

$$\zeta = -\frac{\mathcal{K}\mathcal{T}}{3}, \qquad \eta = -\frac{\mathcal{K}\mathcal{T}}{2}. \tag{3.28}$$

The differentiation of Eq. (3.22) [13,14] along the effective fluid lines leads again to Eq. (2.20), which remains valid in the causal thermal view of scalar-tensor gravity.

[3]In what follows, we drop the ϕ superscript for the effective fluid to simplify the notation.

IV. BIANCHI I COSMOLOGY

Let us consider now spatially homogeneous but anisotropic Bianchi I solutions of Brans-Dicke gravity,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_1^2(t)\mathrm{d}x^2 + a_2^2(t)\mathrm{d}y^2 + a_3^2(t)\mathrm{d}z^2, \tag{4.1}$$

$$\phi = \phi(t). \tag{4.2}$$

Bianchi I cosmologies are much simpler than a general geometry because they allow for shear and bulk viscosity but not for spatial heat fluxes.

The scalar field gradient $\nabla^a\phi$ is timelike and is future-oriented if $\phi_t \equiv \partial_t\phi < 0$. Since

$$u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\nabla^0\phi}{\sqrt{-\nabla^c\phi\nabla_c\phi}} = \frac{g^{0a}\partial_a\phi}{\sqrt{\phi_t^2}} = \frac{-\phi_t}{|\phi_t|} = 1, \tag{4.3}$$

one has $dt = d\tau$ and differentiation with respect to t coincides with differentiation with respect to τ on scalar functions f, $\dot{f} \equiv df/d\tau = df/dt$. We use the average scale factor

$$a(t) \equiv [a_1(t)a_2(t)a_3(t)]^{1/3}, \tag{4.4}$$

the average Hubble function

$$H(t) \equiv \dot{a}/a \tag{4.5}$$

(in terms of which the expansion scalar is $\Theta = 3H$), and the three directional Hubble functions

$$H_{(i)} \equiv \frac{\dot{a}_{(i)}}{a_{(i)}} \qquad (i = 1, 2, 3). \tag{4.6}$$

Assuming the relations (3.4)–(3.6) and using the quantities

$$X \equiv \frac{\tau_0}{3\zeta\mathcal{T}}, \qquad Y \equiv \frac{\tau_1}{\mathcal{K}\mathcal{T}^2}, \qquad Z \equiv \frac{\tau_2}{2\eta\mathcal{T}}, \tag{4.7}$$

the reduced equations of causal thermodynamics (3.16)–(3.18) read, in a Bianchi I universe,

$$\tau_0\dot{P}_\mathrm{v} + \frac{\tau_0}{2X}[\nabla_a(Xu^a)]P_\mathrm{v} = 0, \tag{4.8}$$

$$\tau_1 h_a{}^b\dot{q}_b + \frac{\tau_1}{2Y}[\nabla_b(Yu^b)]q_a - (q_c u^c)u_a = 0, \tag{4.9}$$

$$h_a{}^c h_b{}^d\dot{\pi}_{cd} + \frac{1}{2Z}[\nabla_e(Zu^e)]\pi_{ab} = 0, \tag{4.10}$$

which share a similar structure.

Let us begin by analyzing Eq. (4.8), which is rewritten as

$$\frac{\dot{P}_\mathrm{v}}{P_\mathrm{v}} + \frac{3H}{2} + \frac{\dot{X}}{2X} = 0 \tag{4.11}$$

and integrates to

$$|P_\mathrm{v}| = \frac{P_0}{a^{3/2}\sqrt{|X|}} = \frac{P_0\sqrt{3|\zeta|\mathcal{T}}}{\sqrt{\tau_0}a^{3/2}}, \tag{4.12}$$

where $P_0 > 0$ is an integration constant. Consistency with the assumption $P_\mathrm{v} = -\zeta\Theta = -3\zeta H$ then requires that

$$3|\zeta||H| = \frac{P_0\sqrt{3|\zeta|\mathcal{T}}}{\sqrt{\tau_0}a^{3/2}}, \tag{4.13}$$

where $\zeta = -\mathcal{K}\mathcal{T}/3$, yielding the thermal conductivity

$$\mathcal{K} = \frac{K_0}{\tau_0 a\dot{a}^2} = \frac{P_0^2}{\tau_0 a\dot{a}^2}, \tag{4.14}$$

with $\mathcal{K}_0 = P_0^2$. From this expression we derive the effective fluid temperature by remembering that $\mathcal{K}\mathcal{T} = \sqrt{-\nabla^c\phi\nabla_c\phi}/(8\pi\phi)$:

$$\mathcal{T} = \frac{\tau_0}{8\pi P_0^2}a\dot{a}^2\frac{|\dot{\phi}|}{\phi}. \tag{4.15}$$

This result is particularly remarkable as it yields the first consistent splitting of $\mathcal{K}$ and $\mathcal{T}$, that is impossible to obtain (in a unique way) within the noncausal thermal formulation. Equations (4.14) and (4.15) for $\mathcal{K}$ and $\mathcal{T}$ can be substituted back into Eq. (4.12) for P_v, producing

$$|P_\mathrm{v}| = |H|\frac{|\dot{\phi}|}{8\pi\phi}. \tag{4.16}$$

Let us consider now Eq. (4.9), which we rewrite [using Eq. (3.7)] as

$$\tau_1 h_a{}^b\dot{q}_b + \frac{\tau_1}{2Y}[\nabla_c(Yu^c)]h_a{}^b q_b + \left\{\frac{\tau_1}{2Y}[\nabla_c(Yu^c)] + 1\right\} \times (-q_c u^c)u_a = 0. \tag{4.17}$$

A vector is identically zero if and only if all of its components vanish, hence this equality can be rewritten in terms of its projection in the direction of u^a and onto the 3-space orthogonal to u^a, i.e.,

$$h_a{}^b\tau_1\dot{q}_b + \left\{\frac{\tau_1}{2Y}[\nabla_c(Yu^c)]q_b\right\}h_a{}^b = 0, \tag{4.18}$$

$$\left\{\frac{\tau_1}{2Y}[\nabla_c(Yu^c)] + 1\right\}(-q_c u^c) = 0, \tag{4.19}$$

respectively. These equations admit the trivial solution $q_a = 0$.

Observe that, since $q_c u^c \neq 0$ we have that Eq. (4.19) implies

$$\frac{\tau_1}{2Y}[\nabla_c(Yu^c)] = -1. \tag{4.20}$$

Thus, plugging this condition into Eq. (4.18) yields

$$(\tau_1 \dot{q}_b - q_b) h_a{}^b = 0, \tag{4.21}$$

therefore the vector field $\tau_1 \dot{q}_b - q_b$ is parallel to the time direction, i.e. $\tau_1 \dot{q}_b - q_b = \lambda u_b$. However, for Bianchi I we have that $\dot{u}^a = u^c \nabla_c u^a = \nabla_0 u^a = 0$ since $u^a = \delta^a_0$, hence the second term $-q_b h_a{}^b = -q_a^{(\perp)} = -\mathcal{K}\mathcal{T}\dot{u}_a = 0$. We therefore conclude that $\dot{q}_a$ is parallel to u_a.

Finally, let us discuss Eq. (4.10), which is rewritten as

$$h_a{}^c h_b{}^d \dot{\pi}_{cd} + \frac{1}{2}\left(\frac{\dot{Z}}{Z} + 3H\right)\pi_{ab} = 0, \tag{4.22}$$

or

$$h_a{}^c h_b{}^d \dot{\pi}_{cd} + \pi_{ab}\left(\ln\sqrt{|Z|a^3}\right)^{\bullet} = 0. \tag{4.23}$$

It is shown in Appendix A that $\dot{\pi}_{ab}$ is always spatial, i.e., $h_a{}^c h_b{}^d \dot{\pi}_{cd} = \dot{\pi}_{ab}$, turning this equation into

$$\dot{\pi}_{ab} + \pi_{ab}\left(\ln\sqrt{|Z|a^3}\right)^{\bullet} = 0, \tag{4.24}$$

which admits a solution of the form

$$\pi_{ab} = \frac{C_{ab}}{\sqrt{|Z|a^3}}, \tag{4.25}$$

where C_{ab} is a constant symmetric and purely spatial 2-tensor. Using the definition of Z and Eqs. (4.14) and (4.15), this equation becomes

$$\pi_{ab} = \bar{C}_{ab}\sqrt{\frac{\tau_0}{\tau_2}}H\frac{|\dot{\phi}|}{\phi}, \tag{4.26}$$

absorbing some extra constants into the redefinition $\bar{C}_{ab}$ of C_{ab} (and assuming $\tau_0, \tau_1, \tau_2 > 0$, which is reasonable since they are meant to represent timescales).

V. FLRW COSMOLOGY

Let us consider now spatially flat Friedmann-Lemaître-Robertson-Walker universes, which are special cases of Bianchi I metrics (4.1) with $a_1(t) = a_2(t) = a_3(t) \equiv a(t)$. Due to spatial homogeneity and isotropy, the shear σ_{ab}, anisotropic stresses π_{ab}, and spatial heat flux density $q_a^{(\perp)}$ all vanish, there is no shear viscosity and only bulk viscosity associated with the bulk viscous pressure P_{v} survives.

The expressions (4.16) and (4.15) for the viscous pressure and temperature derived in Bianchi I cosmologies still hold in the degenerate FLRW case. The effective pressure of the ϕ-fluid, read from Eq. (2.11), is clearly

$$8\pi P_{\mathrm{v}} = -\frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right). \tag{5.1}$$

Let us restrict, for simplicity, to FLRW cosmologies in which $\Box\phi = -(\ddot{\phi} + 3H\dot{\phi}) = 0$, corresponding [cf., Eq. (1.3)] to vacuum or conformal matter and zero or quadratic potential $V(\phi)$. In this case, the equation of motion $\Box\phi = 0$ gives the first integral $\dot{\phi} = \mathrm{const}/a^3$. In FLRW space we have

$$\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi = \dot{\phi}^2\nabla_0\nabla_0\phi = \dot{\phi}^2(\ddot{\phi} - \Gamma^0_{00}\dot{\phi}) = \dot{\phi}^2\ddot{\phi}, \tag{5.2}$$

giving

$$P_{\mathrm{v}} = H\frac{|\dot{\phi}|}{8\pi\phi}. \tag{5.3}$$

As a check, this time dependence matches that of Eq. (4.16).

The heat flux density $q^a = (q^0, \mathbf{0})$ has only the time component while for u^a to be future-oriented we need $\dot{\phi} < 0$, thus

$$\begin{aligned}
q^{(\|)} = \mathcal{K}\dot{\mathcal{T}} &= \frac{P_0^2}{\tau_0 a\dot{a}^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\tau_0}{8\pi P_0^2}a\dot{a}^2\frac{\dot{\phi}}{\phi}\right)\\
&= -\frac{1}{8\pi}\frac{1}{\tau_0 a\dot{a}^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tau_0 a\dot{a}^2\frac{\dot{\phi}}{\phi}\right)\\
&= -\frac{1}{8\pi}\frac{1}{\tau_0 a\dot{a}^2}\left[\dot{\tau}_0 a\dot{a}^2\frac{\dot{\phi}}{\phi} + \tau_0\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a}^2)\frac{\dot{\phi}}{\phi} + \tau_0 a\dot{a}^2\left(\frac{\dot{\phi}}{\phi}\right)^{\bullet}\right]\\
&= -\frac{1}{8\pi}\left[\frac{\dot{\tau}_0}{\tau_0}\frac{\dot{\phi}}{\phi} + \frac{1}{a\dot{a}^2}\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a}^2)\frac{\dot{\phi}}{\phi} + \left(\frac{\dot{\phi}}{\phi}\right)^{\bullet}\right]\\
&= -\frac{1}{8\pi}\frac{\dot{\phi}}{\phi}\left[\frac{\dot{\tau}_0}{\tau_0} + \frac{1}{a\dot{a}^2}\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a}^2) + \frac{\phi}{\dot{\phi}}\left(\frac{\dot{\phi}}{\phi}\right)^{\bullet}\right]\\
&= -\frac{\dot{\phi}}{8\pi\phi}\frac{\mathrm{d}}{\mathrm{d}t}\ln\left|\frac{a\dot{a}^2\tau_0\dot{\phi}}{\phi}\right|.
\end{aligned} \tag{5.4}$$

If we now use the assumption $\Box\phi = 0$ we find that $\dot{\phi} = v_0/a^3$, for which we conclude that

$$q^{(\|)} = -\frac{\dot{\phi}}{8\pi\phi}\frac{\mathrm{d}}{\mathrm{d}t}\ln\left|\frac{a\dot{a}^2\tau_0\dot{\phi}}{\phi}\right| = -\frac{\dot{\phi}}{8\pi\phi}\frac{\mathrm{d}}{\mathrm{d}t}\ln\left|\frac{H^2\tau_0}{\phi}\right|. \tag{5.5}$$

Let us consider now the viscous component of the effective energy density (2.8)

$$\rho_{\rm v} = \frac{1}{8\pi\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right) = -\frac{3H\dot{\phi}}{8\pi\phi} = 3P_{\rm v}. \tag{5.6}$$

We also have

$$\rho_{\rm v} = 2\mathcal{K}\dot{\mathcal{T}} = -\frac{\dot{\phi}}{4\pi\phi}\frac{\rm d}{{\rm d}t}\ln\left|\frac{H^2\tau_0}{\phi}\right|. \tag{5.7}$$

For consistency, the two expressions of $\rho_{\rm v}$ (5.6) and (5.7) must be equal, which provides an equation for the relaxation timescale τ_0:

$$-\frac{3H\dot{\phi}}{8\pi\phi} = -\frac{\dot{\phi}}{4\pi\phi}\frac{\rm d}{{\rm d}t}\ln\left|\frac{H^2\tau_0}{\phi}\right|, \tag{5.8}$$

or

$$\frac{3\dot{a}}{2a} = \frac{\rm d}{{\rm d}t}\ln\left|\frac{H^2\tau_0}{\phi}\right|, \tag{5.9}$$

which implies

$$\frac{\rm d}{{\rm d}t}\ln a^{3/2} = \frac{\rm d}{{\rm d}t}\ln\left|\frac{H^2\tau_0}{\phi}\right|,$$

which yields

$$\tau_0 = \frac{C_0 a^{3/2}\phi}{H^2} = \frac{C_0 a^{7/2}\phi}{\dot{a}^2} \tag{5.10}$$

with $C_0 > 0$ an integration constant. Moreover the logarithmic derivative of τ_0 yields

$$\frac{\dot{\tau}_0}{\tau_0} = \frac{3H}{2} - \frac{2\dot{H}}{H} - \frac{|\dot{\phi}|}{\phi} = \frac{\Theta}{2} - \frac{2\dot{\Theta}}{\Theta} - 8\pi\mathcal{K}\mathcal{T}. \tag{5.11}$$

This equation highlights a competition between the first two terms and the third one. In an expanding universe, the expansion terms consistently provide a positive a contribution, whereas the third term always contributes negatively. This interplay serve as a benchmark for understanding when the effective Eckart theory properly approximates the Israel-Stewart generalization introduced here. Indeed, during the relaxation process, when GR is approached, $\mathcal{K}\mathcal{T} \to 0$. In this limit the expansion terms becomes dominant, leading to $\dot{\tau}_0 > 0$, which makes the causal formulation relevant. Conversely, in strong gravity regimes far from GR ($\mathcal{K}\mathcal{T} \to \infty$) the negative term dominates leading to $\dot{\tau}_0 < 0$. In this scenario the Eckart theory provides a good approximation of the causal thermal setup.

Finally, let us divide Eqs. (5.3) and (5.6) to obtain

$$\frac{P_{\rm v}}{\rho_{\rm v}} = \frac{1}{3}, \tag{5.12}$$

that is, the viscous component of the effective ϕ-fluid obeys the radiation equation of state. One cannot ascribe much physical meaning to this feature because $P_{\rm v}$ and $\rho_{\rm v}$ always appear together with their nonviscous counterparts and their splitting into nonviscous and viscous parts is formal. The exception is the case $\omega = 0$, $V = 0$ in which only the viscous components are present (which could be called a "maximally viscous" scenario). $f(R)$ gravity is equivalent to an $\omega = 0$ Brans-Dicke theory, but with a complicated potential $V(\phi)$ [10–12] which reintroduces a nonviscous term. The noncausal thermal view of $f(R)$ cosmology is discussed in [37]. We leave the causal thermal analysis of $f(R)$ gravity for future work.

VI. EXACT SOLUTIONS

In this final section, we consider exact solutions that provide an explicit expression for ϕ and allow for the analytical computation of the thermodynamic quantities.

A. Kasner solution

The Kasner vacuum solution in GR is a special case of the spatially homogeneous and anisotropic Bianchi I geometry, which generalizes the FLRW spatially flat universe. The line element in comoving Cartesian coordinates reads

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2, \tag{6.1}$$

with

$$a_i = \left(\frac{t}{t_0}\right)^{p_i}, \quad i = 1, 2, 3 \tag{6.2}$$

and where t_0 and p_i are constants obeying

$$p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^2 = 1. \tag{6.3}$$

However, in scalar-tensor gravity with constant ω, $V(\phi) = 0$ and $\phi = \phi(t)$ (because of spatial homogeneity), one obtains instead ([38,39])

$$a_i = \left(\frac{t}{t_0}\right)^{\frac{p_i}{1+C}}, \qquad \phi(t) = \phi_0\left(\frac{t}{t_0}\right)^{\frac{C}{1+C}} \tag{6.4}$$

with ϕ_0 and C constants.[4] This gives the average scale factor

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{3(1+C)}}, \tag{6.5}$$

where Eq. (6.3)has been used.

[4]Setting $C = 0$ reduces the solution to GR.

The future-orientedness of $\nabla^a\phi$ requires $\dot{\phi} < 0$, namely

$$\dot{\phi} = \frac{C}{1+C}\frac{\phi_0}{t_0}\left(\frac{t}{t_0}\right)^{-\frac{1}{1+C}} < 0, \tag{6.6}$$

that it is satisfied for

$$-1 < C < 0. \tag{6.7}$$

With these prescriptions, one can now compute the (effective) dissipative quantities $\rho_{\mathrm{v}}, P_{\mathrm{v}}, \mathcal{K}, \mathcal{T}, q^{(\parallel)}$ and $\mathcal{KT}$. However, since $\mathcal{K}$ and $\mathcal{T}$ depend on τ_0, we first compute the latter

$$\tau_0 = 9(C+1)^2 C_0 t_0^2 \phi_0 \left(\frac{t}{t_0}\right)^{\frac{6C+5}{2(C+1)}}, \tag{6.8}$$

$$\rho_{\mathrm{v}} = 3P_{\mathrm{v}} = \frac{|C|}{8\pi(C+1)^2 t^2}, \tag{6.9}$$

$$\mathcal{K} = \frac{P_0^2}{\phi_0 C_0}\left(\frac{t}{t_0}\right)^{-\frac{3+2C}{2(1+C)}}, \tag{6.10}$$

$$\mathcal{T} = \frac{|C| C_0 \phi_0}{8\pi(C+1)P_0^2 t_0}\left(\frac{t}{t_0}\right)^{\frac{1}{2(1+C)}}, \tag{6.11}$$

$$\mathcal{KT} = \frac{|C|}{8\pi t(1+C)}. \tag{6.12}$$

The restriction (6.7) on C ensures the positivity of all these quantities. At late times, both the viscous components and $\mathcal{KT}$ relax to zero, recovering GR. This situation is consistent with a perfect fluid reaching a thermal stable equilibrium. From a thermodynamic perspective, this limit corresponds to the relaxation of a dissipative process driven by the cosmic expansion. Conversely for $t \to 0^+$, all the thermodynamic quantities diverge, with the exception of $\mathcal{T}$ that vanishes. Moreover, τ_0 does not vanish in the $t \to +\infty$ limit for $C \in [-5/6, 0)$, namely in the parameter space closest to GR. This agrees with Eq. (5.11), indicating that the causal model becomes increasingly relevant as the theory approaches GR. Finally, the time component of the heat flux reads

$$q^{(\parallel)} = \frac{|C|}{16\pi(C+1)^2 t^2}, \tag{6.13}$$

that matches $3P_{\mathrm{v}} = \rho_{\mathrm{v}} = 2q^{(\parallel)}$.

B. O'Hanlon and Tupper (O'H-T) solution

The O'H-T vacuum solution [40] represents a flat FLRW power-law universe of Brans-Dicke cosmology obtained for $\omega = \text{const.} > -3/2$, $V = 0$, and

$$a = \left(\frac{t}{t_0}\right)^{q_\pm}, \qquad \phi = \phi_0\left(\frac{t}{t_0}\right)^{s_\pm}, \tag{6.14}$$

with

$$s_\pm = 1 - 3q_\pm, \quad q_\pm = \frac{\omega}{3(\omega+1) \mp \sqrt{3(2\omega+3)}}. \tag{6.15}$$

The request that $\nabla^a\phi$ be future-oriented forces the selection of the $(+)$ solution, yielding

$$\rho_{\mathrm{v}} = 3P_{\mathrm{v}} = \frac{|s(s-1)|}{8\pi t^2}, \tag{6.16}$$

$$\tau_0 = \frac{9C_0\phi_0 t_0^2}{(s-1)^2}\left(\frac{t}{t_0}\right)^{\frac{5+s}{2}}, \tag{6.17}$$

$$\mathcal{K} = \frac{P_0^2}{C_0\phi_0}\left(\frac{t}{t_0}\right)^{\frac{s-3}{2}}, \tag{6.18}$$

$$\mathcal{T} = \frac{C_0\phi_0|s|}{8\pi P_0^2 t_0}\left(\frac{t}{t_0}\right)^{\frac{1-s}{2}}, \tag{6.19}$$

$$\mathcal{KT} = \frac{|s|}{8\pi t}. \tag{6.20}$$

Both ρ_{v} and P_{v} vanish for $s \to 0$ (i.e., $\phi \to \phi_0$) and for $t \to \infty$, stressing their dissipative nature. Notice that $s - 1 = 0$ would require $q = 0$, which is impossible for $\omega \neq 0$. The decoupling of $\mathcal{K}$ and $\mathcal{T}$ shows that, at any given finite time t, $\mathcal{T}$ is the only parameter that provides relaxation toward GR in the $s \to 0$ limit and again exhibits a regular behavior as $t \to 0^+$. Vice-versa, $\mathcal{K}$ vanishes as $t \to +\infty$ and diverges as $t \to 0^+$. For this solution τ_0 never vanishes at late times, whereas it does in the strong gravity regime as $t \to 0^+$ when Eckart theory well approximates Israel-Stewart. Finally, the time component of q^a,

$$q^{(\parallel)} = \frac{|s(s-1)|}{16\pi t^2}, \tag{6.21}$$

shares the same analysis with ρ_{v} and P_{v}.

VII. CONCLUSIONS

We have extended the thermal analogy for the effective ϕ-fluid from Eckart's noncausal formulation to the Israel-Stewart causal framework, adopting the minimal ansatz of promoting the heat flux density from a purely spatial vector to a timelike one, $q_a = (q^{(\parallel)}, q_a^{(\perp)})$. By restricting our analysis to a subclass of theories where the dissipative components still obey Eckart's constitutive relations (2.14)–(2.16), the spatial projection $q_a^{(\perp)}$ preserves the definition of the effective temperature (2.18). Thus, the product $\mathcal{KT}$ maintains its crucial role as the order parameter governing the relaxation toward general relativity. Meanwhile, the time component $q^{(\parallel)}$ enters the Israel-Stewart equations and provides a clear physical interpretation of the "viscous" effective density, $\rho_v = 2q^{(\|)}$. Specializing this setup to Bianchi I and FLRW cosmologies allowed us to uniquely decouple $\mathcal{K}$ and $\mathcal{T}$ for the first time. This approach also naturally yields an expression for the relaxation time τ_0, whose evolution defines the regimes where Eckart's theory remains a valid approximation of the Israel-Stewart model.

A precise mapping of scalar-tensor gravity onto the Israel-Stewart constitutive equations would require higher-order derivatives of ϕ. While these could be introduced via higher-order terms in the Lagrangian without suffering from the Ostrogradsky instability, doing so would fundamentally disrupt the standard effective dissipative fluid structure. Therefore, our ansatz constitutes the minimal viable step toward a consistent causal formalism. Due to the purely formal nature of this analogy, we do not attempt to draw connections to a physical entropy, as such a quantity lacks a rigorous thermodynamic meaning in this effective fluid context. Future works will apply the new causal thermal framework to $f(R)$ gravity and will further investigate its implications for cosmological singularities.

A relevant question is the extent to which the ansatz (3.4)–(3.6) applied to the effective fluid of scalar-tensor gravity, which we ended up working with, is applicable to real fluids. In the end, mathematical statements about the Israel-Stewart (or any other causal) formalism should be corroborated by rigorous mathematical theorems and by kinetic theory. The latter is not available for the effective ϕ-fluid of scalar-tensor gravity because it is not made of particles like a real fluid. The ansatz (3.4)–(3.6) works for the effective fluid, leading to exact solutions, but at this stage one cannot be sure about real fluids. This question will be pursued in the future.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: PROOF OF EQ. (4.24)

First, recall that we are working in Bianchi I and that $u^a = \delta^a{}_0$. Thus we have that

$$\begin{aligned}\dot{\pi}_{ab} &\equiv u^c \nabla_c \pi_{ab} = u^c(\partial_c \pi_{ab} - \Gamma^d{}_{ca}\pi_{db} - \Gamma^d_{cb}\pi_{ad}) \\ &= \partial_0 \pi_{ab} - \Gamma^d{}_{0a}\pi_{db} - \Gamma^d{}_{0b}\pi_{ad},\end{aligned} \tag{A1}$$

The only nonvanishing Christoffel symbols of the Bianchi I metric (4.1) read

$$\Gamma^i{}_{0j} = H_{(i)}\delta^i{}_j,$$

for $i,\ j = 1,\ 2,\ 3$. This implies that

$$\dot{\pi}_{00} = 0 \quad \text{and} \quad \dot{\pi}_{0j} = 0,$$

again with $j = 1,\ 2,\ 3$. Whereas, it is easy to see that

$$\dot{\pi}_{ij} = \partial_0 \pi_{ij} - [H_{(i)} + H_{(j)}]\pi_{ij},$$

with $i,\ j = 1,\ 2,\ 3$. Therefore, $\dot{\pi}_{ab}$ has no nonvanishing time-time or time-space components and $h_a{}^c h_b{}^d \dot{\pi}_{cd} = \dot{\pi}_{ab}$.

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