

Gauge invariance and generalized η regularization

Antonio Padilla^{*} and Robert G. C. Smith[†]

*School of Physics and Astronomy, University of Nottingham,
University Park, Nottingham NG7 2RD, United Kingdom
and The Nottingham Centre of Gravity, University of Nottingham,
Nottingham NG7 2RD, United Kingdom*



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We generalize the η regularization scheme in order to develop a framework for systematically studying regularization of loops in quantum field theory. This allows us to “solve” a set of gauge consistency conditions for families of gauge invariant regularization schemes. We recover several known examples such as dimensional and denominator regularizations, as well as some more general solutions. We also study anomalies in chiral theories in order to carefully describe how our formalism should be properly implemented.

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I. INTRODUCTION

In perturbative quantum field theory (QFT), renormalization is fundamental to our ability to compute measurable quantities. The essence of renormalization is to utilize a regularization prescription that can consistently isolate and remove all infinities. Without a consistent and robust regularization method, these infinities would otherwise prohibit any systematic predictions from loop amplitudes.

There are a number of common regularization schemes, all of which have their advantages and disadvantages [1,2]. When evaluating the strengths and weaknesses of a particular regularization method, an important criterion is the preservation of gauge symmetry. All of the fundamental forces of nature—i.e., gravitational, electromagnetic, weak and strong forces—are governed by basic symmetry principles. For example, the electromagnetic, weak, and strong forces are characterized by the gauge symmetry $U(1) \times SU(2) \times SU(3)$, while the gravitational force is governed by diffeomorphism invariance. In fact, any Lorentz invariant theory of massless fields of spin one or spin two must possess a gauge symmetry [3].

The calculation of perturbative quantum corrections can also give rise to quantum anomalies [1,4,5]. A quantum anomaly is the failure to preserve a classical symmetry in the process of quantization and regularization. This is not necessarily a problem if the classical symmetry is a global

one. On the other hand, if the classical symmetry is local, a gauge anomaly generates a mismatch between the numbers of the degrees of freedom in the classical and quantum corrected theories.

Chiral anomalies are the prototypical example often used to probe the effects of regularization, first discovered in the context of spinor electrodynamics. Classically, there are two conserved currents: an axial current associated with the chiral symmetry and a vector current associated with the gauge symmetry. In 1969 Adler [6] and Bell and Jackiw [7], showed in the computation of decay processes of a neutral pion to two photons that only the vector current is conserved after quantum corrections are taken into account. As a result, the conservation of the axial current is found to be anomalously broken.

Of course, not every regularization scheme is well suited to studying a particular process for a particular QFT. For example, Pauli-Villars regularization runs into issues with broken gauge invariance, in contrast to dimensional regularization [8]. For its part, dimensional regularization is hard to understand intuitively and is tricky to implement in chiral theories due the presence of the dimensionally dependent definition of the γ_5 matrix.

In this paper, we study the problem of gauge invariance in a scheme independent way and in a fixed number of dimensions. The tools for doing this are inspired by the η regularization scheme developed in [9]. η regularization is a generalization to QFT of the method of smoothed asymptotics studied in number theory [10]. Just as smoothed asymptotics offers an intuitive and technically simple approach to the regularization of divergent series, so η regularization does the same for divergent loop integrals. This technical simplicity allowed us to reveal an unexpected relationship between the regularization of divergent series in number theory and gauge invariance in QFT.

^{*}Contact author: antonio.padilla@nottingham.ac.uk

[†]Contact author: robert.smith@nottingham.ac.uk

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We shall generalize η regularization just enough to capture most known regularization schemes, whilst retaining enough simplicity to “solve” a set of gauge consistency conditions. These “solutions” correspond to regularization schemes that preserve gauge invariance, like dimensional regularization. We also study anomalies in chiral theories within this generalized framework. This allows us to demonstrate how η regularization ought to be properly implemented in order to recover the correct results.

This is a first step toward a generalized theory of consistent regulators which may offer a tantalizing glimpse of something more fundamental. Indeed, if string theory or some other fundamental theory is UV finite, it could point toward a preferred regularization scheme in the field theory limit. It is this interplay among regularization, consistency, and UV finiteness that we wish to explore in a series of papers.

The rest of this paper is organized as follows: In the next section we give an introduction to η regularization, generalizing the underlying ideas presented in [9]. We emphasize the role of momentum routing and how this connects to gauge invariance. In Sec. III, we explain how one can go about solving gauge consistency conditions for the regulator. We are able to recover a host of gauge invariant regularization schemes, some of which were already known, some of which are new. In Sec. IV we show how to perform renormalization with η regularization by explicitly computing the β function in QED. In Sec. V we study anomalies in chiral theories before ending with a discussion in Sec. VI.

II. GENERALIZED η REGULARIZATION

A systematic study of regularization in QFT benefits from a canonical form for the diverging integrals that appear in a generic theory. At one loop and for QFTs defined on Minkowski space, this canonical form is provided by the one-fold irreducible loop integrals (ILIs), as advocated by Wu [11–13] but also discussed in earlier works [14]. In four spacetime dimensions, the ILIs of integer spin s are given by

$$I_{-2\alpha}^{\mu_1 \dots \mu_s}(\mathcal{M}^2) = \int \frac{d^4 k}{(2\pi)^4} \frac{k^{\mu_1} \dots k^{\mu_s}}{(k^2 + \mathcal{M}^2 - i\epsilon)^{2+\frac{s}{2}+\alpha}}. \quad (2.1)$$

The subscript (-2α) labels the power counting dimension of energy-momentum with $\alpha = -1$ and $\alpha = 0$ corresponding to quadratic and logarithmically divergent integrals. The mass term $\mathcal{M}^2 = \mathcal{M}^2(m_i^2, p_i^2, \dots)$ is a function of Feynman parameters, external momenta p_i , and corresponding mass scales m_i . We define $k^2 = g_{\mu\nu} k^\mu k^\nu$ where the metric $g_{\mu\nu}$ is written with mostly positive signature. To generalize this canonical form to n loops, one can introduce the n -fold ILIs for which there are no longer the overlapping factors $(k_i - k_j + p)$ in the

denominator of the integrand and no factors of the scalar momentum k^2 in the numerator [13].

Following a similar strategy to [9], we implement a generalized form of η regularization by Wick rotating to Euclidean signature $k_0 \rightarrow ik_4$ and inserting a regularization factor of $\eta(\frac{k}{\mu}, \epsilon)$ into the integrand of the ILI, where k is now the norm of the Euclidean four-momentum, μ is a fixed mass scale, and ϵ is a dimensionless parameter that controls the rate at which the regulator damps the corresponding integrals. This yields $I_{-2\alpha}^{\mu_1 \dots \mu_s} \rightarrow iJ_{-2\alpha}^{\mu_1 \dots \mu_s}[\eta]$, where the η factor need not coincide for different ILIs of different spin. We will see why this must be the case in a moment. In any event, the regularized ILIs are given by

$$J_{-2\alpha}^{\mu_1 \dots \mu_s}[\eta](\mathcal{M}^2) = \int \frac{d^4 k}{(2\pi)^4} \frac{k^{\mu_1} \dots k^{\mu_s}}{(k^2 + \mathcal{M}^2)^{2+\frac{s}{2}+\alpha}} \eta\left(\frac{k}{\mu}, \epsilon\right) \quad (2.2)$$

where the integration is over four-dimensional Euclidean space. Note that the rotational symmetry ensures that the ILIs of odd spin vanish identically.

As in [9], the role of the regulator function η is to suppress the ultraviolet divergences in the integrals. In [9] we took $\eta = \eta(k/\Lambda)$ where Λ is some regularization scale and $\eta(x)$ was assumed to be a smooth function defined on the positive real line. For this to correspond to a good regulator, we further assumed that $\eta(x) \rightarrow 1$ as $x \rightarrow 0^+$ and that $\eta(x)$ was a Schwartz function, vanishing faster than any negative power of x as $x \rightarrow \infty$. In this paper we explore a more general form for our regulator functions, namely $\eta(\frac{k}{\mu}, \epsilon)$ as described above. For $\eta(y, \epsilon)$ to be a good regulator function, we now require it to be smooth in y and to satisfy the following:

- (i) in a neighbourhood of $\epsilon = 0$, we have $\eta(y, \epsilon) \approx 1 + \epsilon^\delta f(y)$ for some $\delta > 0$ and some smooth function $f(y)$; and
- (ii) there exists $a, b, c > 0$, such that for $\epsilon \geq a$, we have $|\eta(y, \epsilon)| < cy^{-2b}$ as $y \rightarrow \infty$.

The first condition states that the regulator function tends smoothly to one as we remove the control parameter, ϵ . The second condition states that for ϵ greater than some critical value $a > 0$, the regulator dies off at infinity faster than some negative power. Clearly this is a much weaker set of conditions than those imposed in [9]. For example, we can now capture the form of the regulator expected from dimensional regularization, for which $\eta(y, \epsilon) = y^{-2\epsilon}$. If we were to identify a regularization scale, $\Lambda = \mu/\epsilon$, the class of regulator functions studied in [9] correspond to the case where $\eta(y, \epsilon) = \eta(y\epsilon)$.

Whenever we compute a correlation function up to one loop in a given QFT, we have to do some work to write it in terms of ILIs. This process is discussed in detail in Appendix A of [11] for generic vacuum polarization functions for gauge fields. We also provide an explicit demonstration for the vacuum polarization function in QED

in Sec. IV. To express a generic amplitude in terms of ILIs, one must perform a series of standard manipulations, including Feynman parametrization [3], which combines denominators into a single power, and momentum routing, which shifts the integration variable in loop momentum space. A subtle but crucial point arises when dealing with traces over gamma matrices or contractions involving the metric tensor and loop momenta. The decomposition into ILIs depends on whether such contractions are performed inside or outside the integral. It is essential that these contractions be carried out within the integrals. This is because metric tensors or traces of gamma matrices cannot, in general, be unambiguously pulled through divergent integrals. This was explicitly demonstrated in [15] for several regularization schemes. We will see how it is also the case for gauge invariant η regularization schemes in Sec. V.

As many of our manipulations are only valid if the integrals are properly regulated, these regularizations should be implicitly understood when arriving at the desired form of regularized ILIs. In this sense, our approach is similar in spirit to the implicit regularization methods introduced in [14]. As stated above, some manipulations involve the use of Feynman parametrization. In the Appendix, we explicitly show how the regulator function changes form as we undo a Feynman parametrization.

The last step of Feynman parametrization is always to shift the momentum integration. This might be a worry, since it is not at all obvious that momentum routing is allowed under η regularization. In [9], we allowed the regulators to differ for integrals of different spin. This was argued for on the grounds of generality, although we will now show that it was crucial to allow for momentum routing.

Consider the following regularized ILIs and impose momentum routing, with $\rho(\frac{k}{\mu}, \epsilon)$ a general smooth regulator,

$$\begin{aligned} J_{-2\alpha}^{\mu_1 \dots \mu_s}[\rho_{-2\alpha}](\mathcal{M}^2) &= \int \frac{d^4 k}{(2\pi)^4} \frac{k^{\mu_1} \dots k^{\mu_s}}{(k^2 + \mathcal{M}^2)^{2+\frac{s}{2}+\alpha}} \rho_{-2\alpha}\left(\frac{k}{\mu}, \epsilon\right) \\ &= \int \frac{d^4 k}{(2\pi)^4} \frac{(k+q)^{\mu_1} \dots (k+q)^{\mu_s}}{((k+q)^2 + \mathcal{M}^2)^{2+\frac{s}{2}+\alpha}} \rho_{-2\alpha}\left(\frac{|k+q|}{\mu}, \epsilon\right) \\ &= \int \frac{d^4 k}{(2\pi)^4} e^{q^\nu \frac{\partial}{\partial k^\nu}} \left[\frac{k^{\mu_1} \dots k^{\mu_s}}{(k^2 + \mathcal{M}^2)^{2+\frac{s}{2}+\alpha}} \rho_{-2\alpha}\left(\frac{k}{\mu}, \epsilon\right) \right]. \end{aligned} \quad (2.3)$$

For this to be true for any value of q , we must have the following constraints for any integer value of $r, s \geq 0$:

$$\int \frac{d^4 k}{(2\pi)^4} \frac{\partial^s}{\partial k^{\nu_1} \dots \partial k^{\nu_r}} \left[\frac{k^{\mu_1} \dots k^{\mu_s}}{(k^2 + \mathcal{M}^2)^{2+\frac{s}{2}+\alpha}} \rho_{-2\alpha}\left(\frac{k}{\mu}, \epsilon\right) \right] = 0. \quad (2.4)$$

Thanks to the rotational symmetry of the scalar integrand, this is trivially true for odd values of $r+s$. However, for even values, we obtain nontrivial relations between ILIs of different spin. For example, for $r=s=1$ and taking $\rho = \eta_{-2\alpha}$, we see that

$$\int \frac{d^4 k}{(2\pi)^4} \frac{\partial}{\partial k^\mu} \left[\frac{k_\nu}{(k^2 + \mathcal{M}^2)^{2+\alpha}} \eta_{-2\alpha}\left(\frac{k}{\mu}, \epsilon\right) \right] = 0. \quad (2.5)$$

For $\alpha \neq -2$, this yields the following relationship between the ILIs of spin two and spin zero,

$$J_{-2\alpha}^{\mu\nu}[\theta_{-2\alpha}](\mathcal{M}^2) = \frac{1}{2(2+\alpha)} g^{\mu\nu} J_{-2\alpha}[\eta_{-2\alpha}](\mathcal{M}^2) \quad (2.6)$$

where

$$\theta_{-2\alpha}(y, \epsilon) = -\frac{(y^2 + y_0^2)^{3+\alpha}}{2(2+\alpha)} \frac{\partial}{y \partial y} \left[\frac{\eta_{-2\alpha}(y, \epsilon)}{(y^2 + y_0^2)^{2+\alpha}} \right]. \quad (2.7)$$

Similarly, for $r=s=2$ and again taking $\rho = \eta_{-2\alpha}$, we find

$$\int \frac{d^4 k}{(2\pi)^4} \frac{\partial^2}{\partial k^{\mu_1} \partial k^{\mu_2}} \left[\frac{k_{\mu_3} k_{\mu_4}}{(k^2 + \mathcal{M}^2)^{2+\alpha}} \eta_{-2\alpha}\left(\frac{k}{\mu}, \epsilon\right) \right] = 0. \quad (2.8)$$

For $\alpha \neq -2, -3$, this gives a complex relationship between the ILIs of spin four, spin two, and spin zero,

$$\begin{aligned} J_{-2\alpha}^{\mu_1 \dots \mu_4}[\kappa_{-2\alpha}] &= \frac{1}{2(3+\alpha)} [J_{-2\alpha}^{\mu_1 \mu_3}[\theta_{-2\alpha}] g^{\mu_2 \mu_4} + J_{-2\alpha}^{\mu_1 \mu_4}[\theta_{-2\alpha}] g^{\mu_2 \mu_3} \\ &\quad + J_{-2\alpha}^{\mu_2 \mu_3}[\theta_{-2\alpha}] g^{\mu_1 \mu_4} + J_{-2\alpha}^{\mu_2 \mu_4}[\theta_{-2\alpha}] g^{\mu_1 \mu_3} + J_{-2\alpha}^{\mu_3 \mu_4}[\eta_{-2\alpha}] g^{\mu_1 \mu_2}] \\ &\quad - \frac{1}{4(2+\alpha)(3+\alpha)} J_{-2\alpha}[\eta_{-2\alpha}] (g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}) \end{aligned} \quad (2.9)$$

where

$$\kappa_{-2\alpha}(y, \epsilon) = \frac{(y^2 + y_0^2)^{4+\alpha}}{4(2+\alpha)(3+\alpha)} \left(\frac{\partial}{y \partial y} \right)^2 \left[\frac{\eta_{-2\alpha}(y, \epsilon)}{(y^2 + y_0^2)^{2+\alpha}} \right]. \quad (2.10)$$

Combining (2.9) with (2.6), we can bypass the ILIs of spin 2, and relate the ILIs of spin four and spin zero directly,

$$\begin{aligned} J_{-2\alpha}^{\mu_1 \dots \mu_4}[\kappa_{-2\alpha}](\mathcal{M}^2) &= \frac{1}{4(2+\alpha)(3+\alpha)} S^{\mu_1 \dots \mu_4} J_{-2\alpha}[\eta_{-2\alpha}](\mathcal{M}^2) \end{aligned} \quad (2.11)$$

where $S^{\mu_1 \dots \mu_4} = g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} + g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}$ is the symmetric product of metrics.

It is not too difficult to convince oneself that if $\eta_{-2\alpha}$ is a good regulator in the way we defined earlier, then the same is true of both $\theta_{-2\alpha}$ and $\kappa_{-2\alpha}$. Furthermore, we note that (2.6) and (2.11) are identical to the gauge consistency conditions derived by Wu [11–13] from generalized Ward identities in a generic gauge theory with a gauge group of arbitrary rank and an arbitrary number of Dirac spinors. This alignment is no coincidence, as discussed in detail in [14]. In [9], we imposed those gauge consistency conditions in η regularization while allowing the regulator to differ for integrals of different spin and different dimension. We now see that this was a necessary ingredient, given the explicit form of $\theta_{-2\alpha}$ and $\kappa_{-2\alpha}$ presented in (2.7) and (2.10) derived using momentum routing.

We end this section with a brief comment on causality. As is well known, to preserve the causal structure, we should implement Feynman's $i\epsilon$ prescription prior to any Wick rotation from Minkowski to Euclidean signature, $k_0 \rightarrow ik_4$. In a generalized η regularization scheme, it should be possible to Wick rotate the integrand as long as the smooth regulator functions do not introduce any extra poles that block the path of the contour rotation.

III. GAUGE INVARIANT η REGULARIZATION

η regularization provides us with a formalism to study regularization schemes in an abstract way. To illustrate this, we consider the question of gauge invariance. It is well known that many popular regularization schemes such as cut-off regularization or Pauli-Villars can run into problems with broken gauge invariance. In this section, we will show how one can use the formalism of η regularization to “solve” for regularization schemes that preserve the underlying gauge symmetries.

Consider a generic gauge theory where the gauge group has dimension d_G and where there are N_f Dirac spinors ψ_n ($n = 1, \dots, N_f$) interacting with the Yang-Mills field A_μ^a ($a = 1, \dots, d_G$). This is described by a Lagrangian,

$$\mathcal{L} = \bar{\psi}_n(i\mathcal{D} - m)\psi_n - \frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a, \quad (3.1)$$

where

$$\begin{aligned} F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf_{abc}A_\mu^b A_\nu^c, \\ D_\mu \psi_n &= (\partial_\mu + igT^a A_\mu^a)\psi_n, \end{aligned} \quad (3.2)$$

and T^a are the generators of the gauge group whose commutator $[T^a, T^b] = if^{abc}T^c$ defines the structure constants f^{abc} . Following [11,12,16] and discussed further in [9], one can compute correlation functions at one loop and impose a set of generalized Ward identities. For example, the vacuum polarization of the gauge field takes the form

$$\Pi_{\mu\nu}^{ab}(p) = \Pi_{\mu\nu}^{(g)ab}(p) + \Pi_{\mu\nu}^{(f)ab}(p), \quad (3.3)$$

where p^μ is the external momentum, $\Pi_{\mu\nu}^{(g)ab}(p)$ are the pure Yang-Mills contributions coming from gauge field loops and ghost loops, and $\Pi_{\mu\nu}^{(f)ab}(p)$ are the contributions from fermion loops, coming from the interaction of the fermions with the gauge field. As is well known, gauge invariance is given in terms of the Ward identities $p^\mu \Pi_{\mu\nu}^{ab} = \Pi_{\mu\nu}^{ab} p^\nu = 0$. However, since our interest lies in the regularization scheme rather than the details of any individual theory, we require these Ward identities to hold for *any* gauge theory and with *any* number of fermions. This suggests that the Ward identities should hold separately for the gauge field and fermionic contributions

$$p^\mu \Pi_{\mu\nu}^{(g)ab} = \Pi_{\mu\nu}^{(g)ab} p^\nu = 0, \quad p^\mu \Pi_{\mu\nu}^{(f)ab} = \Pi_{\mu\nu}^{(f)ab} p^\nu = 0. \quad (3.4)$$

One can apply a similar logic to the Ward identities derived from other correlation functions. In order to satisfy the resulting generalized Ward identities, we arrive at consistency conditions on the ILIs for $\alpha = -1, 0, 1$, which are those derived from momentum routing, in (2.6) and (2.11). Note that for $\alpha > 0$, the integrals are convergent and no regulator is required. In this case, the consistency conditions are satisfied trivially as, of course, they should be. For this, our remaining focus is on the case where $\alpha = 0, -1$, corresponding to logarithmic and quadratically divergent ILIs, respectively.

It is important to note that generic regularization schemes do *not* necessarily satisfy the gauge consistency conditions. For example, consider the following quadratically divergent ILIs using cutoff regularisation

$$\begin{aligned} J_2^{\mu\nu}(\mathcal{M}^2) &= \int^\Lambda \frac{d^4 k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + \mathcal{M}^2)^2} \\ &= \frac{g^{\mu\nu}}{8\pi^2} \left(\frac{\Lambda^2}{8} - \frac{\mathcal{M}^2}{4} \ln\left(\frac{\Lambda^2}{\mathcal{M}^2}\right) \right) + \dots \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} J_2(\mathcal{M}^2) &= \int^\Lambda \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + \mathcal{M}^2)} \\ &= \frac{1}{8\pi^2} \left(\frac{\Lambda^2}{2} - \frac{\mathcal{M}^2}{2} \ln\left(\frac{\Lambda^2}{\mathcal{M}^2}\right) \right) + \dots \end{aligned} \quad (3.6)$$

where the ellipses denote finite terms as $\Lambda \rightarrow \infty$. Since $J_2^{\mu\nu}(\mathcal{M}^2) \neq \frac{1}{2}g^{\mu\nu}J_2(\mathcal{M}^2)$ the gauge consistency conditions are clearly violated. This is well known: cutoff regularization breaks gauge invariance.

However, as is also well known, gauge invariant regulators do exist. In this paper, our goal is to present a systematic way of finding them using η regularization. For a suitably chosen $\eta_{-2\alpha}$, we may be tempted to regard (2.7)

and (2.10) as “solutions” of the gauge consistency conditions. However, these solutions are not particularly useful or illuminating. The reason is that one is always free to shift each of the regulators as follows, without having any impact on the consistency conditions (2.6) and (2.11),

$$\eta_{-2\alpha}(y, \epsilon) \rightarrow \eta_{-2\alpha}(y, \epsilon) + Z_{-2\alpha}^{(0)}(y, \epsilon), \quad (3.7)$$

$$\theta_{-2\alpha}(y, \epsilon) \rightarrow \theta_{-2\alpha}(y, \epsilon) + Z_{-2\alpha}^{(2)}(y, \epsilon), \quad (3.8)$$

$$\kappa_{-2\alpha}(y, \epsilon) \rightarrow \kappa_{-2\alpha}(y, \epsilon) + Z_{-2\alpha}^{(4)}(y, \epsilon), \quad (3.9)$$

where the $Z^{(n)}$ for $n = 0, 2, 4$ are smooth functions defined on $y \geq 0$. These functions should vanish in the asymptotic limit, $Z^{(n)}(y, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. For sufficiently large $\epsilon > 0$, they should decay at large momenta, vanishing faster than some inverse polynomial as $y \rightarrow \infty$. These conditions ensure that the shifts given above preserve the properties of a good regulator defined in the previous section. However, in order to also protect the value of the ILIs, and by association the gauge consistency conditions, we further require

$$J_{-2\alpha}[Z_{-2\alpha}^{(0)}](\mathcal{M}^2) = 0, \quad J_{-2\alpha}^{\mu\nu}[Z_{-2\alpha}^{(2)}](\mathcal{M}^2) = 0, \\ J_{-2\alpha}^{\mu_1 \dots \mu_4}[Z_{-2\alpha}^{(4)}](\mathcal{M}^2) = 0. \quad (3.10)$$

Note that a shift in, say, $\eta_{-2\alpha}$ of the form (3.7) will generate a shift of the form (3.8) and (3.9) in the $\theta_{-2\alpha}$ and $\kappa_{-2\alpha}$ under the definition (2.7) and (2.10).

All of this suggests that the consistency equations (2.6) and (2.11) are equations for the regulators that may be solved directly, allowing for solutions that do not necessarily take the specific form given by (2.7) and (2.10). This is the approach we will take to finding consistent regulators in a familiar form.

To this end, we use the rotational symmetry to write the regulated ILIs for spin two and spin four as scalar integrals,

$$J_{-2\alpha}^{\mu\nu}[\theta_{-2\alpha}](\mathcal{M}^2) \\ = \frac{1}{4} g^{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \frac{k^2}{(k^2 + \mathcal{M}^2)^{3+\alpha}} \theta_{-2\alpha}\left(\frac{k}{\mu}, \epsilon\right), \quad (3.11)$$

$$J_{-2\alpha}^{\mu_1 \dots \mu_4}[\kappa_{-2\alpha}](\mathcal{M}^2) \\ = \frac{1}{4!} S^{\mu_1 \dots \mu_4} \int \frac{d^4 k}{(2\pi)^4} \frac{k^4}{(k^2 + \mathcal{M}^2)^{4+\alpha}} \kappa_{-2\alpha}\left(\frac{k}{\mu}, \epsilon\right). \quad (3.12)$$

Plugging these expressions into the conditions (2.6) and (2.11) and rearranging a little, we obtain the following integral constraints:

$$\int_0^\infty dy \frac{y^3}{(y^2 + y_0^2)^{2+\alpha}} \left[\frac{y^2}{y^2 + y_0^2} \theta_{-2\alpha} - \frac{2}{2+\alpha} \eta_{-2\alpha} \right] = 0, \quad (3.13)$$

$$\int_0^\infty dy \frac{y^5}{(y^2 + y_0^2)^{3+\alpha}} \left[\frac{y^2}{y^2 + y_0^2} \kappa_{-2\alpha} - \frac{3}{3+\alpha} \theta_{-2\alpha} \right] = 0. \quad (3.14)$$

It will be useful to construct solutions to these constraints in the form

$$\frac{y^2}{y^2 + y_0^2} \theta_{-2\alpha} - \frac{2}{2+\alpha} \eta_{-2\alpha} = \frac{(y^2 + y_0^2)^{2+\alpha}}{y^3} \frac{\partial}{\partial y} \psi_{-2\alpha}(y, \epsilon), \quad (3.15)$$

$$\frac{y^2}{(y^2 + y_0^2)} \kappa_{-2\alpha} - \frac{3}{(3+\alpha)} \theta_{-2\alpha} = \frac{(y^2 + y_0^2)^{3+\alpha}}{y^5} \frac{\partial}{\partial y} \phi_{-2\alpha}(y, \epsilon), \quad (3.16)$$

where $\psi_{-2\alpha}(y, \epsilon)$ and $\phi_{-2\alpha}(y, \epsilon)$ are smooth functions defined on $y \geq 0$. For sufficiently large $\epsilon > 0$, these functions should vanish at $y = 0$ and as $y \rightarrow \infty$. Furthermore, in the limit where $\epsilon \rightarrow 0$, recall that all regulators tend to unity, from which we infer that

$$\psi_{-2\alpha}(y, 0) = -\frac{1}{2(2+\alpha)} \frac{y^4}{(y^2 + y_0^2)^{2+\alpha}}, \\ \phi_{-2\alpha}(y, 0) = -\frac{1}{2(3+\alpha)} \frac{y^6}{(y^2 + y_0^2)^{3+\alpha}} \quad (3.17)$$

where we have ignored the possibility of an irrelevant additive constant in each case.

A. Explicit “solutions”

We are now ready to state explicit solutions to the consistency conditions (2.6) and (2.11) using the solution generating method we have just derived.

1. Tao inspired regularization schemes

We begin with the regularization schemes explored in [9], for which the regulators have the form $\eta = \eta(\epsilon y)$. Introducing $x = \epsilon y$, the integral constraints (3.13) and (3.14) now read

$$\epsilon^{2\alpha} \int_0^\infty dx \frac{x^3}{(x^2 + \epsilon^2 y_0^2)^{2+\alpha}} \left[\frac{x^2}{x^2 + \epsilon^2 y_0^2} \theta_{-2\alpha} - \frac{2}{2+\alpha} \eta_{-2\alpha} \right] = 0, \quad (3.18)$$

$$\epsilon^{2\alpha} \int_0^\infty dx \frac{x^5}{(x^2 + \epsilon^2 y_0^2)^{3+\alpha}} \left[\frac{x^2}{x^2 + \epsilon^2 y_0^2} \kappa_{-2\alpha} - \frac{3}{3+\alpha} \theta_{-2\alpha} \right] = 0. \quad (3.19)$$

If we carry out an expansion in $\epsilon \ll 1$, we obtain

$$\begin{aligned}
& \epsilon^{2\alpha} \int_0^\infty dx x^{-1-2\alpha} \left[\theta_{-2\alpha} - \frac{2}{2+\alpha} \eta_{-2\alpha} \right] \\
& + \epsilon^{2(\alpha+1)} y_0^2 \int_0^\infty dx x^{-3-2\alpha} [-(3+\alpha)\theta_{-2\alpha} + 2\eta_{-2\alpha}] \\
& + \mathcal{O}(\epsilon^{2(\alpha+2)}) = 0,
\end{aligned} \quad (3.20)$$

$$\begin{aligned}
& \epsilon^{2\alpha} \int_0^\infty dx x^{-1-2\alpha} \left[\kappa_{-2\alpha} - \frac{3}{3+\alpha} \theta_{-2\alpha} \right] \\
& + \epsilon^{2(\alpha+1)} y_0^2 \int_0^\infty dx x^{-3-2\alpha} [-(4+\alpha)\kappa_{-2\alpha} + 3\theta_{-2\alpha}] \\
& + \mathcal{O}(\epsilon^{2(\alpha+2)}) = 0,
\end{aligned} \quad (3.21)$$

where we have explicitly shown the terms which are nonvanishing as $\epsilon \rightarrow 0$, at least for $\alpha = 0, -1$. For quadratically divergent ILIs with $\alpha = -1$, requiring the divergent parts to vanish order by order in ϵ gives the following constraints:

$$\int_0^\infty dx x [\theta_2 - 2\eta_2] = 0, \quad \int_0^\infty dx x [2\kappa_2 - 3\theta_2] = 0, \quad (3.22)$$

$$\int_0^\infty dx \frac{\theta_2 - \eta_2}{x} = 0, \quad \int_0^\infty dx \frac{\kappa_2 - \theta_2}{x} = 0. \quad (3.23)$$

Similarly, for logarithmically divergent ILIs with $\alpha = 0$, we obtain

$$\int_0^\infty dx \frac{\theta_0 - \eta_0}{x} = 0, \quad \int_0^\infty dx \frac{\kappa_0 - \theta_0}{x} = 0. \quad (3.24)$$

The expressions (3.22) to (3.24) are exactly equivalent to the “enhanced regulator” conditions derived in [9]. Recall that the vanishing Mellin transform given by (3.22) connects gauge invariance in this scheme to the suppression of divergences in number theory.

2. Dimensional regularization

We now turn our attention to a more familiar regularization scheme and demonstrate how it can emerge in our formalism. In dimensional regularization, the ILIs take the form

$$J_{-2\alpha}|_{\dim \text{ reg}}[\eta_{-2\alpha}] = \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \frac{1}{(k^2 + \mathcal{M}^2)^{2+\alpha}} = \int \frac{d^4k}{(2\pi)^4} f(\epsilon) \left(\frac{k}{\mu}\right)^{-2\epsilon} \frac{1}{(k^2 + \mathcal{M}^2)^{2+\alpha}}, \quad (3.25)$$

$$\begin{aligned}
J_{-2\alpha}^{\mu\nu}|_{\dim \text{ reg}}[\theta_{-2\alpha}] &= \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + \mathcal{M}^2)^{3+\alpha}} = \frac{\mu^{2\epsilon}}{4-2\epsilon} g_{\mu\nu} \int \frac{d^{4-2\epsilon}k}{(2\pi)^4} \frac{k^2}{(k^2 + \mathcal{M}^2)^{3+\alpha}} \\
&= \frac{1}{4} g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{f(\epsilon)}{(1-\frac{\epsilon}{2})} \left(\frac{k}{\mu}\right)^{-2\epsilon} \frac{k^2}{(k^2 + \mathcal{M}^2)^{3+\alpha}},
\end{aligned} \quad (3.26)$$

$$\begin{aligned}
J_{-2\alpha}^{\mu_1 \dots \mu_4}|_{\dim \text{ reg}}[\kappa_{-2\alpha}](\mathcal{M}^2) &= \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \frac{k^{\mu_1} \dots k^{\mu_4}}{(k^2 + \mathcal{M}^2)^{4+\alpha}} \\
&= \frac{1}{(4-2\epsilon)(6-2\epsilon)} S^{\mu_1 \dots \mu_4} \int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \frac{k^4}{(k^2 + \mathcal{M}^2)^{4+\alpha}} \\
&= \frac{1}{4!} S^{\mu_1 \dots \mu_4} \int \frac{d^4k}{(2\pi)^4} \frac{f(\epsilon)}{(1-\frac{\epsilon}{2})(1-\frac{\epsilon}{3})} \left(\frac{k}{\mu}\right)^{-2\epsilon} \frac{k^4}{(k^2 + \mathcal{M}^2)^{4+\alpha}},
\end{aligned} \quad (3.27)$$

where

$$f(\epsilon) = \frac{(2\pi)^\epsilon \Omega_{3-2\epsilon}}{\Omega_3} \quad (3.28)$$

and $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the volume of a unit n -sphere. In other words,

$$\eta_{-2\alpha} = f(\epsilon) y^{-2\epsilon}, \quad \theta_{-2\alpha} = \frac{f(\epsilon) y^{-2\epsilon}}{1-\frac{\epsilon}{2}}, \quad \kappa_{-2\alpha} = \frac{f(\epsilon) y^{-2\epsilon}}{(1-\frac{\epsilon}{2})(1-\frac{\epsilon}{3})}. \quad (3.29)$$

To see how this solution might have been obtained from the generating technique described above, we consider the smooth seed functions introduced in (3.15) and (3.16), and make the following simple choice:

$$\begin{aligned}
\psi_{-2\alpha}(y, \epsilon) &= -\frac{A(\epsilon)}{2(2+\alpha)} \frac{y^{4-2\epsilon}}{(y^2 + y_0^2)^{2+\alpha}}, \\
\phi_{-2\alpha}(y, \epsilon) &= -\frac{B(\epsilon)}{2(3+\alpha)} \frac{y^{6-2\epsilon}}{(y^2 + y_0^2)^{3+\alpha}},
\end{aligned} \quad (3.30)$$

where the coefficients A, B depend on the regularization parameter ϵ and are assumed to behave smoothly at the

origin such that $A(0) = B(0) = 1$. It is easy to see that this choice satisfies all the conditions we impose on the seed functions above. It might also have been easily guessed as a deformation of the known solution at $\epsilon = 0$. Plugging (3.30) into (3.15) and (3.16), we find that

$$\begin{aligned} & \frac{y^2}{y^2 + y_0^2} \theta_{-2\alpha} - \frac{2}{2 + \alpha} \eta_{-2\alpha} \\ &= \frac{y^2}{y^2 + y_0^2} A(\epsilon) y^{-2\epsilon} - \frac{2}{2 + \alpha} A(\epsilon) \left(1 - \frac{\epsilon}{2}\right) y^{-2\epsilon}, \quad (3.31) \end{aligned}$$

$$\begin{aligned} & \frac{y^2}{(y^2 + y_0^2)} \kappa_{-2\alpha} - \frac{3}{(3 + \alpha)} \theta_{-2\alpha} \\ &= \frac{y^2}{(y^2 + y_0^2)} B(\epsilon) y^{-2\epsilon} - \frac{3}{(3 + \alpha)} B(\epsilon) \left(1 - \frac{\epsilon}{3}\right) y^{-2\epsilon}. \quad (3.32) \end{aligned}$$

It is easy to read off the solution (3.29) provided we identify $A(\epsilon) = f(\epsilon)/(1 - \frac{\epsilon}{2})$ and $B(\epsilon) = f(\epsilon)/(1 - \frac{\epsilon}{2})(1 - \frac{\epsilon}{3})$. Notice that the solution generating technique correctly identifies the form of the regulators in (3.29) and their ratios.

3. Denominator regularization

Another gauge invariant regularization scheme is denominator regularization [17–20]. In this case the ILIs take the form

$$\begin{aligned} & J_{-2\alpha}|_{\text{den reg}}[\eta_{-2\alpha}] \\ &= \mu^{2\epsilon} f_{(2+\alpha, -2\alpha)}(\epsilon) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + \mathcal{M}^2)^{2+\alpha+\epsilon}} \\ &= \int \frac{d^4 k}{(2\pi)^4} f_{(2+\alpha, -2\alpha)}(\epsilon) \left(\frac{k^2}{\mu^2} + \frac{\mathcal{M}^2}{\mu^2}\right)^{-\epsilon} \frac{1}{(k^2 + \mathcal{M}^2)^{2+\alpha}}, \quad (3.33) \end{aligned}$$

$$\begin{aligned} & J_{-2\alpha}^{\mu\nu}|_{\text{den reg}}[\theta_{-2\alpha}] \\ &= \mu^{2\epsilon} f_{(3+\alpha, -2\alpha)}(\epsilon) \int \frac{d^4 k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + \mathcal{M}^2)^{3+\alpha+\epsilon}} \\ &= \frac{1}{4} g^{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} f_{(3+\alpha, -2\alpha)}(\epsilon) \left(\frac{k^2}{\mu^2} + \frac{\mathcal{M}^2}{\mu^2}\right)^{-\epsilon} \\ &\quad \times \frac{k^2}{(k^2 + \mathcal{M}^2)^{3+\alpha}}, \quad (3.34) \end{aligned}$$

$$\begin{aligned} & J_{-2\alpha}^{\mu_1 \dots \mu_4}|_{\text{den reg}}[\kappa_{-2\alpha}](\mathcal{M}^2) \\ &= \mu^{2\epsilon} f_{(4+\alpha, -2\alpha)}(\epsilon) \int \frac{d^4 k}{(2\pi)^4} \frac{k^{\mu_1} \dots k^{\mu_4}}{(k^2 + \mathcal{M}^2)^{4+\alpha}} \\ &= \frac{1}{4!} S^{\mu_1 \dots \mu_4} \int \frac{d^4 k}{(2\pi)^4} f_{(4+\alpha, -2\alpha)}(\epsilon) \left(\frac{k^2}{\mu^2} + \frac{\mathcal{M}^2}{\mu^2}\right)^{-\epsilon} \\ &\quad \times \frac{k^4}{(k^2 + \mathcal{M}^2)^{4+\alpha}}. \quad (3.35) \end{aligned}$$

In each case, the function $f_{(n,p)}(\epsilon)$ is introduced for integrals with whose denominator integrand has power n , after Feynman parametrization, and whose integral has degree of divergence p . These functions should tend smoothly to unity as $\epsilon \rightarrow 0$ and are fixed by demanding that the Laurent expansion of the corresponding amplitude in ϵ converges for all finite $\epsilon > 0$. In other words, the amplitude should only have a simple pole at $\epsilon = 0$; $f_{(n,p)}(\epsilon)$ should cancel any UV poles that go as $1/(\epsilon - p/2)$ and any IR poles that emerge for $\epsilon \in \mathbb{N}^+$ even in the massless limit [18,20]. The mass here refers to the physical masses in the system, as opposed to the mass $\mathcal{M}$ appearing in the ILIs, which also depends on external momenta and Feynman parameters. In practice, the procedure for computing the $f_{(n,p)}(\epsilon)$ requires us to also compute the integration over the Feynman parameters, as well as the regulated ILIs. In any event, we can read off the form of our η regulators as follows:

$$\begin{aligned} \eta_{-2\alpha} &= \frac{f_{(2+\alpha, -2\alpha)}(\epsilon)}{(y^2 + y_0^2)^\epsilon}, & \theta_{-2\alpha} &= \frac{f_{(3+\alpha, -2\alpha)}(\epsilon)}{(y^2 + y_0^2)^\epsilon}, \\ \kappa_{-2\alpha} &= \frac{f_{(4+\alpha, -2\alpha)}(\epsilon)}{(y^2 + y_0^2)^\epsilon} \end{aligned} \quad (3.36)$$

Similar structures can also be derived from our formalism. To do so, we make the following choice for our seed functions:

$$\begin{aligned} \psi_{-2\alpha}(y, \epsilon) &= -\frac{A(\epsilon)}{2(2 + \alpha)} \frac{y^4}{(y^2 + y_0^2)^{2+\alpha+\epsilon}}, \\ \phi_{-2\alpha}(y, \epsilon) &= -\frac{B(\epsilon)}{2(3 + \alpha)} \frac{y^6}{(y^2 + y_0^2)^{3+\alpha+\epsilon}} \end{aligned} \quad (3.37)$$

where, as before, we assume that the coefficients A, B depend on the regularization parameter ϵ and are assumed to behave smoothly at the origin such that $A(0) = B(0) = 1$. Plugging this into (3.15) and (3.16), we find that

$$\begin{aligned} & \frac{y^2}{y^2 + y_0^2} \theta_{-2\alpha} - \frac{2}{2 + \alpha} \eta_{-2\alpha} \\ &= \frac{y^2}{y^2 + y_0^2} \left[\frac{2 + \alpha + \epsilon}{2 + \alpha} \frac{A(\epsilon)}{(y^2 + y_0^2)^\epsilon} \right] - \frac{2}{2 + \alpha} \left[\frac{A(\epsilon)}{(y^2 + y_0^2)^\epsilon} \right], \quad (3.38) \end{aligned}$$

$$\begin{aligned} & \frac{y^2}{(y^2 + y_0^2)} \kappa_{-2\alpha} - \frac{3}{(3 + \alpha)} \theta_{-2\alpha} \\ &= \frac{y^2}{y^2 + y_0^2} \left[\frac{3 + \alpha + \epsilon}{3 + \alpha} \frac{B(\epsilon)}{(y^2 + y_0^2)^\epsilon} \right] - \frac{3}{3 + \alpha} \left[\frac{B(\epsilon)}{(y^2 + y_0^2)^\epsilon} \right]. \quad (3.39) \end{aligned}$$

This yields (3.36) provided we have the following consistency conditions:

$$\begin{aligned} f_{(3+\alpha,-2\alpha)}(\epsilon) &= \frac{2+\alpha+\epsilon}{2+\alpha} f_{(2+\alpha,-2\alpha)}(\epsilon), \\ f_{(4+\alpha,-2\alpha)}(\epsilon) &= \frac{3+\alpha+\epsilon}{3+\alpha} f_{(3+\alpha,-2\alpha)}(\epsilon) \end{aligned} \quad (3.40)$$

and identify $A(\epsilon) = f_{(2+\alpha,-2\alpha)}(\epsilon)$ and $B(\epsilon) = f_{(3+\alpha,-2\alpha)}(\epsilon)$. These conditions seem to be violated by some of the $f_{(n,p)}(\epsilon)$ seen in the literature [18]. This seems to be because Horowitz advocates for a *minimal* choice of the $f_{(n,p)}(\epsilon)$, avoiding any additional prefactors that do not affect the convergence properties of the amplitude. However, it seems that this is too restrictive. It is clear that the amplitude still converges for all finite $\epsilon > 0$ if we multiply $f_{(n,p)}(\epsilon)$ by some linear function of ϵ . Contrary to [19], our analysis suggests these nonminimal factors are necessary if denominator regularization is to preserve gauge invariance.

4. Schwinger proper time

The next gauge invariant regularization method we will consider is Schwinger proper time. In the Schwinger representation of the propagator, the regularized ILIs take the form

$$\begin{aligned} J_{-2\alpha}|_{\text{Schwinger}}[\eta_{-2\alpha}] &= \frac{1}{(1+\alpha)!} \int_0^\infty \frac{d\tau}{\tau} \rho(\mu^2\tau, \epsilon) \tau^{2+\alpha} \\ &\times \int \frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\mathcal{M}^2)}, \end{aligned} \quad (3.41)$$

$$\begin{aligned} J_{-2\alpha}^{\mu\nu}|_{\text{Schwinger}}[\theta_{-2\alpha}] &= \frac{1}{(2+\alpha)!} \int_0^\infty \frac{d\tau}{\tau} \rho(\mu^2\tau, \epsilon) \tau^{3+\alpha} \\ &\times \int \frac{d^4k}{(2\pi)^4} k^\mu k^\nu e^{-\tau(k^2+\mathcal{M}^2)}, \end{aligned} \quad (3.42)$$

$$\begin{aligned} J_{-2\alpha}^{\mu_1 \dots \mu_4}|_{\text{Schwinger}}[\kappa_{-2\alpha}] &= \frac{1}{(3+\alpha)!} \int_0^\infty \frac{d\tau}{\tau} \rho(\mu^2\tau, \epsilon) \tau^{4+\alpha} \\ &\times \int \frac{d^4k}{(2\pi)^4} k^{\mu_1} \dots k^{\mu_4} e^{-\tau(k^2+\mathcal{M}^2)}, \end{aligned} \quad (3.43)$$

where $\rho(u, \epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$. For sufficiently large $\epsilon > 0$, we also require $\rho(u, \epsilon) \rightarrow 0$ as $u \rightarrow 0$, with the rate at which it vanishes being fast enough to render the integrals convergent. This gives the following form for the regulators:

$$\eta_{-2\alpha} = \frac{1}{(1+\alpha)!} \int_0^\infty \frac{du}{u} \rho(u, \epsilon) [u(y^2 + y_0^2)]^{2+\alpha} e^{-u(y^2+y_0^2)}, \quad (3.44)$$

$$\theta_{-2\alpha} = \frac{1}{(2+\alpha)!} \int_0^\infty \frac{du}{u} \rho(u, \epsilon) [u(y^2 + y_0^2)]^{3+\alpha} e^{-u(y^2+y_0^2)}, \quad (3.45)$$

$$\kappa_{-2\alpha} = \frac{1}{(3+\alpha)!} \int_0^\infty \frac{du}{u} \rho(u, \epsilon) [u(y^2 + y_0^2)]^{4+\alpha} e^{-u(y^2+y_0^2)}. \quad (3.46)$$

To see how this can emerge from our formalism, we first note that our boundary conditions (3.17) can be written in the form

$$\psi_{-2\alpha}(y, 0) = -\frac{y^4}{2(2+\alpha)!} \int_0^\infty \frac{du}{u} u^{2+\alpha} e^{-u(y^2+y_0^2)}, \quad (3.47)$$

$$\phi_{-2\alpha}(y, 0) = -\frac{y^6}{2(3+\alpha)!} \int_0^\infty \frac{du}{u} u^{3+\alpha} e^{-u(y^2+y_0^2)}. \quad (3.48)$$

This inspires the choice

$$\psi_{-2\alpha}(y, \epsilon) = -\frac{y^4}{2(2+\alpha)!} \int_0^\infty \frac{du}{u} \rho(u, \epsilon) u^{2+\alpha} e^{-u(y^2+y_0^2)}, \quad (3.49)$$

$$\phi_{-2\alpha}(y, \epsilon) = -\frac{y^6}{2(3+\alpha)!} \int_0^\infty \frac{du}{u} \rho(u, \epsilon) u^{3+\alpha} e^{-u(y^2+y_0^2)}. \quad (3.50)$$

After plugging this into (3.15) and (3.16), we recover the desired regulators (3.44) to (3.46).

5. A general gauge invariant regulator

Finally, we derive a very general gauge invariant regularization scheme that captures all of the above and more by making the following choice for our seed functions:

$$\begin{aligned} \psi_{-2\alpha}(y, \epsilon) &= \psi_{-2\alpha}(y, 0) \mathcal{F}(\vartheta, \varphi, \epsilon), \\ \phi_{-2\alpha}(y, \epsilon) &= \phi_{-2\alpha}(y, 0) \mathcal{G}(\vartheta, \varphi, \epsilon), \end{aligned} \quad (3.51)$$

where $\psi_{-2\alpha}(y, 0)$ and $\phi_{-2\alpha}(y, 0)$ are, of course, given by (3.17) and $\vartheta = \ln y^2$, $\varphi = \ln(y^2 + y_0^2)$. For sufficiently large $\epsilon > 0$, we require that $y^{-2\alpha} \mathcal{F}(\vartheta, \varphi, \epsilon)$ and $y^{-2\alpha} \mathcal{G}(\vartheta, \varphi, \epsilon)$ should vanish as $y \rightarrow \infty$. We also require that $\mathcal{F}(\vartheta, \varphi, 0) = \mathcal{G}(\vartheta, \varphi, 0) = 1$. Plugging this into (3.15) and (3.16), we find that

$$\begin{aligned} \frac{y^2}{y^2 + y_0^2} \theta_{-2\alpha} - \frac{2}{2+\alpha} \eta_{-2\alpha} \\ = \frac{y^2}{y^2 + y_0^2} \left[\mathcal{F} - \frac{\partial_\varphi \mathcal{F}}{2+\alpha} \right] - \frac{2}{2+\alpha} \left[\mathcal{F} + \frac{\partial_\vartheta \mathcal{F}}{2} \right], \end{aligned} \quad (3.52)$$

$$\begin{aligned} & \frac{y^2}{(y^2 + y_0^2)^{\kappa-2\alpha}} - \frac{3}{(3+\alpha)} \theta_{-2\alpha} \\ &= \frac{y^2}{y^2 + y_0^2} \left[\mathcal{G} - \frac{\partial_\phi \mathcal{G}}{3+\alpha} \right] - \frac{3}{3+\alpha} \left[\mathcal{G} + \frac{\partial_\theta \mathcal{G}}{3} \right]. \end{aligned} \quad (3.53)$$

For consistency, $\mathcal{F} = \frac{1}{3} e^{-3\theta+(2+\alpha)\phi} \partial_\theta \mathcal{H}$ and $\mathcal{G} = -\frac{1}{2+\alpha} e^{-3\theta+(2+\alpha)\phi} \partial_\phi \mathcal{H}$, giving the following general form for gauge invariant regulators:

$$\eta_{-2\alpha} = -\frac{1}{6} e^{-3\theta+(2+\alpha)\phi} [\partial_\theta \mathcal{H} - \partial_\theta^2 \mathcal{H}], \quad (3.54)$$

$$\theta_{-2\alpha} = -\frac{1}{3(2+\alpha)} e^{-3\theta+(2+\alpha)\phi} \partial_\theta \partial_\phi \mathcal{H}, \quad (3.55)$$

$$\kappa_{-2\alpha} = -\frac{1}{(2+\alpha)(3+\alpha)} e^{-3\theta+(2+\alpha)\phi} [\partial_\phi \mathcal{H} - \partial_\phi^2 \mathcal{H}]. \quad (3.56)$$

Note that $\mathcal{H}(\vartheta, \varphi, 0) = e^{3\vartheta-(2+\alpha)\varphi}$. Meanwhile, for sufficiently large $\epsilon > 0$, we require $e^{-3\vartheta+2\varphi} \partial_i \mathcal{H}(\vartheta, \varphi, \epsilon) \rightarrow 0$ at large $\vartheta \sim \varphi$.

IV. η REGULARIZATION AND RENORMALIZATION

To illustrate how renormalization is implemented in η regularization, we compute the one-loop vacuum polarization in QED. This not only demonstrates how divergences are tamed by the η prescription but also allows us to explicitly derive the QED β function in an elegant fashion.

We start with the bare Lagrangian for QED,

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (4.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$. The photon self-energy is given by

$$i\Pi^{\mu\nu}(p) = -(-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[\gamma^\mu \frac{i}{k - \not{p} - m + i\epsilon} \gamma^\nu \frac{i}{k - m + i\epsilon} \right]. \quad (4.2)$$

After Wick rotating to Euclidean signature, taking traces and performing Feynman parametrizations, we find that

$$\Pi^{\mu\nu}(p) = -4e^2 \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx \frac{2(k-p)^{(\mu} k^{\nu)} - g^{\mu\nu} [(k-p) \cdot k + m^2]}{[(k-xp)^2 + \mathcal{M}^2]^2}, \quad (4.3)$$

where

$$\mathcal{M}^2 = m^2 + x(1-x)p^2$$

and the integral is taken over four-dimensional Euclidean space. Using the rotational symmetry of the integral, we find that it can be rewritten as

$$\begin{aligned} \Pi^{\mu\nu} = & -4e^2 \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx [2\mathcal{J}_2^{\mu\nu} - g^{\mu\nu} \mathcal{J}_{2\lambda}^\lambda \\ & - [2x(1-x)(p^\mu p^\nu - p^2 g^{\mu\nu}) + \mathcal{M}^2 g^{\mu\nu}] \mathcal{J}_0] \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} \mathcal{J}_{-2\alpha}^{\mu\nu} &= \frac{(k-xp)^\mu (k-xp)^\nu}{[(k-xp)^2 + \mathcal{M}^2]^{3+\alpha}}, \\ \mathcal{J}_{-2\alpha} &= \frac{1}{[(k-xp)^2 + \mathcal{M}^2]^{2+\alpha}}. \end{aligned} \quad (4.5)$$

We now put this in ILI form using the fact that $\mathcal{J}_{2\lambda}^\lambda = \mathcal{J}_2 - \mathcal{M}^2 \mathcal{J}_0$. This gives

$$\begin{aligned} \Pi^{\mu\nu} = & -4e^2 \int_0^1 dx [2J_2^{\mu\nu}[\theta_2](\mathcal{M}^2) - J_2[\eta_2](\mathcal{M}^2) g^{\mu\nu} \\ & - 2x(1-x)(p^\mu p^\nu - p^2 g^{\mu\nu}) J_0[\eta_0](\mathcal{M}^2)]. \end{aligned} \quad (4.6)$$

If we apply the gauge consistency condition (2.6), the quadratically divergent terms cancel, leaving a logarithmic divergent result

$$\Pi^{\mu\nu} = -(p^\mu p^\nu - p^2 g^{\mu\nu}) \Pi(p^2) \quad (4.7)$$

with

$$\Pi(p^2) = -8e^2 \int_0^1 dx x(1-x) J_0[\eta_0](\mathcal{M}^2). \quad (4.8)$$

At one loop, the dressed photon propagator goes as

$$iD^{\mu\nu} = -\frac{i}{p^2} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \frac{1}{1 - \Pi(p^2)} \quad (4.9)$$

and the amplitude for electron scattering as

$$\mathcal{A} = (-ie^2) j_\mu(q_1, q'_1) \left(-\frac{i}{p^2} g^{\mu\nu} \frac{1}{1 - \Pi(p^2)} \right) j_\nu(q_2, q'_2) \quad (4.10)$$

where $j_\mu(q, q') = \bar{u}(q')\gamma_\mu u(q)$ are the currents for the external fields and $p = q_1 - q'_1 = q'_2 - q_2$ the momentum transfer. We are now ready to renormalize. To this end, we introduce a renormalized charge $e_R(\Lambda) = e/\sqrt{1 - \Pi(-\Lambda^2)}$ so that the amplitude scales as $e_R^2(\Lambda)/p^2$ near $p^2 = -\Lambda^2$. More generally, we find that

$$\mathcal{A} \sim \frac{e_R^2(\Lambda)}{p^2} \left(\frac{1 - \Pi(-\Lambda^2)}{1 - \Pi(p^2)} \right) \approx \frac{e_R^2(\Lambda)}{p^2} \frac{1}{1 - \Pi_R(p^2)} \quad (4.11)$$

where, as usual, we work at leading order in e^2 (or equivalently e_R^2), so that

$$\begin{aligned} \Pi_R(p^2) &= \Pi(p^2) - \Pi(-\Lambda^2) \\ &= -8e_R^2 \int_0^1 dx x(1-x) [J_0[\eta_0](\mathcal{M}) - J_0[\eta_0](\mathcal{M}_\Lambda)] \end{aligned} \quad (4.12)$$

and $\mathcal{M}_\Lambda^2 = m^2 - \Lambda^2 x(1-x)$. Note that as we remove the regulator, this integral remains finite

$$\Pi_R(p^2) \rightarrow -\frac{e_R^2}{2\pi^2} \int_0^1 dx x(1-x) \ln(\mathcal{M}_\Lambda^2/\mathcal{M}^2) \quad (4.13)$$

and vanishes for $p^2 = -\Lambda^2$.

In this particular example, we have seen both logarithmically and quadratically divergent ILIs. The latter dropped out thanks to the gauge consistency condition and the corresponding use of enhanced regulators. As for the remaining logarithmic ILIs, our renormalisation procedure amounts to a renormalization of the ILIs

$$J_0[\eta_0](\mathcal{M}) \rightarrow J_0[\eta_0](\mathcal{M}) - J_0[\eta_0](\mathcal{M}_\Lambda) \quad (4.14)$$

giving rise to finite integrals even as we remove the regulator. Note that the “counterterm ILI,” $J_0[\eta_0](\mathcal{M}_\Lambda^2)$, has the same regularization factor as its parent, but a different mass, set by the scale of the subtraction point.

We are now ready to compute the β function for the electron charge, using the fact that

$$e_R^{-2}(\Lambda) = e^{-2} - \chi(-\Lambda^2) \quad (4.15)$$

where we define $\chi(p^2) = \frac{\Pi(p^2)}{e^2} = -8 \int_0^1 dx x(1-x) \times J_0[\eta_0](\mathcal{M}^2)$. Since the bare coupling e is independent of the renormalization scale, we find the following expression for the running of the renormalized coupling:

$$\beta(e_R) \equiv \frac{de_R(\Lambda)}{d \ln \Lambda} = a e_R^3. \quad (4.16)$$

Now the prefactor $a = -\Lambda^2 \chi'(-\Lambda^2)$ is finite, even as we remove the regulators. Indeed, by explicit computation, we find that

$$\begin{aligned} a &\rightarrow -16 \int_0^1 dx x(1-x) J_{-2}[1](\mathcal{M}_\Lambda^2) \Lambda^2 x(1-x) \\ &= \frac{1}{2\pi^2} \int_0^1 dx x(1-x) \left(1 - \frac{m^2}{\mathcal{M}_\Lambda^2} \right) \end{aligned} \quad (4.17)$$

where we have used the fact that $J_{-2}[1](\mathcal{M}_\Lambda^2) = \frac{1}{32\pi^2 \mathcal{M}_\Lambda^2}$. We see how the problem is reduced to one for convergent ILIs. For asymptotically high energies, $\Lambda \gg m$, we recover the well known result $a = \frac{1}{12\pi^2}$.

In general, the η regularization procedure regulates each ILI independently with a damping factor that is smooth and Lorentz invariant. For gauge symmetric regulators, only logarithmic divergences remain. Counterterms are now easily introduced via counterterm ILIs with the same structure, the same regularization factor, but a different mass and opposite overall sign. As usual in renormalization, the choice of mass sets the subtraction point. As the residual divergence was logarithmic, the renormalized ILIs are guaranteed to be finite. Indeed, it is notable that we were able to compute the β function for QED without ever computing a divergent integral.

V. η REGULARIZATION AND ANOMALIES

In quantum field theory, classical symmetries are often broken by quantum effects. In the case of a global symmetry, this is a physical effect and one that is always expected to occur if the theory is coupled to gravity [21]. Gauge anomalies, on the other hand, can be fatal, since they allow new degrees of freedom to start propagating in the effective field theory, often leading to violations of unitarity. The classic playground for anomalies are chiral models in an even number of spacetime dimensions, in which a gauge field is coupled to massless Dirac fermions [1]. Classically, there are two conserved currents: a vector current associated with the gauge symmetry and an axial current associated with the chiral symmetry. In two dimensions, two point functions reveal both symmetries to be anomalous, although somewhat unusually, this does not lead to any inconsistencies [22]. In four dimensions, three point functions reveal a chiral anomaly, while the gauge symmetry remains intact [6,7]. Of course, for non-Abelian theories, the go-to regularization scheme is dimensional regularization [8] which is tricky to implement for chiral models. Indeed, the definition of the chiral model makes use of the γ_5 matrix whose properties are ambiguous when the number of dimensions is analytically continued (see, for example, [23–31]).

η regularization has the advantage that the dimension of spacetime always remains fixed. Nevertheless, we have seen how it is capable of capturing the key ingredients of

dimensional regularization and other regularization schemes that preserve gauge invariance. It is therefore important to better understand how the anomalies of the chiral model should be treated in our panoptic framework. Note that γ_5 ambiguities for regularization schemes in fixed dimension have been considered in [15,32,33].

To simplify our analysis while still capturing the important physics, we consider the chiral Schwinger model. This is given by the Lagrangian (3.1) for a single massless Dirac fermion in two spacetime dimensions. Classically, there are two conserved currents: the vector current, $j_\mu^V = \bar{\psi}\gamma_\mu\psi$ and the axial current, $j_\mu^A = \bar{\psi}\gamma_\mu\gamma_5\psi$ where we define $\gamma_5 = -i\gamma^0\gamma^1$ in two dimensions. To investigate the anomalies, we consider correlation functions of these currents, specifically

$$\begin{aligned}\Pi_{\mu\nu}^{VV}(p) &= \langle j_\mu^V(p) j_\nu^V(-p) \rangle, \\ \Pi_{\mu\nu}^{VA}(p) &= \langle j_\mu^V(p) j_\nu^A(-p) \rangle, \\ \Pi_{\mu\nu}^{AA}(p) &= \langle j_\mu^A(p) j_\nu^A(-p) \rangle.\end{aligned}\quad (5.1)$$

The classical Ward identities give $p^\mu \Pi_{\mu\nu}^{VV} = p^\nu \Pi_{\mu\nu}^{VV} = 0$. These Ward identities are anomalous. It is well known that there is a gauge anomaly coming from the generation of a mass term for the gauge field [22]. While this results in the nonconservation of $\Pi_{\mu\nu}^{VV}$ at one loop, in this particular case it does not lead to a violation of unitarity. Here we will focus on the two point function with mixed vertices, $\Pi_{\mu\nu}^{VA}$ for which we expect only the axial part to be anomalous $p^\mu \Pi_{\mu\nu}^{VA} = 0$, $p^\nu \Pi_{\mu\nu}^{VA} \neq 0$. At one loop, we have that

$$i\Pi_{\mu\nu}^{VA} = - \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left(\gamma_\mu \frac{i}{\not{k} - \not{p} + i\epsilon} \gamma_\nu \gamma_5 \frac{i}{\not{k} + i\epsilon} \right). \quad (5.2)$$

The first step in η regularization is to cast this into an ILI form. To this end we perform a Wick rotation $k_0 \rightarrow ik_2$, $\gamma^0 \rightarrow -i\gamma^2$, and $\gamma_5 \rightarrow \gamma^1\gamma^2$ to Euclidean signature along with a Feynman parametrization, giving

$$\begin{aligned}\Pi_{\mu\nu}^{VA} &= \int \frac{d^2k}{(2\pi)^2} \int_0^1 dx \text{Tr}(\gamma_5 \gamma_\sigma \gamma_\mu \gamma_\rho \gamma_\nu) \\ &\times \frac{(k - xp + xp)^\sigma (k - xp + (x-1)p)^\rho}{[(k - xp)^2 + \mathcal{M}^2]^2}\end{aligned}\quad (5.3)$$

where $\mathcal{M}^2 = x(1-x)p^2$. At this stage, the temptation might be to pull the trace operator $\text{Tr}(\gamma_5 \gamma_\sigma \gamma_\mu \gamma_\rho \gamma_\nu)$ out of the integral and write the correlation function as a product of the trace operator and the corresponding ILIs. However, one cannot pull the trace through the divergent integral without leaving it unchanged [15]. As we will see, the correct thing to do is to leave trace operator *inside* the integral, decompose it in terms of the metric and express the final result in terms of the ILIs. To this end, we note that

$$\begin{aligned}\text{Tr}(\gamma_5 \gamma_\sigma \gamma_\mu \gamma_\rho \gamma_\nu) &= 2(-\epsilon_{\sigma\mu} g_{\rho\nu} + \epsilon_{\sigma\rho} g_{\mu\nu} - \epsilon_{\sigma\nu} g_{\mu\rho} - \epsilon_{\mu\rho} g_{\sigma\nu} \\ &+ \epsilon_{\mu\nu} g_{\sigma\rho} - \epsilon_{\rho\nu} g_{\sigma\mu})\end{aligned}\quad (5.4)$$

where $\epsilon_{\mu\nu}$ is the totally antisymmetric Levi-Civita tensor in two dimensions. Using the rotational symmetry of the integral, the correlation function can be written as

$$\begin{aligned}\Pi_{\mu\nu}^{VA} &= -4 \int \frac{d^2k}{(2\pi)^2} \int_0^1 dx \epsilon_{\sigma\nu} \left[\mathcal{J}_{0\mu}^\sigma - \frac{1}{2} \mathcal{J}_{0\lambda}^\lambda \delta_\mu^\sigma \right. \\ &\left. - x(1-x) \left(p^\sigma p_\mu - \frac{1}{2} p^2 \delta_\mu^\sigma \right) \mathcal{J}_{-2} \right]\end{aligned}\quad (5.5)$$

where, in two dimensions, we have

$$\begin{aligned}\mathcal{J}_{-2\alpha}^{\mu\nu} &= \frac{(k - xp)^\mu (k - xp)^\nu}{[(k - xp)^2 + \mathcal{M}^2]^{2+\alpha}}, \\ \mathcal{J}_{-2\alpha} &= \frac{1}{[(k - xp)^2 + \mathcal{M}^2]^{1+\alpha}}.\end{aligned}\quad (5.6)$$

To put this in ILI form, we once again use the fact that $\mathcal{J}_{0\lambda}^\lambda = \mathcal{J}_0 - x(1-x)p^2 \mathcal{J}_{-2}$ so that we can write

$$\begin{aligned}\Pi_{\mu\nu}^{VA} &= -4 \int_0^1 dx \epsilon_{\sigma\nu} \left[J_{0\mu}^\sigma[\theta_0](\mathcal{M}^2) \right. \\ &\left. - \frac{1}{2} J_0[\eta_0](\mathcal{M}^2) \delta_\mu^\sigma - x(1-x)(p^\sigma p_\mu - p^2 \delta_\mu^\sigma) J_{-2} \right].\end{aligned}\quad (5.7)$$

Here we have introduced the ILIs in two dimensions along with their corresponding regulators

$$J_{-2\alpha}^{\mu_1 \dots \mu_2}[\eta](\mathcal{M}^2) = \int \frac{d^2k}{(2\pi)^2} \frac{k^{\mu_1} \dots k^{\mu_s}}{[k^2 + \mathcal{M}^2]^{1+\frac{s}{2}+\alpha}} \eta\left(\frac{k}{\mu}, \epsilon\right). \quad (5.8)$$

Recall that the final ILI in (5.7) is convergent with $J_{-2}[1] = 1/4\pi\mathcal{M}^2$. We now use momentum routing in two dimensions to derive a consistency condition for the logarithmically divergent ILIs,

$$\begin{aligned}\int \frac{d^2k}{(2\pi)^2} \frac{\partial}{\partial k^\mu} \left[\frac{k_\nu}{k^2 + \mathcal{M}^2} \rho\left(\frac{k}{\mu}, \epsilon\right) \right] &= 0 \\ \Rightarrow J_0^{\mu\nu}[\theta_0](\mathcal{M}^2) &= \frac{1}{2} J_0[\eta_0](\mathcal{M}^2) g^{\mu\nu}.\end{aligned}\quad (5.9)$$

Plugging all of this into (5.7) we arrive at the final expression,

$$\Pi_{\mu\nu}^{VA} = \frac{\epsilon_{\sigma\nu}}{\pi} (\hat{p}^\sigma \hat{p}_\mu - \delta_\mu^\sigma) \quad (5.10)$$

where $\hat{p}^\mu = p^\mu/p$. It is easy to check that this correlation function generates an axial anomaly but not a vector anomaly, consistent with our expectations,

$$p^\mu \Pi_{\mu\nu}^{VA} = 0, \quad p^\nu \Pi_{\mu\nu}^{VA} = -\frac{\epsilon_{\mu\nu} p^\nu}{\pi}. \quad (5.11)$$

It is now clear η regularization avoids some of the confusion and ambiguity associated with analytically continued dimensions in chiral theories. However, it is important that η regularization is properly implemented. To recover the correct result for the anomaly, it was crucial for us to fully decompose the integrand in the correlation function into the integrands of the ILIs, taking all the relevant contractions with the metric *inside* the integral. Had we pulled the trace operator out of the integral before doing these decompositions and contractions, we would have got the wrong result.

To see this explicitly, consider what would happen had we adopted the wrong approach and pulled the trace operator through the integral before carrying out all the relevant contractions. We find that the correlation function would take the form

$$\begin{aligned} \Pi_{\mu\nu}^{VA}|_{\text{wrong}} &\equiv \text{Tr}(\gamma_5 \gamma_\sigma \gamma_\mu \gamma_\rho \gamma_\nu) \int \frac{d^2 k}{(2\pi)^2} \int_0^1 dx \\ &\times \frac{(k - xp + xp)^\sigma (k - xp + (x-1)p)^\rho}{[(k - xp)^2 + \mathcal{M}^2]^2} \end{aligned} \quad (5.12)$$

$$\begin{aligned} &= \text{Tr}(\gamma_5 \gamma_\sigma \gamma_\mu \gamma_\rho \gamma_\nu) \int \frac{d^2 k}{(2\pi)^2} \int_0^1 dx \\ &\times [\mathcal{J}_0^{\sigma\rho} - x(1-x)p^\sigma p^\rho \mathcal{J}_{-2}] \end{aligned} \quad (5.13)$$

$$\begin{aligned} &\rightarrow \text{Tr}(\gamma_5 \gamma_\sigma \gamma_\mu \gamma_\rho \gamma_\nu) \int_0^1 dx [J_0^{\sigma\rho}[\theta_0](\mathcal{M}^2) \\ &- x(1-x)p^\sigma p^\rho J_{-2}(\mathcal{M}^2)] \end{aligned} \quad (5.14)$$

where in the last line we have introduced the ILIs and the corresponding regulators, θ_0 , *before* carrying out the relevant contractions coming from the trace. This will be key in giving us a different answer to (5.7). After rewriting the trace using (5.4) and using the fact that $J_{0\lambda}^\lambda[\theta_0](\mathcal{M}^2) = J_0[\theta_0](\mathcal{M}^2) - x(1-x)p^2 J_{-2}(\mathcal{M}^2)$, we arrive at the following expression:

$$\begin{aligned} \Pi_{\mu\nu}^{VA}|_{\text{wrong}} &= -4 \int_0^1 dx \epsilon_{\sigma\nu} [J_{0\mu}^\sigma[\theta_0](\mathcal{M}^2) - \frac{1}{2} J_0[\theta_0](\mathcal{M}^2) \delta_\mu^\sigma \\ &- x(1-x)(p^\sigma p_\mu - p^2 \delta_\mu^\sigma) J_{-2}]. \end{aligned} \quad (5.15)$$

This looks very similar to (5.7), albeit with one crucial difference: the choice of regulator in the second term corresponding to scalar ILI. The difference is entirely due to the fact that we have written the expression in ILI form before carrying out our contractions. The difference is also important. Recall that the gauge consistency condition is given by (5), and so

$$\Pi_{\mu\nu}^{VA}|_{\text{wrong}} = \Pi_{\mu\nu}^{VA} - 2\epsilon_{\mu\nu} \int_0^1 dx [J_0[\eta_0](\mathcal{M}^2) - J_0[\theta_0](\mathcal{M}^2)]. \quad (5.16)$$

Since $\eta_0 \neq \theta_0$ in general, we see explicitly how the order of operations in implementing η regularization matters. To recover the right result for the anomalies, we should take care to perform all the relevant contractions inside the integral before casting the result into its ILI decomposition. Of course, the underlying reason for this is that one cannot move metric contractions through a divergent integral without generating possible corrections [15].

VI. DISCUSSION

Originally inspired by regularization techniques in number theory, η regularization has provided us with a tool to investigate regularization schemes in quantum field theory in a systematic way, at least at one loop. Here we have shown how it can be used to “solve” gauge consistency conditions, delivering old and new gauge invariant regularization schemes. This includes dimensional regularization, denominator regularization, Schwinger proper time and a generalization that captures all of the above and more. We have also shown how to carry out renormalization in this scheme by explicitly computing the β function for QED. A careful implementation of η regularization also yields the correct results for anomalies in chiral theories.

The extension of η regularization to higher loops is of paramount importance. Given the role of ILIs in developing the formalism, we expect an extension to higher loops to closely follow the analysis of Wu [11–13]. These extensions are nontrivial, not least because of the presence of overlapping divergences that begin at two loops. Extending η regularization to higher loops would also allow us to investigate unitarity, potentially deriving and solving new consistency conditions associated with same. We would also like to use η regularization to “solve” for regularization schemes that preserve supersymmetry.

However, our ultimate goal is to use η regularization to gain a window into how nature protects us from infinity. For example, superstring theory is said to be UV finite [34–37]. When we carefully take the field theory limit, this UV finiteness should manifest itself in some way. This question was explored in [38] where the heavy string oscillator modes provide the appropriate counterterms in the field theory limit. Here we imagine another perspective, whereby the field theory inherits some sort of stringy regulator. What would this regulator look like and can it be generalized without spoiling its most important ingredients? Presumably, if string theory is consistent, a string inspired regulator should satisfy all the necessary consistency conditions anticipated from bottom up investigations like the one presented here. Furthermore, it is well known that worldsheet modular invariance is key to ensuring that string

theory remains UV finite. This suggests that string inspired regulators might be expected to feature this in some way, perhaps through some subtle UV/IR mixing that leads to unexpected cancellations as in [39–41].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: UNDOING THE FEYNMAN PARAMETRIZATION WITH η FACTORS

In the main text, we introduced our η regulators after performing a range of manipulations on the integrals, most notably, Feynman parametrizations. Such manipulations are not valid for divergent integrals so one is implicitly assuming that some form of regularization is always present and that this will result in the proposed η regularization of the ILIs at the end of the calculation. We shall now investigate how the regularization scheme might look prior to Feynman parametrization for certain integrals.

To this end, we consider a contribution to a correlation function coming from a regularized scalar ILI

$$\mathcal{A} = (n-1)! \int_x \int_k \frac{1}{(k^2 + \mathcal{M}^2)^n} \eta\left(\frac{k}{\mu}, \epsilon\right). \quad (\text{A1})$$

Here $\int_x = \int_0^1 dx_1 \cdots \int_0^1 dx_n \delta(\sum_i x_i - 1)$ denotes the integration over n Feynman parameters, x_i , subject to the constraint $\sum_i x_i = 1$. Furthermore, $\mathcal{M}^2 = \sum_i x_i (p_i^2 + m_i^2) - (\sum_i x_i p_i)^2$ is a function of these parameters, the particle masses, m_i and external momenta, p_i . $\int_k$ denotes the integration over the loop momentum, which could be in any number of dimensions.

To undo the Feynman parametrization, it is convenient to rewrite the integral in Schwinger form. To do so, we first use momentum routing to shift the loop momentum

$$\mathcal{A} = (n-1)! \int_x \int_k \frac{1}{((k + \sum_i x_i p_i)^2 + M^2)^n} \times \eta\left(\frac{|k + \sum_i x_i p_i|}{\mu}, \epsilon\right) \quad (\text{A2})$$

$$= (n-1)! \int_x \int_k \frac{1}{(\sum_i x_i \Delta_i)^n} \eta\left(\frac{|k + \sum_i x_i p_i|}{\mu}, \epsilon\right) \quad (\text{A3})$$

where $\Delta_i = (k + p_i)^2 + m_i^2$, before integrating in a Schwinger parameter, s

$$\mathcal{A} = \int_0^\infty ds s^{n-1} \int_x \int_k e^{-s(\sum_i x_i \Delta_i)} \eta\left(\frac{|k + \sum_i x_i p_i|}{\mu}, \epsilon\right). \quad (\text{A4})$$

We now introduce a set of n variables $s_i = sx_i$ allowing us to combine the integration of the Schwinger parameter and the Feynman parameters,

$$\mathcal{A} = \int_0^\infty ds_1 \cdots \int_0^\infty ds_n \int_k e^{-(\sum_i s_i \Delta_i)} \eta\left(\frac{|k + \sum_i s_i p_i|}{\mu}, \epsilon\right). \quad (\text{A5})$$

Finally, we absorb the Δ_i into a redefinition of the $s_i \rightarrow s_i/\Delta_i$ so that we have

$$\mathcal{A} = \int_k \frac{1}{\Delta_1 \cdots \Delta_n} \tilde{\eta} \quad (\text{A6})$$

where

$$\tilde{\eta} = \int_0^\infty ds_1 e^{-s_1} \cdots \int_0^\infty ds_n e^{-s_n} \eta\left(\frac{|k + \sum_i s_i p_i / \Delta_i|}{\mu}, \epsilon\right). \quad (\text{A7})$$

This is the form of regularization factor prior to Feynman parametrization. Note that $\tilde{\eta} \rightarrow 1$ and $\eta \rightarrow 1$, as of course it should.

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