

Asymptotics of superfluid Bjorken flow

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We consider the dynamics of an expanding superfluid modeled by Mueller-Israel-Stewart theory coupled to a complex scalar field with a $U(1)$ symmetry that is spontaneously broken. This is a manageable theoretical setting for explorations of the chiral phase transition of expanding quark-gluon plasma. We study the late proper-time behavior of Bjorken flow in this physical system and find that asymptotic solutions can be expressed as a transseries of a novel form, which contains factors like $\tau^{-\ln \tau}$. This transseries describes how the information encoded in the initial data is diluted in the course of dissipative evolution. These solutions retain memory of the symmetry-breaking transition and describe two qualitatively different late-time behaviors of the dynamical variables, depending on condensate relaxation rate: either a purely damped falloff or damped oscillations. The possibility that such oscillations could be imprinted in the observed outcomes of heavy ion collision experiments is the main physical insight that follows from our analysis.

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I. INTRODUCTION

Fluid dynamics can be viewed as a systematic effective field theory approach using fundamental conservation laws to describe long wavelength excitations of systems out of equilibrium [1,2]. In certain settings, such as near critical loci in the phase diagram, these excitations include Goldstone modes of spontaneously broken symmetries, as occurs in superfluids, including those present in nuclear matter [3–5].

The focus of this paper is on the physics of quark-gluon plasma (QGP) in the context of heavy ion collision experiments, where critical dynamics constitute an important part in modern experimental searches for the QCD critical point [6]. In this context it is useful to consider the idealized case of QCD in the chiral limit, where spontaneous breaking of chiral symmetry leads to the appearance of massless Goldstone bosons which can be identified with the pions. Even though in the real world chiral symmetry is explicitly broken by quark masses, there are many features of QCD that can be understood from the perspective that

views the explicit breaking of chiral symmetry in the light quark sector as a perturbation. This perspective has also been explored in the context of QGP physics, in part to explain the enhanced production of soft pions as a signature of the chiral phase transition [7–12].

In this study, instead of considering the non-Abelian chiral symmetry group of massless QCD, we consider a simple model of a complex scalar field with dynamics invariant under a $U(1)$ symmetry, spontaneously broken by the ground state. The resulting Goldstone boson is coupled to an expanding conformal fluid described using Mueller-Israel-Stewart (MIS) theory [13,14]. Following earlier work [15,16], we consider the combined system undergoing Bjorken flow, where the dynamics is captured by a system of three ordinary differential equations. The nondissipative version of this theory has been shown to lead to causal and stable evolution in flat space, see [17] for details. The causality and stability of the dissipative theory was studied at the linear level in Ref. [15].

The physical picture of the typical evolution of the Bjorken superfluid is as follows. In analogy to the QGP, the system is initialized with a temperature in the unbroken phase, i.e., $T > T_c$. The condensate rapidly rolls down to its minimum value, which in the unbroken phase is $\rho = 0$. The temperature evolution in this regime is thus unaffected by the condensate, evolving in accordance with the MIS attractor [18]. The system remains in this regime for some time, which in Ref. [16], was referred to as the *attractor time*. Once the temperature drops below the critical value

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T_c , the mass term in the potential becomes negative, triggering spontaneous symmetry breaking and the condensate rolls down to the minimum of the potential in the broken phase. Despite the continued expansion, the system then freezes to a constant nonzero value of the condensate, as well as a constant nonzero value of the temperature, in contrast to the typical Bjorken evolution $T \propto \tau^{-1/3}$ at late times, see, e.g., Ref. [19]. A complete description of the attractor in this case requires a formulation in terms of an extended phase space characterized by four-dimensional constant proper time slices, adapting the approach of Ref. [20] to the dynamical system considered here. The attractor is then viewed as a hypersurface in the extended phase space.

While earlier studies relied mostly on numerical work, here we apply asymptotic methods to study the physics at late times. From a geometrical perspective, we determine the form of the parametric representation of the attractor hypersurface [21] in the extended phase space in the late time regime. The power series asymptotic behavior of the superfluid system studied here has novel features, notably the occurrence of logarithms. Their appearance can in fact be traced to a simplified system where the condensate is frozen to its asymptotic value; in that case one can reduce the equations of motion to a single second order ODE for the effective temperature, where logarithms appear in the asymptotics even in the case of an ideal fluid. The asymptotic late time solution can be seen to take the form

$$T(\tau) \sim \sum_{n=0}^{\infty} \sum_{m=0}^n t_{n,m} \tau^{-n} \ln^m(\Lambda\tau), \quad (1)$$

where the $t_{n,m}$ are constant coefficients determined by the evolution equations and Λ is an integration constant. It turns out that the asymptotic solution of the full superfluid system retains the form of Eq. (1). While the form of this series may appear unusual, series of this form have appeared in other contexts—further details are provided in the Appendix B.

It is well known that the late proper time expansion is divergent and captures the physics in a rather subtle way, with a part of the information carried by power law terms, while the rest is exponentially suppressed [18] (see also the reviews [22–24]). These exponentially suppressed terms are in correspondence with the quasinormal modes of the microscopic system. In the case of systems previously studied, these modes were connected with relaxation of the shear stress and were necessary to ensure the causality as well as the stability of the equilibrium state [25,26]. In the superfluid system the nonhydrodynamic quasinormal modes include ones connected to the relaxation of the symmetry breaking condensate, and depend on its relaxation rate, denoted by C_κ below. The mathematical structure of late time asymptotic solutions is that of a transseries, and in this work we establish its form in the case of

superfluid Bjorken flow. This transseries turns out to have novel features, but expresses the same physical idea as in other examples [18,27–29]. A physically interesting point that emerges from the present analysis is that the exponentially damped terms in the superfluid system can include oscillating modes that describe the dynamics of the condensate. The precise form of the transseries reveals that there is a critical value of the condensate relaxation rate below which oscillations are present. Outside this parameter domain the system becomes overdamped and the oscillations are no longer present. This fact can be verified numerically and such analysis confirms the conclusions implied by the transseries solution. If present, such oscillations could leave an imprint in the observables measured in heavy-ion collision experiments. We note however that in the vicinity of a symmetry breaking transition one can expect that fluctuations may become significant; for a discussion in the context of Bjorken flow see, e.g., [30]. Our present analysis does not take this effect into account.

The outline of the paper is as follows: in Sec. II, we outline the superfluid construction in Bjorken flow. We then discuss the frozen condensate limit in Sec. III, where we are firmly in the broken regime, but the condensate dynamics play no role. This serves as a warm-up to the full dynamical case in Sec. IV, where we analyze the asymptotics of the resulting theory. In Sec. V, we present some results concerning the large order behavior of the series appearing in the lowest transseries sectors and show that the pattern of branch point singularities in the Borel plane matches the structure of the transseries.

II. SUPERFLUID BJORKEN FLOW

The chiral symmetry group of QCD with two massless flavors of quarks is $SU(2) \times SU(2) \sim O(4)$. Spontaneous breaking of this symmetry, signaled by a nonzero value of the quark condensate, leads to the appearance of a triplet of Goldstone bosons that are interpreted as the chiral limit analog of the physical pion triplet. In this paper, following [15], we consider a simpler model involving a global $U(1)$ symmetry, where only one Goldstone boson appears. Obviously, this cannot be used directly as a model of pion physics, but it may serve as an illustration for the dynamics of the condensate interacting with the QGP, which is the focus of interest here.

In the ideal fluid limit, the system of interest can be described by the effective action [15,16]

$$S = \int d^4x \sqrt{-g} \left[p(T) - \frac{1}{2} (D_\mu \Sigma)(D^\mu \Sigma)^\dagger - V(\Sigma, T, \mu) \right], \quad (2)$$

where $\Sigma = \rho e^{i\psi}$ is the complex scalar field with ρ denoting the condensate and ψ the $U(1)$ phase. The potential is given by

$$V(\rho, T, \mu) = \frac{m_0(T - T_c)}{2} \rho^2 + \frac{\lambda}{4} \rho^4. \quad (3)$$

Here T is the (dynamical) effective temperature and T_c is the critical temperature. Note that the mass term explicitly changes sign as the system temperature drops below the critical temperature and into the broken phase. In what follows, we will concern ourselves with the dynamics of the condensate, ρ , at zero chemical potential, which due to the Josephson constraint fixes the phase to be a constant (see [15] for the formulation of the present theory with a dynamical phase). Note that the case of explicit symmetry breaking can be modeled by adding to the potential additional terms linear in Σ , which would correspond to a small quark mass that would break chiral symmetry explicitly in QCD, see, e.g., [7,8].

The equations of motion express the conservation of energy-momentum, the dynamics of the order parameter and the relaxation of the shear-stress according to MIS theory. The energy momentum tensor, $T^{\mu\nu} = T_{\text{ideal}}^{\mu\nu} + \Pi^{\mu\nu}$, decomposed into ideal and dissipative contributions, respectively, is conserved

$$\nabla_\mu T^{\mu\nu} = 0, \quad (4)$$

$$T_{\text{ideal}}^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \varepsilon u^\mu u^\nu + p \Delta^{\mu\nu} + \nabla^\mu \rho \nabla^\nu \rho - g^{\mu\nu} \left(\frac{1}{2} \nabla_\mu \rho \nabla^\mu \rho + V \right), \quad (5)$$

where ε is the energy density, p is the pressure, and u^μ is the four velocity (timelike normalized $u^\mu u_\mu = -1$) and $\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$. Note that the dissipative tensor is both orthogonal to the fluid velocity and traceless, i.e., $\Pi^{\mu\nu} u_\mu = 0 = \Pi^{\mu\nu} g_{\mu\nu}$. In what follows, we will consider a conformal fluid with the equation of state, $\varepsilon = 3p \sim T^4$.

The equation of motion for the order parameter is assumed to be

$$-\nabla_\mu \nabla^\mu \rho + \frac{\partial V}{\partial \rho} = -\kappa u^\mu \nabla_\mu \rho. \quad (6)$$

This differs from the Euler-Lagrange equation following from the action Eq. (2) by the addition of the dissipative term on the right-hand side. This term describes the relaxation of the condensate, where κ is a transport coefficient that captures the relaxation rate of the condensate to the minimum of the potential. It will be shown below that this term affects the asymptotic late time behavior by damping oscillations of the condensate.

The final element is the MIS relaxation equation [13,14,31]

$$\tau_\pi \left(u \cdot \nabla + \frac{4}{3} \nabla \cdot u \right) \Pi^{\mu\nu} + \Pi^{\mu\nu} = -2\eta \sigma^{\mu\nu}, \quad (7)$$

where η is the shear viscosity, τ_π is the relaxation timescale,

and

$$\sigma^{\mu\nu} = \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{1}{3} \Delta^{\mu\nu} \nabla_\alpha u^\alpha \quad (8)$$

is the shear tensor. We adopt the temperature dependence of the transport coefficients appropriate for high temperatures

$$\eta = \frac{4}{3} C_\eta T^3, \quad \tau_\pi = C_\tau T^{-1}, \quad \kappa = C_\kappa T. \quad (9)$$

At lower temperatures, one would expect these to be modified by mass corrections, m_0 , see footnote 2 in [15].

In our study we consider the system undergoing Bjorken flow, which captures a number of kinematic as well as dynamical features of the QGP created in heavy ion collisions. The symmetries of Bjorken flow [19] are implemented in the standard way by passing to Milne coordinates (τ, η) with the metric

$$ds^2 = -d\tau^2 + dx_\perp^2 + \tau^2 d\eta^2, \quad (10)$$

and imposing the condition that all fields depend only on τ . We parametrize the dissipation tensor via the dimensionless quantity χ defined by $\Pi_\eta^\eta = -\chi(\varepsilon + p)$. We can relate this to pressure anisotropy of Bjorken flow, defined in terms of eigenvalues of the energy-momentum tensor $T_\nu^\mu \equiv \text{diag}(-\mathcal{E}, \mathcal{P}_T, \mathcal{P}_T, \mathcal{P}_L)$ using

$$\mathcal{A} \equiv \frac{\mathcal{P}_T - \mathcal{P}_L}{\mathcal{P}}, \quad \mathcal{P} = \frac{1}{3}(\mathcal{P}_L + 2\mathcal{P}_T), \quad (11)$$

can be expressed as $\mathcal{A} = \frac{3}{2} \frac{\varepsilon + p}{\mathcal{P}} \chi$.

The dissipative equations of motion (6), (4), and (7) then take the form [15,16]

$$\rho'' + \frac{\rho'}{\tau} + \lambda \rho^3 + m_0(T - T_c)\rho = -C_\kappa T \rho', \quad (12a)$$

$$\frac{\tau T'}{T} = \frac{1}{3}(\chi - 1) + m_0 \frac{\rho^2 + 2\tau \rho \rho'}{8T^3} + \frac{\tau}{4T^3} C_\kappa \rho'^2, \quad (12b)$$

$$\tau \chi' + \frac{4}{3} \left(\chi - \frac{C_\eta}{C_\tau} \right) + 4\chi \frac{\tau T'}{T} + \frac{\tau}{C_\tau} \chi T = 0, \quad (12c)$$

where the prime denotes the derivative with respect to τ , e.g., $\rho' = \partial_\tau \rho$.

In the above discussion, we have neglected the dynamics of the $U(1)$ phase. Note that in the current homogeneous Bjorken setting, it was shown in [15] that the phase dynamics decouple early due to the expansion. Since at present we are primarily concerned with homogeneous late time dynamics, we will work at zero chemical potential. Note also that the Bjorken expansion acts in some sense as a nonlinear quench, so in the expanding $O(4)$ theory we expect that the pion dynamics will play an important role at later times, as in simulations in the quenched scenarios [11,12].

In order to simplify analytic calculations, we will adopt a specific choice of potential parameters. A convenient choice that we will use unless noted otherwise is given by

$$\lambda = \frac{1}{6T_c}, \quad m_0 = \frac{T_c}{3}. \quad (13)$$

In what follows, we also set $T_c = 1$, i.e., we measure in units of T_c . These choices are motivated only by the desire to display formulas in a readable form. The results presented in the following can also be obtained for general values of the parameters appearing in the potential; some results that do not assume the values in Eq. (13) are presented in the Appendix C.

For analytic computations, we will keep the transport coefficients arbitrary. However, in numerical calculations we will take

$$C_\eta = \frac{1}{4\pi}, \quad C_\tau = \frac{2 - \ln 2}{2\pi}, \quad (14)$$

which are the values appropriate for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [32]. We will consider various values of C_κ , since the superfluid system exhibits qualitatively different behavior depending on the value of this parameter, as we show in the following.

III. THE FROZEN CONDENSATE LIMIT

Before undertaking the analysis of the complete set of equations (12), it will be helpful to first gain some intuition by considering the limit $C_\kappa \rightarrow \infty$, in which the condensate assumes a constant value $\rho(\tau) = \rho_0$ and the evolution equations simplify considerably. To get a consistent set of equations one also needs to take a suitable limit for parameters appearing in the potential energy, such that $m_0 \rightarrow 0$ with $m_0 \rho_0^2$ remaining constant. This reduces the order of the equations of motion for $T = T(\tau)$ and leads to

$$\begin{aligned} \frac{C_\tau \tau T''}{T} + \frac{3C_\tau \tau (T')^2}{T^2} + \left(\frac{11C_\tau}{3T} + \tau \right) T' + \frac{T}{3} - \frac{4(C_\eta - C_\tau)}{9\tau} \\ = \xi^3 \frac{(3C_\tau \tau T' + 4C_\tau T + 3\tau T^2)}{72\tau T^4}, \end{aligned} \quad (15)$$

where $\xi^3 \equiv 3\rho_0^2 m_0$. If the constant value of the condensate ρ_0 is set to zero, this is the same equation for the effective temperature as considered in Ref. [18].

We first consider the ideal fluid limit of Eq. (15), i.e., the limit $C_\tau \rightarrow 0$ and $C_\eta \rightarrow 0$. In the presence of a frozen

condensate one finds a closed form solution

$$T(\tau) = \frac{1}{2} \left(\xi^3 + \frac{8\Lambda^2}{\tau} \right)^{1/3}, \quad (16)$$

where Λ is a constant of integration. Note that the temperature does not tend to zero at late times, but to a value set by the condensate.

In the Navier-Stokes limit, $C_\tau \rightarrow 0$ with $C_\eta \neq 0$ and vanishing condensate $\xi \rightarrow 0$, one has the well-known exact solution for viscous Bjorken flow

$$T(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}} \left(1 - \frac{2C_\eta}{3} \frac{1}{(\Lambda\tau)^{2/3}} \right). \quad (17)$$

With a nonvanishing condensate, the Navier-Stokes equation for Bjorken flow cannot be solved in closed form. One can however calculate the late time asymptotic solution, which takes the form of Eq. (1); specifically

$$\begin{aligned} T(\tau) = \frac{\xi}{2} + \frac{4C_\eta \ln(\Lambda\tau)}{9\tau} - (2 + 2\ln(\Lambda\tau) + \ln^2(\Lambda\tau)) \frac{32C_\eta^2}{81\xi} \tau^{-2} \\ + \mathcal{O}(\tau^{-3}), \end{aligned} \quad (18)$$

where Λ is again an integration constant. As in the case of an ideal fluid, the temperature does not vanish at late times, with the limiting value set by the condensate. This is expected, since dissipative effects become negligible as the system approaches equilibrium. Furthermore, the late time expansion contains all negative integer powers of the proper time, but the result is no longer just a power series: we see a novel structure where the coefficients of powers of τ are polynomials in $\ln(\Lambda\tau)$. This can be determined by examining the possible form of correction to the leading order constant solution. It is immediately established that a pure power law correction cannot satisfy Eq. (18) at late times, but one with a logarithmic enhancement does. This pattern can then be seen to hold at higher orders.

We now return to Eq. (15) with nonzero C_τ . The asymptotic late time solution can again be seen to take the form given in Eq. (1). The coefficients appearing in Eq. (1) are determined by Eq. (15), apart from the constant $t_{1,0}$, which remains undetermined and is thus a constant of integration. One can however absorb it into the scale of the logarithm, so that Λ is in fact the only integration constant that appears in the asymptotic solution. At leading orders one finds

$$t_{0,0} = \frac{1}{2}, \quad t_{1,1} = \frac{4C_\eta}{9}, \quad t_{2,2} = -\frac{32C_\eta^2}{81}, \quad t_{2,1} = -\frac{64C_\eta^2}{81}, \quad t_{2,0} = -\frac{8}{81}(8C_\eta^2 - 3C_\eta C_\tau). \quad (19)$$

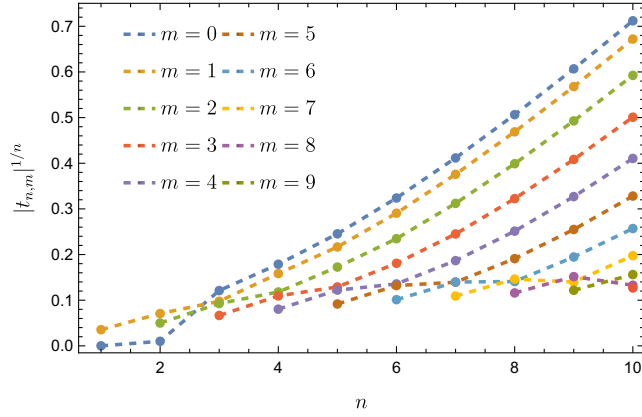


FIG. 1. Factorial growth of the coefficients in Eq. (1) of the frozen condensate case, where each curve corresponds to a specific log sector, with increasing power of $\ln(\Lambda\tau)$ going from left to right.

The asymptotic series Eq. (1) can be rearranged into sectors grouping terms proportional to a given power of the logarithm,

$$T(\tau) \sim \sum_{m=0}^{\infty} \ln^m(\Lambda\tau) \left[\sum_{n=0}^{\infty} t_{n,m} \tau^{-n} \right], \quad (20)$$

where we have defined $t_{n,m} = 0$ for $m > n$. The power series appearing in each log sector can be seen to have a vanishing radius of convergence; they are in fact factorially divergent, as seen in Fig. 1. This is typical of late proper time expansions in Bjorken flow, and the hydrodynamic gradient expansion in general.

As is well known, asymptotic series can be considered as elements of a transseries, which include terms beyond all orders of the original expansion. In the context of relativistic hydrodynamics such contributions are exponentially suppressed and depend on the nonhydrodynamic modes, which are required by causality. In the present case they also depend on the symmetry breaking sector. Proceeding as in Ref. [18], one finds that the first transseries sector assumes the form

$$\delta T(\tau) \sim \sigma \tau^{-7/3} e^{-\frac{\xi\tau}{2C_\tau} - \frac{2C_\eta \ln^2(\Lambda\tau)}{9C_\tau}} \left(\xi - \frac{32C_\eta^2 \ln^2(\Lambda\tau)}{81C_\tau} \frac{1}{\tau} - \frac{32C_\eta(4C_\eta + 9C_\tau) \ln(\Lambda\tau)}{81C_\tau} \frac{1}{\tau} \dots \right). \quad (21)$$

The constant σ is the transseries parameter that plays the role of the second integration constant of Eq. (15). The series appearing in Eq. (20) as well as the transseries sector given in Eq. (21) have a novel form compared to other cases studied in the context of Bjorken flow. The appearance of logarithms in the perturbative sector of a transseries was previously noted for instance in the case of the Painlevé I

equation [33], but to the best of our knowledge the $\ln^2(\Lambda\tau)$ term that appears in the transseries element in Eq. (21) has not been observed previously. The expansion coefficients appearing in the infinite series above are expected to satisfy resurgence relations, which connect them to the late coefficients of the perturbative series Eq. (20), in a similar way to what happens in the case without a condensate [18,27]. These relations are essential for an ambiguity-free result of Borel resummation applied sector by sector to the full transseries (see, e.g., [34]). We leave a careful analysis of these matters to future work.

IV. LATE TIME ASYMPTOTICS

Having determined the form of the late time dynamics in the simpler setting of the frozen condensate limit in the previous Section, we now turn our attention to the full Bjorken superfluid asymptotics. Numerical solutions to Eq. (12) have been previously studied in Refs. [15,16]. Here we turn to the main focus of the present article, which is an analytic description of these solutions at late time.

The form of the asymptotic solutions is found to be

$$T(\tau) \sim \sum_{n=0}^{\infty} \sum_{m=0}^n t_{n,m} \tau^{-n} \ln^m(\Lambda\tau) \equiv \Phi^{(0)}(\tau), \quad (22)$$

with asymptotic solutions for ρ and χ of the same structure, with coefficients r_{mn} and x_{mn} . These coefficients can be determined from the equations of motion. Adopting the parameter values listed in Eq. (13) leads to the leading order values of $t_{0,0} = 1/2$, $x_{0,0} = 0$ and $r_{0,0} = 1$. Full expressions with arbitrary parameters can be found in the Appendix C. Note that $t_{1,0}$ is an arbitrary integration constant¹; as in the case of the frozen condensate in Sec. III, it can be absorbed into the logarithms—this has already been done in Eq. (22), where this integration constant is denoted by Λ . We will return to the infinite series $\Phi^{(0)}$ in the following Section.

The constant Λ is the only part of the initial data of the system Eq. (12) that survives to asymptotically late times. The remaining initial data to which solutions of Eq. (12) are entitled appears in the form of contributions which are beyond the gradient expansion, that is, contributions that are suppressed more strongly than any power of the proper time. Such contributions enter as exponentially damped corrections to classical asymptotics, with the complex frequencies that are in one to one correspondence with nonhydrodynamic quasinormal modes of the system. In the case at hand one finds

$$T(\tau) \sim \Phi^{(0)}(\tau) + \sum_{k=0,\pm} \sigma_k \tau^{\beta_k} e^{-\gamma_k \tau - \gamma'_k \ln^2(\Lambda\tau)} \Phi_k^{(1)}(\tau) + \dots \quad (23)$$

¹Note that we could instead choose, e.g., $r_{1,0}$ as the integration constant.

The series

$$\Phi_k^{(1)} \sim 1 + \mathcal{O}\left(\frac{\ln(\Lambda\tau)^2}{\tau}\right) \quad (24)$$

are infinite series defining the first transseries sector. There are of course further such contributions, as is always the case for nonlinear systems. One can obtain the transseries sectors for ρ and χ in the same way as one can obtain the

perturbative sectors for them from the perturbative series $\Phi^{(0)}$.

The parameters of the purely damped sector are given by

$$\beta_0 = -\frac{7}{3}, \quad \gamma_0 = \frac{1}{2C_\kappa}, \quad \gamma'_0 = \frac{C_\eta}{6C_\tau}, \quad (25)$$

and parameters of the oscillatory sectors are

$$\beta_\pm = -\frac{1}{2} \pm \frac{3C_\kappa}{2\sqrt{9C_\kappa^2 - 64}}, \quad \gamma_\pm = \frac{C_\kappa}{4} \pm \frac{1}{12}\sqrt{9C_\kappa^2 - 64}, \quad \gamma'_\pm = \frac{C_\eta}{12} \left(C_\kappa \pm \frac{3C_\kappa^2 + 16}{\sqrt{9C_\kappa^2 - 64}} \right). \quad (26)$$

Equations (25) and (26) describe the leading contributions to a transseries solution of Eq. (12), with the power law asymptotic series $\Phi^{(0)}(\tau)$ acting as the perturbative sector. The coefficients γ_k appearing in the exponentials coincide with the location of branch point singularities in the Borel plane, as seen in Fig. 4.

The coefficients σ_k are the transseries parameters which encode the initial data. For the system Eq. (12), one has four real integration constants. One of them survives in the perturbative series Eq. (22), while the remaining ones are exponentially suppressed at late times. The exponential corrections reflect the quasinormal modes of the theory: one [corresponding to $k = 0$ in Eq. (23)] is a purely damped mode that is always present in MIS theory. The remaining two are not purely damped in general: they are complex conjugate and introduce oscillatory behavior of solutions whenever their real part does not vanish.

From Eq. (26) it follows that there is a critical value of C_κ , namely $C_\kappa^{(\text{crit})} = 8/3$ with the choice of parameters Eq. (13), beyond which the transseries contains no oscillating contributions; a formula valid for arbitrary parameters is given in the Appendix C. This is a definite prediction coming out of the asymptotic analysis. It can be verified by inspecting numerical solutions of Eq. (12) for various values of the condensate diffusion parameter C_κ , as illustrated in Fig. 2. In fact, the form of Eq. (23) describes the late time behavior of the solutions rather well, as seen in Fig. 3. To obtain this plot, the infinite series were truncated at order $1/\tau^3$ in the perturbative sector and at order $1/\tau$ in the first transseries sector. Note that the transseries parameters need to satisfy $\sigma_+^* = \sigma_-$ so as to render a real solution.

The presence of the oscillations discussed above could leave an observable imprint in the spectra of hadrons emerging from a heavy ion collision. This possibility also highlights the significance of the condensate relaxation rate, which plays a key role in determining whether this phenomenon occurs. It would be very interesting to understand the microscopic origin of this quantity. We would also like to emphasize that the oscillations of the condensate are completely distinct from oscillations of the phase, which

are expected to leave an imprint of the emitted pion spectra [11,12]. The condensate oscillations discussed here affect not just the condensate, but also the temperature and the pressure anisotropy. Given that the latter couples directly to the perturbation of the postfreezeout distribution function [35], one would expect that these oscillations would affect all particle spectra.

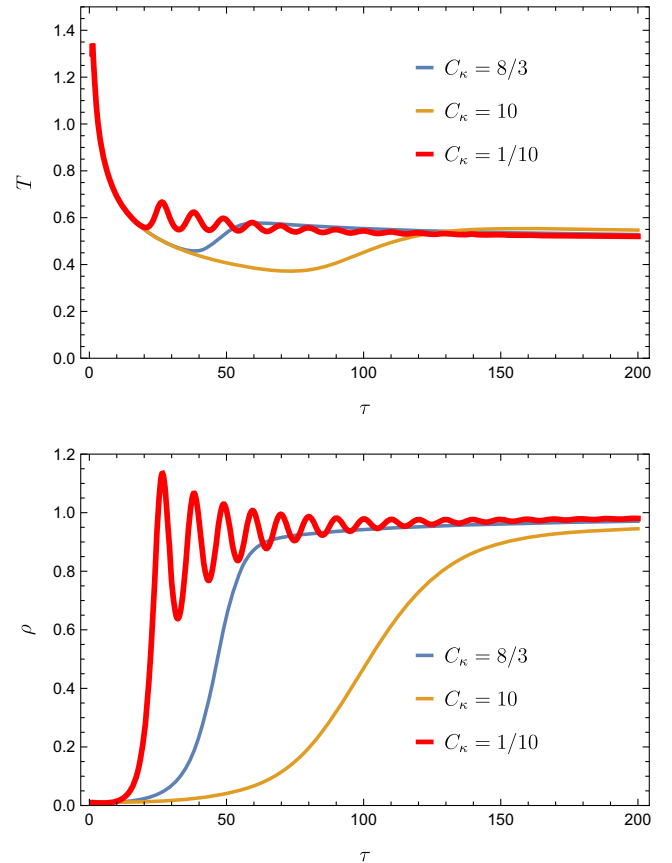


FIG. 2. Temperature and the condensate evaluated by numerically solving Eq. (12) for the same set of initial conditions and parameters, for three characteristic values of the condensate relaxation rate C_κ . Note that the critical value of C_κ corresponds to $8/3$ with the choice of parameters Eq. (13).

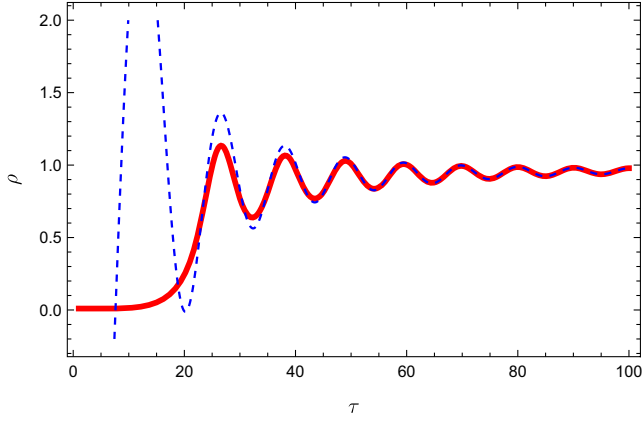


FIG. 3. Red curve is the numerical solution with $C_\kappa = 1/10$ in Fig. 2, while the blue dashed curve is the result of fitting Λ and the transseries parameters σ_k appearing in Eq. (23). Note that the oscillations are due entirely to the exponential contributions to Eq. (23) and is completely invisible to the perturbative series.

V. LARGE ORDER BEHAVIOR

We now turn to the infinite series Φ_k appearing in the transseries solution Eq. (23). As in similar situations (see, e.g., the reviews [22–24]), these are formal asymptotic series and need to be resummed using appropriate techniques, or truncated. In the present case there is a novel feature in the form of the logarithmic terms seen in Eq. (23) that require additional attention. In this article we will only give a brief account of these issues, leaving a more complete analysis to future work.

The formal series $\Phi^{(0)}$ can be grouped into log sectors as explained in Sec. III in the case of constant condensate. The series appearing in the lowest few log sectors can easily be calculated and turn out to diverge factorially. This is almost certainly true for all sectors. Such formal series can be given meaning using Borel summation, which has been applied to some models of relativistic hydrodynamics without a condensate in Refs. [18,27].

As an illustration, we now consider the formal series in the sector with $m = 0$ in Eq. (23)

$$S(\tau) \equiv \sum_{n=1}^{\infty} t_{n,0} \tau^{-n}. \quad (27)$$

Its Borel transform is defined as

$$\tilde{S}(z) = \sum_{n=1}^{\infty} \frac{t_{n,0}}{n!} z^n, \quad (28)$$

where z is the Borel plane variable. The Borel sum of the formal series in Eq. (27) is then given in terms of the Laplace transform of $\tilde{S}$. It has the property that its asymptotic expansion reproduces the original series Eq. (27), so it can be regarded as its resummation [34].

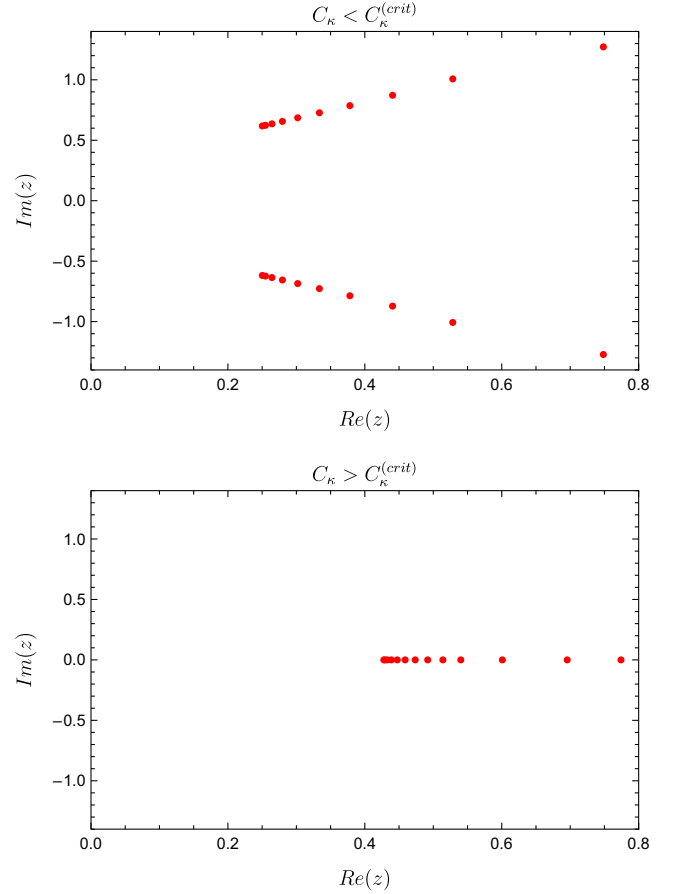


FIG. 4. Poles of the analytically continued Borel transform of the series $t_{k,0}$ with $k = 1, \dots, 58$. The condensation of poles is indicative of branch point singularities. The pattern depends on whether the value of C_κ exceeds the critical value defined in the text, see Eq. (C5).

The Borel transform $\tilde{S}(z)$ converges near the origin and can be analytically continued away from it using Padé approximants to locate its singularities in the Borel plane. This analysis can be done in the same way as, e.g., in Refs. [18,27] and reveals dense sequences of poles indicative of branch points, seen in Fig. 4. Analogous Borel-Padé plots for $t_{k,1}$ are almost indistinguishable from the plots for $t_{k,0}$ and indicate branch points in the same locations. This is very important, since the cancellation of resummation ambiguities relies on balancing contributions from different transseries sectors.

It is expected on general grounds that the pattern of singularities seen in the Borel plane should match the exponential terms of the transseries [18,27,28]. In Fig. 4 one can see the branch points at $z = \gamma_\pm$, but the branch point corresponding to γ_0 is not actually visible, since it is shadowed by the branch points closer to the origin of the Borel plane. This is a consequence of the specific values of the transport coefficients that were adopted for the calculation of the series. The branch point at $z = \gamma_0$ becomes visible in the Borel-Padé plot in Fig. 4 if one increases the

relaxation time C_τ so as to bring γ_0 closer to the origin. In principle it should be visible for any value of C_τ , but only with a much larger number of series coefficients.

We note finally that the occurrence of oscillations (or a lack thereof) is reflected by the presence (or absence) of branch points away from the real axis, depending on the value of C_κ . For $C_\kappa < C_\kappa^{(\text{crit})}$ there are two branch points (at conjugate positions in the Borel plane), located at $\gamma_\pm$; for larger values of C_κ , these branch points approach the real axis and beyond the critical value all the branch points are on the real axis; in Fig. 4 only the one closest to the origin is visible.

VI. OUTLOOK

We have studied the late time asymptotic behavior of Bjorken flow in a system describing an expanding QGP fluid interacting with a condensate resulting from the spontaneous symmetry breaking of a global $U(1)$ symmetry. We have found that the late time behavior of the effective temperature and pressure anisotropy are rather different from what is known from similar studies of MIS theory: the well-known Bjorken falloff $T \sim \Lambda^{2/3} \tau^{-1/3}$ is replaced by a series involving integer powers and logs of the proper time. The series can be decomposed into log sectors, each containing a series with vanishing radius of convergence.

The late time behavior of the hydrodynamic gradient expansion retains information about the underlying microscopic theory, which is not captured by truncations of the perturbative series $\Phi^{(0)}(\tau)$. This is well known in the case of nonhydrodynamic quasinormal modes of MIS theory [18], as well $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [28] and various examples of kinetic theory [36,37]. These modes play the role of a conduit by means of which information about the initial state is relayed to the late time domain. In the case of the superfluid model discussed in this work, the nonhydrodynamic modes include those associated with the spontaneous breaking of the $U(1)$ symmetry. Such contributions are exponentially suppressed relative to the perturbative sector, but they need not be small on physically relevant time scales. It can also happen that such terms carry qualitatively significant physical effects. An example of this arises for instance in the context of QGP dynamics, where the dynamics in the transverse plane enters through exponentially suppressed corrections to Bjorken flow [38]. This is also the case here: we have demonstrated that exponentially suppressed contributions that are beyond all orders of the gradient expansion can carry qualitatively important physical effects, namely the oscillations associated with relaxation of the condensate. Such oscillating quasinormal modes have previously been encountered in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [27,28,39,40]. Whether they occur in the superfluid system is determined by a condition on the condensate relaxation rate, which is one of the main

implications of the form of the transseries established in the present study.

Our work leaves some open questions of mathematical nature. While we have established the form of the transseries and showed that it captures the late time behavior of numerical solutions, we have been working mostly with low-order truncations of the relevant asymptotic series. We have not made any attempts to sum these series or investigate resurgence relations that could be expected to relate expansion coefficients in different sectors (see e.g., Refs. [27,29,34]). It would be interesting to explore this in the future.

The present work was focused on the case of Bjorken flow, but it would also be interesting to investigate the asymptotic late time behavior in other situations. A natural conformal setting would be the de Sitter geometry first discussed in [41], corresponding to spherical slicing of $dS_3 \times \mathbb{R}$, and recently generalized to include also hyperbolic slicing [42] (see also recent applications [43–45]). Numerical studies of superfluid Gubser flow have already been carried out in [16]. We leave the asymptotic analysis of flows in the Gubser or Grozdanov backgrounds to future work.

There are also interesting physical questions which can be addressed following the results reported here. We have considered the simplest possible example of QGP dynamics in the presence of a spontaneously broken symmetry. An important generalization would be to include the full content of the $SU(2) \times SU(2)$ chiral symmetry group of massless QCD, as well as explicit symmetry breaking in the potential. It would then be fascinating to study the possibility that damped oscillations of the condensate could be imprinted upon the spectrum of hadrons emerging from the collision. This would also include additional oscillations not studied here, due to the non-Abelian phase present in the $SU(2) \times SU(2) \sim O(4)$ case. Such oscillations have been seen in the pion correlation function in the instantaneously quenched scenario [11,12]. This would provide strong motivation for exploring the microscopic physics responsible for the condensate relaxation rate, whose potential significance is perhaps the most important physical outcome of our work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: SOLUTIONS

Explicitly, the asymptotic expansion of the temperature, condensate and for the nice parameters in Eq. (13) is given by

$$T(\tau) = \frac{1}{2} + \frac{C_\eta \ln(\Lambda\tau)}{3\tau} + \frac{1}{72\tau^2} [(-32C_\eta^2 + 9C_\eta C_\kappa + 16C_\eta C_\tau) - (32C_\eta^2 + 9C_\eta C_\kappa) \ln(\Lambda\tau) - 12C_\eta^2 \ln^2(\Lambda\tau)] + \mathcal{O}(\tau^{-3}), \quad (\text{A1})$$

$$\rho(\tau) = 1 - \frac{C_\eta \ln(\Lambda\tau)}{3\tau} + \frac{1}{72\tau^2} (32C_\eta^2 + 27C_\eta C_\kappa - 16C_\eta C_\tau + (32C_\eta^2 - 27C_\eta C_\kappa) \ln(\Lambda\tau) + 8C_\eta^2 \ln^2(\Lambda\tau)) + \mathcal{O}(\tau^{-3}), \quad (\text{A2})$$

$$\chi(\tau) = \frac{8C_\eta}{3\tau} - \frac{16C_\eta}{9\tau^2} (C_\tau + C_\eta \ln(\Lambda\tau)) + \mathcal{O}(\tau^{-3}). \quad (\text{A3})$$

APPENDIX B: DOUBLE LOGS

A series in logs and double logs appears in the context of the renormalization group equation in QCD [46–49], which we will briefly outline here. If we consider the beta function in the usual renormalisation group approach, truncated to two loop order

$$\mu g'(\mu) = -\beta_0 g^2 - \beta_1 g^3, \quad (\text{B1})$$

we can solve this asymptotically in $\tau \equiv \ln \mu/\Lambda$. We know that the (QCD) equation for the beta function should hold at low orders in perturbation theory the higher the energy is, which corresponds to the limit $\tau \rightarrow \infty$ and $g \rightarrow 0$. Expanding in these limits, one finds an asymptotic solution of the form

$$\begin{aligned} \tau g_0(\tau) &= \sum_{n=0}^{\infty} \sum_{m=0}^n a_{nm} \tau^{-n} \ln^m \tau, \\ &= \frac{1}{\beta_0} + \frac{-2\beta_1 \ln \beta_0 - \beta_1 \ln \tau}{\beta_0^3 \tau} + \frac{\beta_1^2 \ln^2 \tau + (4\beta_1^2 \ln \beta_0 - \beta_1^2) \ln \tau + (4\ln^2 \beta_0 - 2\ln \beta_0 - 1)\beta_1^2}{\beta_0^5 \tau^2} + \dots \end{aligned} \quad (\text{B2})$$

Curiously, in this case one can compute an exact closed form expression to the truncated equation Eq. (B1) in terms of the Lambert W function [47]

$$g(\tau) = -\frac{\beta_0/\beta_1}{1 + W_0(\beta_1^{-1} e^{-1-\beta_1^{-1}\beta_0^2\tau})}. \quad (\text{B3})$$

Expanding order by order in large τ , we find exact agreement with the asymptotic expansion.

Another instance where a series of similar form has appeared is in recent studies of nonthermal attractors [50]. There it arises as an asymptotic late time solution of an ordinary differential equation that controls prescaling in a system undergoing homogeneous isotropization,

$$\frac{\beta''(t)}{\beta(t)} = \frac{\beta'(t)^2}{\beta(t)^2} + \frac{14\beta'(t)}{t} - \frac{(7\beta(t) + 1)^2}{t^2}. \quad (\text{B4})$$

By inspection, the leading asymptotic contribution is $\beta_\infty = -1/7$. Subsequent subleading terms follow a double series in $\ln(Q\tau)$ and $\ln \ln(Q\tau)$, where Q is an integration constant. Upon setting $\tau = \ln(Q\tau)$, this double series takes the form of Eq. (1).

APPENDIX C: GENERAL EXPRESSIONS

In the main text, we reported on the form of the transseries Eq. (23) with a particular choice of parameters Eq. (13). For completeness, here we provide the transseries for ρ with arbitrary parameters. The purely decaying mode is given by

$$\begin{aligned}
\delta\rho &= \tau^\beta e^{\frac{(m_0 T_c - \rho_0^2 i)}{2C_\tau} \tau - \frac{\rho_0 \lambda C_\tau}{3m_0 C_\tau} \ln^2(\Lambda\tau)} \left(1 - \frac{C_\eta^2 \lambda}{9m_0 C_\tau} \frac{\ln^2(\Lambda\tau)}{\tau} + \dots \right), \\
\beta &= \frac{A}{B}, \\
A &= 2\lambda\rho_0^2 m_0^4 (-120T_c^4 (2C_\kappa C_\tau - 1) + 3\rho_0^2 m_0 T_c (5C_\kappa C_\tau + 1) - 5C_\tau^2 \rho_0^2 m_0^2 + 48C_\tau^2 m_0 T_c^3) \\
&\quad + m_0^5 T_c (48T_c^4 (2C_\kappa C_\tau - 1) - 3\rho_0^2 m_0 T_c (5C_\kappa C_\tau + 1) + 4C_\tau^2 \rho_0^2 m_0^2 - 48C_\tau^2 m_0 T_c^3) \\
&\quad + 48\lambda^4 \rho_0^8 m_0 (5T_c (2C_\kappa C_\tau - 1) + C_\tau^2 m_0) - 96\lambda^3 \rho_0^6 m_0^2 T_c (5T_c (2C_\kappa C_\tau + C_\tau^2 m_0) \\
&\quad + 48\lambda^5 \rho_0^{10} (1 - 2C_\kappa C_\tau) - 3\lambda^2 \rho_0^4 m_0^3 (\rho_0^2 m_0 (5C_\kappa C_\tau + 1) - 160T_c^3 (2C_\kappa C_\tau - 1))), \\
B &= 6(\lambda\rho_0^2 - m_0 T_c) (-m_0^4 (T_c^4 (4 - 4C_\kappa C_\tau) + C_\tau^2 \rho_0^2 m_0^2) - 8\lambda\rho_0^2 m_0^3 T_c^2 (2T_c (C_\kappa C_\tau - 1) + C_\tau^2 m_0) \\
&\quad - 8\lambda^3 \rho_0^6 m_0 (2T_c (C_\kappa C_\tau - 1) + C_\tau^2 m_0) + 8\lambda^2 \rho_0^4 m_0^2 T_c (3T_c (C_\kappa C_\tau - 1) + 2C_\tau^2 m_0) + 4\lambda^4 \rho_0^8 (C_\kappa C_\tau - 1)). \quad (C1)
\end{aligned}$$

Surprising as it may be, for the nice parameters Eq. (13), we find that the above expression reduces to $\beta = -7/3$ for arbitrary transport coefficients.

The oscillatory modes are

$$\delta\rho = \tau^\beta e^{-\gamma_\pm \tau - \gamma'_\pm \ln^2(\Lambda\tau)} (1 + \dots), \quad (C2)$$

$$\begin{aligned}
\gamma_\pm &= -\frac{C_\kappa}{2\sigma} (\lambda\rho_0^2 - \sigma T_c) \mp \frac{\sqrt{C_\kappa^2 (\lambda\rho_0^2 - \sigma T_c)^6 - \rho_0^2 \sigma^2 (\lambda\rho_0^2 - \sigma T_c)^2 (8\lambda^3 \rho_0^4 + \sigma^4 + 8\lambda \sigma^2 T_c^2 - 16\lambda^2 \rho_0^2 \sigma T_c)}}{2\sigma (\lambda\rho_0^2 - \sigma T_c)^2}, \\
\gamma'_\pm &= \frac{C_\eta \gamma_\pm E}{6\rho_0 (\lambda\rho_0^2 - \sigma T_c) F}, \quad (C3)
\end{aligned}$$

$$\begin{aligned}
E &= -4\gamma_\pm^2 \sigma (C_\kappa C_\tau + 1) (\sigma T_c - \lambda\rho_0^2)^3 (5\sigma T_c - 13\lambda\rho_0^2) - 20C_\kappa \gamma_\pm (3\lambda\rho_0^2 - \sigma T_c) (\lambda\rho_0^2 - \sigma T_c)^4 \\
&\quad + 4C_\tau \gamma_\pm^3 \sigma^2 (5\sigma T_c - 11\lambda\rho_0^2) (\lambda\rho_0^2 - \sigma T_c)^2 - C_\tau \gamma_\pm \rho_0^2 \sigma^2 (104\lambda^4 \rho_0^6 + \lambda \sigma^3 (9\rho_0^2 \sigma - 56T_c^3) \\
&\quad + 216\lambda^2 \rho_0^2 \sigma^2 T_c^2 - 264\lambda^3 \rho_0^4 \sigma T_c - 7\sigma^5 T_c) - \rho_0^2 \sigma (120\lambda^4 \rho_0^6 + \lambda \sigma^3 (11\rho_0^2 \sigma - 56T_c^3) \\
&\quad + 232\lambda^2 \rho_0^2 \sigma^2 T_c^2 - 296\lambda^3 \rho_0^4 \sigma T_c - 7\sigma^5 T_c) (\lambda\rho_0^2 - \sigma T_c), \\
F &= 12\gamma_\pm^2 \sigma (C_\kappa C_\tau + 1) (\lambda\rho_0^2 - \sigma T_c)^3 + 8C_\kappa \gamma_\pm (\lambda\rho_0^2 - \sigma T_c)^4 + 16C_\tau \gamma_\pm^3 \sigma^2 (\lambda\rho_0^2 - \sigma T_c)^2 \\
&\quad + 2C_\tau \gamma_\pm \rho_0^2 \sigma^2 (8\lambda^3 \rho_0^4 + \sigma^4 + 8\lambda \sigma^2 T_c^2 - 16\lambda^2 \rho_0^2 \sigma T_c) \\
&\quad + \rho_0^2 \sigma (8\lambda^3 \rho_0^4 + \sigma^4 + 8\lambda \sigma^2 T_c^2 - 16\lambda^2 \rho_0^2 \sigma T_c) (\lambda\rho_0^2 - \sigma T_c), \\
\beta &= \frac{C}{6D}, \\
C &= 4\lambda\rho_0^2 \sigma^4 (2\gamma_\pm T_c^2 (4\gamma_\pm T_c (11C_\kappa C_\tau + 8) + 5C_\kappa T_c^2 - 42C_\tau \gamma_\pm^2) \\
&\quad + 3C_\kappa \gamma_\pm \lambda\rho_0^4 + 4\lambda\rho_0^2 T_c^2 (50T_c - 33C_\tau \gamma_\pm)) \\
&\quad - 8\lambda^4 \rho_0^8 \sigma (\gamma_\pm^2 (11C_\kappa C_\tau + 8) + 5C_\kappa \gamma_\pm T_c + 16\lambda\rho_0^2) \\
&\quad + 16\lambda^3 \rho_0^6 \sigma^2 (C_\tau \gamma_\pm (22C_\kappa \gamma_\pm T_c - 7\gamma_\pm^2 - 11\lambda\rho_0^2) + T_c (5C_\kappa \gamma_\pm T_c + 16\gamma_\pm^2 + 35\lambda\rho_0^2)) \\
&\quad - 16\lambda^2 \rho_0^4 \sigma^3 T_c (T_c (5C_\kappa \gamma_\pm T_c + 24\gamma_\pm^2 + 60\lambda\rho_0^2) - 3C_\tau \gamma_\pm (-11C_\kappa \gamma_\pm T_c + 7\gamma_\pm^2 + 11\lambda\rho_0^2)) \\
&\quad + \sigma^6 (3\gamma_\pm \rho_0^2 (C_\tau \gamma_\pm (C_\kappa T_c - \gamma_\pm) + 2T_c (2C_\kappa T_c + \gamma_\pm)) + 2\lambda\rho_0^4 (30T_c - 13C_\tau \gamma_\pm) + 48T_c^5) \\
&\quad + \sigma^5 (-8\gamma_\pm T_c^3 (C_\tau \gamma_\pm (11C_\kappa T_c - 14\gamma_\pm) + T_c (C_\kappa T_c + 8\gamma_\pm)) \\
&\quad - 3\gamma_\pm \lambda\rho_0^4 (\gamma_\pm (C_\kappa C_\tau + 2) + 8C_\kappa T_c) + 16\lambda\rho_0^2 T_c^3 (11C_\tau \gamma_\pm - 20T_c) - 36\lambda^2 \rho_0^6) \\
&\quad + 8C_\kappa \gamma_\pm \lambda^5 \rho_0^{10} + 4\rho_0^2 \sigma^7 T_c (5C_\tau \gamma_\pm - 6T_c),
\end{aligned}$$

$$\begin{aligned}
D = & 4C_\kappa\gamma_\pm(\lambda\rho_0^2 - \sigma T_c)^4(3C_\tau\gamma_\pm\sigma + 2\lambda\rho_0^2 - 2\sigma T_c) \\
& + \sigma(\lambda\rho_0^2 - \sigma T_c)(16C_\tau\gamma_\pm^3\sigma(\lambda\rho_0^2 - \sigma T_c)^2 + 2C_\tau\gamma_\pm\rho_0^2\sigma(8\lambda^3\rho_0^4 + \sigma^4 + 8\lambda\sigma^2T_c^2 - 16\lambda^2\rho_0^2\sigma T_c) \\
& + 12\gamma_\pm^2(\lambda\rho_0^2 - \sigma T_c)^3 + \rho_0^2(\lambda\rho_0^2 - \sigma T_c)(8\lambda^3\rho_0^4 + \sigma^4 + 8\lambda\sigma^2T_c^2 - 16\lambda^2\rho_0^2\sigma T_c)).
\end{aligned} \tag{C4}$$

Finally, for general values of the potential parameters, the critical value of the transport coefficient is

$$(C_\kappa^{(\text{crit})})^2 = \frac{\rho_0^2 m_0^2 (m_0^4 + 8\lambda(\lambda\rho_0^2 - m_0 T_c)^2)}{(\lambda\rho_0^2 - m_0 T_c)^4}. \tag{C5}$$

This reduces to 8/3 for the nice parameters in Eq. (13).

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