Characteristic decomposition for relativistic numerical simulations. II. Magnetohydrodynamics

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The characteristic decomposition for general relativistic magnetohydrodynamics in the comoving frame of the fluid has been known for a long time. However, it has not been known in the coordinate frame of the simulation and in terms of the conserved variables evolved in typical numerical simulations. This paper applies the method of quasi-invertible transformations developed in Paper I to derive this decomposition. Among other benefits, this will allow us to use the most accurate known computational methods, such as full-wave Riemann solvers. The results turn out to be simpler than expected based on earlier attempts.

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I. INTRODUCTION

Relativistic flows with strong magnetic and gravitational fields are associated with extremely energetic astrophysical phenomena, such as pulsar winds, anomalous x-ray pulsars, soft gamma-ray repeaters, gamma-ray bursts, and jets in active galactic nuclei. The merger of two neutron stars or a neutron star and a black hole is another arena where we need to be able to deal with this kind of physics.

Much of the time, the magnetic field can be treated in the ideal magnetohydrodynamic limit (MHD), where the fluid is treated as a perfect conductor. General relativity (GR) describes the gravitational field and modifies the equations of MHD from their nonrelativistic form. General relativistic magnetohydrodynamics (GRMHD) is used both to model LIGO mergers involving neutron stars and to match observations with the Event Horizon Telescope.

Numerical simulations with GRMHD are much more challenging than purely hydrodynamic simulations in GR. There are more equations and they are more complicated. In addition, there are physical instabilities that can require very high resolution to study accurately. And the conservation law that the divergence of the **B**-field must vanish can be difficult for the evolution equations to satisfy numerically. The numerical methods currently used are variants of high-resolution shock-capturing methods first developed for hydrodynamics.

There is a serious remaining theoretical gap in the implementation of all GRMHD codes, both with finite-difference and discontinuous Galerkin techniques. The gap is the lack of a complete characteristic decomposition for the system of equations. In numerical relativity, the evolution of fluids, magnetic fields, and the metric itself is done with hyperbolic evolution equations. Hyperbolicity means that at every point the field admits a decomposition into independent waves, each with its own wave speed. Formally, the decomposition is carried out by solving an eigenvalue problem. The most accurate numerical algorithms typically make use in some way of this characteristic decomposition. These uses include accurate imposition of boundary conditions and treatment of shock waves. Moreover, the most robust and accurate prescriptions for numerical fluxes across subdomain boundaries are based on characteristic decomposition.

The characteristic decomposition of the MHD equations in general relativity was first studied by Bruhat [1] and then by Anile and Pennisi [2]. The Anile and Pennisi work uses essentially the same form of the equations as Bruhat, except for changing one of the independent thermodynamic variables from enthalpy to pressure. This work was reproduced in Anile's book [3], from which the results have been widely used in the literature. We note that, although Bruhat was the first to completely analyze the MHD characteristic speeds in full general relativity directly from the characteristic matrix, her work has never been cited in this connection by subsequent authors, with the exception of Ref. [4].

A key result of the Anile and Pennisi work is that there is a formulation of the MHD equations that is strongly hyperbolic, a necessary condition for having a well-posed initial-value problem and thus being suitable for numerical computations. However, the equations studied in this early work are not in the form typically used in current numerical work.

Virtually all numerical codes today use equations in conservation form: A set of evolved variables is chosen so that their spatial derivatives appear only in the divergences

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of fluxes. Equations in conservation form allow a wide variety of robust and accurate treatments of shocks and other discontinuities in the flow. As shown in Ref. [5], the equations in conservation form differ from those of Bruhat and Anile by the addition of multiples of the divergence constraint $\nabla \cdot \mathbf{B} = 0$. This does not change the solutions, but does change the principal part of the equations (the highest derivative terms of the evolved variables). Changing the principal part changes the eigenvalues and eigenvectors of the characteristic matrix, and hence can affect the hyperbolicity of the system. In fact, Ref. [5] shows that the usual conservative system, the so-called Valencia formulation (see, e.g., the review in [6]), is only weakly hyperbolic and thus does not have a well-posed initial-value problem.

One of the challenges of MHD simulations, both as relativistic and in the Newtonian limit, is to guarantee that the divergence constraint remains satisfied during the evolution. Various computational techniques have been developed to ensure this, one of which is divergence cleaning [7–9]. In this method, we introduce a scalar field with an evolution equation that damps the divergence of the magnetic field and also advects it off the grid. Interestingly, Ref. [10] has shown that the divergence cleaning formulation of the Valencia scheme *is* strongly hyperbolic and thus has a well-posed initial-value problem, making it attractive for numerical simulations.

In Paper I [11], we introduced a new method for finding characteristic decompositions of fluid flows in relativity, which we describe further below. The goal of the current paper is to apply this method to GRMHD. Besides the work of Anile [3], the previous works on the characteristic decomposition problem for GRMHD that are most related to this paper are those of Antón *et al.* [12], Ibáñez *et al.* [13], Schoepe *et al.* [5], and Hilditch and Schoepe [10]. We will discuss the connections at appropriate points in the text. A detailed comparison with the work of Antón *et al.* [12] is given in Appendix A 1 and with that of Schoepe *et al.* [5] in Appendix A 2. For other references to earlier work, see, for example, the introduction to Ref. [12].

As discussed in more detail in Paper I [11], in the $3+1$ decomposition we introduce a coordinate frame and consider the evolution with respect to the $t = \text{const}$ slices of this frame. The tangent vectors $t^a = (\partial/\partial t)^a$ to the t-coordinate lines are related to the normal n^a to the slices by

$$t^a = \alpha n^a + \beta^a. \tag{1.1}$$

Here α is the lapse function and β^a is the shift vector.[1]

A special case is to choose the comoving or Lagrangian frame, where the shift vector is chosen so that the spatial coordinates are comoving with the fluid. In this case t^a is proportional to u^a, the four-velocity of the fluid. As noted in Paper I and by many previous authors, the characteristic decomposition for hydrodynamics or MHD is most easily determined in this frame, since it is the natural frame associated with the fluid flow. Ultimately, however, we will want the characteristic decomposition in the $3+1$ coordinate frame and for the conserved variables in that frame. We noted in Paper I that we can first find the decomposition in the Eulerian frame, which is the frame of observers at rest in the $t = \text{const}$ slices and whose four-velocity is simply n^a. Then Eq. (3.7) of Paper I shows that the eigenvalues in the $3+1$ coordinate frame are simply related to those in the Eulerian frame, while the eigenvectors are the same. So it is sufficient to find the decomposition in the Eulerian frame.

In Paper I, we introduced a method of doing this entire procedure for hydrodynamics by first finding the decomposition for a set of primitive variables in the comoving frame and then transforming the eigensystem to the conserved Eulerian variables. The key difference between this algorithm and previous transformation techniques is the use of a *quasi-invertible transformation*. The main technical innovation in the current paper is the extension of the quasi-invertible transformation procedure from the relatively simple case of the fluid velocity to the more challenging case of the magnetic field. We will first apply the technique to the nonconservative Anile formulation, even though this formulation is not used in numerical work. This will make contact with previous literature and is also useful because many of the results are identical for the conservative divergence cleaning formulation, which we study afterward.

This paper is organized as follows: In Sec. II we derive a system of equations equivalent to those of Anile [3] for the nonconservative formulation of GRMHD. We then carry out the comoving decomposition in Sec. II A in enough detail to make the subsequent developments clear. Section II B contains the main technical accomplishment of the paper, the quasi-invertible transformation from the comoving variables $\{u^a, b^a\}$ to the Eulerian variables $\{v^a, B^a\}$ and vice versa. This transformation is then applied to the comoving decomposition to get the decomposition in the Eulerian frame. Section III introduces the conserved variables. The Jacobian transformation matrix from primitive to conserved variables is given in Sec. III B. The inverse transformation is given in Sec. III C in a form much simpler than in previous work. These transformations are then applied to the Eulerian Anile eigenvectors to express them in terms of conserved variables. These are not the eigenvectors for the conservative evolution equations, but make contact with earlier work and will be useful in presenting the correct conservative decomposition.

[1] In this paper, Roman letters $a, b, \ldots$ from the beginning of the alphabet label 4D abstract spacetime indices. Roman letters $i, j, \ldots$ from the middle of the alphabet label 3D spatial indices in some particular coordinate system.

We present this decomposition for the conservative formulation with divergence cleaning in Sec. IV.

II. NONCONSERVATIVE FORMULATION OF MAGNETOHYDRODYNAMICS

In the MHD approximation, the electromagnetic field is completely determined by the comoving magnetic field

$$b^a = -{}^*F^{ab}u_b, \qquad {}^*F^{ab} = u^b b^a - u^a b^b. \tag{2.1}$$

Here ${}^*F^{ab}$ is the dual of the electromagnetic field tensor:

$${}^*F^{ab} = \frac{1}{2}\epsilon^{abcd}F_{cd}. \tag{2.2}$$

Note that all electromagnetic field quantities in this paper are renormalized by a factor $1/\sqrt{4\pi}$, the standard convention in numerical work. Also, note that the Levi-Civita tensor ϵ_{abcd} is defined with the convention [14] that in flat spacetime $\epsilon_{0123} = +1$. The Maxwell equation $\nabla_a {}^*F^{ab} = 0$ becomes[2]

$$-\nabla_a(u^b b^a - u^a b^b) = 0. \tag{2.3}$$

The projection of this equation along u_b gives the divergence constraint, which can be written in several equivalent forms:

$$\nabla_a b^a + u^a u^b \nabla_b b_a = 0, \tag{2.4a}$$

$$\nabla_a b^a - b^a u^b \nabla_b u_a = 0, \tag{2.4b}$$

$$h^{ab}\nabla_b b_a = 0. \tag{2.4c}$$

Here h^{ab} is the projection tensor that projects orthogonal to u^a:

$$h^{ab} = g^{ab} + u^a u^b. \tag{2.5}$$

Equation (2.4b) follows from Eq. (2.4a) using $u^a b_a = 0$. Equation (2.4c) follows from Eq. (2.4a) using $\nabla_a b^a = (h^{ab} - u^a u^b)\nabla_a b_b$. As we will see below, the divergence constraint plays an important role in distinguishing different formulations of the MHD equations.

We will start out using u^a and b^a as two of the primitive variables. In addition, we need two variables describing the thermodynamic state for a perfect fluid. Of course, at some stage we have to enforce the constraints

$$u^a u_a = -1, \qquad u^a b_a = 0, \tag{2.6}$$

reducing the number of independent variables from 10 to 8.

It does not make much difference which two thermodynamic variables we choose as primitive variables; it is easy to transform from one set to any other. We will choose the pair (p, ϵ), where p is the pressure and ϵ the specific internal energy density. This choice simplifies the algebra slightly. Later we will replace p by ρ, the rest-mass density, since (ρ, ϵ) are more convenient in astrophysical codes.

Associated with the fluid are also the total energy density e, the specific entropy s, and the specific enthalpy h. These quantities are related by the following equations:

$$e = \rho(1 + \epsilon), \tag{2.7}$$

$$h = 1 + \epsilon + p/\rho, \tag{2.8}$$

$$T ds = d\epsilon + p d(1/\rho) \qquad \text{(First Law)}, \tag{2.9}$$

$$p = p(\rho, \epsilon) \qquad \text{(Equation of state)}, \tag{2.10}$$

$$dp = \frac{\partial p}{\partial \rho}\bigg|_\epsilon d\rho + \frac{\partial p}{\partial \epsilon}\bigg|_\rho d\epsilon \equiv \chi d\rho + \kappa d\epsilon. \tag{2.11}$$

Here T is the temperature. The quantities χ and κ are related to the speed of sound c_s by

$$c_s^2 = \frac{1}{h}\left(\chi + \frac{p}{\rho^2}\kappa\right) \tag{2.12}$$

(cf. e.g., Appendix A of Paper 1).

We now present a formulation of the MHD equations equivalent to that of Anile [3], but in slightly simpler form. In particular, there are only eight dynamical equations since the constraints $u^a u_a = -1$ and $u^a b_a = 0$ are explicitly enforced. The formulation here is equivalent to that in Ref. [5].

A basic equation for fluid flow is the equation of mass conservation,

$$u^a \nabla_a \rho + \rho \nabla_a u^a = 0. \tag{2.13}$$

The next fundamental equation is the energy equation. This can be derived from the projection of the conservation equation $\nabla_a T^{ab}$ along u_b, where T^{ab} is the total stress-energy tensor. It is simpler to proceed as follows: Since the fluid is taken to be perfect, the flow is isentropic (except at shocks). Accordingly, we set $ds = 0$ in Eq. (2.9) and get

$$\frac{d\epsilon}{d\tau} = \frac{p}{\rho^2}\frac{d\rho}{d\tau}. \tag{2.14}$$

Here τ is the proper time along the fluid worldlines. Rewrite $d/d\tau = u^a \nabla_a$ and use Eq. (2.13) so that the energy

[2] We insert the minus sign in front of this equation so that the corresponding term on the diagonal of the matrix A^a in Eq. (2.26) is positive, the same as the sign of the other diagonal terms. This simplifies the orthogonality relation for the eigenvectors [cf. Eq. (4.39) of Paper I].

equation becomes

$$u^a\nabla_a\epsilon + \frac{p}{\rho}\nabla_a u^a = 0. \tag{2.15}$$

To use p as a primitive variable, we need its evolution equation. Equation (2.11) gives

$$\begin{aligned}\frac{dp}{d\tau} &= \chi\frac{d\rho}{d\tau} + \kappa\frac{d\epsilon}{d\tau}\\ &= \left(\chi + \kappa\frac{p}{\rho^2}\right)\frac{d\rho}{d\tau}\\ &= hc_s^2\frac{d\rho}{d\tau},\end{aligned} \tag{2.16}$$

where we have used Eqs. (2.14) and (2.12). So finally,

$$u^a\nabla_a p + \rho hc_s^2\nabla_a u^a = 0. \tag{2.17}$$

The total stress-energy tensor for the matter plus magnetic field is

$$T^{ab} = \rho h^* u^a u^b + p^* g^{ab} - b^a b^b, \tag{2.18}$$

where

$$\rho h^* = \rho h + b^2, \qquad p^* = p + \frac{1}{2}b^2. \tag{2.19}$$

The Euler equation follows from projecting the conservation equation $\nabla_b T^{ab} = 0$ orthogonal to u^a using h^{ab}. This gives

$$\begin{aligned}&\rho h^* h^d{}_b u^a\nabla_a u^b + h^{da}\nabla_a p\\ &\quad + h^d{}_b b^a(\nabla^b b_a - \nabla_a b^b) - b^d\nabla_a b^a = 0.\end{aligned} \tag{2.20}$$

The term proportional to $\nabla_a b^a$ in this equation turns out to be problematic in determining the characteristic decomposition below because the derivative contains pieces both along and perpendicular to u^a. Accordingly, the characteristic matrix is not completely orthogonal to u^a. We eliminate this problematic term by forming the combination

$$b^a\nabla_b T_a{}^b + \rho h^*[\text{lhs of Eq. (2.4b)}] = 0. \tag{2.21}$$

This equation simplifies to

$$\nabla_a b^a = -\frac{1}{\rho h}b^a\nabla_a p. \tag{2.22}$$

Note that the u^a component of the derivative on the right-hand side of Eq. (2.22) drops out because it is projected along b^a. Substituting Eq. (2.22) in Eq. (2.20) gives the final form

$$\begin{aligned}&\rho h^* h^d{}_b u^a\nabla_a u^b + h^{da}\nabla_a p\\ &\quad + b^a\left[h^d{}_b(\nabla^b b_a - \nabla_a b^b) + \frac{b^d}{\rho h}\nabla_a p\right] = 0.\end{aligned} \tag{2.23}$$

This equation is equivalent to Eq. (2.75) of Ref. [3] when that equation is projected with h_{ab}; the component of that equation along u_a is zero.

We used the component of the Maxwell equation (2.3) along u_b in Eq. (2.21), so we only need the projection with h_{cb}. This gives

$$-h_{cb}(b^a\nabla_a u^b - u^a\nabla_a b^b) + b_c\nabla_a u^a = 0. \tag{2.24}$$

This equation is equivalent to Eq. (2.76) of Ref. [3] when that equation is projected with h_{ab} and Eq. (2.17) is used. Again, note that the component of Eq. (2.76) in Ref. [3] along u_a is zero.

To enforce the condition $u^a u_a = -1$, in the above equations we use the identity

$$\nabla_a u^a = h^a{}_b\nabla_a u^b. \tag{2.25}$$

Then Eqs. (2.23), (2.24), (2.17), and (2.15) can be written as a quasilinear system [Eq. (2.9) of Paper I] for the variables (u^a, b^a, p, ϵ), where the matrices A^a are

$$A^a = \begin{bmatrix} u^a h_c{}^b & (b^b h_c{}^a - b^a h_c{}^b)/(\rho h^*) & h_c{}^a/(\rho h^*) + b_c b^a/(\rho^2 h h^*) & 0\\ -b^a h_c{}^b + b_c h^{ba} & h_c{}^b u^a & 0 & 0\\ \rho hc_s^2 h^{ba} & 0 & u^a & 0\\ (p/\rho)h^{ba} & 0 & 0 & u^a\end{bmatrix}. \tag{2.26}$$

A. Comoving decomposition

1. Characteristic speeds

As explained in Sec. II of Paper I, to find the characteristic speeds in the comoving frame, we introduce a vector

$$q_a = y_u u_a + \zeta_a, \tag{2.27}$$

where y_u is the comoving eigenvalue (characteristic speed) and ζ_a is a unit vector orthogonal to u_a defining the direction of the decomposition. We then set to zero the determinant of the characteristic matrix $A^a q_a$. The determinant is evaluated in Appendix B, and we find up to a constant factor the well-known result

$$\det(A^a q_a) = a^2(B^2 - \rho h^* a^2)\{a^2 G b^2 - \rho h a^4 + c_s^2[\rho h a^2(a^2 + G) - B^2 G]\}, \tag{2.28}$$

where we use the common notation

$$a = q^a u_a, \qquad B = q^a b_a, \qquad G = q^a q_a. \tag{2.29}$$

From Eq. (2.28), we see that there are three classes of characteristic speeds:

(i) There is a doubly degenerate eigenvalue given by $a = u^a q_a = y_u = 0$.

(ii) Two *Alfvén waves* have speeds given by $B = \pm\sqrt{\rho h^*} a$. With our conventions, the choice $B = -\sqrt{\rho h^*} a$ leads to $y_u = +b^a \zeta_a / \sqrt{\rho h^*}$, which we will take to be the "first" Alfvén eigenvector.

(iii) Four *magnetosonic waves* have speeds given by the roots of

$$N \equiv a^2 G b^2 - \rho h a^4 + c_s^2[\rho h a^2(a^2 + G) - B^2 G] = 0, \tag{2.30}$$

a quartic in q^a.

2. Right eigenvectors in the comoving frame

The right eigenvectors follow from solving $(A^a q_a)X = 0$. Take X to be of the form

$$X = \begin{bmatrix} X_b \\ Y_b \\ X_7 \\ X_8 \end{bmatrix}. \tag{2.31}$$

Here the unknown vectors X_b and Y_b can be taken to be orthogonal to u^b. Using the explicit form (2.26) of A^a and the definitions (2.29), we get the equations

$$aX_c + \frac{1}{\rho h^*}\left(b^b Y_b h_{ac} q^a - B Y_c + q^a h_{ac} X_7 + \frac{B b_c}{\rho h} X_7\right) = 0, \tag{2.32a}$$

$$B X_c - b_c q^a X_a - a Y_c = 0, \tag{2.32b}$$

$$\rho h c_s^2 q^a X_a + a X_7 = 0, \tag{2.32c}$$

$$\frac{p}{\rho} q^a X_a + a X_8 = 0. \tag{2.32d}$$

Of course, these equations are not linearly independent when we insert a q^a corresponding to an eigenvalue.

Eigenvectors for the degenerate eigenvalues. Setting $q^a u_a = 0$ and solving Eqs. (2.32) from the bottom up, we get in turn X_8 arbitrary, $q^a X_a = 0$, $X_c = 0$, $X_7 = 0$. Then we can take the two linearly independent eigenvectors to have either $Y_c = 0$ or $X_8 = 0$:

$$X_{\text{entropy}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad X_{\text{constraint}} = \begin{bmatrix} 0 \\ h_{ab} q^b \\ 0 \\ 0 \end{bmatrix}. \tag{2.33}$$

Here we have followed the standard convention of naming these eigenvectors the entropy and constraint eigenvectors. The entropy eigenvector emerges naturally when using entropy as one of the two thermodynamic variables in the analysis, while the constraint eigenvector is associated with the divergence constraint equation.

Alfvén eigenvectors. Inserting $B = -\sqrt{\rho h^*} a$ in Eqs. (2.32), we find $X_8 = X_7 = q^a X_a = b^b Y_b = 0$ and $Y_c = -\sqrt{\rho h^*} X_c$. Thus, X_c and Y_c are both orthogonal to q^a and b^a. Since they must also be orthogonal to u^a, we can take

$$X_{\text{Alf}} = \begin{bmatrix} \epsilon_{abcd} u^b b^c q^d \\ -\sqrt{\rho h^*}\epsilon_{abcd} u^b b^c q^d \\ 0 \\ 0 \end{bmatrix}. \tag{2.34}$$

The other eigenvector has a positive square root and a corresponding change in q^d.

We can write the Alfvén eigenvectors in an alternative form by introducing two "tangential" vectors $\mathfrak{t}^a_{(1,2)}$ that are orthogonal to q^a and u^a. (These vectors are tangential because $q^a = y_u u^a + \zeta^a$ implies that they are spatial in the comoving frame and orthogonal to the normal vector ζ^a.) Requiring X_c to be a linear combination of these tangential vectors that is orthogonal to b^a gives for the case $B = -\sqrt{\rho h^*} a$

$$X_{\text{Alf}} = \begin{bmatrix} b_2 \mathfrak{t}^a_{(1)} - b_1 \mathfrak{t}^a_{(2)} \\ -\sqrt{\rho h^*}(b_2 \mathfrak{t}^a_{(1)} - b_1 \mathfrak{t}^a_{(2)}) \\ 0 \\ 0 \end{bmatrix}, \tag{2.35}$$

where $b_1 = b_a \mathfrak{t}^a_{(1)}$, $b_2 = b_a \mathfrak{t}^a_{(2)}$. This expression agrees with that of Ref. [5].

Magnetosonic eigenvectors. To solve Eqs. (2.32) in the magnetosonic case, we can proceed as follows:

(i) Use Eqs. (2.32b) and (2.32c) to eliminate Y_a and X_7 from Eq. (2.32a).
(ii) Dot the resulting equation with b^c to get an expression for $b^c X_c$ and substitute this expression back into the equation.
(iii) Solve the resulting equation for X_c. Simplify the result using the eigenvalue equation (2.30): Solve that equation for b^2 and eliminate b^2 from the equation for X_c. Discard the overall normalization factor (which is proportional to $X^a q_a$). The result is surprisingly simple:

$$X_a = BGb_a - \rho h a^2 h_{ab} q^b. \tag{2.36}$$

(iv) Use the solution (2.36) to determine the other unknowns. The resulting eigenvector is

$$X_{\rm mag} = \begin{bmatrix} BGb_a - \rho h a^2 h_{ab} q^b \\ \rho h a[(a^2+G)b_a - Bh_{ab}q^b] \\ \rho h a\mathcal{G} \\ pa\mathcal{G}/(\rho c_s^2) \end{bmatrix}, \tag{2.37}$$

where, using Eq. (2.30),

$$\mathcal{G} = [\rho h a^2(a^2+G) - B^2 G]c_s^2/a^2 = \rho h a^2 - Gb^2. \tag{2.38}$$

3. Left comoving eigenvectors

The left eigenvectors follow from solving $L(A^a q_a) = 0$ analogously to the steps to find the right eigenvectors in Sec. II A 2. Take L to be of the form

$$L = \begin{bmatrix} L_b & Y_b & L_7 & L_8 \end{bmatrix}, \tag{2.39}$$

with L_b and Y_b orthogonal to u^b. Using the explicit form (2.26) of A^a and the definitions (2.29), we get the equations

$$aL_c - BY_c + h_{ca}q^a(b^b Y_b + \rho h c_s^2 L_7 + pL_8/\rho) = 0, \tag{2.40a}$$

$$BL_c - b_c q^a L_a - \rho a h^* Y_c = 0, \tag{2.40b}$$

$$Bb^a L_a + \rho h(q^a L_a + \rho a h^* L_7) = 0, \tag{2.40c}$$

$$aL_8 = 0. \tag{2.40d}$$

Degenerate eigenvectors. There are two linearly dependent eigenvectors corresponding to the degenerate eigenvalue given by $a = 0$. In this case, we see from Eqs. (2.40b) and (2.40c) that $L_c = 0$. Then one solution has $Y_c = 0$ while the other has $L_7 = L_8 = 0$:

$$\begin{aligned} L_{\rm entropy} &= \begin{bmatrix} 0 & 0 & -p/\rho & \rho h c_s^2 \end{bmatrix}, \\ L_{\rm constraint} &= \begin{bmatrix} 0 & h_{ab}q^b & 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.41}$$

Alfvén eigenvectors. The left Alfvén eigenvectors are very similar to the right eigenvectors. The L_c and Y_c parts are once again orthogonal to q^a and b^a. The eigenvector for $B = -\sqrt{\rho h^*} a$ is

$$L_{\rm Alf} = \begin{bmatrix} \sqrt{\rho h^*}\epsilon_{abcd}u^b b^c q^d & -\epsilon_{abcd}u^b b^c q^d & 0 & 0 \end{bmatrix}. \tag{2.42}$$

The other eigenvector has no minus sign and the appropriate change in q^a. An alternative representation analogous to Eq. (2.35) is

$$L_{\rm Alf} = \begin{bmatrix} \sqrt{\rho h^*}(b_2 \mathbb{t}^a_{(1)} - b_1 \mathbb{t}^a_{(2)}) & -(b_2 \mathbb{t}^a_{(1)} - b_1 \mathbb{t}^a_{(2)}) & 0 & 0 \end{bmatrix}. \tag{2.43}$$

Magnetosonic eigenvectors. By steps similar to those resulting in Eq. (2.37), we find from Eqs. (2.40) that the magnetosonic eigenvectors are

$$L_{\rm mag} = \begin{bmatrix} B(a^2+G)b_a/a - \rho h^* a h_{ab} q^b \\ (a^2+G)b_a - Bh_{ab}q^b \\ \mathcal{G}/(\rho h c_s^2) \\ 0 \end{bmatrix}^T. \tag{2.44}$$

It is easy to check that the eigenvectors are all mutually orthogonal: The matrix of the left row eigenvectors multiplied by the matrix of the corresponding right column vectors is diagonal (see Sec. IV F of Paper I). Showing orthogonality among the four different magnetosonic eigenvectors is actually a little more complicated and requires the explicit expressions for the eigenvalues. The diagonal elements of the matrix product correspond to the norms of each eigenvector. We will not bother to renormalize the eigenvectors to unit norm, since in practice it is easier to renormalize the eigenvectors numerically if necessary.

B. Eulerian eigensystem

Most relativistic MHD codes use as primitive variables a fluid velocity and magnetic field that are purely spatial, rather than the pair (u^a, b^a). We will use the Eulerian velocity and magnetic field, that is, the quantities measured by an observer whose four-velocity is n^a, the unit normal to the $t = \text{const}$ hypersurface. This is the observer at rest in the $3+1$ coordinate system defined with respect to n^a.

In covariant form, these quantities are defined as

$$v^a = \gamma^{ab} u_b / W, \qquad B^a = n_b {}^*F^{ba} = W b^a + u^a (n_b b^b), \tag{2.45}$$

with the inverse relations

$$u^a = W(v^a + n^a), \qquad b^a = h^{ab} B_b / W. \tag{2.46}$$

Here

$$W = -n_a u^a = \frac{1}{\sqrt{1 - v^a v_a}} \tag{2.47}$$

is a generalized gamma factor, and

$$\gamma^{ab} = g^{ab} + n^a n^b \tag{2.48}$$

is the projection tensor orthogonal to n^a that serves as the corresponding spatial three-metric.

1. Transformation between (v^a, B^a) *and* (u^a, b^a)

As shown in Paper I, to get both the left and the right transformed eigenvectors, we need the Jacobian matrix for the transformation from the eight variables $U_{\rm old} = (u^a, b^a)$ to the six variables $U = (v^a, B^a)$, as well as the matrix for the inverse transformation. Since these Jacobian matrices are effectively 8×6 and 6×8, they cannot be true matrix inverses. Recall the detailed discussion in Paper I for the case without a magnetic field. We invoked a theorem from linear algebra to conclude that, starting with the matrix $\partial U_{\rm old}/\partial U = \partial u^c/\partial v^b$, we could find a left inverse $\partial U/\partial U_{\rm old} = \partial v^a/\partial u^c$ such that

$$\frac{\partial v^a}{\partial u^c}\frac{\partial u^c}{\partial v^b} = \gamma^a{}_b. \tag{2.49}$$

The left inverse is not unique, and we were able to fix this nonuniqueness by imposing the additional "inverse" requirement

$$\frac{\partial u^a}{\partial v^c}\frac{\partial v^c}{\partial u^b} = h^a{}_b. \tag{2.50}$$

This relation plays a crucial role in showing that the transformation is quasi-invertible, as defined and discussed in Sec. VII B of Paper I and also below. The explicit Jacobian matrices satisfying these conditions are

$$\frac{\partial u^a}{\partial v^b} = W\gamma^a{}_b + W^2 u^a v_b, \tag{2.51}$$

$$\frac{\partial v^a}{\partial u^b} = \frac{1}{W}\delta^a{}_b + \frac{1}{W^2} u^a n_b. \tag{2.52}$$

Formally, we have satisfied the quasi-invertible condition of Eq. (7.8) in Paper I,

$$A^a_{\rm old} = A^a \frac{\partial U}{\partial U_{\rm old}} \quad \Leftrightarrow \quad A^a = A^a_{\rm old}\frac{\partial U_{\rm old}}{\partial U}, \tag{2.53}$$

by the conditions

$$\frac{\partial U}{\partial U_{\rm old}}\frac{\partial U_{\rm old}}{\partial U} = \mathbb{1}, \tag{2.54}$$

$$\frac{\partial U_{\rm old}}{\partial U}\frac{\partial U}{\partial U_{\rm old}} = \mathbb{1}. \tag{2.55}$$

We now need to generalize these transformations to include the magnetic field. We will see that, while we are always guaranteed to be able to find a left inverse $\partial U/\partial U_{\rm old}$ for $\partial U_{\rm old}/\partial U$ as in Eq. (2.54), we will not be able to make this left inverse unique by imposing Eq. (2.54). Fortunately, we can find an alternative criterion that still gives us a quasi-invertible transformation. To see how to proceed, we will attempt to enforce Eq. (2.55) and see where it breaks down.

We start by differentiating $U_{\rm old} = (u^a, b^a)$. Since in Eqs. (2.45) and (2.46) the relations between u^a and v^a are independent of the magnetic field, Eqs. (2.51) and (2.52) are still valid. In addition, contributions to the Jacobian matrices like $\partial v^a/\partial b^b$ and $\partial u^a/\partial B^b$ are zero. To find the nonzero terms, start with Eq. (2.46) in the form

$$b^a = h^a{}_c \gamma^c{}_b B^b / W. \tag{2.56}$$

Differentiating gives

$$\frac{\partial b^a}{\partial B^b} = \frac{1}{W} h^a{}_c \gamma^c{}_b. \tag{2.57}$$

Similarly, since $\partial W/\partial v^b = W^3 v_b$, we get

$$\begin{aligned}\frac{\partial b^a}{\partial v^b} &= -W v_b h^a{}_c B^c + \frac{1}{W}\left(\frac{\partial u^a}{\partial v^b} u_c + \frac{\partial u_c}{\partial v^b} u^a\right) B^c \\ &= u^a B_b - W B^a v_b + W B^c v_c (\gamma^a{}_b + W u^a v_b),\end{aligned} \tag{2.58}$$

where we have used Eq. (2.51) and made some simple substitutions.

Now Eq. (2.54) becomes [cf. (2.49) with no magnetic field]

$$\begin{aligned}\frac{\partial(v^a, B^a)}{\partial(u^c, b^c)}\frac{\partial(u^c, b^c)}{\partial(v^b, B^b)} &= \begin{bmatrix} \frac{\partial v^a}{\partial u^c} & 0 \\ \frac{\partial B^a}{\partial u^c} & \frac{\partial B^a}{\partial b^c} \end{bmatrix} \begin{bmatrix} \frac{\partial u^c}{\partial v^b} & 0 \\ \frac{\partial b^c}{\partial v^b} & \frac{\partial b^c}{\partial B^b} \end{bmatrix} \\ &= \begin{bmatrix} \gamma^a{}_b & 0 \\ 0 & \gamma^a{}_b \end{bmatrix}.\end{aligned} \tag{2.59}$$

This gives two conditions on the magnetic field part of the transformation:

$$\frac{\partial B^a}{\partial u^c}\frac{\partial u^c}{\partial v^b}+\frac{\partial B^a}{\partial b^c}\frac{\partial b^c}{\partial v^b}=0, \tag{2.60}$$

$$\frac{\partial B^a}{\partial b^c}\frac{\partial b^c}{\partial B^b}=\gamma^a{}_b. \tag{2.61}$$

Let us first consider Eq. (2.61) and determine $\partial B^a/\partial b_c$. The most general expression for this quantity can be written in term of n^a, u^a, B^a, and the metric. Also, from Eq. (2.45) we expect the magnetic field to enter the Jacobian matrix at most linearly. So we set

$$\frac{\partial B^a}{\partial b^c}=A\delta^a{}_c+Bu^au_c+Cu^an_c+Dn^au_c+En^an_c \\ +FB^au_c+Gu^aB_c+HB^an_c+In^aB_c. \tag{2.62}$$

(Note that the scalars $A, B, \ldots$ used in this subsection are not related to the same symbols used elsewhere in the paper.) Substitute this expression and Eq. (2.57) in Eq. (2.61). Thus, find

$$\frac{\partial B^a}{\partial b^c}=W\delta^a{}_c+Bu^au_c+u^an_c+Dn^au_c+FB^au_c. \tag{2.63}$$

We will return to Eq. (2.60) in a moment. Note that the "inverse" requirement (2.55) multiplies the matrices in Eq. (2.59) in the opposite order. This gives the two conditions

$$\frac{\partial b^a}{\partial v^c}\frac{\partial v^c}{\partial u^b}+\frac{\partial b^a}{\partial B^c}\frac{\partial B^c}{\partial u^b}\stackrel{?}{=}0, \tag{2.64}$$

$$\frac{\partial b^a}{\partial B^c}\frac{\partial B^c}{\partial b^b}=h^a{}_b. \tag{2.65}$$

The question mark over the equals sign in Eq. (2.64) is to remind us that we will find that we cannot, in fact, accomplish this requirement.

Turning our attention first to Eq. (2.65) and using Eqs. (2.57) and (2.63), we find $B=F=0$ in Eq. (2.63).

Next, set

$$\frac{\partial B^a}{\partial u^c}=\bar{A}\delta^a{}_c+\bar{B}u^au_c+\bar{C}u^an_c+\bar{D}n^au_c+\bar{E}n^an_c \\ +\bar{F}B^au_c+\bar{G}u^aB_c+\bar{H}B^an_c+\bar{I}n^aB_c. \tag{2.66}$$

Substitute this expression into Eq. (2.64) and choose the coefficients to zero the expression. However, *do not try to zero the terms proportional to* u^a. You will find that it is not possible to do so and simultaneously zero all the other terms. Instead, use the corresponding coefficients to help zero the remaining terms. We will explain at the end of this subsection why this procedure works.

Proceeding in this way, we find

$$\frac{\partial B^a}{\partial u^c}=\bar{D}n^au_c+\bar{E}n^an_c+\bar{I}n^aB_c \\ +\frac{1}{W}\left[B^a(Wu_c-n_c)-(B^bu_b)(W\delta^a{}_c+u^an_c)\right]. \tag{2.67}$$

Finally, substitute the results so far into Eq. (2.60). Thus, find $D=W\bar{I}$ and $\bar{E}=0$.

We are free to set the remaining coefficients $\bar{D}$, $\bar{I}$, and D to zero so that

$$\frac{\partial}{\partial u^b}(B^an_a)=\frac{\partial}{\partial b^b}(B^an_a)=0. \tag{2.68}$$

Thus, the final form of the transformation matrices is

$$\frac{\partial B^a}{\partial u^b}=\frac{1}{W}\left[B^a(Wu_b-n_b)-(B^bu_b)(W\delta^a{}_b+u^an_b)\right], \tag{2.69}$$

$$\frac{\partial B^a}{\partial b^b}=W\delta^a{}_b+u^an_b. \tag{2.70}$$

We now explain why the above procedure works. Recall that a quasi-invertible transformation ensures that the matrices of right and left eigenvectors are inverses:

$$\mathbb{1}=L\cdot X=L_{\rm old}(A^a_{\rm old}n_a)\frac{\partial U_{\rm old}}{\partial U}\frac{\partial U}{\partial U_{\rm old}}X_{\rm old}. \tag{2.71}$$

Here we have used the transformation rules for going from the comoving to the Eulerian eigenvectors, Eqs. (5.3) and (7.13) of Paper I. Notice that the factor $(A^a_{\rm old}n_a)$ gives zero when acting on u^a. Accordingly, we can allow the right-hand side of Eq. (2.55) to differ from the identity by an off-diagonal term proportional to u^a and Eq. (2.71) will still hold. [In fact, instead of zero, the right-hand side of Eq. (2.64) turns out to be u^ab_b.] Since the inverse of a matrix is unique, we are assured that any procedure that finds the correct inverse L of the matrix X must be valid. Essentially what we have done is to require that the quasi-invertible condition (2.53) only needs to hold when projected along n_a.

2. *Eigenvalues for the Eulerian frame*

In the Eulerian frame, the vector q_a defined in Eq. (2.27) becomes

$$q_a=yn_a+s_a, \tag{2.72}$$

where s_a is a unit spatial vector in the direction of the decomposition. Here y is now the eigenvalue in the Eulerian frame.

The degenerate eigenvalues corresponding to $a = q^a u_a = 0$ have the value in the Eulerian frame

$$y = v^a s_a. \tag{2.73}$$

For the Alfvén eigenvalues, the condition $B = -\sqrt{\rho h^*} a$ gives

$$y_{\mathrm{Alf}} = v^a s_a + \frac{B^a s_a}{W^2(\sqrt{\rho h^*} + B^a v_a)}. \tag{2.74}$$

The other eigenvector has the opposite sign for the square root. The magnetosonic eigenvalues require the solution of the quartic equation (2.30) expressed in Eulerian variables.

3. Right eigenvectors in the Eulerian frame

The transformation from the comoving to the Eulerian right eigenvectors is given by the Jacobian matrix $\partial(v^a, B^a)/\partial(u^b, b^b)$. This matrix acts on the vector part of each eigenvector. The scalar part of the eigenvectors is unchanged.

Degenerate eigenvectors. Applying this transformation to the degenerate eigenvectors (2.33) and setting $q^a = yn^a + s^a$, we get

$$\mathbf{x}_{\mathrm{entropy}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_{\mathrm{constraint}} = \begin{bmatrix} 0 \\ s^a - (s^b v_b) v^a \\ 0 \\ 0 \end{bmatrix}. \tag{2.75}$$

Alfvén eigenvectors. For the Alfvén eigenvectors, we have not found a simple way of transforming the expression (2.35). Thus, we start with Eq. (2.34). The transformation of the first component of that expression consists of the following steps:

(i) The Jacobian matrix (2.52) gives

$$\frac{\partial v^a}{\partial u^e} \epsilon^e{}_{bcd} u^b b^c q^d = \frac{1}{W} \epsilon^a{}_{bcd} u^b b^c q^d + \frac{1}{W^2} u^a n_e \epsilon^e{}_{bcd} u^b b^c q^d. \tag{2.76}$$

(ii) Replace b^a and u^a by their Eulerian equivalents B^a and v^a, Eq. (2.46). Also set $q^a = y_{\mathrm{Alf}} n^a + s^a$.

(iii) Expand B^a and v^a in the orthonormal Eulerian spatial basis:

$$B^a = B_n s^a + B_1 t^a_{(1)} + B_2 t^a_{(2)},$$
$$v^a = v_n s^a + v_1 t^a_{(1)} + v_2 t^a_{(2)}. \tag{2.77}$$

The $(s^a, t^a_{(1)}, t^a_{(2)})$ components of the result are all proportional to $\epsilon_{abcd} n^a s^b t^c_{(1)}, t^d_{(2)}$, which is unity.

(iv) Substitute for y_{Alf} from Eq. (2.74). Simplify the result using $B_n v_n + B_1 v_1 + B_2 v_2 = B^a v_a$ and $v_n^2 + v_1^2 + v_2^2 = 1 - 1/W^2$.

Repeat the procedure to find the second component of the eigenvector using Eqs. (2.69) and (2.70). The result is

$$\mathbf{x}_{\mathrm{Alf}} = \begin{bmatrix} B_n(B_1 v_2 - B_2 v_1)/r_1 W^2 \\ (r_2 B_2 - r_3 v_2)/r_1 W^2 \\ (-r_2 B_1 + r_3 v_1)/r_1 W^2 \\ 0 \\ r_4 v_2 + B_2 \sqrt{\rho h^*} \\ -r_4 v_1 - B_1 \sqrt{\rho h^*} \\ 0 \\ 0 \end{bmatrix}, \tag{2.78}$$

where

$$r_1 = B^a v_a + \sqrt{\rho h^*}, \qquad r_2 = B_n v_n - r_1,$$
$$r_3 = B_n^2 + r_1 B^a v_a W^2, \qquad r_4 = B^a B_a + r_1 B^a v_a W^2. \tag{2.79}$$

In Eq. (2.78) the first six components are the $(s^a, t^a_{(1)}, t^a_{(2)})$ components of each of the vectors.

Equation (22) in Ref. [13] gives an implicit expression for $\mathbf{x}_{\mathrm{Alf}}$ that uses (B^a, v^a, s^a) as a basis instead of the orthonormal basis $(s^a, t^a_{(1)}, t^a_{(2)})$. We have computed that expression explicitly and have verified that our expression is equivalent up to the normalization, but is somewhat simpler.[3]

Magnetosonic eigenvectors. Next we consider the magnetosonic eigenvectors. We need to apply the Jacobian matrix $\partial(v^a, B^a)/\partial(u^b, b^b)$ to the expression (2.37). For the first component of Eq. (2.37), we find that

[3]There is a typo in the (2,2) term of the characteristic matrix in [13]. The term $B^i B_j$ should be $-B^i B_j$.

$$\frac{\partial v^a}{\partial u^c}(BGb^c - \rho h a^2 h^{cb} q_b) = \frac{BG(b^b n_b u^a + W b^a) - \rho h a^2 (n^b q_b u^a + W q^a)}{W^2}. \tag{2.80}$$

Replace b^a and u^a by their Eulerian equivalents B^a and v^a, Eq. (2.46). Also set $q^a = y_{\rm mag} n^a + s^a$. Thus, get

$$\frac{BGB^a - \rho h a^2 (s^a - y_{\rm mag} v^a)}{W^2}. \tag{2.81}$$

Transforming the second component of Eq. (2.37) in the same way with Eqs. (2.69) and (2.70), we get

$$\frac{\rho h a}{W}\left[B^a(GW - y_{\rm mag} a) - W^2(B - aB^b v_b)(s^a - y_{\rm mag} v^a)\right]. \tag{2.82}$$

Making the replacement $B = aB^b v_b + B^b s_b/W$ gives the final expression

$$\mathbf{x}_{\rm mag} = \begin{bmatrix} \dfrac{BGB^a - \rho h a^2 W(s^a - y_{\rm mag} v^a)}{W^2} \\ \rho h a[B^a(1 - y_{\rm mag} v_n) - B_n(s^a - y_{\rm mag} v^a)] \\ \rho h a \mathcal{G} \\ p a \mathcal{G}/(\rho c_s^2) \end{bmatrix}. \tag{2.83}$$

The above equation agrees with Eq. (25) in [13]. The authors remark, "The expressions of the eigenvectors have been obtained after tedious algebraic manipulations." And this for only the right eigenvectors, and not in terms of the conserved variables.

We can also express the vector parts of Eq. (2.83) in the $(s^a, t^a_{(1)}, t^a_{(2)})$ basis to give the result

$$\mathbf{x}_{\rm mag} = \begin{bmatrix} [BGB_n - \rho h a^2 W(1 - y_{\rm mag} v_n)]/W^2 \\ [BGB_1 + \rho h a^2 W y_{\rm mag} v_1]/W^2 \\ [BGB_2 + \rho h a^2 W y_{\rm mag} v_2]/W^2 \\ 0 \\ \rho h a[B_1(1 - y_{\rm mag} v_n) + v_1 B_n y_{\rm mag}] \\ \rho h a[B_2(1 - y_{\rm mag} v_n) + v_2 B_n y_{\rm mag}] \\ \rho h a \mathcal{G} \\ p a \mathcal{G}/(\rho c_s^2) \end{bmatrix}. \tag{2.84}$$

4. Left eigenvectors in the Eulerian frame

The transformation from comoving to Eulerian frames for the left eigenvectors is more complicated than the one for the right eigenvectors. The reason is that the transformation is not invertible, but only quasi-invertible. As shown in Paper I, the transformation is given by

$$L = L_{\rm old}(A^a_{\rm old} n_a)\frac{\partial U_{\rm old}}{\partial U} \tag{2.85}$$

[Eq. (7.13) of Paper I]. Here the matrix $\partial U_{\rm old}/\partial U$ must be chosen to satisfy the quasi-invertible condition and takes the form

$$\frac{\partial U_{\rm old}}{\partial U} = \begin{bmatrix} \dfrac{\partial u^c}{\partial v^b} & 0 & 0 & 0 \\ \dfrac{\partial b^c}{\partial v^b} & \dfrac{\partial b^c}{\partial B^b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.86}$$

Using Eqs. (2.26), (2.51), (2.57), and (2.58), we find after some simplifications

$$(A^a_{\rm old} n_a)\frac{\partial U_{\rm old}}{\partial U} = \begin{bmatrix} C^{ab} & -\dfrac{1}{\rho h^* W}[(b^c n_c)h^{ab} + W H^a b^b] & \dfrac{1}{\rho^2 h h^*}[(b^c n_c) b^a - \rho h W H^a] & 0 \\ 0 & -(\gamma^{ab} + W u^a v^b) & 0 & 0 \\ -\rho h c_s^2 W^3 v^b & 0 & -W & 0 \\ -\dfrac{p}{\rho} W^3 v^b & 0 & 0 & -W \end{bmatrix}. \tag{2.87}$$

Here

$$C^{ab} = \frac{1}{\rho h^*}\left\{[(b^c n_c)^2 - \rho h^* W^2]h^{ab} + W(b^c n_c)H^a b^b + W^3 H^a(b^2 v^b + \rho h^* n^b) + W^2(b^c n_c) b^a v^b\right\},$$
$$H^a = u^a - n^a/W. \tag{2.88}$$

Degenerate eigenvectors. Applying the transformation (2.87) to the eigenvectors (2.41) corresponding to the degenerate eigenvalues gives

$$\mathbf{l}_{\mathrm{entropy}} = \begin{bmatrix} 0 & 0 & -p/\rho & \rho h c_s^2 \end{bmatrix},$$
$$\mathbf{l}_{\mathrm{constraint}} = \begin{bmatrix} 0 & \gamma_{ab} q^b & 0 & 0 \end{bmatrix}. \tag{2.89}$$

Alfvén eigenvectors. Applying the transformation (2.87) to the Alfvén eigenvectors (2.42) is straightforward but a little lengthy. We have verified our calculations with computer algebra. The transformation consists of the following steps:

(i) Multiply the eigenvector (2.42) into the matrix (2.87) and then convert all quantities to their Eulerian counterparts using Eq. (2.46).
(ii) Substitute $q^a = y_{\mathrm{Alf}} n^a + s^a$ with y_{Alf} from Eq. (2.74).
(iii) Expand the vectors B^a and v^a when contracted into ϵ_{abcd} in the $(s^a, t^a_{(1)}, t^a_{(2)})$ basis.
(iv) Project the vector pieces of the eigenvector along $(s^a, t^a_{(1)}, t^a_{(2)})$ and simplify. A useful relation for this is

$$B^a B_a = [\rho h^* - (B^a v_a)^2] W^2 - \rho h W^2. \tag{2.90}$$

(v) Divide out a common scale factor of $\epsilon_{abcd} n^a s^b t^c_{(1)} t^d_{(2)}$.

The result is

$$\mathbf{l}_{\mathrm{Alf}} = \begin{bmatrix} l_1(B_2 v_1 - B_1 v_2) \\ l_1(B_n v_2 - B_2 v_n) + B_2(b^2 + \rho h W^2) \\ l_1(B_1 v_n - B_n v_1) - B_1(b^2 + \rho h W^2) \\ \sqrt{\rho h^*} v_n (B_1 v_2 - B_2 v_1) \\ -\sqrt{\rho h^*} B_2 (1 - v_n^2) - v_2(\sqrt{\rho h^*} B_n v_n + b^2) \\ \sqrt{\rho h^*} B_1 (1 - v_n^2) + v_1(\sqrt{\rho h^*} B_n v_n + b^2) \\ B_2 v_1 - B_1 v_2 \\ 0 \end{bmatrix}^T, \tag{2.91}$$

where

$$l_1 = B_n \sqrt{\rho h^*} - \rho h v_n W^2. \tag{2.92}$$

Magnetosonic eigenvectors. For the magnetosonic eigenvectors, we multiply Eq. (2.44) into Eq. (2.87). We use the definitions (2.29) and then convert the remaining quantities to their Eulerian counterparts using Eq. (2.46). Simplify the eigenvector using the eigenvalue equation (2.30) to replace a term $B^2 G$. The manipulations are straightforward but a little lengthy, and we have again verified the results with computer algebra. The result is

$$\mathbf{l}_{\mathrm{mag}} = \begin{bmatrix} a(B^a B_a + \rho h W^2) s_a - (S v_a + T B_a)/a \\ (a B_n + K B^b v_b) v_a + K B_a / W^2 + B_n s_a / W \\ K + \dfrac{G B B_n}{\rho h a^2} \\ 0 \end{bmatrix}^T, \tag{2.93}$$

where

$$K = a y_{\mathrm{mag}} - G W = -W(1 - v_n y_{\mathrm{mag}}),$$
$$S = B_n G W^2 B^a v_a + a K B^a B_a,$$
$$T = B_n (a^2 + G) - a K B^a v_a. \tag{2.94}$$

5. Transformation to (ρ, ϵ)

Since many numerical codes use the pair (ρ, ϵ) as primitive variables rather than (p, ϵ), we transform the Eulerian eigenvectors determined above to these variables. (The transformation procedure is similar for any other pair of thermodynamic variables.) The transformation is invertible:

$$\frac{dU_{\mathrm{old}}}{dU} = \frac{\partial(p, \epsilon)}{\partial(\rho, \epsilon)} = \begin{bmatrix} \chi & \kappa \\ 0 & 1 \end{bmatrix}, \tag{2.95}$$

$$\frac{dU}{dU_{\mathrm{old}}} = \frac{\partial(\rho, \epsilon)}{\partial(p, \epsilon)} = \begin{bmatrix} 1/\chi & -\kappa/\chi \\ 0 & 1 \end{bmatrix}. \tag{2.96}$$

The part of the transformation acting on the vector part of the eigenvectors is the identity matrix, so we have not bothered to write it explicitly. Applying Eq. (2.96) to the Eulerian right eigenvectors above, we find that $\mathbf{x}_{\mathrm{constraint}}$ and $\mathbf{x}_{\mathrm{Alf}}$ are unchanged, while

$$\mathbf{x}_{\mathrm{entropy}} = \begin{bmatrix} 0 \\ 0 \\ -\kappa \\ \chi \end{bmatrix}, \qquad \mathbf{x}_{\mathrm{mag}} = \begin{bmatrix} \dfrac{B G B^a - \rho h a^2 W (s^a - y_{\mathrm{mag}} v^a)}{W^2} \\ \rho h a \left[B^a (1 - y_{\mathrm{mag}} s^b v_b) - B^b s_b (s^a - y_{\mathrm{mag}} v^a) \right] \\ \rho a \mathcal{G} / c_s^2 \\ p a \mathcal{G} / (\rho c_s^2) \end{bmatrix}. \tag{2.97}$$

Note that we have not bothered to introduce a separate notation to distinguish (p,ϵ) eigenvectors from (ρ,ϵ) eigenvectors since these are only being used as intermediate quantities on the way to using conserved variables.

Similarly, applying Eq. (2.95) to the left eigenvectors leaves $\mathbf{l}_{\text{constraint}}$ unchanged. The entropy eigenvector becomes

$$\mathbf{l}_{\text{entropy}} = \begin{bmatrix} 0 & 0 & -\chi p/\rho & \chi\rho \end{bmatrix}. \tag{2.98}$$

Equation (2.91) for $\mathbf{l}_{\text{Alf}}$ is unchanged, except that the last two elements become

$$\begin{bmatrix} \chi(B_2 v_1 - B_1 v_2) \\ \kappa(B_2 v_1 - B_1 v_2) \end{bmatrix}^T. \tag{2.99}$$

Equation (2.93) for $\mathbf{l}_{\text{mag}}$ is unchanged, except that the last two elements become

$$\begin{bmatrix} \chi(K + GBB_n/\rho h a^2) \\ \kappa(K + GBB_n/\rho h a^2) \end{bmatrix}^T. \tag{2.100}$$

III. CONSERVED FORMULATION

In this section we introduce the conserved variables, which are almost universally used in practical codes. Our ultimate goal is to give the complete characteristic decomposition for the evolution of these variables. However, this decomposition *cannot* be found by simply transforming our previous results to these new variables. The reason is that as shown in Ref. [5], the formulation of GRMHD in terms of conservation equations is *not* simply a linear combination of the Anile evolution equations. The two formulations differ by multiples of the divergence constraint. Since the divergence constraint contains derivatives, the principal parts of the equations differ, and hence so do the eigensystems.

Reference [5] also showed that the popular "Valencia" conservation formulation of GRMHD is only weakly hyperbolic: For certain directions of the spatial normal chosen for making the characteristic decomposition, there is not a complete set of eigenvectors. This implies that the system is not well posed and is not a good candidate for numerical simulations. Accordingly, in the next section we will focus on a divergence-cleaning conservation formulation that is well posed.

In the remainder of this section we will define the usual set of conserved variables. We then give the transformation matrix between these variables and the primitive variables, and also give the inverse matrix. Next we transform the Anile eigenvectors to the conserved variables. We must emphasize again that these eigenvectors are not the eigenvectors for the evolution system of the conserved variables. We present them to make contact with previous literature, and also because the actual eigenvectors we will want in the end are very similar to these ones.

In Appendix A 1 we discuss the relation between our approach and the method of Ref. [12].

A. Conserved variables

The primitive variables in the Eulerian frame are

$$U_{\text{old}} = [v^a, B^a, \rho, \epsilon]. \tag{3.1}$$

We will want to transform from these variables to the conserved variables

$$\begin{aligned} S^a &= \rho h^* W^2 v^a - (\alpha b^0) b^a, \\ B^a &= B^a, \\ D &= \rho W, \\ \tau &= \rho h^* W^2 - p^* - (\alpha b^0)^2 - \rho W. \end{aligned} \tag{3.2}$$

Here

$$\alpha b^0 = -n_a b^a = u_a B^a = W(v_i B^i). \tag{3.3}$$

Alternative expressions in terms of the Eulerian magnetic field are

$$\begin{aligned} S^a &= (\rho h W^2 + B^b B_b) v^a - (B^b v_b) B^a, \\ \tau &= \rho h W^2 - p - \rho W + \left(1 - \frac{1}{2W^2}\right) B^a B_a - \frac{1}{2}(B^a v_a)^2. \end{aligned} \tag{3.4}$$

B. Transformation matrix for right eigenvectors

We will get the eigenvectors for the conservative formulation by transforming from a primitive variable formulation rather than deriving them directly from the equations of motion in conservation form. Using U_{old} from Eq. (3.1), and denoting the conserved variables by $U = (S^a, B^a, D, \tau)$, the transformation matrix $\partial U/\partial U_{\text{old}}$ consists of the following derivatives:

$$\frac{\partial(S^a,B^a,D,\tau)}{\partial(v^b,B^b,\rho,\epsilon)} = \begin{bmatrix} \rho h W^2(\gamma^a{}_b + 2W^2 v^a v_b) + (B^c B_c)\gamma^a{}_b - B^a B_b & 2v^a B_b - B^a v_b - (v^c B_c)\gamma^a{}_b & (1+\epsilon+\chi)W^2 v^a & (\rho+\kappa)W^2 v^a \\ 0 & \gamma^a{}_b & 0 & 0 \\ \rho W^3 v_b & 0 & W & 0 \\ \rho(2hW-1)W^3 v_b + (B^c B_c)v_b - (v^c B_c)B_b & (2-1/W^2)B_b - (v^c B_c)v_b & (1+\epsilon+\chi)W^2 - \chi - W & (\rho+\kappa)W^2 - \kappa \end{bmatrix}. \tag{3.5}$$

Here we have used the expressions (3.4), and the derivatives of ρh have been computed using $\rho h = \rho + \rho\epsilon + p$. It is convenient when using this matrix to replace $1+\epsilon$ by $h - p/\rho$ since this makes it easier to introduce the sound speed (2.12).

C. Transformation matrix for left eigenvectors

We need the inverse of the Jacobian matrix (3.5) to compute the left eigenvectors. The inverse can be found by computing the necessary partial derivatives directly. We use the idea in Refs. [12,15] of defining an intermediate variable

$$Z = \rho h W^2. \tag{3.6}$$

Write the primitive variables in terms of the conserved variables and Z:

$$\rho = \frac{D}{W}, \tag{3.7}$$

$$\epsilon = -1 + \frac{Z}{DW} - \frac{pW}{D}, \tag{3.8}$$

$$p = Z - \tau - D - \frac{(B^a S_a)^2}{2Z^2} + \left(1 - \frac{1}{2W^2}\right)B^a B_a, \tag{3.9}$$

$$v^a = \frac{S^a}{Z + B^a B_a} + \frac{B^a(B^b S_b)}{Z(Z + B^a B_a)}. \tag{3.10}$$

The last two equations follow from Eqs. (3.4).

As an example of the calculations, consider the derivatives of the primitive variables $(v^a, B^a, \rho, \epsilon)$ with respect to τ, keeping S^a, B^a, and D constant. Differentiating Eq. (3.10) and simplifying with the first equation in (3.4), we get

$$\frac{\partial v^a}{\partial\tau} = -\frac{B^a(B^b S_b) + Z^2 v^a}{Z^2(Z + B^a B_a)}\frac{\partial Z}{\partial\tau}. \tag{3.11}$$

Since $\partial W/\partial v^a = W^3 v_a$, we find

$$\frac{\partial W}{\partial\tau} = \frac{W[Z(1-W^2) - W^2(B^a v_a)^2]}{Z(Z + B^a B_a)}\frac{\partial Z}{\partial\tau}. \tag{3.12}$$

Next, Eq. (3.7) gives

$$\frac{\partial\rho}{\partial\tau} = -\frac{D}{W^2}\frac{\partial W}{\partial\tau}. \tag{3.13}$$

Equation (3.9) gives, after some simplifications,

$$\frac{\partial p}{\partial\tau} = -1 + \frac{Z + b^2}{Z + B^a B_a}\frac{\partial Z}{\partial\tau}. \tag{3.14}$$

Then Eq. (3.8) gives after simplification

$$\frac{\partial\epsilon}{\partial\tau} = \frac{W}{D} - \frac{p}{D}\frac{\partial W}{\partial\tau}. \tag{3.15}$$

The final relation we need is one for $\partial Z/\partial\tau$. We get this by using the equation of state information, $p(\rho,\epsilon)$:

$$\frac{\partial p}{\partial\tau} = \chi\frac{\partial\rho}{\partial\tau} + \kappa\frac{\partial\epsilon}{\partial\tau}. \tag{3.16}$$

Equating this expression to Eq. (3.14) and solving the resulting linear equation for $\partial Z/\partial\tau$ gives

$$\frac{\partial Z}{\partial\tau} = \frac{(\kappa+\rho)(Z + B^a B_a)}{\mathcal{D}}, \tag{3.17}$$

where

$$\mathcal{D} = \rho[b^2 - c_s^2(B^a v_a)^2 + \rho h c_s^2 + Z(1 - c_s^2)]. \tag{3.18}$$

Of course, $\partial B^a/\partial\tau = 0$, as are $\partial B^a/\partial D$ and $\partial B^a/\partial S^b$.

Using a similar procedure for the other variables, we find

$$\frac{\partial v^a}{\partial D} = -\frac{B^a(B^b S_b) + Z^2 v^a}{Z^2(Z + B^a B_a)}\frac{\partial Z}{\partial D}, \tag{3.19}$$

$$\frac{\partial W}{\partial D} = \frac{W[Z(1-W^2) - W^2(B^a v_a)^2]}{Z(Z + B^a B_a)}\frac{\partial Z}{\partial D}, \tag{3.20}$$

$$\frac{\partial\rho}{\partial D} = \frac{1}{W} - \frac{D}{W^2}\frac{\partial W}{\partial D}, \tag{3.21}$$

$$\frac{\partial p}{\partial D} = -1 + \frac{Z+b^2}{Z+B^aB_a}\frac{\partial Z}{\partial D}, \tag{3.22}$$

$$\frac{\partial \epsilon}{\partial D} = \frac{(D+p)W^2 - Z}{D^2 W} - \frac{p}{D}\frac{\partial W}{\partial D}, \tag{3.23}$$

$$\frac{\partial Z}{\partial D} = \frac{[(\kappa(W-h) + \rho h c_s^2 + \rho W](Z+B^aB_a)}{W\mathcal{D}}, \tag{3.24}$$

$$\frac{\partial v^a}{\partial S^b} = \frac{Z\gamma^a{}_b + B^aB_b}{Z(Z+B^aB_a)} - \frac{B^a(B^bS_b) + Z^2v^a}{Z^2(Z+B^aB_a)}\frac{\partial Z}{\partial S^b}, \tag{3.25}$$

$$\begin{aligned}\frac{\partial W}{\partial S^b} &= \frac{W^3[(B^av_a)B_b + Zv_b]}{Z(Z+B^aB_a)}\\ &\quad + \frac{W[Z(1-W^2) - W^2(B^av_a)^2]}{Z(Z+B^aB_a)}\frac{\partial Z}{\partial S^b},\end{aligned} \tag{3.26}$$

$$\frac{\partial \rho}{\partial S^b} = -\frac{D}{W^2}\frac{\partial W}{\partial S^b}, \tag{3.27}$$

$$\frac{\partial p}{\partial S^b} = \frac{(B^aB_a)v_b - (B^av_a)B_b}{Z+B^aB_a} + \frac{Z+b^2}{Z+B^aB_a}\frac{\partial Z}{\partial S^b}, \tag{3.28}$$

$$\frac{\partial \epsilon}{\partial S^b} = -\frac{W}{D}v_b - \frac{p}{D}\frac{\partial W}{\partial S^b}, \tag{3.29}$$

$$\begin{aligned}\frac{\partial Z}{\partial S^b} &= \{-[\kappa(Z+B^aB_a) + \rho(B^aB_a + Zc_s^2)]v_b\\ &\quad + \rho(B^av_a)(1-c_s^2)B_b\}/\mathcal{D},\end{aligned} \tag{3.30}$$

$$\begin{aligned}\frac{\partial v^a}{\partial B^b} &= \frac{Z\gamma^a{}_b(B^av_a) + B^aS_b - 2Zv^aB_b}{Z(Z+B^aB_a)}\\ &\quad - \frac{B^a(B^bS_b) + Z^2v^a}{Z^2(Z+B^aB_a)}\frac{\partial Z}{\partial B^b},\end{aligned} \tag{3.31}$$

$$\begin{aligned}\frac{\partial W}{\partial B^b} &= \frac{W^3(B^av_a)(S_b + Zv_b) - 2ZW(W^2-1)B_b}{Z(Z+B^aB_a)}\\ &\quad + \frac{W[Z(1-W^2) - W^2(B^av_a)^2]}{Z(Z+B^aB_a)}\frac{\partial Z}{\partial B^b},\end{aligned} \tag{3.32}$$

$$\frac{\partial \rho}{\partial B^b} = -\frac{D}{W^2}\frac{\partial W}{\partial B^b}, \tag{3.33}$$

$$\begin{aligned}\frac{\partial p}{\partial B^b} &= \frac{[(B^aB_a) + Z(2W^2-1) + W^2(B^av_a)^2]B_b}{W^2(Z+B^aB_a)}\\ &\quad - \frac{Z(B^av_a)v_b}{Z+B^aB_a} + \frac{Z+b^2}{Z+B^aB_a}\frac{\partial Z}{\partial B^b},\end{aligned} \tag{3.34}$$

$$\frac{\partial \epsilon}{\partial B^b} = -\frac{B_b + W^2(B^av_a)v_b}{WD} - \frac{p}{D}\frac{\partial W}{\partial B^b}, \tag{3.35}$$

$$\begin{aligned}\frac{\partial Z}{\partial B^b} &= -\{[\rho Z(2W^2 - 1 - 2(W^2-1)c_s^2)\\ &\quad + \kappa(Z+B^aB_a) + \rho(B^aB_a)]B_b\\ &\quad + W^2(B^av_a)[\kappa(Z+B^aB_a) + \rho(B^aB_a + Zc_s^2)]v_b\\ &\quad - \rho W^2(B^av_a)(1-c_s^2)S_b\}/W^2\mathcal{D},\end{aligned} \tag{3.36}$$

$$\frac{\partial B^a}{\partial B^b} = \gamma^a{}_b. \tag{3.37}$$

D. Right eigenvectors in conserved variables

We use Eq. (5.3) of Paper I with the matrix (3.5) and the eigenvectors in Sec. II B 5 to express the right eigenvectors in terms of the conserved variables. (Once again: these quantities are not the eigenvectors for the conservative evolution equations.) Using $\mathbf{R}$ to denote the right eigenvectors in conserved variables, we find

$$\mathbf{R}_{\rm entropy} = \begin{bmatrix} hW(\kappa - \rho c_s^2)v^a \\ 0 \\ \kappa \\ \kappa(hW-1) - \rho hWc_s^2 \end{bmatrix}, \qquad \mathbf{R}_{\rm Alf} = \begin{bmatrix} -2\sqrt{\rho h^*}B_{21}(B_n + r_1v_nW^2) \\ -\sqrt{\rho h^*}[B_nB_{32} + B_1B_{21} + r_1W^2(B_2 + v_1B_{21} + v_nB_{32})] \\ \sqrt{\rho h^*}[B_nB_{31} - B_2B_{21} + r_1W^2(B_1 - v_2B_{21} + v_nB_{31})] \\ 0 \\ \sqrt{\rho h^*}B_2 + v_2r_4 \\ -\sqrt{\rho h^*}B_1 - v_1r_4 \\ -\rho WB_{21} \\ -B_{21}W(2W\sqrt{\rho h^*}r_1 - \rho) \end{bmatrix},$$

$$\mathbf{R}_{\rm constraint} = \begin{bmatrix} -B^bv_bs^a + (2B_n - v_nB^bv_b)v^a - v_nB^a/W^2 \\ s^a - v_nv^a \\ 0 \\ B_n(2 - 1/W^2) - 2v_nB^av_a \end{bmatrix}, \qquad \mathbf{R}_{\rm mag} = \begin{bmatrix} m_{1\,\rm s}s^a + m_{1\rm v}v^a + m_{1\rm B}B^a \\ \rho ha[B^a(1 - y_{\rm mag}v_n) - B_n(s^a - y_{\rm mag}v^a)] \\ -\rho BGB_n/a - \rho^2ah(ya - GW) \\ m_4 \end{bmatrix}. \tag{3.38}$$

Here r_1 and r_4 are defined in Eq. (2.79), while

$$B_{21} = B_2 v_1 - B_1 v_2, \qquad B_{31} = B_n v_1 - B_1 v_n, \qquad B_{32} = B_n v_2 - B_2 v_n, \tag{3.39}$$

and

$$\begin{aligned} m_{1\mathrm{s}} &= \rho haW(BB^a v_a - \rho h^* a),\\ m_{1\mathrm{v}} &= \rho h\{B_n B^a v_a(ya + 2GW) - 2aB_n^2 - aW[ya(B^a B_a/W^2 + \rho h) + (1 - 1/c_s^2)\mathcal{G}W]\},\\ m_{1\mathrm{B}} &= \rho h[B(ya - GW) + 2B_n(a^2 + G)]/W,\\ m_4 &= (\rho/a)\{a^2 B^2 h - 2B_n^2 h(a^2 + G) + B^2 W(2yah - G) + B_n B(G + 2hWG - 2yah)\\ &\quad + \mathcal{G}Wa^2[hW(1 - c_s^2) - 1]/c_s^2 + \rho ha^3[y - 2yh^* W + a(W - h^*)]\}. \end{aligned} \tag{3.40}$$

While these expressions are somewhat complicated, they are much simpler than expected based on previous attempts. See, for example, Ref. [12] and the discussion in Appendix A 1.

E. Left eigenvectors in conserved variables

Similarly, applying the transformation matrix defined by the elements in Sec. III C to the left Anile eigenvectors of Sec. II B 4, we find

$$\mathbf{L}_{\mathrm{entropy}} = \frac{1}{\rho h c_s^2}\begin{bmatrix} Wv_b & \gamma_{ba}b^a & h - W & -W \end{bmatrix}, \qquad \mathbf{L}_{\mathrm{constraint}} = \frac{1}{1 - v_n^2}\begin{bmatrix} 0 & s_b & 0 & 0 \end{bmatrix},$$

$$\mathbf{L}_{\mathrm{Alf}} = \frac{1}{\sqrt{\rho h^*}}\begin{bmatrix} B_{21}y_{\mathrm{Alf}}\\ B_2 + B_{32}y_{\mathrm{Alf}}\\ -B_1 - B_{31}y_{\mathrm{Alf}}\\ -B_{21}y_{\mathrm{Alf}}\sqrt{\rho h^*}\\ -(B_2 + B_{32}y_{\mathrm{Alf}})\sqrt{\rho h^*}\\ (B_1 + B_{31}y_{\mathrm{Alf}})\sqrt{\rho h^*}\\ -B_{21}\\ -B_{21} \end{bmatrix}^T, \qquad \mathbf{L}_{\mathrm{mag}} = \begin{bmatrix} as_b - BGWB_b/Za + f_{1\mathrm{v}}v_b\\ Bs_b + g_{1\mathrm{B}}B_b + g_{1\mathrm{v}}v_b\\ h_1 + \mathcal{G}(\kappa - \rho c_s^2)/\rho^2 c_s^2\\ h_1 \end{bmatrix}^T, \tag{3.41}$$

where

$$f_{1\mathrm{v}} = W(-G + BGB_nW/Za^2 + \mathcal{G}W^2(\kappa + \rho)/Z\rho c_s^2), \tag{3.42}$$

$$g_{1\mathrm{B}} = \mathcal{G}\kappa W/\rho Zc_s^2 - (a^2 + G)/W, \tag{3.43}$$

$$g_{1\mathrm{v}} = B^a v_a W^2 g_{1\mathrm{B}} + W(aB + \mathcal{G}BW^2/Za), \tag{3.44}$$

$$h_1 = -(f_{1\mathrm{v}} + ya). \tag{3.45}$$

Note that entries 4–6 in $\mathbf{L}_{\mathrm{Alf}}$ are just $-\sqrt{\rho h^*}$ times entries 1–3. Note also that finding the relatively simple expressions for $\mathbf{L}_{\mathrm{mag}}$ required considerable human ingenuity, showing that sometimes such ingenuity can surpass brute force computer algebra, at least for now.

IV. CONSERVATIVE DIVERGENCE CLEANING FORMULATION

The divergence cleaning formulation of MHD is typically presented as a set of equations in flux-conservative form. To apply our methods, however, we need to start with an equivalent representation in the comoving formulation. Fortunately, Ref. [10] has done this for us. The derivation below is patterned after that work, which shows that these equations are linear combinations of the flux-conservative ones. Recall, as mentioned in Paper I, that the eigenvectors of Eq. (2.11) and the determinant in Eq. (2.12) of Paper I are invariant under forming linear combinations of the rows of the matrix $A^a q_a$.

In the divergence cleaning formulation, the Maxwell equation (2.3) is modified to become

$$\nabla_a(u^b b^a - u^a b^b - g^{ab}\phi) = -\kappa_\tau n^b \phi. \tag{4.1}$$

The constant κ_τ is the inverse of the timescale for driving the divergence constraint exponentially toward zero. Projecting this equation along u_b now gives the evolution equation for ϕ instead of the divergence constraint (2.4c):

$$u^a\nabla_a\phi + \kappa_\tau W\phi + h^{ab}\nabla_a b_b = 0. \tag{4.2}$$

Projecting Eq. (4.1) perpendicular to u_b gives the replacement for Eq. (2.24):

$$\begin{aligned}&-h_{cb}(b^a\nabla_a u^b - u^a\nabla_a b^b) + b_c\nabla_a u^a\\&\quad + h_{cb}\nabla^b\phi - \kappa_\tau h_{cb}n^b\phi = 0.\end{aligned} \tag{4.3}$$

With the divergence cleaning prescription added to the Maxwell equation, we cannot derive the energy equation corresponding to Eq. (2.17) simply by using the First Law (2.9) as we did in the nonconservative Anile formulation. Instead, we have to go back to first principles and use the conservation equation

$$u_a\nabla_b T^{ab} = 0. \tag{4.4}$$

Subtracting the modified Maxwell equation (4.1) projected along b_b from this equation and simplifying, we find

$$\begin{aligned}&\frac{1}{\chi}u^a\left\{\left(h-\frac{p}{\rho}\right)\nabla_a p - \left[\kappa\left(h-\frac{p}{\rho}\right)-\chi\rho\right]\nabla_a\epsilon\right\}\\&\quad+\rho h\nabla_a u^a - b^a(\nabla_a\phi - \kappa_\tau n_a\phi) = 0.\end{aligned} \tag{4.5}$$

Now rewrite Eq. (2.13) in terms of (p, ϵ) using Eq. (2.11):

$$\frac{1}{\chi}u^a(\nabla_a p - \kappa\nabla_a\epsilon) + \rho\nabla_a u^a = 0. \tag{4.6}$$

Use this equation to eliminate $u^a\nabla_a\epsilon$ from Eq. (4.5). Simplify and find

$$u^a\nabla_a p + \rho h c_s^2\nabla_a u^a - \frac{\kappa}{\rho}b^a(\nabla_a\phi - \kappa_\tau n_a\phi) = 0. \tag{4.7}$$

Similarly, using Eq. (4.7) to eliminate $u^a\nabla_a p$ from Eq. (4.6) and simplifying gives

$$u^a\nabla_a\epsilon + \frac{1}{\rho}(p\nabla_a u^a - b^a\nabla_a\phi + \kappa_\tau\phi b^a n_a) = 0. \tag{4.8}$$

Finally, we get the Euler equation from the projection of $\nabla_b T^{ab} = 0$ orthogonal to u^a. This gives Eq. (2.20) as before. Now, however, we cannot eliminate the problematic term $\nabla_b b^b$ using the divergence constraint as we did in Eq. (2.21). This equation is no longer valid in the presence of the scalar field ϕ, being replaced by Eq. (4.1). So instead we write

$$\nabla_a b^a = (h^{ab} - u^a u^b)\nabla_a b_b = h^{ab}\nabla_a b_b + u^a b^b\nabla_a u_b. \tag{4.9}$$

It is convenient not to introduce another term containing $u^a\nabla_a u_b$, so consider

$$\begin{aligned}0 &= b^a\nabla_b T_a{}^b\\&= b^a\nabla_a p + (\rho h + b^2)b^a u^b\nabla_b u_a - b^2\nabla_a b^a\\&= b^a\nabla_a p + \rho h\nabla_a b^a - (\rho h + b^2)h^{ab}\nabla_b b_a,\end{aligned} \tag{4.10}$$

where we have used Eq. (4.9) to get the last equation. We now eliminate $\nabla_a b^a$ from Eq. (2.20) using this relation and find

$$\begin{aligned}&\rho h^* h^d{}_b u^a\nabla_a u^b + h^{da}\nabla_a p\\&+ b^a\left[h^d{}_b(\nabla^b b_a - \nabla_a b^b) + \frac{b^d}{\rho h}\nabla_a p\right] - \frac{h^*}{h}b^d h^{ab}\nabla_b b_a = 0.\end{aligned} \tag{4.11}$$

This equation is identical to Eq. (2.23) except for the last term, which is proportional to the divergence constraint.

To enforce the condition $u^a u_a = -1$, in the above equations we again use the identity (2.25). Then Eqs. (4.11), (4.3), (4.7), (4.8), and (4.2) can be written as a quasilinear system [Eq. (2.9) of Paper I] for the variables $(u^a, b^a, p, \epsilon, \phi)$, where the matrices A^a are

$$A^a = \begin{bmatrix} u^a h_c{}^b & (b^b h_c{}^a - b^a h_c{}^b)/(\rho h^*) - b_c h^{ba}/(\rho h) & h_c{}^a/(\rho h^*) + b_c b^a/(\rho^2 h h^*) & 0 & 0\\ -b^a h_c{}^b + b_c h^{ba} & h_c{}^b u^a & 0 & 0 & h_c{}^a\\ \rho h c_s^2 h^{ba} & 0 & u^a & 0 & -\kappa b^a/\rho\\ (p/\rho)h^{ba} & 0 & 0 & u^a & -b^a/\rho\\ 0 & h^{ba} & 0 & 0 & u^a \end{bmatrix}. \tag{4.12}$$

A. Characteristic speeds

The characteristic speeds are the zeros of the determinant of the characteristic matrix $A^a q_a$. The determinant is evaluated in Appendix B 2, and we find up to a constant factor

$$\det(A^a q_a) = aG(B^2 - \rho h^* a^2)\{a^2 G b^2 - \rho h a^4 + c_s^2[\rho h a^2(a^2 + G) - B^2 G]\}. \tag{4.13}$$

Comparing with Eq. (2.28), we see that the Alfvén and magnetosonic speeds are the same as in the nonconservative formulation. However, the entropy eigenvalue ($a = 0$) is no longer degenerate with the constraint eigenvalue. The constraint eigenvalue is replaced by two eigenvalues, reflecting the increase from an eight- to a nine-dimensional system. We call these the scalar eigenvalues since they arise from introducing the scalar field. They are the roots of $G = q^a q_a = 1 - y^2 = 0$. So $y_{\mathrm{scalar}} = \pm 1$, which are the wave speeds for any choice of unit timelike and normal vectors to define q^a.

B. Comoving eigensystem

1. Right eigenvectors in the comoving frame

The right eigenvectors follow from solving $(A^a q_a)X = 0$. Take X to be of the form (2.31) with an extra component X_9 at the end. Then the analog of Eqs. (2.32) is

$$aX_c + \frac{1}{\rho h^*}\left(b^b Y_b h_{ac} q^a - B Y_c + q^a h_{ac} X_7 + \frac{B b_c}{\rho h} X_7 - \frac{h^* b_c}{h} q^b Y_b\right) = 0, \tag{4.14a}$$

$$B X_c - b_c q^a X_a - a Y_c - h_{ac} q^a X_9 = 0, \tag{4.14b}$$

$$\rho h c_s^2 q^a X_a + a X_7 - \frac{\kappa B}{\rho} X_9 = 0, \tag{4.14c}$$

$$\frac{p}{\rho} q^a X_a + a X_8 - \frac{B}{\rho} X_9 = 0, \tag{4.14d}$$

$$q^a Y_a + a X_9 = 0. \tag{4.14e}$$

These equations differ from (2.32) in the extra last term in the first four equations as well as the new fifth equation.

Entropy, Alfvén, and magnetosonic eigenvectors. The (nondegenerate) entropy eigenvector corresponding to $a = 0$, as well as the Alfvén and magnetosonic vectors, are the same as the expressions given in Sec. II A 2 with a zero appended. This follows since Eq. (4.14e) has a solution $X_9 = q^a Y_a = 0$. Then Eqs. (4.14) reduce to Eqs. (2.32) and have the same solutions found there, since those solutions satisfy $q^a Y_a = 0$.

Scalar eigenvectors. It remains to find the two scalar eigenvectors corresponding to the eigenvalues given by $G = 1 - y^2 = 0$. Since the equations are linearly dependent when the eigenvalue is inserted, it is easy to get $0 = 0$. One way of proceeding is as follows:

(i) Use Eq. (4.14e) to eliminate X_9 from the other equations.
(ii) Use Eq. (4.14c) to eliminate X_7 from Eq. (4.14a).
(iii) Dot Eq. (4.14a) with b^c and solve the resulting equation for $q^c Y_c$.
(iv) Dot Eq. (4.14a) with q^c, use the expression for $q^c Y_c$ from the previous step, and solve for $b^c Y_c$.
(v) Also use the expression for $q^c Y_c$ in Eq. (4.14b) and solve the equation for Y_c in terms of X_c, $q^c X_c$, and $b^c X_c$.
(vi) In Eq. (4.14a), make the substitutions for $q^c Y_c$, $b^c Y_c$, and Y_c. Dot with b^c and solve for $b^c X_c$. Substitute this back in the equation and solve for X_c. An overall factor of $q^c X_c$ is an arbitrary normalization. Choose it to remove the denominator of X_c to find

$$X_c = \kappa_\rho B h_{cb} q^b + \rho a^2 (1 - c_s^2) b_c, \tag{4.15}$$

where

$$\kappa_\rho \equiv \kappa + \rho c_s^2. \tag{4.16}$$

(vii) Use this expression to backsolve for the other quantities.

The final expression for the eigenvector is

$$X_{\mathrm{scalar}} = \begin{bmatrix} \kappa_\rho B h_{ab} q^b + \rho a^2(1 - c_s^2) b_a \\ -\kappa_\rho a B b_a + h_{ab} q^b[\kappa_\rho B^2/a + \rho^2 h a (1 - c_s^2)] \\ -\kappa_\rho a B \rho h \\ -aB[p + \rho(h - \chi)] \\ -\rho^2 h a^2 (1 - c_s^2) \end{bmatrix}. \tag{4.17}$$

This agrees with Eq. (36) in [10] when expressed in a spatial basis in the comoving frame.

2. Left eigenvectors in the comoving frame

The equations $L(A^a q_a) = 0$ for the left eigenvectors are very similar to the case of the Anile formulation treated in Sec. II A 3. There is an additional scalar L_9 in the equation corresponding to (2.39). The equations are the same as those in Eq. (2.40) except that there is an additional term in Eq. (2.40b) of the form

$$h^* h_{ca} q^a (L^b b_b / h - \rho L_9), \tag{4.18}$$

and there is a new fifth equation

$$B(\kappa L_7 + L_8) - \rho a L_9 - \rho Y^b q_b = 0. \tag{4.19}$$

Entropy eigenvector. The entropy eigenvector corresponding to $a = 0$ is again the only one with $L_8 \neq 0$. Equation (2.40c) again implies $L_b = 0$, and solving the remaining equations gives

$$L_{\mathrm{entropy}} = \begin{bmatrix} 0 & h_{ab}q^b\chi B/G & -p/\rho & \rho h c_s^2 & 0 \end{bmatrix}. \tag{4.20}$$

Scalar eigenvectors. For the other eigenvectors, we set $L_8 = 0$ and use the remaining two scalar equations to eliminate L_7 and L_9 from the two vector equations. For the scalar eigenvector ($G = 0$), the Anile case (2.41) suggests the ansatz $L_b = 0$. We then find

$$L_{\mathrm{scalar}} = \begin{bmatrix} 0 & h_{ab}q^b & 0 & 0 & -a \end{bmatrix}. \tag{4.21}$$

Alfvén eigenvectors. Similarly, for the Alfvén eigenvector we try the ansatz that L_b and Y_b are both orthogonal to q^a and b^a. Then we find that for $B = -\sqrt{\rho h^*}a$ the eigenvector is the same as Eq. (2.42) with a zero appended:

$$L_{\mathrm{Alf}} = \begin{bmatrix} \sqrt{\rho h^*}\epsilon_{abcd}u^b b^c q^d & -\epsilon_{abcd}u^b b^c q^d & 0 & 0 & 0 \end{bmatrix}. \tag{4.22}$$

The other eigenvector has no minus sign and the appropriate change in q^d. There is an alternative representation in the form of Eq. (2.43) with a zero appended.

Magnetosonic eigenvectors. For the magnetosonic eigenvectors, since Eq. (2.40) is the same in the divergence cleaning case, we make the ansatz that the corresponding quantity L_b is the same too, namely the first element in Eq. (2.44). We look for Y_b as a linear combination of b_b and $h_{ba}q^a$ and find a solution very similar to Eq. (2.44). The differences are shown boxed in the following equation:

$$L_{\mathrm{mag}} = \begin{bmatrix} B(a^2+G)b_a/a - \rho h^* a h_{ab}q^b \\ (a^2+G)b_a - B h_{ab}q^b\left[1 - \boxed{\kappa_\rho \mathcal{G}_\rho/(\rho G)}\right] \\ \mathcal{G}_\rho \\ 0 \\ -\boxed{\mathcal{G}_\rho B[c_s^2/a + \kappa_\rho a/(\rho G)]} \end{bmatrix}^T, \tag{4.23}$$

where

$$\mathcal{G}_\rho = \mathcal{G}/\rho h c_s^2. \tag{4.24}$$

C. Eigenvectors in the Eulerian frame

1. Right eigenvectors in the Eulerian frame

As for the nonconservative formulation, the transformation from the comoving to the Eulerian right eigenvectors is given by the Jacobian matrix $\partial(v^a, B^a)/\partial(u^b, b^b)$. This matrix acts on the vector part of each eigenvector. The scalar part of the eigenvectors is unchanged. Since all the comoving right eigenvectors in the divergence cleaning formulation are the same as for the nonconservative formulation (with a zero appended) except X_{scalar}, the eigenvectors in the Eulerian frame are the same as Eqs. (2.75), (2.78), and (2.84) with a zero appended. The scalar eigenvector is the only one that is different. There are two eigenvectors corresponding to the roots $y = \pm 1$ of $G = 1 - y^2 = 0$:

$$\mathbf{x}_{\mathrm{scalar}} = \begin{bmatrix} [\kappa_\rho BWF^a + (1-c_s^2)\rho a^2 B^a]/W^2 \\ n^b q_b \kappa_\rho B B^a/W + [\kappa_\rho B B_n + (1-c_s^2)\rho^2 h a^2 W]F^a/a \\ -\kappa_\rho \rho h B a \\ -[p + \rho(h-\chi)]Ba \\ -\rho^2 h a^2(1-c_s^2) \end{bmatrix}, \tag{4.25}$$

where $F^a = \gamma^{ab}q_b + (n^b q_b)v^a$. Note that $F^a = s^a - y v^a$ and $n^b q_b = -y$ with $y = \pm 1$, but we will not make these substitutions yet.

Applying the transformation (2.96) to Eq. (4.25) to change the thermodynamic variables from (p, ϵ) to (ρ, ϵ), we find that only the third entry changes:

$$-\kappa_\rho \rho h B a \to -(\kappa + \rho)\rho B a. \tag{4.26}$$

Note that the authors of Ref. [10] were not able to find a simple expression for $\mathbf{x}_{\mathrm{scalar}}$ using their dual frames approach.

2. Left eigenvectors in the Eulerian frame

The transformation (2.87) for the divergence cleaning case has some additional terms. First, expand (2.87) to a 5×5 matrix by padding it with zeros. Then add the following terms:

$$\begin{bmatrix} 0 & Wb^a v^b/(\rho h) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -WH^a \\ 0 & 0 & 0 & 0 & -\kappa(b^a n_a)/\rho \\ 0 & 0 & 0 & 0 & -(b^a n_a)/\rho \\ 0 & -Wv^b & 0 & 0 & -W \end{bmatrix}. \tag{4.27}$$

Applying this transformation to the left eigenvectors in Sec. IV B 2 and changing from (p,ϵ) to (ρ,ϵ), we find

$$\mathbf{l}_{\mathrm{entropy}} = \left[0 \quad B\gamma_{ab}q^b/G \quad -pW/\rho \quad \rho hWc_s^2 \quad b^a n_a - Bq^a n_a/G\right], \tag{4.28}$$

$$\mathbf{l}_{\mathrm{scalar}} = \left[0 \quad \gamma_{ab}q^b \quad 0 \quad 0 \quad -q^a n_a\right]. \tag{4.29}$$

Since the comoving Alfvén eigenvector (4.22) is essentially the same as Eq. (2.42), we find that the transformation to the Eulerian frame gives the same form as Eq. (2.91) for the first eight components. The change to (ρ,ϵ) transforms the seventh and eighth components to Eq. (2.99). Only the ninth (last) component is new, and it takes the simple form

$$-(B_2v_1 - B_1v_2). \tag{4.30}$$

The magnetosonic eigenvectors turn out to be the same as Eq. (2.93) except for some small changes shown boxed below.

$$\mathbf{l}_{\mathrm{mag}} = \begin{bmatrix} a(B^aB_a+\rho hW^2)s_a-(Sv_a+TB_a)/a \\ (aB_n+KB^bv_b)v_a+\dfrac{KB_a}{W^2}+\left(\dfrac{B_n}{W}\boxed{-\dfrac{\mathcal{G}_\rho\kappa_\rho B}{G\rho}}\right)s_a \\ K+\dfrac{GBB_n}{\rho ha^2} \\ 0 \\ \boxed{\dfrac{BK(G\rho-\mathcal{G}_\rho\kappa_\rho)+GB_n[\rho(a^2+G)-\mathcal{G}_\rho\kappa]}{G\rho a}} \end{bmatrix}^T. \tag{4.31}$$

Transforming from (p,ϵ) to (ρ,ϵ) changes the third and fourth terms to those shown in Eq. (2.100).

D. Eigenvectors for conserved variables

1. Right eigenvectors for conserved variables

As mentioned already in Sec. IV C 1, with the exception of the scalar eigenvectors, the right eigenvectors are the same as for the Anile case with a zero appended. Since the transformation matrix (3.5) is the same, the eigenvectors for the conserved variables are the same as in Eq. (3.38), with the exception of $\mathbf{R}_{\mathrm{constraint}}$, which gets replaced by the scalar eigenvectors.

The scalar eigenvectors transform to

$$\mathbf{R}_{\mathrm{scalar}} = \begin{bmatrix} -\dfrac{1}{aW}[(y\kappa_B + 2\kappa_\rho aBB_n)B^a + W^2B\kappa_B(s^a+yv^a) - 2W\kappa_\rho BB_n^2v^a] \\ \kappa_\rho yBB^a/W + (s^a - yv^a)[\kappa_\rho BB_n + (1-c_s^2)\rho^2a^2hW]/a \\ \kappa_\rho y\rho B - (1-c_s^2)\rho aB_n \\ \{\kappa_\rho B[2B_n^2 + \rho a(ah^* - y)] - \kappa_B B - 2\kappa_{Bv}yW + (1-c_s^2)\rho^2a^2B_n\}/a \\ -(1-c_s^2)\rho^2a^2h \end{bmatrix}, \tag{4.32}$$

with $y = \pm 1$. Here

$$\begin{aligned} \kappa_B &= \kappa_\rho B^2 + (1-c_s^2)\rho^2a^2h, \\ \kappa_{Bv} &= \kappa_B B^a v_a - \kappa_\rho\rho aBh^*. \end{aligned} \tag{4.33}$$

2. Left eigenvectors for conserved variables

For the divergence cleaning case, the field ϕ is both a primitive and a conservative variable. Thus, the inverse transformation matrix derived in Sec. III C simply gets an extra row and column of zeros, with a 1 in the lower-right corner. Applying the transformation to the left eigenvectors given in Sec. IV C 2, we get

$$\mathbf{L}_{\mathrm{entropy}} = \frac{1}{\rho hc_s^2}\begin{bmatrix} Wv_b \\ \gamma_{ba}b^a - (B_n/GW)s_b \\ h - W \\ -W \\ WB^av_a - B_nv_n/GW \end{bmatrix}^T, \tag{4.34}$$

$$\mathbf{L}_{\mathrm{scalar}} = \frac{1}{1-v_n^2}\begin{bmatrix} 0 \\ s_b \\ 0 \\ 0 \\ y_{\mathrm{scalar}} \end{bmatrix}^T. \tag{4.35}$$

Since $\mathbf{l}_{\mathrm{Alf}}$ is the same as for the Anile case except for the additional term (4.30), the eigenvector $\mathbf{L}_{\mathrm{Alf}}$ for the conserved variables is the same as in Eq. (3.41) with the additional element $-B_{21}\sqrt{\rho h^*}$ appended.

The magnetosonic eigenvectors are almost identical to those in Eq. (3.41). The only differences are exactly the two additional terms shown boxed in Eq. (4.31). The first additional term gets added to the second element in Eq. (3.41), while the second additional term becomes the new fifth element.

E. Eigenvectors for the Valencia formulation

It is perhaps surprising that the eigenvalues and eigenvectors for the divergence cleaning formulation are so similar to those of the Anile formulation. Note that the Valencia formulation does not follow from setting the scalar field ϕ to zero in the divergence cleaning system. Nevertheless, the eigenvalues again are almost all identical to those in Eq. (2.28). However, Ref. [5] shows that even in the comoving frame, the eigenvectors differ significantly from those given in Sec. II A. The expected symptoms of weak hyperbolicity, such as lack of convergence, have not been reported in simulations using the Valencia formulation. This could be because of the difficulties in showing strict convergence in simulations containing magnetic field instabilities or shocks. Many numerical implementations are able to use constrained transport [16–18], which guarantees conservation of $\nabla \cdot \mathbf{B}$ to machine precision. If the constraint were identically satisfied, this could explain why the system is actually strongly hyperbolic in the constraint-satisfying subspace. Then simulations would have to be at machine precision to see signs of weak hyperbolicity. This speculation is obviously difficult to demonstrate in practice.

Since the Valencia formulation is weakly hyperbolic and not all simulation schemes allow constrained transport, including relativistic discontinuous Galerkin schemes, we have not worked out its decomposition here. However, the methods of this paper could be used to find the decomposition in terms of the conserved variables used in simulations.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have presented a straightforward method to derive the characteristic decomposition for GRMHD. The resulting eigenvectors are simple enough to be used in numerical simulations. This simplicity is somewhat surprising given the complexity of previous attempts.

The key innovation that makes these results possible is the introduction of a quasi-invertible transformation, as described in Sec. II B. This transformation allows us to transform between the variables $\{u^a, b^a\}$ in the comoving frame and the variables $\{v^a, B^a\}$ in the Eulerian frame in such a way that the correct eigenvectors are obtained. Together with the new transformation law for left eigenvectors (2.85), the quasi-invertible transformation allows the solution of the problem.

We have first derived the decomposition in terms of conserved variables for the nonconserved formulation of Anile. This decomposition is not intended to be used in simulations because this formulation is not equivalent to the flux-conservative formulation used in practice [5]. We give this decomposition solely to make contact with previous work, and because the results are very close to the formulation that uses a divergence-cleaning scalar field with the flux-conservative form.

We are currently implementing this decomposition in the SPECTRE code [19] and will test its performance with a full-wave Riemann solver such as the Marquina scheme [20,21].

An important future step is to study the impact of degeneracies on the decomposition. Degeneracies have been carefully studied and circumvented in the comoving [3] formulation and also in the work of Ref. [12]. However, such a treatment will have to be redone for the conservative case treated here and will be the subject of a future paper. Degeneracies only occur for very specific orientations of the magnetic field and so hopefully at only a small number of grid points in a realistic simulation. In this case, as a temporary measure we can get the eigenvectors numerically from the characteristic matrix itself.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: COMPARISON WITH PREVIOUS METHODS

1. The method of Antón *et al.*

In Ref. [12], Antón *et al.* found the characteristic decomposition for MHD in special relativity by a transformation method. They start with the covariant Anile eigenvectors for the ten variables $U_{\rm cov} = (u^a, b^a, p, s)$. The particular choice of the two thermodynamic variables is not relevant for the discussion here. They use the same set of eight conserved variables $U_{\rm cons}$ that we use, defined in Sec. III. They derive the eigensystem along the x-axis, so they can eliminate B^x since it satisfies the trivial evolution equation $\partial B^x/\partial t = 0$, leaving seven variables. They then compute the 7×10 transformation matrix $\partial U_{\rm cons}/\partial U_{\rm cov}$ for the right eigenvectors by direct differentiation, using the constraints (2.6) and the 0-component of Eq. (2.3), which is the divergence constraint.

For the inverse transformation, needed for the left eigenvectors, they introduce an intermediate set of seven primitive variables $V = (u^x, u^y, u^z, b^y, b^z, p, \rho)$. Here, the vector quantities are a subset of the covariant variables that are easy to work with in special relativity. The inverse transformation is then essentially computed by using the matrix product

$$\frac{\partial U_{\rm cov}}{\partial V}\frac{\partial V}{\partial U_{\rm cons}}. \tag{A1}$$

For this procedure to work, we require that the 10×7 matrix defined by the product (A1) serve as the matrix inverse of $\partial U_{\rm cons}/\partial U_{\rm cov}$. But as we have seen, a rectangular matrix may only have a one-sided inverse in general. We have not checked whether the matrices given in [12] function as inverses, since the calculation seems daunting with the information given. However, even if the method is consistent, it does not seem possible to apply it in general relativity. Here the relation between Eulerian and comoving frames is more complicated, which is why we introduced the idea of quasi-invertibility.

More important, as mentioned earlier, the Anile equations do not correspond [5] to the conservation equations, which are those typically used in numerical simulations. Simply transforming the Anile equations to conserved variables does not lead to the correct decomposition.

Finally, the transformations in Ref. [12] are extremely complicated, leading to eigenvectors covering several journal pages. For example, their equations (163)–(166) correspond to our short expression for $\mathbf{L}_{\rm entropy}$ in Eq. (3.41). Their equations contain derivatives of the quantity Z, which are evaluated only for the case of an ideal fluid in the thesis [15]. We have evaluated these derivatives for a general equation of state in Sec. III C. We have substituted our expressions into their equations (163)–(166) and managed to simplify the equations. After many cancellations, two of their equations reduce to the first and fourth elements of $\mathbf{L}_{\rm entropy}$. We have not tried to track down the origin of the differences in the other two elements, since our methods lead to much simpler expressions as it is. Note that three of the elements in $\mathbf{L}_{\rm entropy}$ are the same as in hydrodynamics—the $\mathbf{B}$ field cancels out.

2. The method of Schoepe *et al.*

Schoepe *et al.* [5] introduced an algorithm for characteristic decomposition in relativity that relies on a dual frames approach. The two frames in the algorithm are again the comoving and the Eulerian frames of the fluid. The formalism is derived from the dual foliation formalism [23], but because the comoving frame does not correspond to a foliation, the name was changed to dual frames. Note that this treatment is not related to the earlier use of "dual frames" in Ref. [24], which forms the basis of the SPEC numerical relativity code [25].

In Refs. [5,10], the dual-frame formalism is used very successfully to address the hyperbolicity of various formulations of relativistic fluid equations. In particular, they show the surprising result that the most commonly used formulation of GRMHD is only weakly hyperbolic, and so does not have a well-posed initial value problem.

For numerical work, however, there are several drawbacks to the dual-frames approach. First, the spatial basis in the Eulerian frame is treated in a nonstandard way to simplify the algebra, and one has to translate those results to the standard Eulerian coordinate basis. Also, from the point of view of numerical work, nonstandard choices of primitive variables are made. These are relatively minor drawbacks. From our perspective, more serious is that the computations are quite complicated and require computer algebra essentially throughout, especially for GRMHD. Thus, extending the dual-frames results from the decomposition for the primitive variables to that of the conserved variables in a useful form for numerical work is a somewhat daunting task that remains to be carried out. By contrast, the quasi-invertible transformation technique introduced here leads to much simpler calculations and allows the complete determination of the GRMHD decomposition in term of conserved variables, as shown in this paper.

APPENDIX B: EVALUATING THE CHARACTERISTIC DETERMINANT

1. Nonconservative formulation

In the usual comoving approach in the literature, one typically deals with a 10×10 characteristic determinant corresponding to the variables (u^a, b^a) and two thermodynamic variables. The standard form of the determinant is given in [3], with the remark "Now it is easy to show that…," and the value of the determinant is written down without derivation. Subsequent authors have also just quoted the answer, either repeating Anile's remark or without comment. The determinant was actually first evaluated in 1966 by Bruhat [1], with a derivation.

In our case, the characteristic matrix has rank eight since we have explicitly enforced the two constraints on u^a and b^a. Equation (2.26) gives

$$A^a q_a = \begin{bmatrix} u^a q_a h_c{}^b & q_a(b^b h_c{}^a - b^a h_c{}^b)/(\rho h^*) & q_a[h_c{}^a/(\rho h^*) + b_c b^a/(\rho^2 h h^*)] & 0 \\ q_a(-b^a h_c{}^b + b_c h^{ba}) & h_c{}^b u^a q_a & 0 & 0 \\ \rho h c_s^2 h^{ba} q_a & 0 & u^a q_a & 0 \\ (p/\rho) h^{ba} q_a & 0 & 0 & u^a q_a \end{bmatrix}. \tag{B1}$$

Formally, this is a 10×10 matrix with rank eight, so we must be careful in evaluating the determinant. Since the determinant is a scalar, we can evaluate it in the local comoving frame of the fluid, where $h^b{}_c \to \delta^i{}_j$, a 3×3 matrix. Using the definitions (2.29), we get

$$\det(A^a q_a) = a \det \begin{bmatrix} a\delta^i{}_j & (-B\delta^i{}_j + b^i q_j)/(\rho h^*) & (Bb_j + \rho h q_j)/(\rho^2 h h^*) \\ -B\delta^i{}_j + q^i b_j & a\delta^i{}_j & 0 \\ \rho h c_s^2 q^i & 0 & a \end{bmatrix}. \tag{B2}$$

We evaluate the determinant using the formula for the determinant of a partitioned matrix:

$$\det \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \det S \det(P - QS^{-1}R). \tag{B3}$$

In our case,

$$\begin{aligned} P &= a\delta^i{}_j, \\ Q &= \left[(-B\delta^i{}_j + b^i q_j)/(\rho h^*) \quad (Bb_j + \rho h q_j)/(\rho^2 h h^*) \right], \\ R &= \begin{bmatrix} -B\delta^i{}_k + q^i b_k \\ \rho h c_s^2 q^i \end{bmatrix}, \\ S &= \begin{bmatrix} a\delta^k{}_i & 0 \\ 0 & a \end{bmatrix}. \end{aligned} \tag{B4}$$

Here we have arranged the tensor indices to facilitate the matrix multiplications in Eq. (B3). We find

$$\begin{aligned} &QS^{-1} \\ &= \left[(-B\delta^k{}_j + b^k q_j)/(\rho h^\star a) \quad (Bb_j + \rho h q_j)/(\rho^2 h h^\star a) \right] \end{aligned} \tag{B5}$$

and

$$\begin{aligned} &QS^{-1}R = (B^2\delta^i{}_j + w^i q_j + z^i b_j)/(\rho h^* a), \\ &\quad w^i \equiv Bb^i - (b^2 + \rho h c_s^2) q^i, \qquad z^i \equiv (1 - c_s^2) B q^i, \end{aligned} \tag{B6}$$

and hence

$$\det(A^a q_a) = a^5 \det(T\delta^i{}_j + w^i q_j + z^i b_j)/(\rho h^* a)^3, \tag{B7}$$

where

$$T \equiv -(B^2 - \rho h^* a^2). \tag{B8}$$

The remaining determinant in Eq. (B7) is given by the identity

$$\begin{aligned} &\det[T\delta^i{}_j + w^i q_j + z^i b_j] \\ &\quad = T[T^2 + T(w^i q_i + z^i b_i) + w^i q_i z^j b_j - w^i b_i z^j q_j], \end{aligned} \tag{B9}$$

which we prove below. Now we have

$$\begin{aligned} w^i q_i &= B^2 - q^i q_i (b^2 + \rho h c_s^2), \\ z^j b_j &= (1 - c_s^2) B^2, \\ w^i b_i &= -\rho h c_s^2 B, \\ z^j q_j &= (1 - c_s^2) B q^j q_j. \end{aligned} \tag{B10}$$

In these equations, we can go back from the local rest frame of the fluid to a general frame by the replacement

$$q^i q_i \to h_{ab} q^a q^b = G + a^2. \tag{B11}$$

Putting all the pieces together, Eq. (B7) becomes

$$\begin{aligned} \det(A^a q_a) = a^2 (B^2 - \rho h^* a^2) \{ &a^2 G b^2 - \rho h a^4 \\ &+ c_s^2 [\rho h (a^2 + G) - B^2 G] \} / (\rho h^*)^2. \end{aligned} \tag{B12}$$

Dropping the constant denominator gives Eq. (2.28) of the main text.

To prove the identity (B9), which is a generalization of the matrix determinant lemma, note that the determinant is a scalar. The only scalars that can be formed out of T, w^i, z^i, q_i, and b_i are T, $w^i q_i$, $w^i b_i$, $z^i q_i$, and $z^i b_i$. (Quantities like $w^i z_i$ cannot appear because the determinant only multiplies and adds entries together; it cannot raise or lower indices.) Each term in the determinant consists of three elements multiplied together. These must be three T's, or two T's and a pair of vector elements like w^i and q_j, or one T and a set of four elements from distinct vectors. No elements from the same vector can be repeated, since the determinant uses only one element from each row or column at a time. This is also the reason that none of the allowed scalar products can appear squared. Thus, the form of the determinant must be

$$\begin{aligned} T\big[&a_0 T^2 + T(a_1 w^i q_i + a_2 z^i b_i + a_3 w^i b_i + a_4 z^j q_j) \\ &+ a_5 w^i q_i z^j b_j + a_6 w^i b_i z^j q_j \big]. \end{aligned} \tag{B13}$$

The only way that a term like q_i can appear is multiplied by a w^j, since the determinant selects these terms together. Similarly b_i must appear together with z^j. Thus, $a_3 = a_4 = 0$. Now choose $w^i = (w^1, w^2, 0)$, $z^i = (z^1, z^2, 0)$, $b_j = (b_1, b_2, 0)$, and $q_j = (q_1, 0, 0)$ and evaluate the determinant to determine $a_0, a_1, \ldots$. This gives the result (B9).

2. Divergence cleaning formulation

Equation (4.12) gives in the comoving frame

$$\det(A^a q_a) = \det \begin{bmatrix} a\delta^i{}_j & (-B\delta^i{}_j + b^i q_j)/(\rho h^*) - q^i b_j/(\rho h) & (Bb_j + \rho h q_j)/(\rho^2 h h^*) & 0 & 0 \\ -B\delta^i{}_j + q^i b_j & a\delta^i{}_j & 0 & 0 & q_j \\ \rho h c_s^2 q^i & 0 & a & 0 & -\kappa B/\rho \\ (p/\rho) q^i & 0 & 0 & a & -B/\rho \\ 0 & q^i & 0 & 0 & a \end{bmatrix}. \tag{B14}$$

Multiply the last column by q^i/a and subtract from the second column to eliminate q^i in the bottom row. This gives

$$\det(A^a q_a) = a^2 \det \begin{bmatrix} a\delta^i{}_j & (-B\delta^i{}_j + b^i q_j)/(\rho h^*) - q^i b_j/(\rho h) & (Bb_j + \rho h q_j)/(\rho^2 h h^*) \\ -B\delta^i{}_j + q^i b_j & a\delta^i{}_j - q^i q_j/a & 0 \\ \rho h c_s^2 q^i & \kappa B q^i/(\rho a) & a \end{bmatrix}. \tag{B15}$$

Following the same procedure as we used to evaluate Eq. (B2), we see that Eq. (B4) is replaced by

$$\begin{aligned} &P = a\delta^i{}_j, \qquad Q = \left[\, (-B\delta^i{}_j + b^i q_j)/(\rho h^*) - q^i b_j/(\rho h) \quad (Bb_j + \rho h q_j)/(\rho^2 h h^*) \,\right], \\ &R = \begin{bmatrix} -B\delta^i{}_k + q^i b_k \\ \rho h c_s^2 q^i \end{bmatrix}, \qquad S = \begin{bmatrix} a\delta^k{}_i - q^k q_i/a & 0 \\ \kappa B q^k/(\rho a) & a \end{bmatrix}. \end{aligned} \tag{B16}$$

Using the matrix determinant lemma, we find that

$$\begin{aligned} \det(S) &= a \det(a\delta^k{}_i - q^k q_i/a) = a^2\theta, \\ \theta &\equiv a^2 - q_i q^i. \end{aligned} \tag{B17}$$

Using, for example, partitioned matrix techniques again, we can compute

$$S^{-1} = \frac{1}{a\theta} \begin{bmatrix} \theta\delta^i{}_j + q^i q_j & 0 \\ -\kappa B q^i/\rho & \theta \end{bmatrix}. \tag{B18}$$

So

$$QS^{-1} = \begin{bmatrix} -\rho h[\kappa B q^k q_j + \rho\theta(B\delta^k{}_j - b^k q_j) + (\kappa B^2 + \rho^2 h^* a^2) q^k b_j]/(\rho^3 h h^* \theta a) \\ (Bb_j + \rho h q_j)/(\rho^2 h h^* a) \end{bmatrix}^T. \tag{B19}$$

We then find that $QS^{-1}R$ is given by the same expression as Eq. (B6), and hence the only difference from Eq. (B7) is that $\det(S) = a^5$ is replaced by $a^2 \det(S) = a^4\theta$. Since $\theta = a^2 - q_i q^i = -G$ by Eq. (B11), we get Eq. (B12) but with the initial factor of a^2 replaced by aG. This gives Eq. (4.13) of the main text.

[1] Y. Bruhat, Etude des equations des fluides chargés relativistes inductifs et conducteurs, Commun. Math. Phys. **3**, 334 (1966).
[2] A. M. Anile and S. Pennisi, On the mathematical structure of test relativistic magnetofluiddynamics, Ann. de l'I.H.P. Phys. théor. **46**, 27 (1987), available at http://www.numdam.org/item/AIHPA_1987__46_1_27_0.
[3] A. M. Anile, *Relativistic Fluids and Magneto-Fluids: With Applications in Astrophysics and Plasma Physics* (Cambridge University Press, New York, 1989).
[4] M. H. P. M. van Putten, Maxwell's equations in divergence form for general media with applications to MHD, Commun. Math. Phys. **141**, 63 (1991).
[5] A. Schoepe, D. Hilditch, and M. Bugner, Revisiting hyperbolicity of relativistic fluids, Phys. Rev. D **97**, 123009 (2018).
[6] J. A. Font, Numerical hydrodynamics and magnetohydrodynamics in general relativity, Living Rev. Relativity **11**, 7 (2008).
[7] P. Mösta, B. C. Mundim, J. A. Faber, R. Haas, S. C. Noble, T. Bode, F. Löffler, C. D. Ott, C. Reisswig, and E. Schnetter, GRHydro: A new open-source general-relativistic magnetohydrodynamics code for the Einstein toolkit, Classical Quantum Gravity **31**, 015005 (2014).
[8] S. L. Liebling, L. Lehner, D. Neilsen, and C. Palenzuela, Evolutions of magnetized and rotating neutron stars, Phys. Rev. D **81**, 124023 (2010).
[9] A. J. Penner, General relativistic magnetohydrodynamic Bondi-Hoyle accretion, Mon. Not. R. Astron. Soc. **414**, 1467 (2011).
[10] D. Hilditch and A. Schoepe, Hyperbolicity of divergence cleaning and vector potential formulations of general relativistic magnetohydrodynamics, Phys. Rev. D **99**, 104034 (2019).
[11] S. A. Teukolsky, preceding article, Characteristic decomposition for relativistic numerical simulations: I. Hydrodynamics, Phys. Rev. D **113**, 124015 (2026).
[12] L. Antón, J. A. Miralles, J. M. Martí, J. M. Ibáñez, M. A. Aloy, and P. Mimica, Relativistic magnetohydrodynamics: Renormalized eigenvectors and full wave decomposition Riemann solver, Astrophys. J. Suppl. Ser. **188**, 1 (2010).
[13] J.-M. Ibáñez, I. Cordero-Carrión, M.-Á. Aloy, J.-M. Martí, and J.-A. Miralles, On the convexity of relativistic ideal magnetohydrodynamics, Classical Quantum Gravity **32**, 095007 (2015).
[14] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[15] L. Antón, Magnetohidrodinámica relativista numérica: Aplicaciones en relatividad especial y general, Ph.D. thesis, Universitat de València, 2008, (in Spanish).
[16] C. R. Evans and J. F. Hawley, Simulation of magnetohydrodynamic flows: A constrained transport model, Astrophys. J. **332**, 659 (1988).
[17] D. S. Balsara and D. S. Spicer, A staggered mesh algorithm using high order Godunov fluxes to ensure solenoidal magnetic fields in magnetohydrodynamic simulations, J. Comput. Phys. **149**, 270 (1999).
[18] G. Tóth, The $\nabla \cdot B = 0$ constraint in shock-capturing magnetohydrodynamics codes, J. Comput. Phys. **161**, 605 (2000).
[19] N. Deppe, W. Throwe, L. E. Kidder, N. L. Vu, K. C. Nelli, C. Armaza, M. S. Bonilla, F. Hébert, Y. Kim, P. Kumar, G. Lovelace, A. Macedo, J. Moxon, E. O'Shea, H. P. Pfeiffer, M. A. Scheel, S. A. Teukolsky, N. A. Wittek *et al.*, `SpECTRE v2025.08.19`, 10.5281/zenodo.16906840 (2025).
[20] R. Donat and A. Marquina, Capturing shock reflections: An improved flux formula, J. Comput. Phys. **125**, 42 (1996).
[21] M. A. Aloy, J. M. Ibáñez, J. M. Martí, and E. Müller, GENESIS: A high-resolution code for three-dimensional relativistic hydrodynamics, Astrophys. J. Suppl. Ser. **122**, 151 (1999).
[22] J. M. Martin-Garcia, xact: Efficient tensor computer algebra, https://josmar493.dreamhosters.com (2025).
[23] D. Hilditch, Dual foliation formulations of general relativity, arXiv:1509.02071.
[24] M. A. Scheel, H. P. Pfeiffer, L. Lindblom, L. E. Kidder, O. Rinne, and S. A. Teukolsky, Solving Einstein's equations with dual coordinate frames, Phys. Rev. D **74**, 104006 (2006).
[25] http://www.black-holes.org/SpEC.html.