Operator lift of the Reshetikhin-Turaev formalism to Khovanov-Rozansky topological quantum field theory

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(Received 13 October 2025; accepted 5 January 2026; published 26 January 2026)

Topological quantum field theory (TQFT) is a powerful tool to describe homologies, which normally involve complexes and a variety of maps/morphisms, what makes a functional integration approach with a sum over a single kind of map seemingly problematic. In TQFT this problem is overcome by exploiting the rich set of zero modes of BRST operators, which appear sufficient to describe complexes. We explain what this approach looks like for the important class of Khovanov-Rozansky (KR) cohomologies, which categorify the observables (Wilson lines or knot polynomials) in 3D Chern-Simons theory. We develop a construction of odd differential operators, associated with all link diagrams, including tangles with open ends. These operators become nilpotent only for a diagram with no external legs, but even for open tangles one can develop a factorization formalism, which preserves Reidemeister/topological invariance—the symmetry of the problem. This technique seems much more "physical" than the conventional language of homological algebra and should have many applications to various problems beyond Chern-Simons theory. We also hope that this language will provide efficient algorithms, and finally allow one to computerize the calculation of KR cohomologies—for closed diagrams and for open tangles.

DOI: 10.1103/dhrr-wjgb

I. INTRODUCTION

The Khovanov-Rozansky (KR) theory [1–5] is known to "categorify," or refine or generalize a theory of HOMFLY polynomial invariants [6] of knots and links. Physically HOMFLY polynomials are treated as averages of knotted Wilson loop operators in the 3d Chern-Simons (CS) theory with gauge group $SU(n)$ calculated with the help of the quantum group theory, a so-called Reshetikhin-Turaev (RT) formalism [7,8]. It is expected that the CS theory is a string field theory [9] for topological strings, or M-theory in more general terms. The never fading interest to knot/link theory categorification in the physicist community [10–22] is provided by expectations to catch an echo of the high energy M theory in this deformation.

In this brief note we would like to initiate an attempt to translate the Khovanov-Rozansky theory from the homological algebra language to more pedestrian terms of ordinary differential operators more often used in the physicist community. We believe these new terms would have better algorithmizability properties and would allow one to calculate KR invariants of links on machines with higher efficiency.

In short, to a given link diagram L (possibly a tangle diagram with open outgoing and incoming legs) we would associate a differential operator $\mathscr{O}_L$ acting on functions of even (bosonic) and odd (fermionic, Grassmann) variables following the rules:

(1) We associate with the crossing (valence-2,2) diagrams two odd operator $\mathscr{O}_v^{\pm}$, depending on two pairs of x and $\bar{x}$ variables on incoming and outgoing lines, respectively. There is also a triplet of additional odd variables that are associated with the vertex itself we will denote by Greek letters θ, η, and ϵ. Operators $\mathscr{O}_v^{\pm}$ are ordinary differential operators in the odd variables one could rewrite in terms of elementary ones: we denote by $\hat{\theta}$ an operator of multiplication by odd variable θ, and by $\hat{\theta}^{\dagger}$ an operator of differentiation with respect to θ.

The label $\pm$ refers to two different orientations, and there is also a pair of conjugate odd parameters ϵ that allow one to distinguish between orientations. All the operators $\mathscr{O}_v^{\pm}$ anticommute at different vertices and in this sense are *local*.

(2) The squares $\mathscr{O}^2 \sim \sum_{i=1}^2 (w(x_i) - w(\bar{x}_i))$ with some function (superpotential) w. In the simplest case of

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the fundamental representation of $\mathfrak{sl}_n$ all x are numbers and we choose $w(x) = x^{n+1}$. A choice of a generic polynomial would lead to so called *equivariant* KR cohomology [23,24].

(3) To consider generic link diagrams (tangles) L drawn inside a disk, we have to take into account the disk boundary. In the 3D CS theory such a diagram would correspond to a collection of possibly open Wilson lines in a ball, and on the ball boundary we would put necessarily Wess-Zumino-Witten theory to preserve topological invariance. Here we treat the disk boundary as a multivalence vertex and associate to it a generic operator $\mathscr{O}_\partial(x_l, \bar{x}_l)$ depending only on x_l and $\bar{x}_l$ for all incoming and outgoing legs. This operator is constrained solely to the following natural conditions:

$$\{\mathscr{O}_\partial, \mathscr{O}_v^\pm\} = 0, \quad \forall\ v, \quad \mathscr{O}_\partial^2 = \sum_l (-w(x_l) + w(\bar{x}_l)), \tag{1.1}$$

as for the "boundary vertex" outgoing legs are inflowing edges and vice versa. Then with arbitrary link diagram (tangle) L one can associate

$$\mathscr{O}_L(x_e, \hat{\theta}_v, \hat{\eta}_v, \hat{\epsilon}_v, \hat{\theta}_v^\dagger, \hat{\eta}_v^\dagger, \hat{\epsilon}_v^\dagger | x_l, \bar{x}_l) = \sum_{\text{vertices of } L} \mathscr{O}_v^\pm + \mathscr{O}_\partial, \tag{1.2}$$

which depends on x_e on the internal edges of L (each x_e appears as incoming for one $\mathscr{M}_v^\pm$ and outgoing for another), on triplets of θ_v, η_v, ϵ_v and their conjugates at vertices, and also on x_l and $\bar{x}_l$ for the incoming and outgoing legs. For the square we have,

$$\mathscr{O}_L^2 = 0. \tag{1.3}$$

(4) What are invariants of topological diagram moves are "invariant wave functions'—zero modes of $\mathscr{O}_L$ modulo "gauge transform" g_ω,

$$\mathscr{O}_L \Psi_L = 0, \qquad g_\omega(\Psi_L) = \Psi_L + \mathscr{O}_L \omega. \tag{1.4}$$

Equivalently we say that wave functions are cohomologies $H^*(\mathscr{O}_L)$ of $\mathscr{O}_L$. Since $\mathscr{O}_L$ is the differential operator with respect to the odd variables only, this is a simple equation, solved by methods of linear algebra. Products of R matrices are obtained by an equally straightforward factorization of Ψ_L modulo $\mathscr{O}_L$. This whole procedure can be regarded as a generalization of KR cohomology construction—a necessary one if we want to deal with tangles.

(5) For a closed diagram $\bar{L}$ the boundary is empty $\mathscr{O}_\partial = 0$ and one can consider cohomologies of $\mathscr{O}_{\bar{L}}$ without boundary contribution. They can be related to the more familiar cohomologies of the KR *bi* complex (or double complex), associated to the collection $\{\Gamma\}$ of Murakami-Ohtsuki-Yamada (MOY) resolutions [25] of L. That construction is actually two step, with the first evaluation the "vertical cohomologies" H_Γ^* for a given MOY resolution Γ of L, and then those of the induced "horizontal" complex, made from these H_Γ^* for different Γ. The corresponding Poincare polynomial $P_{\bar{L}}$ is known to generalize the fundamental HOMFLY polynomial, obtained from RT calculus.

We believe this reformulation of the KR theory would make it easier to compare with other physical Morse/Floer approaches to categorified $\mathfrak{su}_n$ link invariants originating from direct compactifications of M theory [26–34]. It is always easy to reformulate the problem of finding cohomologies of operator $\mathscr{O}_L$ as a search for ground state wave functions of a Laplacian-like Hamiltonian [35][1]:

$$\mathcal{H}_L = \frac{1}{2}(\mathscr{O}_L \mathscr{O}_L^\dagger + \mathscr{O}_L^\dagger \mathscr{O}_L),$$
$$H^*(\mathscr{O}_L) \coloneqq \frac{\mathrm{Ker}\, \mathscr{O}_L}{\mathrm{Im}\, \mathscr{O}_L} \cong \{\Psi | \mathcal{H}_L \Psi = 0\}. \tag{1.5}$$

There is almost no chance that $\mathcal{H}_L$ would be somewhat reminiscent of a Hamiltonian H_M in M theory. Nevertheless, one might expect in fact those Hamiltonians are connected by a RG flow trajectory in the space of theories preserving zero modes. In this case an equivalence of the two approaches would be transparent. In the modern literature a rigorous proof of this equivalence or observing explicit discrepancies remains an open question.

We should stress these approaches have rather different spirits. In the KR story operators $\mathscr{O}_L$ are constructed from local mutually anticommuting quantities $\mathscr{O}_v^\pm$ corresponding to diagram vertices, whereas on the other side the Morse instanton flow could connect diagram "colorings" (wave functions) differing nonlocally. However, on the other side, Morse/Floer M theory approaches are usually designed to make defect operators corresponding to link manifestly topological, so the invariance with respect to homotopies of L is apparent, whereas for the KR approach this invariance is a deep theorem.

Also one could look at this reformulation of the KR theory in terms of (1.5) as a Hamiltonian evolution setup of KR foam TQFTs [36–41]. Yet in comparison to those picturesque approaches we believe the language of

[1]Here operator $\mathscr{O}_L^\dagger$ corresponds to $\mathscr{O}_L$ conjugated with respect to some nondegenerate norm on polynomials of even and odd variables. Seemingly the simplest choice would be to promote variables x_k to Heisenberg operators $\hat{x}_k = x_k \cdot$ so that trivial polynomial 1 corresponds to vacuum $|0\rangle$, $\langle 0|0\rangle = 1$. In this representation $\hat{x}_k^\dagger = \partial_{x_k}$.

differential equations could be more efficient in computational applications.

For the sake of brevity we will omit many peculiarities to be considered elsewhere. Here we only take a glimpse on the following questions:

(i) HOMFLY polynomials are known as Jones polynomials at $n = 2$. Respectively, the KR theory is expected to reduce directly to Khovanov's categorification of Jones polynomials [42–44]. Could this reduction be made more manifest in terms of operators $\mathscr{O}$?

(ii) Decategorification. The ways to show directly that operators $\mathscr{O}_L$ are decategorified directly to $U_q(\mathfrak{sl}_n)$ R matrices and how constrained the inverse process is.

(iii) Krasner [45] proposed a drastic simplification of computing KR complex cohomologies for bipartite knots [46–48]. Could we observe this simplification in terms of operators $\mathscr{O}$?

The paper is organized as follows. In Sec. II we review the structure of matrix factorization as a differential operator. In Sec. III we give a brief review of the KR double complex structure. In Sec. IV we discuss a construction of vertical morphisms, and in Sec. V a construction of horizontal morphisms. In Sec. VI we present a construction of operators $\mathscr{O}_L$ and in Sec. VII we indicate that they define link invariants. Finally, Sec. VIII is devoted to examples and miscellaneous questions.

II. MATRIX FACTORIZATIONS (MF)

Matrix factorization (MF) is a correction to the supercharge of 2D $\mathcal{N} = (2,2)$ supersymmetric topological string theory due to a worldsheet boundary [49–56]. See also Appendix A.

For us matrix factorization $\mathscr{M}$ is a differential operator acting on a ring $\mathcal{R}$ of even, bosonic variables x_k and odd, fermionic variables θ_i (by straight brackets $[*,*]$ we imply commutators, and by curly brackets $\{,*\}$ we imply anticommutators):

$$[x_k, x_l] = 0, \qquad [x_k, \theta_i] = 0, \qquad \{\theta_i, \theta_j\} = 0. \tag{2.1}$$

Let us denote operators of multiplication by θ_i as $\hat{\theta}_i \coloneqq \theta_i\cdot$ and operators of differentiation with respect to θ_i as $\hat{\theta}_i^\dagger \coloneqq \partial/\partial\theta_i$. These operators commute with x_k and form the odd Heisenberg algebra,

$$\{\hat{\theta}_i, \hat{\theta}_j\} = \{\hat{\theta}_i^\dagger, \hat{\theta}_j^\dagger\} = 0, \qquad \{\hat{\theta}_i, \hat{\theta}_j^\dagger\} = \delta_{ij}. \tag{2.2}$$

We define a MF operator $\mathscr{M}$ as an arbitrary differential operator, linear in $\hat{\theta}_i$, $\hat{\theta}_i^\dagger$,

$$\mathscr{M} = \sum_i A_i\hat{\theta}_i + B_i\hat{\theta}_i^\dagger, \tag{2.3}$$

where A_i, B_i are arbitrary polynomials in even variables x_k.

It is easy to derive the basic property of the MF operator,

$$\mathcal{M}^2 = \left(\sum_i A_i B_i\right) \cdot \mathbb{1}. \tag{2.4}$$

This property of MF explains its name, as a polynomial in the rhs might not factorize in a pair of polynomials, yet it decomposes into a sum and factorizes as a square of a linear operator. Surely, there are generalizations when A_i and B_i are matrices of polynomials rather than just vectors [49], however we will not need this prescription here.

Defined this way MF operator is equivalent to the following MF [1]:

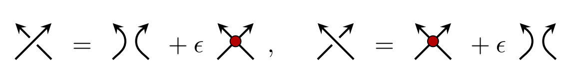
$$\mathscr{M} \cong \bigotimes_i \left(\mathbb{Q}[x_1, x_2, \ldots] \underset{B_i}{\overset{A_i}{\rightleftarrows}} \mathbb{Q}[x_1, x_2, \ldots] \right). \tag{2.5}$$

III. KR DOUBLE COMPLEX

Construction of [1] can be rewritten in the proposed language of differential operators in the following way.

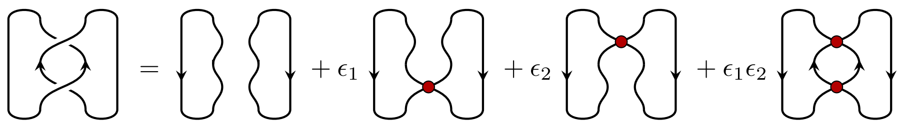
(1) The starting point is a link diagram L with two types of valence-4 vertices depicting resolved intersections of link strands after projecting to a plane. A resolved intersection vertex saves the information of which link strand went above and which one went below before projection. If a diagram has no external edges, we call is *closed* and denoted by $\bar{L}$.

(2) Substitute each intersection with a formal sum of implanted resolutions,



(3.1)

where each fermionic odd variable ϵ is unique for each intersection. For example,



(3.2)

(3) Graphs consisting of oriented edges and four-valent vertices, resolved as introduced above, will be called *MOY diagrams* in analogy with [25]. We will denote resolutions by letter Γ,

$$L = \sum_{\alpha}^{2^{\#(\text{vertices})}} \Gamma_\alpha. \tag{3.3}$$

(4) There is a basic parameter n—a number of colors in categorified $U_q(\mathfrak{su}_n)$.

Here we should stress that so far rules were applicable both to the RT and the KR formalisms. This indicates that the HOMFLY polynomial constructed with the RT formalism is an Euler characteristic for the KR complex and its construction procedure is similar. Further we list points specific for the KR construction.

(5) To each MOY diagram Γ we assign a ring $\mathcal{R}_\epsilon$ of variables x_k, θ_i mixed with odd variables ϵ_α. After that we can associate with Γ an operator $\mathscr{M}_\Gamma$ of degree

$$d = \text{qdeg}\mathscr{M} = n+1. \tag{3.4}$$

It depends on even variables x_e at all edges as a polynomial and is a differential operator in terms of θ_v associated to all vertices (see details in Sec. IV).

(6) For closed diagrams $\bar{\Gamma}$ operators $\bar{\mathscr{M}}_{\bar{\Gamma}} \coloneqq \mathscr{M}_{\bar{\Gamma}}$ are nilpotent, $\bar{\mathscr{M}}_{\bar{\Gamma}}^2 = 0$, i.e., become *differentials*

(7) To *closed* link diagram $\bar{L}$ we assign a KR double complex $(\mathcal{R}_\epsilon^{*,*}, \bar{\mathscr{M}}, \mathfrak{D})$. Cohomologies are link invariants.[2]

(8) KR complex is a double complex ($d = n+1$),

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r] & \mathcal{R}_\epsilon^{*,*}(\bar{\Gamma}_1) \arrow[r, "\mathfrak{D}"] \arrow[d, "\bar{\mathscr{M}}_{\bar{\Gamma}_1}"] & \mathcal{R}_\epsilon^{*,*+1}(\bar{\Gamma}_2) \arrow[r, "\mathfrak{D}"] \arrow[d, "\bar{\mathscr{M}}_{\bar{\Gamma}_2}"] & \mathcal{R}_\epsilon^{*,*+2}(\bar{\Gamma}_3) \arrow[r] \arrow[d, "\bar{\mathscr{M}}_{\bar{\Gamma}_3}"] & \cdots \\
\cdots \arrow[r] & \mathcal{R}_\epsilon^{*+d,*}(\bar{\Gamma}_1) \arrow[r, "\mathfrak{D}"] \arrow[d] & \mathcal{R}_\epsilon^{*+d,*+1}(\bar{\Gamma}_2) \arrow[r, "\mathfrak{D}"] \arrow[d] & \mathcal{R}_\epsilon^{*+d,*+2}(\bar{\Gamma}_3) \arrow[r] \arrow[d] & \cdots \\
 & \cdots & \cdots & \cdots &
\end{tikzcd} \tag{3.5}$$

This double complex is *equivalent* to a double complex on a ring $\mathcal{R}_\epsilon$ with two kinds of differentials $\bar{\mathscr{M}}_{\bar{\Gamma}}$ and $\mathfrak{D}$.

(9) The vertical degree qdeg is fixed for x_k and θ_i elements of each $\mathcal{R}$,

$$\text{qdeg}x_k = 2, \quad \text{qdeg}\theta_i = 1-n \;\text{ or }\; 3-n. \tag{3.6}$$

(10) The horizontal degree is defined by the powers of ϵ, $\text{qdeg}\epsilon_i = -2$ or 0, $\text{tdeg}\epsilon_i = 1$.

(11) Horizontal differential reads (see details in Sec. V),

$$\mathfrak{D} = \sum_{\alpha \in \text{intersections}} \chi^{(\alpha)}\epsilon_\alpha \cdot, \qquad \text{qdeg}\mathfrak{D} = 0, \tag{3.7}$$

where $\chi^{(\alpha)}$ are local MF homomorphisms,

$$[\chi^{(\alpha)}, \chi^{(\beta)}] = 0, \qquad \chi^{(\alpha)}\bar{\mathscr{M}}_{\bar{\Gamma}_j} = \bar{\mathscr{M}}_{\bar{\Gamma}_{j+1}}\chi^{(\alpha)}. \tag{3.8}$$

Apparently, $\mathfrak{D}^2 = 0$.

(12) To calculate cohomologies of a double complex one descends first on vertical cohomologies using (3.8) and then considers a new induced monocomplex,

[2] In principle we are able to extend this construction to an open link diagram L by considering a new *variable* diagram G such that together with L they form a closed diagram $\overline{L \cup G}$. The resulting double complex $\overline{L \cup G}$ depends explicitly on G, yet some conclusions, homotopies of the double complex, could be drawn for G staying arbitrary. For instance, Reidemeister moves would produce double complex homotopies of this type.

$$\cdots \longrightarrow H^{z,f}\left(\overline{\mathscr{M}}_{\bar{\Gamma}_1}\right) \xrightarrow{\tilde{\mathfrak{D}}} H^{z+1,f+1}\left(\overline{\mathscr{M}}_{\bar{\Gamma}_2}\right) \xrightarrow{\tilde{\mathfrak{D}}} H^{z+2,f+2}\left(\overline{\mathscr{M}}_{\bar{\Gamma}_3}\right) \longrightarrow \cdots \tag{3.9}$$

(13) A Khovanov-Rozansky categorified HOMFLY polynomial is a Poincaré polynomial of the double complex,

$$P(q,t) = \sum_{z,f} q^z t^f \dim H^{z,f}(\bar{\mathscr{M}}, \tilde{\mathscr{D}}). \tag{3.10}$$

The HOMFLY polynomial corresponds to $P(q,-1)$. In Morse/Floer approaches (see, e.g., [29]) to the Khovanov-Rozansky construction grading z corresponds to the central charge of the superalgebra, and grading f corresponds to the fermion number operator in the respective TQFT.

IV. VERTICAL MORPHISMS

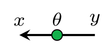
A. Constructing MF from MOY

A MOY diagram [25] is a graph consisting of oriented edges ↑ and (2,2)-valent vertices. Edges correspond to fundamental or antifundamental factors in tensor power objects in the RT formalism [7,8]. And the four-valent vertices correspond to projectors to the antisymmetric representation. We will use both equivalent notations when a four-valent vertex is short and when it is stretched to an edge of different color: $=$.

As an addition to the canonical MOY construction we would like to allow one to add *anywhere* on any edge simple two-valent vertices. These two-valent vertices might seem redundant at first sight, moreover MOY equivalence (4.13a), which we will discuss in what follows allows one to eliminate or add such a type of a vertex freely. Nevertheless, they turn out to be relevant for uniformizing ring representations of MOY diagrams to make horizontal morphisms explicitly local (see Sec. V).

We construct a ring $\mathcal{R}$ as a polynomial ring from distinct *even* variables assigned to *edges* and *odd* variables assigned to *vertices* of a MOY graph. See Appendix A. To any segment of an oriented edge $\circ\xrightarrow{x_k}\circ$ between two vertices of any type we assign the respective even variable x_k.

To a two-valent vertex between two segments x and y we assign an odd variable θ and the respective *local* MF operator,



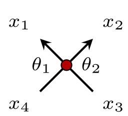
$$\Longrightarrow \quad \mathscr{M}_{x,y} = \pi_{xy}\hat{\theta} + (x-y)\hat{\theta}^\dagger, \tag{4.1}$$

$$\mathscr{M}^2_{x,y} = x^{n+1} - y^{n+1}, \quad \text{here } \mathrm{qdeg}\theta = -\mathrm{qdeg}\theta^\dagger = 1-n, \tag{4.2}$$

From the factorization condition (2.4) we easily find polynomial π_{xy},

$$\pi_{xy} = \frac{x^{n+1} - y^{n+1}}{x-y}. \tag{4.3}$$

To a four-valent vertex we assign two odd variables θ_1, θ_2 and the following *local* MF operator:



$$\Longrightarrow \quad \mathscr{M}_{4\mathrm{v}}\begin{bmatrix} x_1 & x_2 \\ x_4 & x_3 \end{bmatrix} = u_1\hat{\theta}_1 + (x_1+x_2-x_3-x_4)\hat{\theta}_1^\dagger + u_2\hat{\theta}_2 + (x_1x_2 - x_3x_4)\hat{\theta}_2^\dagger \tag{4.4}$$

$$\mathscr{M}^2_{4\mathrm{v}} = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}, \qquad \text{here } \mathrm{qdeg}\theta_1 = 1-n, \qquad \mathrm{qdeg}\theta_2 = 3-n. \tag{4.5}$$

Necessary expressions for $u_{1,2}$ could be easily derived following polynomial factorization procedure discussed in Appendix B 2. Note that neither $\frac{x_1^{n+1}+x_2^{n+1}-x_3^{n+1}-x_4^{n+1}}{x_1+x_2-x_3-x_4}$ nor $\frac{x_1^{n+1}+x_2^{n+1}-x_3^{n+1}-x_4^{n+1}}{x_1x_2-x_3x_4}$ are polynomials, therefore MF in this case is a more sophisticated construction and involves two terms in (2.4)—and thus two Grassmann parameters $\theta_{1,2}$. For concreteness, we recall explicit solutions derived in [1],

$$u_1(x_1,x_2,x_3,x_4) = \frac{g(x_1+x_2, x_1x_2) - g(x_3+x_4, x_1x_2)}{x_1+x_2-x_3-x_4},$$
$$u_2(x_1,x_2,x_3,x_4) = \frac{g(x_3+x_4, x_1x_2) - g(x_3+x_4, x_3x_4)}{x_1x_2-x_3x_4},$$
$$g(p,q) = \left(\frac{1}{2}(p-\sqrt{p^2-4q})\right)^{n+1} + \left(\frac{1}{2}(p+\sqrt{p^2-4q})\right)^{n+1}. \tag{4.6}$$

Quite often we will omit denoting fermion variables on diagrams and two-valent vertices as well if there is no abuse of notations. If we label an edge with two or more different even variables this implies that there are two-valent vertices in between them.

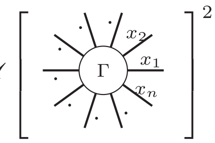
The total MF operator is constructed as a sum of all local MF operators associated to two-valent and four-valent vertices,

$$\mathscr{M}_\Gamma = \sum_{v\,\in\,\text{vertices}} \mathscr{M}_v. \tag{4.7}$$

As MF operators for different vertices depend on sets of differential operators $\hat{\theta}$ and $\hat{\theta}^\dagger$ assigned solely to a particular vertex, and therefore independent across vertices, we conclude that for distinct vertices v and v' MF operators anticommute,

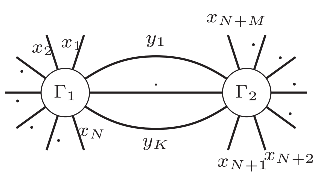
$$\{\mathscr{M}_v, \mathscr{M}_{v'}\} = 0. \tag{4.8}$$

Then for a generic tangle T with open ends labeled by x_k we find,

$\mathscr{M}$ 

$$= \sum_{v\in\text{vertices}} \mathscr{M}_v^2 = \sum_k s_k x_k^{n+1}\,, \tag{4.9}$$

where a sign s_k is $+1$ if an edge k is flowing outward Γ and -1 if it flows inward Γ. It is simple to prove this relation inductively. Apparently, it is correct for basic two-valent (4.2) and four-valent (4.5) vertices. Now as a induction step let us consider two arbitrary diagrams Γ_1 and Γ_2 for which (4.9) is satisfied and join them into a diagram $\Gamma_1 \cup \Gamma_2$ by edges labeled by y_i. Those edges that remain disconnected we will mark by variables x_i,



,

$$\mathscr{M}^2_{\Gamma_1} = \sum_{i=1}^{N} s_i x_i^{n+1} + \sum_{i=1}^{K} \sigma_i y_i^{n+1}, \quad \mathscr{M}^2_{\Gamma_2} = \sum_{i=N+1}^{M} s_i x_i^{n+1} - \sum_{i=1}^{K} \sigma_i y_i^{n+1}\,,$$
$$\mathscr{M}^2_{\Gamma_1\cup\Gamma_2} = \mathscr{M}^2_{\Gamma_1} + \mathscr{M}^2_{\Gamma_2} = \sum_{i=1}^{N+M} s_i x_i^{n+1}\,. \tag{4.10}$$

The signs in front of y terms for L_1 and L_2 are opposite since if for L_1 this is an incoming edge then for L_2 this is an outgoing edge and vice versa. And, in general, contributions of internal edges cancel each other for the MF operator squared.

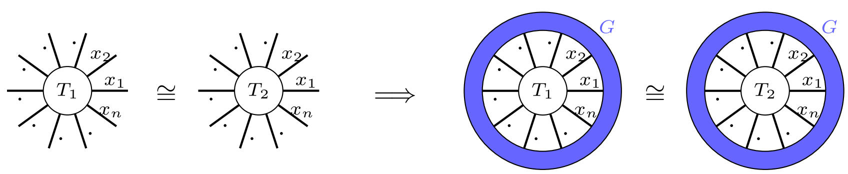
It is easy to see that for any *closed* MOY diagram $\bar{\Gamma}$ without external edges

$$\bar{\mathscr{M}}^2_{\bar{\Gamma}} = 0, \tag{4.11}$$

so that the vertical morphisms in (3.5) are differentials indeed, and the double complex makes sense.

B. MOY local equivalences

We would like to consider in what follows pictorial equivalences between generic MOY graphs with open legs at the boundary $T_1 \cong T_2$. Precisely, by such an equivalence we imply that two tangles can be implanted in an *arbitrary* graph G, so that both $T_1 + G$ and $T_2 + G$ are closed MOY diagrams $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. This prescription is equivalent to saying that external legs are connected to a multivalent vertex G at infinity. We say that $T_1 \cong T_2$ when respective vertical cohomologies are isomorphic for any choice of G,

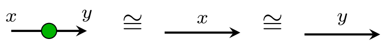


$, \quad \forall G\,,$

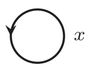
$$\mathscr{M}_{\bar{\Gamma}_i} = \mathscr{M}_{T_i}(x_1,\ldots,x_n) + \mathscr{M}_G(x_1,\ldots,x_n), \quad \{\mathscr{M}_G, \mathscr{M}_{T_i}\} = 0, \quad -\mathscr{M}_G^2 = \mathscr{M}_{T_1}^2 = \mathscr{M}_{T_2}^2\,,$$
$$H^*\left(\mathscr{M}_{\bar{\Gamma}_1}\right) \cong H^*\left(\mathscr{M}_{\bar{\Gamma}_2}\right), \quad \forall G\,. \tag{4.12}$$

Naturally, we will use direct sum and tensor product signs for pictures to indicate that respective cohomologies decompose as vector spaces also for arbitrary G.

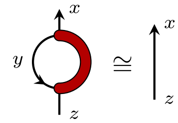
MF operators $\mathscr{M}$ exhibit a list of local equivalences known as MOY moves. We add to this list a relation allowing one to remove or to put a two-valent vertex to any place in the MOY graph,



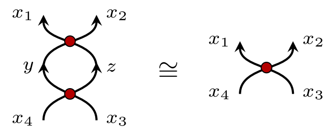
(4.13a)



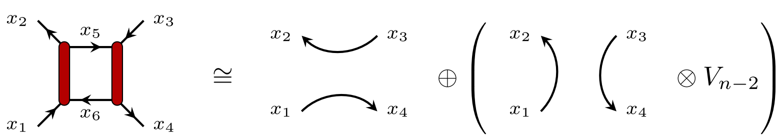
$$\cong V_n, \quad \mathrm{ind}\, V_n = [n]_q\,, \tag{4.13b}$$



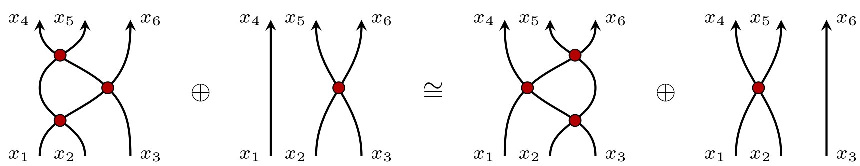
$$\otimes V_{n-1}, \quad \mathrm{ind}\, V_{n-1} = [n-1]_q\,, \tag{4.13c}$$



$$\otimes V_2, \quad \mathrm{ind}\, V_2 = [2]_q\,, \tag{4.13d}$$



$$,\quad \mathrm{ind}\, V_{n-2} = [n-2]_q\,, \tag{4.13e}$$



(4.13f)

Proofs of these relations are rather straightforward and we will not review all of them here. A proof for (4.13a) is given in Sec. IV C 2. A proof for (4.13e) is discussed in Sec. VIII D. We do not consider a proof for (4.13c) in this note, however it follows directly from the proof of invariance with respect to the Reidemeister I move in Sec. VII.

The MOY moves are enough to reduce *any* MOY graph to its q dimension computed via RT formalism. This theorem is proven in [57] when edges of MOY diagrams are colored with $\square$ and $\wedge^2\,\square$ only, and it is proven in [5] for coloring by generic antisymmetric rep $\wedge^k\,\square$.[3] For us this theorem implies immediately,[4]

$$\mathrm{ind}[\Gamma] \coloneqq \sum_z q^z \dim H^z(\mathscr{M}[\Gamma]) = \mathrm{qdim}[\Gamma]. \tag{4.14}$$

Relation (4.14) implies in turn that the Euler characteristic of the KR double complex given by a specific value $P_{\bar{L}}(q,-1)$ of the Poincaré polynomial coincides indeed with the respective HOMFLY polynomial.

C. Solving MFs

1. Localization to quotient rings

Let us start with just a few words about rather efficient methodology to construct cohomologies of MF-like operators. We separate one of the odd variables θ and denote the rest of the even and odd variables as formal letter V. Consider the following odd MF-like operator:

[3]For the particular case of $n=3$ the proof is rather elementary. In this case $\wedge^2\,\square \cong \overline{\square}$ and MOY diagrams become simply three-valent bipartite graphs implying that all the faces are $2m$-gons. For such a graph drawn on a sphere according to the Euler theorem we have $F-E+V=2$ where F, E, and V are numbers of faces, edges, and vertices, respectively. For trivalent graphs $V=\frac{2}{3}E$, so $F=2+\frac{1}{3}E$. If all the faces of the graph were $2m$-gons then for the number of faces we would have $F=\frac{1}{m}E$. If there are no faces with $m<3$ then $F\leq\frac{1}{3}E$ that contradicts $F=2+\frac{1}{3}E$. The latter statement implies that for any MOY graph for $n=3$ we are able to find a face that is a two-gon or a four-gon. So each MOY graph for $n=3$ could be reduced to a number value via a sequence of moves (4.13b)–(4.13e).

[4]Here we call q dimension an "index" following the physical intuition. As we expect that parameter t plays the role of fugacity for the fermion number, and a specification $t=-1$ refers to the supersymmetric Witten index [13,29].

$$M = M_0(V) + A(V)\hat{\theta} + B(V)\hat{\theta}^\dagger. \qquad (4.15)$$

Here we do not require that M_0 has the form of a matrix factorization operator, there is no requirement for it to be linear in other odd variables either. However it is expected that A, B, and M_0 commute mutually. It is also natural to split respective dependence on θ in wave functions $\Psi = \Psi_0(V) + \theta\Psi_1(V)$. The action of operator M on Ψ reads,

$$\begin{aligned} M\Psi = (M_0(V)\Psi_0(V) + B(V)\Psi_1(V)) \\ + \theta(-M_0(V)\Psi_1(V) + A(V)\Psi_0(V)). \end{aligned} \qquad (4.16)$$

So for zero modes of M we see that the wave function might have only one independent component, say, Ψ_0 then $\Psi_1 = -B^{-1}M_0\Psi_0$. On the other hand any B-generated ideal in the first component lies in the image of M,

$$B(V)X(V) - \theta M_0(V)X(V) = M(\theta X(V)), \quad \forall\, X. \qquad (4.17)$$

So we conclude:

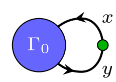
Cohomologies of MF-like operator (4.15) localize to quotient ring $S \coloneqq \mathbb{Q}[V]/(B(V)\mathbb{Q}[V])$.

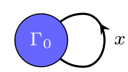
Actually, by localizing the θ^0 component of (4.16) to S we would derive a localized equation for zero modes $M_0|_S\Psi_0|_S = 0$. To acquire the localized operator $M_0|_S$ one should resolve constraint $B = 0$ and substitute it back in M_0, which will *decrease* the number of free variables and has a good chance to make it *simpler*. So our general strategy to define cohomologies of MF operators is to perform a chain of quotient ring localizations until the localized zero-mode equation becomes trivial to solve.

2. Blinking two-valent vertex

Relations (4.13a)–(4.13f) are conceptually very simple. Still, their operator realization is somewhat verbose. This is a well-known phenomenon in homological algebra, but not quite usual for physically oriented literature. Operator language is already a considerable simplification, because operators themselves are simple and intuitive. Still, the second step—equivalence of their zero modes, moreover, of their universality classes—remains somewhat tedious and lengthy. Clearly, one should look for a better and more concise terminology and reasoning. From this point of view quite representative is already the most trivial relation (4.13a), for which we need at least a two-page derivation, despite the obvious simplicity of the operators (4.20) and relative simplicity of the main claim (4.21). This example is not quite exhaustive because of the lack of the second resolution. However, as we will see in Sec. VII, this does not lead to serious complication—after the material of the present subsection is consumed and thought through.

So, we would like to prove relation (4.13a). Consider two closed MOY graphs $\bar{\Gamma}_{1,2}$,

$\bar{\Gamma}_1 =$ 

, $\bar{\Gamma}_2 =$ 

. (4.18)

We assume that graphs differ by the fact that on some edge there is a two-valent vertex with associated variable θ dividing an edge marked with variable x in $\bar{\Gamma}_2$ into two edges marked by two variables x and y in $\bar{\Gamma}_1$. Clearly—and formally, according to (4.13a)—the insertion of the two-valent green vertex changes nothing, we call it "blinking." However, to validate this fact, we need to perform a whole exercise—we would like to show that cohomologies of $\mathscr{M}_{\bar{\Gamma}_{1,2}}$ are isomorphic,

$$H^*(\mathscr{M}_{\bar{\Gamma}_1}) \underset{\Phi_2}{\overset{\Phi_1}{\rightleftarrows}} H^*(\mathscr{M}_{\bar{\Gamma}_2}) \quad . \qquad (4.19)$$

Let us mark the set of all variables corresponding to vertices and internal edges of graph Γ_0 as V. Then for MF operators we have,

$$\begin{aligned} &\mathscr{M}_{\bar{\Gamma}_1} = \mathscr{M}_{\Gamma_0}(V, x, y) + \pi_{xy}\hat{\theta} + (x - y)\hat{\theta}^\dagger, \\ &\mathscr{M}_{\bar{\Gamma}_2} = \mathscr{M}_{\Gamma_0}(V, x, x). \end{aligned} \qquad (4.20)$$

In these terms we can explicitly decompose the first and the second space of polynomials as $\Psi_0(V, x, y) + \theta\Psi_1(V, x, y)$ and $\psi(V, x)$, respectively. In these terms isomorphism Ψ acts as,

$$\Phi_1[\Psi_0(V,x,y)+\theta\Psi_1(V,x,y)]=\Psi_0(V,x,x),$$
$$\Phi_2[\psi(V,x)]=\left(1-\theta\frac{\mathscr{M}_{\Gamma_0}(V,x,y)-\mathscr{M}_{\Gamma_0}(V,x,x)}{x-y}\right)\psi(V,x). \tag{4.21}$$

Proof. Let us consider first,

$$\begin{aligned}\Phi_1[\mathscr{M}_{\bar\Gamma_1}(\Psi_0(V,x,y)+\theta\Psi_1(V,x,y))]&=\Phi[(\mathscr{M}_{\Gamma_0}(V,x,y)\Psi_0(V,x,y)+(x-y)\Psi_1(V,x,y))+\theta(\ldots)]\\&=\mathscr{M}_{\Gamma_0}(V,x,x)\Psi_0(V,x,x)=\mathscr{M}_{\bar\Gamma_2}\Phi_1[\Psi_0(V,x,y)+\theta\Psi_1(V,x,y)].\end{aligned} \tag{4.22}$$

In simpler words we have shown that

$$\Phi_1\circ\mathscr{M}_{\bar\Gamma_1}=\mathscr{M}_{\bar\Gamma_2}\circ\Phi_1, \tag{4.23}$$

so Φ_1 is a homomorphism of complexes.

Now consider,

$$\begin{aligned}\mathscr{M}_{\bar\Gamma_1}\Phi_2[\psi(V,x)]&=(\mathscr{M}_{\Gamma_0}(V,x,y)+\pi_{xy}\hat\theta+(x-y)\hat\theta^\dagger)\left(1-\theta\frac{\mathscr{M}_{\Gamma_0}(V,x,y)-\mathscr{M}_{\Gamma_0}(V,x,x)}{x-y}\right)\psi(V,x)\\&=\theta\left(\mathscr{M}_{\Gamma_0}(V,x,y)\frac{\mathscr{M}_{\Gamma_0}(V,x,y)-\mathscr{M}_{\Gamma_0}(V,x,x)}{x-y}-\underbrace{\frac{\mathscr{M}_{\Gamma_0}(V,x,y)^2}{x-y}}_{=-\pi_{xy}}+\underbrace{\frac{\mathscr{M}_{\Gamma_0}(V,x,x)^2}{x-y}}_{=0}\right)\psi(V,x)\\&\quad+\mathscr{M}_{\Gamma_0}(V,x,x)\psi(V,x)=\Phi_2[\mathscr{M}_{\bar\Gamma_2}\psi(V,x)].\end{aligned} \tag{4.24}$$

So we have shown that Φ_2 is also a homomorphism,

$$\mathscr{M}_{\bar\Gamma_1}\circ\Phi_2=\Phi_2\circ\mathscr{M}_{\bar\Gamma_2}. \tag{4.25}$$

It remains to show that Φ_1 is invertible on cohomologies and Φ_2 is its inverse. Apparently,

$$\Phi_1[\Phi_2[\psi(V,x)]]=\psi(V,x). \tag{4.26}$$

On the other hand the fact that $\Psi\in\mathrm{Ker}\mathscr{M}_{\bar\Gamma_1}$ imposes on functions two constraints,

$$\Psi_1(V,x,y)=-\frac{\mathscr{M}_{\Gamma_0}(V,x,y)}{x-y}\Psi_0(V,x,y),\qquad \mathscr{M}_{\Gamma_0}(V,x,x)\Psi_0(V,x,x)=0. \tag{4.27}$$

Then we derive,

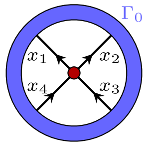
$$\begin{aligned}\Psi_0(V,x,y)+\theta\Psi_1(V,x,y)&=\left(1-\theta\frac{\mathscr{M}_{\Gamma_0}(V,x,y)-\mathscr{M}_{\Gamma_0}(V,x,x)}{x-y}\right)\Psi_0(V,x,x)+\mathscr{M}_{\bar\Gamma_1}\left(\frac{\Psi_0(V,x,y)-\Psi_0(V,x,x)}{x-y}\right)\\&=\Phi_2[\Phi_1[\Psi_0(V,x,y)+\theta\Psi_1(V,x,y)]]+\mathscr{M}_{\bar\Gamma_1}(\ldots).\end{aligned} \tag{4.28}$$

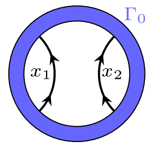
The latter relation becomes an equality modulo $\mathrm{Im}\,\mathscr{M}_{\bar\Gamma_1}$ and indicates that Φ_1 is indeed an isomorphism of cohomologies, and $\Phi_2=\Phi_1^{-1}$,

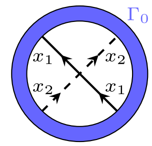
$$\Phi_1\circ\Phi_2=\mathbb{1}_{H^*(\mathscr{M}_{\bar\Gamma_2})},\qquad \Phi_2\circ\Phi_1=\mathbb{1}_{H^*(\mathscr{M}_{\bar\Gamma_1})}. \tag{4.29}$$

3. Splitting four-valent vertex

A rather useful tool for calculating MF cohomologies is a splitting process for four-valent vertices allowing one to represent cohomologies of the graph with a four-valent vertex as *constrained* cohomologies of a graph with the vertex eliminated. Consider the following diagrams:

$\bar{\Gamma}_1 =$ 

, $\bar{\Gamma}_+ =$ 

, $\bar{\Gamma}_- =$ 

(NO VERTEX!!!) . (4.30)

In diagram $\bar{\Gamma}_\pm$ we have identified edges of diagram Γ_0 accordingly and identified respective variables. In diagram $\bar{\Gamma}_-$ edges are of the same type, one of them is simply marked as dashed to indicate that identification is criss-cross without any intersection or vertex.

Again let us denote the set of internal variables for subgraph Γ_0 as V. Respective MF operators read,

$$\begin{aligned} &\mathscr{M}_{\bar{\Gamma}_1} = \mathscr{M}_{\Gamma_0}(V, x_1, x_2, x_3, x_4) + u_1\hat{\theta}_1 + s_1\hat{\theta}_1^\dagger + u_2\hat{\theta}_2 + s_2\hat{\theta}_2^\dagger, \\ &\mathscr{M}_{\bar{\Gamma}_+} = \mathscr{M}_{\Gamma_0}(V, x_1, x_2, x_2, x_1), \qquad \mathscr{M}_{\bar{\Gamma}_-} = \mathscr{M}_{\Gamma_0}(V, x_1, x_2, x_1, x_2), \end{aligned} \tag{4.31}$$

where $s_1 = x_1 + x_2 - x_3 - x_4$, $s_2 = x_1x_2 - x_3x_4$ and u_i are defined by (4.6). Consider a set of two-dimensional vectors of (x, y)-polynomial components constrained to coincide in points $x = y$,

$$\Xi(x, y) \coloneqq \left\{ \begin{pmatrix} \xi_+(V, x, y) \\ \xi_-(V, x, y) \end{pmatrix} \middle| \xi_+(V, z, z) = \xi_-(V, z, z), \ \forall \ z \right\}. \tag{4.32}$$

Then we would like to prove that there is the following isomorphism[5]:

$$H^*(\mathscr{M}_{\bar{\Gamma}_1}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} H^*\left(\Xi(x_1, x_2), \mathrm{diag}\left(\mathscr{M}_{\bar{\Gamma}_+}, \mathscr{M}_{\bar{\Gamma}_-}\right)\right) \quad . \tag{4.33}$$

In particular, we have,

$$\begin{aligned} &\Phi[\Psi_0(V, x_1, x_2, x_3, x_4) + \theta_1\Psi_1(V, x_1, x_2, x_3, x_4) + \theta_2\Psi_2(V, x_1, x_2, x_3, x_4) + \theta_1\theta_2\Psi_{12}(V, x_1, x_2, x_3, x_4)] \\ &= (\Psi_0(x_1, x_2, x_2, x_1), \Psi_0(x_1, x_2, x_1, x_2)), \\ &\Phi^{-1}[(\xi_+(x_1, x_2), \xi_-(x_1, x_2))] = \left(1 - \theta_1 v_1 \mathscr{M}_{\bar{\Gamma}_1} - \theta_2 v_2 \mathscr{M}_{\bar{\Gamma}_1} + \theta_1\theta_2 \frac{u_2 + \mathscr{M}_{\bar{\Gamma}_1} v_2 \mathscr{M}_{\bar{\Gamma}_1}}{s_1}\right) \\ &\quad \times \left(\frac{1}{2}\left(1 - \frac{x_3 - x_4}{x_1 - x_2}\right)\xi_+(x_1, x_2) + \left(1 + \frac{x_3 - x_4}{x_1 - x_2}\right)\frac{1}{2}\xi_-(x_1, x_2)\right), \end{aligned} \tag{4.34}$$

where linear maps $v_{1,2}$ are defined in (B7).

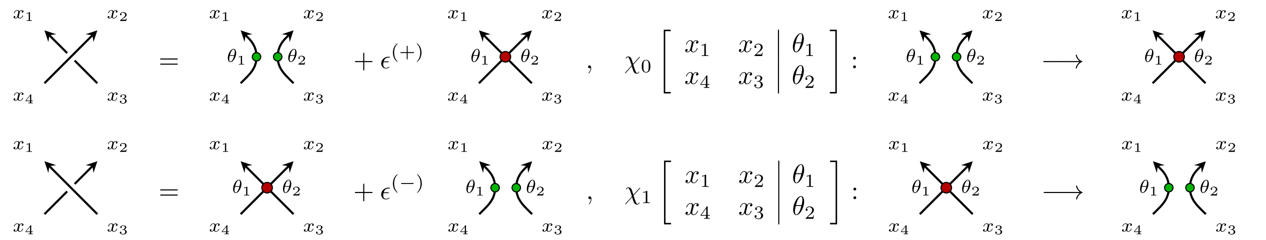
The proof of the statement that Φ is an isomorphism of cohomologies and Φ^{-1} is its inverse is totally analogous to the case of a blinking two-valent vertex (see Sec. IV C 2) and exploits quadratic quotients discussed in Appendix B 2. We will omit it here and present its details elsewhere.

V. HORIZONTAL MORPHISMS

Horizontal morphisms are fixed *uniquely* up to homotopy by locality and homomorphism requirement (3.8). See also [3].

To describe horizontal morphisms as ring morphisms it would be more natural to assign variables to MOY diagrams appearing in decomposition (3.1) in a uniform way. To do so we assign even variables to the edges of the very link diagram in question, and assign a couple of odd variables to both intersection resolutions via a single four-valent vertex or a pair of two-valent vertices. Then we would like to search for horizontal morphisms in terms of a uniform variable choice, there are two types of them,

[5]At the rhs inside the cohomology, we indicate the space at the first place and the differential acting on this space at the second place.



$$\mathrm{qdeg}\,\epsilon^{(+)} = -2, \quad \mathrm{tdeg}\,\epsilon^{(+)} = +1, \quad \mathrm{qdeg}\,\epsilon^{(-)} = 0, \quad \mathrm{tdeg}\,\epsilon^{(-)} = +1\,. \tag{5.1}$$

If the uniform choice of variables is made all the rings $\mathcal{R}[\Gamma]$ for various MOY diagrams Γ are mutually isomorphic naturally.

One is able to derive expressions for morphisms $\chi_{0,1}$ explicitly based on three properties:

(1) Parity. Horizontal differential (3.7) is expected to be an odd, fermionic operator, so that its nilpotency is a natural property. Since ϵ is a fermion, it implies that morphisms $\chi_{0,1}$ are even linear operators.

(2) Degree. In Morse/Floer theory approaches to link invariant categorification homological degrees appear as eigenvalues z and f of commuting quantum charges in topological string theory. These charges commute with the Hamiltonian as well. So a differential concentrated in the Morse theory on instanton flows has also fixed degrees $z=0$ and $f=1$ as a linear operator on the Hilbert space. Then we expect that morphisms $\chi_{0,1}$ would also have a fixed q degree equal to $+2$ and 0, respectively, so that in total $\mathrm{qdeg}\mathfrak{D}=0$. Let us argue briefly this degree choice. Assume that $\psi\notin\ker\mathfrak{D}$ and consider a simple complex homotopic to zero,

$$\left[\begin{array}{c} 0 \longrightarrow \psi \xrightarrow{\mathfrak{D}} \mathfrak{D}\psi \longrightarrow 0 \end{array}\right] \sim 0\,.$$

Its index reads $q^{\mathrm{qdeg}\psi}(1-q^{\mathrm{qdeg}\mathfrak{D}})$. We expect naturally that the index is an invariant and equals 0 for a zero complex, this reasoning imposes a constraint $\mathrm{qdeg}\mathfrak{D}=0$.

(3) Locality. This property seems to be the most important singling out the KR construction among the others. We assume that for the uniform choice of variables morphisms $\chi_{0,1}$ are local: they depend only on variables x_1, x_2, x_3, x_4, θ_1 and θ_2 assigned to a particular intersection (5.1).

Let us calculate morphism χ_0 explicitly. First we separate dependence on $\theta_{1,2}$ in the ring basis,

$$\mathcal{R} = (\underbrace{R \overset{U_0\circlearrowleft}{\oplus} R_{12}\theta_1\theta_2}_{\text{parity}=+1}) \oplus (\underbrace{R_1\theta_1 \overset{U_1\circlearrowleft}{\oplus} R_2\theta_2}_{\text{parity}=-1}), \tag{5.2}$$

where R, R_1, R_2, R_{12} are rings of variables x_i and odd variables coming from other intersections. We combined these subrings in pairs according to their parity. Since χ_0 preserves parity a generic 4×4 linear transform χ_0 on this space is split in two 2×2 matrix blocks U_0 and U_1. We can naturally represent such a map in the form of a differential operator,

$$\begin{aligned}\chi_0 = {}&(U_0)_{11}\hat\theta_1^\dagger\hat\theta_1\hat\theta_2^\dagger\hat\theta_2 + (U_0)_{22}\hat\theta_1\hat\theta_1^\dagger\hat\theta_2\hat\theta_2^\dagger + (U_0)_{21}\hat\theta_1\hat\theta_2 + (U_0)_{12}\hat\theta_2^\dagger\hat\theta_1^\dagger \\ &+(U_1)_{11}\hat\theta_1\hat\theta_1^\dagger\hat\theta_2^\dagger\hat\theta_2 + (U_1)_{12}\hat\theta_1\hat\theta_2^\dagger + (U_1)_{21}\hat\theta_2\hat\theta_1^\dagger + (U_1)_{22}\hat\theta_1^\dagger\hat\theta_1\hat\theta_2\hat\theta_2^\dagger.\end{aligned} \tag{5.3}$$

Matrix elements of these matrices are polynomials in variables $x_{1,2,3,4}$ of fixed degree due to the q-degree constraint,[6]

[6]Here we used q-degree values for θ_i from (4.1) and (4.4). For a map $X\colon A\to B$ to be of degree $+2$ one should obtain the following degrees for matrix elements:

$$\mathrm{qdeg}X_{ij} = 2 + \mathrm{qdeg}A_j - \mathrm{qdeg}B_i. \tag{5.4}$$

$$\mathrm{qdeg}U_0=\begin{pmatrix} 2 & 2(2-n)\\ 2(n-1) & 0\end{pmatrix},\quad \mathrm{qdeg}U_1=\begin{pmatrix} 2 & 2\\ 0 & 0\end{pmatrix}. \tag{5.5}$$

Let us assign respective variable homogeneous polynomials to these matrix elements,

$$U_0=\begin{pmatrix} p_1 & 0\\ g_1 & 1\end{pmatrix},\qquad U_1=\begin{pmatrix} p_2 & p_3\\ r_1 & r_2\end{pmatrix}, \tag{5.6}$$

where polynomials $p_{1,2,3}$ are of degree 1, polynomial g_1 is of degree $n-1$, and $r_{1,2}$ are of degree 0, in other words the latter are just numbers. Apparently, we would be able to define this differential modulo an overall numerical scale, so we fix it by choosing matrix element $(U_0)_{22}$ of degree 0 to be 1. Respectively, matrix element $(U_0)_{12}$ has a negative degree for generic $n>2$, therefore it should be 0.

We describe local matrix factorizations for the two resolutions in the first line of (5.1) explicitly as,

$$\begin{aligned}\mathscr{M}_{\Gamma_0}&=\pi_{14}\hat{\theta}_1+\underbrace{(x_1-x_4)}_{x_{14}}\hat{\theta}_1^\dagger+\pi_{23}\hat{\theta}_2+\underbrace{(x_2-x_3)}_{x_{23}}\hat{\theta}_2^\dagger,\\ \mathscr{M}_{\Gamma_1}&=u_1\hat{\theta}_1+\underbrace{(x_1+x_2-x_3-x_4)}_{s_1}\hat{\theta}_1^\dagger+u_2\hat{\theta}_2+\underbrace{(x_1x_2-x_3x_4)}_{s_2}\hat{\theta}_2^\dagger.\end{aligned} \tag{5.7}$$

For this matrix factorization operators homomorphism property descends to the following equation:

$$\chi_0\mathscr{M}_{\Gamma_0}=\mathscr{M}_{\Gamma_1}\chi_0. \tag{5.8}$$

This relation produces a collection of subconstraints,

$$-p_2s_1+p_1x_{14}-r_2s_2=0, \tag{5.9a}$$

$$p_3s_1+p_1x_{23}-r_1s_2=0, \tag{5.9b}$$

$$-g_1s_2-p_1u_1+p_2\pi_{14}-p_3\pi_{23}=0, \tag{5.9c}$$

$$-p_3x_{14}-p_2x_{23}+s_2=0, \tag{5.9d}$$

$$r_1x_{14}-r_2x_{23}-s_1=0, \tag{5.9e}$$

$$-g_1x_{14}+p_2u_2-r_2u_1-\pi_{23}=0, \tag{5.9f}$$

$$-g_1x_{23}-p_3u_2-r_1u_1+\pi_{14}=0. \tag{5.9g}$$

Let us start with (5.9e). Since $s_1=x_{14}+x_{23}$ this constraint fixes uniquely $r_1=1$, $r_2=-1$. After substituting this solution from (5.9a), (5.9b) we reexpress p_2 and p_3 in terms of p_1,

$$p_2=\frac{s_2+p_1x_{14}}{s_1},\qquad p_3=\frac{s_2-p_1x_{23}}{s_1}. \tag{5.10}$$

However for these relations to make sense the pole in the denominators should be canceled. This imposes constraints on p_1,

$$(s_2+p_1x_{14})|_{s_1=0}=(s_2-p_1x_{23})|_{s_1=0}=0. \tag{5.11}$$

Then we find,

$$p_1|_{x_1=x_3+x_4-x_2}=-\frac{s_2}{x_{14}}\bigg|_{x_1=x_3+x_4-x_2}=\frac{s_2}{x_{23}}\bigg|_{x_1=x_3+x_4-x_2}=x_4-x_2. \tag{5.12}$$

The most generic solution to this constraint for p_1 is the following:

$$p_1=(x_4-x_2)+\mu s_1, \tag{5.13}$$

And we have found a generic solution indeed since different values of μ correspond to homotopies of the double complex.

Dealing in the same way with χ_1, eventually we would arrive to the following morphisms as in [1] with free parameters μ and λ,

$$\begin{aligned}\chi_0&=(\mu(x_1+x_2-x_3-x_4)-x_2+x_4)\hat{\theta}_1^\dagger\hat{\theta}_1\hat{\theta}_2^\dagger\hat{\theta}_2+\hat{\theta}_1\hat{\theta}_1^\dagger\hat{\theta}_2\hat{\theta}_2^\dagger-\left((\mu-1)u_2+\frac{u_1+x_1u_2-\pi_{23}}{x_1-x_4}\right)\hat{\theta}_1\hat{\theta}_2\\ &\quad+(x_4+\mu(x_1-x_4))\hat{\theta}_1\hat{\theta}_1^\dagger\hat{\theta}_2^\dagger\hat{\theta}_2+(\mu(x_2-x_3)-x_2)\hat{\theta}_1\hat{\theta}_2^\dagger-\hat{\theta}_2\hat{\theta}_1^\dagger+\hat{\theta}_1^\dagger\hat{\theta}_1\hat{\theta}_2\hat{\theta}_2^\dagger,\\ \chi_1&=\hat{\theta}_1^\dagger\hat{\theta}_1\hat{\theta}_2^\dagger\hat{\theta}_2+(\lambda(x_3+x_4-x_1-x_2)+x_1-x_3)\hat{\theta}_1\hat{\theta}_1^\dagger\hat{\theta}_2\hat{\theta}_2^\dagger-\left(\lambda u_2+\frac{u_1+x_1u_2-\pi_{23}}{x_4-x_1}\right)\hat{\theta}_1\hat{\theta}_2\\ &\quad+\hat{\theta}_1\hat{\theta}_1^\dagger\hat{\theta}_2^\dagger\hat{\theta}_2+(x_3+\lambda(x_2-x_3))\hat{\theta}_1\hat{\theta}_2^\dagger+\hat{\theta}_2\hat{\theta}_1^\dagger+(x_1+\lambda(x_4-x_1))\hat{\theta}_1^\dagger\hat{\theta}_1\hat{\theta}_2\hat{\theta}_2^\dagger.\end{aligned} \tag{5.14}$$

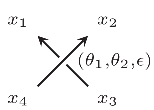
The fact that homomorphism (commutation) relations (3.8) are enough under mild assumptions to derive horizontal morphisms uniquely is a strong indication that modern approaches via foam TQFT [36–39] are equivalent to the homological algebra approach of KR. Indeed TQFT cobordisms saturating matrix elements of the horizontal morphisms correspond to evolution operators in the TQFT Hilbert space. Evolution operators commute necessarily with supercharges we identified with MF operators in Appendix A. Then commutation (homomorphism) relations (3.8) constrain simultaneously both TQFT cobordism functors and horizontal morphisms in the double complex to be compatible with supercharges (vertical morphisms) and, therefore, equivalent to each other.

VI. OPERATORS $\mathscr{O}_L$

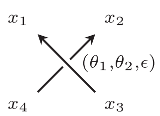
It is well known that cohomologies of a double complex could be rewritten as cohomologies of the total complex [58]. However in the particular case of the KR double complex the total complex could be constructed directly in terms of link diagrams L omitting an additional passage to MOY diagrams.

(1) We assign to all edges k of a link diagram independent even variables x_k, to each intersection α we assign a triplet of independent odd variables $\theta_1^{(\alpha)}$, $\theta_2^{(\alpha)}$, $\epsilon^{(\alpha)}$.

(2) To each intersection we assign a *differential operator* in the following way:



$$, \quad \mathscr{O}_{\times} = \mathscr{M}_{\Gamma_0}\hat{\epsilon}^\dagger\hat{\epsilon} + \mathscr{M}_{\Gamma_1}\hat{\epsilon}\hat{\epsilon}^\dagger + \epsilon\chi_0\,,$$



$$, \quad \mathscr{O}_{\times} = \mathscr{M}_{\Gamma_1}\hat{\epsilon}^\dagger\hat{\epsilon} + \mathscr{M}_{\Gamma_0}\hat{\epsilon}\hat{\epsilon}^\dagger + \epsilon\chi_1\,, \tag{6.1}$$

where $\mathscr{M}_{\Gamma_{0,1}}$ and $\chi_{0,1}$ are given by expressions (5.7) and (5.14).

(3) The total differential operator for link subdiagram ℓ when we neglect external legs is defined as,

$$\mathscr{O}_\ell = \sum_{v\in \text{vertices of } L} \mathscr{O}_v. \tag{6.2}$$

(4) It is simple to derive that for a link diagram ℓ with open external edges labeled by even variables x_k,

$$\mathscr{O}_\ell^2 = \left(\sum_k s_k x_k^{n+1}\right) \times \mathbb{1}, \tag{6.3}$$

where sign s_k equals $+1$ if edge k is flowing outward L and -1 if it flows inward L.

(5) To be able to discuss tangles one should take into account a contribution from the tangle boundary. There are two equivalent ways to deal with the boundary ∂L. One way is to consider the link diagram on a sphere and treat the boundary as another multivalent vertex. The other way is to imply that the link L is implanted into another bigger link L' with all the legs jacked into ports of L' so that the total link $L\cup L'$ is closed. In both cases we assign to the boundary ∂L a *class* of operators $\mathscr{O}_{\partial L}$ with the following properties:
   (a) $\mathscr{O}_{\partial L}$ depends explicitly on even variables x_k corresponding to external legs of L. It is allowed to depend on other even and odd variables not presented in L.
   (b) For any internal vertex v: $\{\mathscr{O}_{\partial L}, \mathscr{O}_v\} = 0$.
   (c) $\mathscr{O}_{\partial L}^2 = -(\sum_k s_k x_k^{n+1}) \times \mathbb{1}$.

Then to a link L we assign the following operator:

$$\mathscr{O}_L = \mathscr{O}_\ell + \mathscr{O}_{\partial L}. \tag{6.4}$$

It is easy to see that $\mathscr{O}_L^2 = 0$.

(6) Link L invariants are defined as "invariant wave functions"—cohomologies $H^{*,*}(\mathscr{O}_L)$.

(7) For a closed link diagram $\bar{L}$ $\mathscr{O}_{\partial\bar{L}} = 0$, and operator $\mathscr{O}_{\bar{L}}$ is a differential. Its cohomologies $H^{*,*}(\mathscr{O}_{\bar{L}})$ are bigraded naturally by qdeg and tdeg, they are isomorphic to the respective KR double complex cohomologies and are a *link* $\bar{L}$ *invariant*.

In what follows we will be interested in equivalence statements for tangles, for examples, we would like to say that tangle homotopy (Reidemeister moves) are tangle equivalences. In all sorts of such statements that $L \cong L'$ with $\partial L = \partial L'$ we imply that respective cohomology groups are isomorphic $H^{*,*}(\mathscr{O}_L) \cong H^{*,*}(\mathscr{O}_{L'})$ for any choice of the boundary correction operator $\mathscr{O}_{\partial L} = \mathscr{O}_{\partial L'}$.

Now let us indicate that this structure behaves well under tangle gluing. Consider two link diagrams L_1 and L_2 and assume that we glued them into $L_1 \cup L_2$ by possibly jacking mutually some of the external legs of these diagrams. For the glued diagram we have,

$$\mathscr{O}_{L_1\cup L_2} = \mathscr{O}_{\ell_1} + \mathscr{O}_{\ell_2} + [\mathscr{O}_{\partial(L_1\cup L_2)}], \tag{6.5}$$

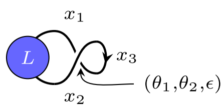
where by square brackets […] we indicate that the whole class of boundary correcting operators is considered. However it is easy to show that $\mathscr{O}_{\ell_2} + [\mathscr{O}_{\partial(L_1\cup L_2)}] \in [\mathscr{O}_{\partial L_1}]$, so that a class $[\mathscr{O}_{L_1\cup L_2}]$ is a subclass in $[\mathscr{O}_{L_1}]$. Therefore any equivalence proven for subtangle L_1 would extend automatically to the glued tangle $L_1 \cup L_2$.

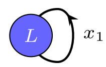
VII. INVARIANTS OF TANGLES

Now we would like to argue that tangle wave functions —cohomologies of $\mathscr{O}_L$—remain invariant under homotopic moves. Canonically this requires us to demonstrate the object is invariant with respect to basic Reidemeister moves (see Fig. 1). With respect to RI one expects an isomorphism of cohomologies with a degree shift in general.

Let us demonstrate explicitly the invariance of $H^{*,*}(\mathscr{O}_L)$ with respect to RI here. Remaining equivalences are proven in a similar fashion and we leave them out of the scope of the present note for the sake of brevity.

We would like to prove the following equivalence of cohomologies of $\mathscr{O}$-operators for homotopically equivalent links,

$\bar{L}_1 =$ 

, $\bar{L}_2 =$ 

. (7.1)

We denote by V all the internal variables in diagram L. Here we have,

$$\mathscr{O}_{\bar{L}_1} = \mathscr{O}_L(V,x_1,x_2) + \mathscr{M}_{\Gamma_0}\hat{\epsilon}^\dagger\hat{\epsilon} + \mathscr{M}_{\Gamma_1}\hat{\epsilon}\hat{\epsilon}^\dagger + \hat{\epsilon}\chi_0,$$
$$\mathscr{O}_{\bar{L}_2} = \mathscr{O}_L(V,x_1,x_1), \tag{7.2}$$

where

$$\mathscr{M}_{\Gamma_0} = \pi_{12}\hat{\theta}_1 + \underline{(x_1-x_2)\hat{\theta}_1^\dagger} + x_3^n\hat{\theta}_2,$$
$$\mathscr{M}_{\Gamma_1} = u_1(x_1,x_3,x_3,x_2)\hat{\theta}_1 + \underline{(x_1-x_2)\hat{\theta}_1^\dagger} + u_2(x_1,x_3,x_3,x_2)\hat{\theta}_2 + x_3x_{12}\hat{\theta}_2^\dagger. \tag{7.3}$$

Let us separate variable ϵ, then the action of $\mathscr{O}_{\bar{L}_1}$ acts as,

$$\mathscr{O}_{\bar{L}_1}(\Psi_0+\epsilon\Psi_1) = (\mathscr{O}_L(V,x_1,x_2)+\mathscr{M}_{\Gamma_0})\Psi_0 + \epsilon(-\mathscr{O}_L(V,x_1,x_2)\Psi_1 + \mathscr{M}_{\Gamma_1}\Psi_1 + \chi_0\Psi_0). \tag{7.4}$$

Now we would like to derive cohomologies Φ_1 and Φ_2 of operators $\mathscr{O}_L(V,x_1,x_2)+\mathscr{M}_{\Gamma_0}$ and $\mathscr{O}_L(V,x_1,x_2)+\mathscr{M}_{\Gamma_1}$, respectively. Applying (4.19) we derive,

$$\Phi_1 = \underbrace{\left(1-\theta_1\frac{\mathscr{O}_L(V,x_1,x_2)-\mathscr{O}_L(V,x_1,x_1)}{x_1-x_2}\right)\theta_2}_{S_1} \times (\varphi(x_3)\psi(V,x_1) \bmod x_3^n),$$
$$\Phi_2 = \underbrace{\left(1-\theta_1\frac{\mathscr{O}_L(V,x_1,x_2)-\mathscr{O}_L(V,x_1,x_1)}{x_1-x_2} - x_3\theta_1\hat{\theta}_2^\dagger\right)\theta_2}_{S_2} \times (\varphi(x_3)\psi(V,x_1) \bmod \; p_{n-1}(x_3,x_1)), \tag{7.5}$$

where φ are arbitrary polynomials in one variable x_3, and $\psi(V,x_1)$ is cohomology of $\mathscr{O}_L(V,x_1,x_1)=\mathscr{O}_{\bar{L}_2}$, and

$$p_{n-1}(x_3,x_1) \coloneqq u_2(x_1,x_3,x_3,x_1) = x_3^{n-1} + \sum_{i=0}^{n-2} a_i(x_1)x_3^i. \tag{7.6}$$

It is easy to see that $\chi_0 S_1\varphi\psi = S_2\varphi\psi$. Let us choose bases in cohomologies:

$$H^*(\mathscr{O}_L(V,x_1,x_2)+\mathscr{M}_{\Gamma_0})\colon\; S_1\psi, S_1x_3\psi, \ldots, S_1x_3^{n-2}\psi, S_1p_{n-1}(x_3,x_1)\psi,$$
$$H^*(\mathscr{O}_L(V,x_1,x_2)+\mathscr{M}_{\Gamma_1})\colon\; S_2\psi, S_2x_3\psi, \ldots, S_2x_3^{n-2}\psi. \tag{7.7}$$

So we consider the wave function in the following form:

$$\Psi_{\bar{L}_1}(V,x_1,x_2,x_3) = \alpha S_1 p_{n-1}(x_3,x_1)\psi + \sum_{k=0}^{n-2}\beta_k S_1 x_3^k\psi + \epsilon\sum_{k=0}^{n-2}\gamma_k S_2 x_3^k\psi, \tag{7.8}$$

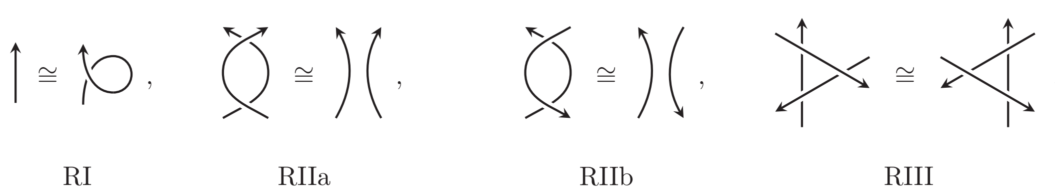


FIG. 1. Reidemeister moves.

and substitute it back into (7.4). This substitution constrains $\beta_k = 0$ and leaves α, γ_k free parameters, on the other hand we find,

$$\epsilon \sum_{k=0}^{n-2} \gamma_k S_2 x_3^k \psi = \mathscr{O}_{\bar{L}_1} \sum_{k=0}^{n-2} \gamma_k S_1 x_3^k \psi \in \operatorname{Im} \mathscr{O}_{\bar{L}_1}. \tag{7.9}$$

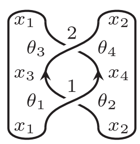
So we conclude that the sole cohomology of $\mathscr{O}_{\bar{L}_1}$ is isomorphic to cohomology of $\mathscr{O}_{\bar{L}_2}$ (up to degree shifts),

$$\begin{aligned} &\Psi_{\bar{L}_1} \in H^{*,*}(\mathscr{O}_{\bar{L}_1}) \cong H^{*,*}(\mathscr{O}_{\bar{L}_2}) \ni \Psi_{\bar{L}_2}, \\ &\Psi_{\bar{L}_1}(V, x_1, x_2, x_3, \theta_1, \theta_2, \epsilon) = \left(1 - \theta_1 \frac{\mathscr{O}_L(V, x_1, x_2) - \mathscr{O}_L(V, x_1, x_1)}{x_1 - x_2}\right) \theta_2 p_{n-1}(x_3, x_1) \times \Psi_{\bar{L}_2}(V, x_1) + \epsilon \cdot 0. \end{aligned} \tag{7.10}$$

VIII. MISCELLANEOUS EXERCISES

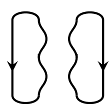
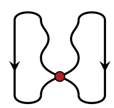
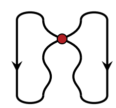
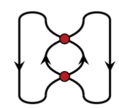
A. Poincaré polynomial for the Hopf link

Here we would present an exercise computation of the Poincaré polynomial for the Hopf link with generic n. First we decompose the Hopf link into MOY diagrams,



$$= \left[\begin{tikzcd} & \epsilon_1 \Gamma_{10} \arrow[dr, "\epsilon_2 \chi_2"] & \\ \Gamma_{00} \arrow[ur, "\epsilon_1 \chi_1"] \arrow[dr, "\epsilon_2 \chi_2"'] & & \epsilon_1 \epsilon_2 \Gamma_{11} \\ & \epsilon_2 \Gamma_{01} \arrow[ur, "\epsilon_1 \chi_1"'] & \end{tikzcd}\right], \tag{8.1}$$

where the following MOY diagrams are used:

$\Gamma_{00} =$ , $\Gamma_{10} =$ , $\Gamma_{01} =$ , $\Gamma_{11} =$ . (8.2)

Now we would like to solve for zero modes of MF operators applying techniques of Sec. IV C 2 and of Sec. IV C 3. For the first MF operator we derive,

$$\mathscr{M}_{\Gamma_{00}}\Big|_{\substack{x_3 = x_1, x_4 = x_2 \\ \theta_3 = 0, \theta_4 = 0}} = x_1^n \hat{\theta}_1 + x_2^n \hat{\theta}_2. \tag{8.3}$$

We easily find cohomology of this differential concentrated at fermion monomial $\theta_1 \theta_2$ and reconstruct complete cohomological wave functions by "dressing" the fermion monomial with (4.21),

$$\sum_{i,j=0}^{n-1} a_{ij} x_1^i x_2^j \theta_1 \theta_2 \xrightarrow{(4.21)} \Psi_{\Gamma_{00}} = \sum_{i,j=0}^{n-1} a_{ij} x_1^i x_2^j \Omega_{00}, \qquad \Omega_{00} = \theta_1 \theta_2 + \theta_3 \theta_2 + \theta_1 \theta_4 + \theta_3 \theta_4. \tag{8.4}$$

For Γ_{10} we find,

$$\mathscr{M}_{\Gamma_{10}}\Big|_{\substack{x_3 = x_1, x_4 = x_2 \\ \theta_3 = 0, \theta_4 = 0}} = (n+1)(x_1^n + x_1^{n-1} x_2 + \cdots + x_2^n) \hat{\theta}_1 + (n+1)(x_1^{n-1} + x_1^{n-2} x_2 + \cdots + x_2^{n-1}) \hat{\theta}_2. \tag{8.5}$$

Cohomologies of this differential are concentrated again in the $\theta_1 \theta_2$ term and are isomorphic to $\mathbb{Q}[x_1, x_2]$ modulo $x_1^n = 0$, $x_2^{n-1} = 0$. So we "dress" these wave functions to derive the respective matrix factorization cohomology with (4.21),

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-2} b_{ij}x_1^i x_2^j \theta_1\theta_2 \xrightarrow{(4.21)} \Psi_{\Gamma_{10}} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-2} b_{ij}x_1^i x_2^j \Omega_{10},$$
$$\Omega_{10} = \Big(\theta_1\theta_2 + x_2\theta_1\theta_3 - \theta_2\theta_3 + x_3\theta_1\theta_4 - \theta_2\theta_4 + (x_3-x_2)\theta_3\theta_4 + \frac{(u_1(x_3,x_4,x_2,x_1) + x_2u_2(x_3,x_4,x_2,x_1)) - (u_1(x_3,x_2,x_2,x_1) + x_2u_2(x_3,x_2,x_2,x_1))}{x_4-x_2}\theta_1\theta_2\theta_3\theta_4\Big). \tag{8.6}$$

Similarly for Γ_{01} we derive cohomology concentrated at $\theta_3\theta_4$:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-2} c_{ij}x_1^i x_2^j \theta_3\theta_4 \xrightarrow{(4.21)} \Psi_{\Gamma_{01}} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-2} c_{ij}x_1^i x_2^j \Omega_{01},$$
$$\Omega_{01} = \Big(\theta_3\theta_4 - x_2\theta_1\theta_3 + \theta_1\theta_4 - x_3\theta_2\theta_3 + \theta_2\theta_4 + (x_3-x_2)\theta_1\theta_2 + \frac{(u_1(x_1,x_2,x_4,x_3) + x_2u_2(x_1,x_2,x_4,x_3)) - (u_1(x_1,x_2,x_2,x_3) + x_2u_2(x_1,x_2,x_2,x_3))}{x_2-x_4}\theta_1\theta_2\theta_3\theta_4\Big). \tag{8.7}$$

Eventually for Γ_{11} we find,

$$\mathscr{M}_{\Gamma_{11}}\Big|_{\substack{x_3=x_1,\ x_4=x_2\\ \theta_3=0,\ \theta_4=0}} = \mathscr{M}_{\Gamma_{11}}\Big|_{\substack{x_3=x_2,\ x_4=x_1\\ \theta_3=0,\ \theta_4=0}} = \mathscr{M}_{\Gamma_{10}}\Big|_{\substack{x_3=x_1,\ x_4=x_2\\ \theta_3=0,\ \theta_4=0}}. \tag{8.8}$$

This implies, actually, that the cohomology of this operator is representable in the form of $f(x_1,x_2) + (x_3-x_4)g(x_1,x_2)$ where both f and g are cohomologies of $\mathscr{M}_{\Gamma_{10}}$. So in this case we obtain the following cohomological wave function:

$$\Psi_{\Gamma_{11}} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-2}(d_{ij}x_1^i x_2^j + (x_3-x_4)e_{ij}x_1^i x_2^j)\Omega_{11},$$
$$\Omega_{11} = \Big(\theta_1\theta_2 - \theta_2\theta_3 + \theta_1\theta_4 + \theta_3\theta_4 + \frac{u_1(x_1,x_2,x_4,x_3) - u_1(x_3,x_4,x_2,x_1)}{x_1x_2 - x_3x_4}\theta_1\theta_2\theta_3\theta_4\Big). \tag{8.9}$$

For the action of horizontal morphisms on forms we find,

$$\begin{aligned}
\chi_1(\Omega_{00}) &= \Omega_{10},\\
\chi_2(\Omega_{00}) &= \Omega_{01} + \mathscr{M}_{\Gamma_{01}}(\theta_1\theta_2\theta_3),\\
\chi_2(\Omega_{10}) &= \frac{1}{2}(x_1-x_2+x_3-x_4)\Omega_{11} + \frac{1}{2}\mathscr{M}_{\Gamma_{11}}(\theta_1\theta_2\theta_3) + \frac{1}{2}\mathscr{M}_{\Gamma_{11}}(\theta_1\theta_3\theta_4),\\
\chi_1(\Omega_{01}) &= \frac{1}{2}(x_1-x_2+x_3-x_4)\Omega_{11} - \frac{1}{2}\mathscr{M}_{\Gamma_{11}}(\theta_1\theta_2\theta_3) + \frac{1}{2}\mathscr{M}_{\Gamma_{11}}(\theta_1\theta_3\theta_4).
\end{aligned} \tag{8.10}$$

Using these explicit expressions it is easy to calculate that the cohomological wave function of $\mathscr{O}_{\mathrm{Hopf}}$ reads,

$$\Psi = \sum_{i=0}^{n-1} a_i x_1^i x_2^{n-1}\Omega_{00} + \epsilon_1\epsilon_2 \sum_{i=0}^{n-1}\sum_{j=0}^{n-2} b_{ij}x_1^i x_2^j \Omega_{11}, \tag{8.11}$$

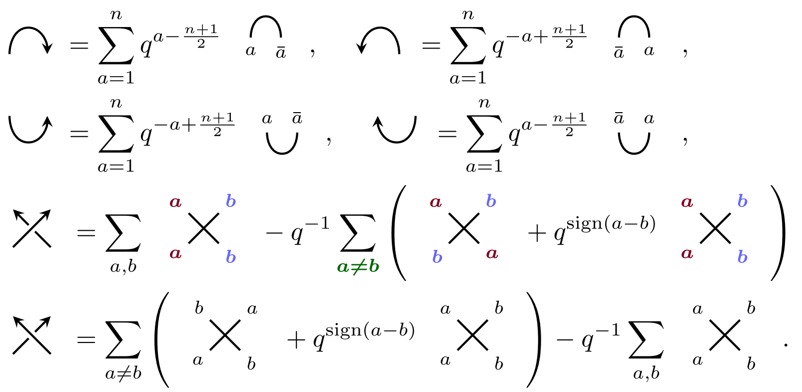
We calculate that the Poincaré polynomial for the Hopf link coincides with the known expression from the literature ([3], Table I),

$$P_{\mathrm{Hopf}}(q,t) = q^{n+1}[n]_q + t^2q^{-1}[n]_q[n-1]_q. \tag{8.12}$$

B. Decategorification

The "golden rule" of any refinement (categorification) procedure is that at any moment refinement could be reversed, i.e., at any stage of the categorification process we could see a simple reduction to decategorified objects. Here we would like to argue that calculating invariant zero modes of operator $\mathscr{O}_L$ decategorifies indeed to the R-matrix calculations in the RT formalism. In the RT formalism the computation of diagram L reduces to rephrasing its elements in terms of functors, simply R matrix and cup/cap braid closure tensors, of the braided tensor category of $U_q(\mathfrak{su}_n)$ [7,8].

Fortunately, explicit expressions for the R matrix and caps/cups are known for fundamental reps $\square$ of generic $U_q(\mathfrak{su}_n)$. Here we present them in the useful graphical language,



(8.13)

Here in the rhs indices a correspond to weights of vectors of $\square$, and barred indices $\bar{a}$ correspond to $\overline{\square}$. These diagrams represent propagation of weights across the strands of a tangle, so the R matrices correspond to permutations of weights and cups/caps to conjugations. To derive a closed link invariant using this graphical language elements of the tangle diagram are substituted by alternative expressions from the rhs, so the diagram is expanded in a formal sum of graphs. In the graphs weight indices on all edges should coincide, then a graph contribution is equal to 1, otherwise it is zero.

Let us *stress* here that naturally there are no limits in summations for R matrices in (8.13). These expressions are universal and n independent. We might assume that the n dependence is concentrated in closing caps and cups.

Comparing diagrammatic expansion for the unknot with respective cohomologies θ, $x\theta$, $x^2\theta$, … one observes that it is natural to identify weight labels a on graphs with "Fourier modes" x^a of wave functions in cohomologies of $\mathscr{O}_L$.

Let us consider a single intersection as an elementary tangle, then we could construct an operator for it [where labeling matches (6.1)]:

$$\mathscr{O}_0 = \mathscr{O}_{\times} + \mathscr{O}_\partial \left[\begin{array}{cc} x_1 & x_2 \\ x_4 & x_3 \end{array}\right]. \tag{8.14}$$

For matrix factorization parts we apply simplifying relations of Secs. IV C 2 and IV C 3, then for cohomologies the following ansatz is applicable:

$$\Psi = \underset{H^*\left(\mathscr{O}_\partial\left[\begin{array}{cc} x_1 & x_2 \\ x_1 & x_2 \end{array}\right]\right)}{\underset{\Cap}{\psi(x_1,x_2)}} + \epsilon\left(\frac{1}{2}\left(1+\frac{x_3-x_4}{x_1-x_2}\right) \underset{H^*\left(\mathscr{O}_\partial\left[\begin{array}{cc} x_1 & x_2 \\ x_2 & x_1 \end{array}\right]\right)}{\underset{\Cap}{\xi_-(x_1,x_2)}} + \frac{1}{2}\left(1-\frac{x_3-x_4}{x_1-x_2}\right) \underset{H^*\left(\mathscr{O}_\partial\left[\begin{array}{cc} x_1 & x_2 \\ x_1 & x_2 \end{array}\right]\right)}{\underset{\Cap}{\xi_+(x_1,x_2)}}\right) + \dots, \tag{8.15}$$

where dotted terms contain nonzero powers of $\theta_{1,2}$ necessarily and are uniquely defined by functions ψ, $\xi_\pm$. Now we see clearly that the pattern of (8.15) matches the pattern of weight distributions in the R matrix (8.13). Diagrams in the rhs of the R-matrix expression define weight matchings: straight (no permutation, same as x_k for ψ), criss-cross (permutation σ_{12}, same as x_k for ξ_-), straight (same as x_k for ξ_+), marked additionally by the respective colors. In addition functions $\xi_\pm$ are constrained $\xi_+(x,x) = \xi_-(x,x)$, that usually (yet not always) leads to a simple solution $\xi_\pm(x_1,x_2) = (x_1-x_2)g_\pm(x_1,x_2)$ for unconstrained $g_\pm$. However this solution implies that wave functions $\xi_\pm$ in matching points $x_1 = x_2$ *vanish* that correspond to constraint $a \neq b$ in the second term of the R matrix (8.13), also we note that the coefficient $(-q^{-1})$ coincides exactly with the index of ϵ.

Pattern matching is a sole observation, therefore it is not as strong as the theorem of MOY moves (4.13a)–(4.13e). Nevertheless we expect this observation to be a straightforward indication that zero modes of operator $\mathscr{O}_L$ are a categorification of the usual RT formalism indeed.

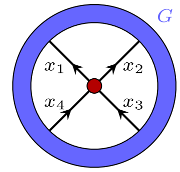
C. Reduction to Khovanov complex at $n=2$

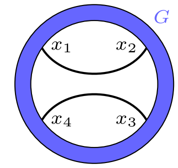
Here we would like to discuss a reduction of Khovanov-Rozansky double complex to a simple Khovanov mono-complex [42] categorifying Jones polynomials at $n=2$.

First, we would like to prove that the MOY edge is actually irrelevant and is equivalent to a pair of a cap and a cup,

$$H^*(\mathscr{M}_{\bar{\Gamma}_1}) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} H^*(\mathscr{M}_{\bar{\Gamma}_2}) \quad , \tag{8.16}$$

where $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are given by

$\bar{\Gamma}_1 =$ 

, $\bar{\Gamma}_2 =$ 

. (8.17)

In the $n=2$ case, we have,

$$u_1 = x_1^2 - x_2x_1 + x_3x_1 + x_4x_1 + x_2^2 + x_3^2 + x_4^2 + x_2x_3 + x_2x_4 + 2x_3x_4,$$
$$u_2 = -3x_3 - 3x_4. \tag{8.18}$$

We again denote the set of internal variables for a subgraph G as V. Respective MF differentials read,

$$\mathscr{M}_{\bar{\Gamma}_1} = \mathscr{M}_G(V, x_1, x_2, x_3, x_4) + u_1\hat{\theta}_1 - 3(x_3 + x_4)\hat{\theta}_2 + (x_1 + x_2 - x_3 - x_4)\hat{\theta}_1^\dagger + (x_1x_2 - x_3x_4)\hat{\theta}_2^\dagger,$$
$$\mathscr{M}_{\bar{\Gamma}_2} = \mathscr{M}_G(V, x_1, x_2, x_3, x_4) + \bar{\pi}_{12}\hat{\xi}_1 + \bar{\pi}_{34}\hat{\xi}_2 + (x_1 + x_2)\hat{\xi}_1^\dagger - (x_3 + x_4)\hat{\xi}_2^\dagger, \tag{8.19}$$

where

$$\bar{\pi}_{ij} \coloneqq \frac{x_i^3 + x_j^3}{x_i + x_j} = x_i^2 - x_ix_j + x_j^2. \tag{8.20}$$

We note that terms in $\mathscr{M}_{\bar{\Gamma}_1}$ in front of $\hat{\theta}_2$ and $\hat{\theta}_1^\dagger$ are linear in x_i, so it would be natural to have them as new fermionic variables. Moreover, it would be nice to make them "orthogonal" in xs. So, we perform the following canonical change of fermionic operators:

$$\hat{\xi}_1 = \hat{\theta}_1 - \frac{1}{3}\hat{\theta}_2^\dagger, \qquad \hat{\xi}_2 = \frac{1}{3}\hat{\theta}_2^\dagger, \qquad \hat{\xi}_1^\dagger = \hat{\theta}_1^\dagger, \qquad \hat{\xi}_2^\dagger = \hat{\theta}_1^\dagger + 3\hat{\theta}_2, \tag{8.21}$$

so that

$$\{\hat{\xi}_i, \hat{\xi}_j^\dagger\} = \delta_{ij}, \qquad \{\hat{\xi}_i, \hat{\xi}_j\} = \{\hat{\xi}_i^\dagger, \hat{\xi}_j^\dagger\} = 0. \tag{8.22}$$

Eventually, we arrive at the following matrix factorization operator:

$$\mathscr{M}_{\bar{\Gamma}_1} = \mathscr{M}_G(V, x_1, x_2, x_3, x_4) + (\bar{\pi}_{12} + (x_3 + x_4)s)\hat{\xi}_1 + (\bar{\pi}_{34} + (x_1 + x_2)s)\hat{\xi}_2 + (x_1 + x_2)\hat{\xi}_1^\dagger - (x_3 + x_4)\hat{\xi}_2^\dagger, \tag{8.23}$$

where

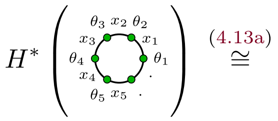
$$s \coloneqq x_1 + x_2 + x_3 + x_4. \tag{8.24}$$

It is easy to observe that,

$$\mathscr{M}_{\bar{\Gamma}_1} = \mathscr{M}_{\bar{\Gamma}_2} + [\mathscr{M}_{\bar{\Gamma}_2}, s\hat{\xi}_1\hat{\xi}_2] = e^{-s\hat{\xi}_1\hat{\xi}_2}\mathscr{M}_{\bar{\Gamma}_2}e^{s\hat{\xi}_1\hat{\xi}_2}. \tag{8.25}$$

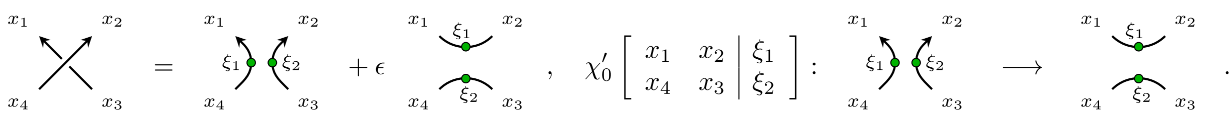
Relation (8.25) implies that map $e^{s\hat{\xi}_1\hat{\xi}_2}$ provides the isomorphism of the cohomologies (8.16).

This isomorphism implies that all the MOY diagrams are decomposed in collections of disjoint cycles with only two-valent vertices. It is trivial to compute respective cohomologies to be a 2D vector space,



$$\cong H^*\left(x^2\hat{\theta}\right) = \mathrm{Span}\{\theta, x\theta\} \cong \mathbb{Q}[x]/\langle x^2\rangle\,. \tag{8.26}$$

Second, due to the homomorphism requirement (3.8), the new horizontal morphism



(8.27)

can be obtained from χ_0 given by (5.14) with $\mu = 1$ and $\lambda = 0$ fixed for simplicity,

$$\chi_0' = e^{s\hat{\xi}_1\hat{\xi}_2}\chi_0 = \chi_0 - \frac{1}{3}s\hat{\xi}_1\hat{\xi}_2\hat{\xi}_2^\dagger\hat{\xi}_1^\dagger + s\hat{\xi}_1\hat{\xi}_2. \tag{8.28}$$

When the free ends of tangles in (8.27) are closed, this horizontal morphism would reduce to

$$\mathbb{Q}[x_1, x_3]/\langle x_1^2, x_3^2, x_3 = -x_1\rangle \xrightarrow{x_1 - x_3} \mathbb{Q}[x_1]/\langle x_1^2\rangle \otimes \mathbb{Q}[x_3]/\langle x_3^2\rangle \tag{8.29}$$

or

$$\mathbb{Q}[x_1]/\langle x_1^2\rangle \otimes \mathbb{Q}[x_2]/\langle x_2^2\rangle \xrightarrow{1} \mathbb{Q}[x_1, x_2]/\langle x_1^2, x_2^2, x_2 = -x_1\rangle \tag{8.30}$$

depending on the choice of the closure. These differentials reproduce the action of Khovanov multiplication and coproduct morphisms [42]. Matrix elements of horizontal morphisms then could be calculated in the language of Bar-Natan's TQFT [59]. Maps $\chi_{0,1}'$ are identified with "pair-of-pants" cobordisms. The complete set of rules for this TQFT is depicted in Fig. 2, and it turns out to be algorithmic enough to boost computations of Khovanov polynomials with the help of computers [60]. A generalization of Bar-Natan's TQFT to foam TQFT [36–41] is much more involved and much less algorithmic due to allowed surface junctions (see Appendix A), nevertheless we hope a little bit more "old fashioned" approach with wave functions and differential equations might prove useful on this route.

In general, in a knot complex, we have full smoothings glued from several)(and $\asymp$ resolutions. The cut-and-join operator χ_0' acts inside local regions of these full smoothings, when one)(resolution transforms to $\asymp$ resolution, and depends locally only on two respective fermionic variables, see (8.27).

Our goal is to get rid of odd variables, as proposed in Sec. IV C 2, in order to reduce the problem to the already analyzed case. The isomorphisms Φ_1 commute with χ_0' because we get rid of "far" variables and act on Grassmann variables not coincident with χ_0' fermions. Proceeding this way, we transfer to new simple vertical morphisms, whose cohomologies are tensor products of two-dimensional spaces, and to horizontal morphisms (8.28) giving rise to Khovanov operators.

It is important to note that this reduction breaks the locality of the KR double complex. Before the reduction, all the morphisms act on local four-valent vertices, whereas the reduction forces the morphisms to act on a whole cycle or a pair of cycles. The horizontal morphism could switch "spin'—a choice of a single cycle wave function "1" or "x"—flowing in the cycle, and switching spins in one isolated vertex might affect rather distant parts of the link diagram. We believe this nonlocality is a result of entanglement [61,62]: despite we start with a system of mutually "non-interacting" operators $\mathscr{O}_v^\pm$ corresponding to local vertices, the true invariant wave function is not a pure state, rather it is highly entangled and therefore nonlocal at all.

D. Bipartite vertex reduction

Certain simplification of KR complexes was performed by Krasner [45] for bipartite knots and links [46–48].

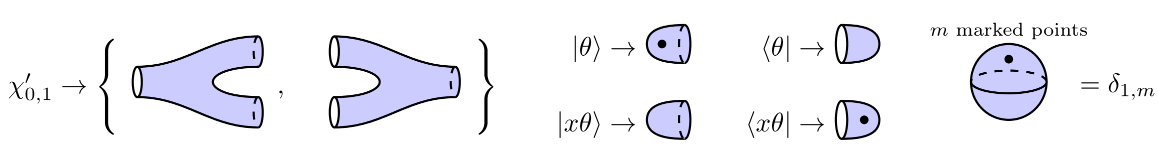
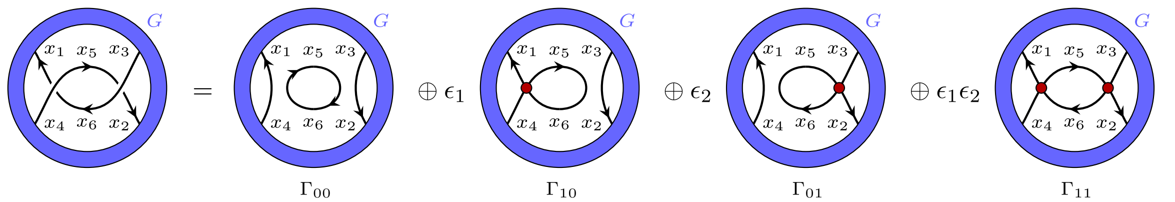


FIG. 2. Rules of TQFT for $n = 2$.

For bipartite links strands could be bent in a specific way to form "locks" of antiparallel strands with two intersections—bipartite vertices. Applying the RT formalism it is possible to show that such a bipartite lock can be decomposed over untangled intersection resolutions, that allows one to decompose a bipartite link into a sum over diagrams of nonintersecting cycles. As it was shown in [45] this RT formalism simplification is lifted to the KR complex via the Gauss elimination algorithm. Here we would like to discuss prospects of this simplification from the point of view of operators $\mathscr{O}_L$.

We consider a bipartite vertex as a sole tangle. We add the boundary correction operator as an external closing link denoted by G,



(8.31)

For MF operators, we have,

$$\begin{aligned}
\mathscr{M}_{\Gamma_{00}} &= \mathscr{M}_G(x_1,x_2,x_3,x_4) + \pi_{14}\hat\theta_1 + x_{14}\hat\theta_1^\dagger + \pi_{56}\hat\theta_2 + x_{56}\hat\theta_2^\dagger + \pi_{23}\hat\theta_3 + x_{23}\hat\theta_3^\dagger + \pi_{56}\hat\theta_4 + x_{65}\hat\theta_4^\dagger,\\
\mathscr{M}_{\Gamma_{10}} &= \mathscr{M}_G(x_1,x_2,x_3,x_4) + u_1\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_1 + s_1\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_1^\dagger + u_2\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_2 + s_2\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_2^\dagger\\
&\quad + \pi_{23}\hat\theta_3 + x_{23}\hat\theta_3^\dagger + \pi_{56}\hat\theta_4 + x_{65}\hat\theta_4^\dagger,\\
\mathscr{M}_{\Gamma_{01}} &= \mathscr{M}_G(x_1,x_2,x_3,x_4) + \pi_{14}\hat\theta_1 + x_{14}\hat\theta_1^\dagger + \pi_{56}\hat\theta_2 + x_{56}\hat\theta_2^\dagger\\
&\quad + u_1\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_3 + s_1\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_3^\dagger + u_2\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_4 + s_2\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_4^\dagger,\\
\mathscr{M}_{\Gamma_{11}} &= \mathscr{M}_G(x_1,x_2,x_3,x_4) + u_1\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_1 + s_1\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_1^\dagger + u_2\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_2 + s_2\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix}\hat\theta_2^\dagger\\
&\quad + u_1\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_3 + s_1\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_3^\dagger + u_2\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_4 + s_2\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}\hat\theta_4^\dagger,
\end{aligned} \tag{8.32}$$

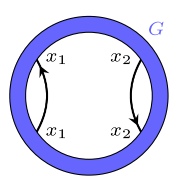
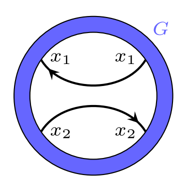
where we denoted

$$s_1\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix} \coloneqq x_2 + x_6 - x_3 - x_5, \qquad s_2\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix} \coloneqq x_2x_6 - x_3x_5. \tag{8.33}$$

Horizontal differentials in this system read,

$$\mathfrak{D} = \mathfrak{D}_G + \hat\epsilon_1\chi_0\begin{bmatrix} x_1 & x_5\\ x_4 & x_6\end{bmatrix} + \hat\epsilon_2\chi_0\begin{bmatrix} x_2 & x_6\\ x_3 & x_5\end{bmatrix}. \tag{8.34}$$

Also we would like to introduce the following notations:

$\mathscr{M}_+ \coloneqq \mathscr{M}_G(x_1,x_2,x_2,x_1) =$ , $\mathscr{M}_- \coloneqq \mathscr{M}_G(x_1,x_2,x_1,x_2) =$ . (8.35)

All the MOY diagrams in the rhs of (8.31) could be simplified by MOY moves (4.13a)–(4.13e). However to define the action of horizontal morphisms on these graphs it is required to calculate wave functions explicitly. Those wave functions could be calculated using methods discussed in Sec. IV C 2 and in Sec. IV C 3. In particular for Γ_{11} one has to apply the MOY move (4.13e). Let us discuss this move in detail.

At the first step we split the left four-valent vertex of Γ_{11} using techniques of Sec. IV C 3. First consider root $x_4 = x_1$, $x_6 = x_5$,

$$\mathscr{M}_{\Gamma_{11}}\big|_{\substack{x_4=x_1\\ x_6=x_5}} = \mathscr{M}_G(x_1,x_2,x_3,x_1) + u_1\begin{bmatrix} x_2 & x_5 \\ x_3 & x_5 \end{bmatrix}\hat{\theta}_3 + (x_2-x_3)\hat{\theta}_3^\dagger + u_2\begin{bmatrix} x_2 & x_5 \\ x_3 & x_5 \end{bmatrix}\hat{\theta}_4 + x_5(x_2-x_3)\hat{\theta}_4^\dagger. \tag{8.36}$$

This MF operator could be reduced further with respect to root $x_2 = x_3$ according to Sec. IV C 2,

$$\mathscr{M}_{\Gamma_{11}}\big|_{\substack{x_4=x_1\\ x_6=x_5\\ x_3=x_2}} = \mathscr{M}_+ + p_{n-1}(x_5,x_2)\hat{\theta}_4, \tag{8.37}$$

where p_{n-1} is a polynomial given by (7.6).

Now we consider the other root $x_4 = x_5$, $x_6 = x_1$ at the first step,

$$\mathscr{M}_{\Gamma_{11}}\big|_{\substack{x_4=x_5\\ x_6=x_1}} = \mathscr{M}_G(x_1,x_2,x_3,x_5) + u_1\begin{bmatrix} x_2 & x_1 \\ x_3 & x_5 \end{bmatrix}\hat{\theta}_3 + s_1\begin{bmatrix} x_2 & x_1 \\ x_3 & x_5 \end{bmatrix}\hat{\theta}_3^\dagger + u_2\begin{bmatrix} x_2 & x_1 \\ x_3 & x_5 \end{bmatrix}\hat{\theta}_4 + s_2\begin{bmatrix} x_2 & x_1 \\ x_3 & x_5 \end{bmatrix}\hat{\theta}_4^\dagger. \tag{8.38}$$

This MF operator could be reduced further again by splitting the second four-valent vertex, there are two roots,

$$\mathscr{M}_{\Gamma_{11}}\big|_{\substack{x_4=x_5=x_6=x_1\\ x_3=x_2}} = \mathscr{M}_+, \qquad \mathscr{M}_{\Gamma_{11}}\big|_{\substack{x_4=x_5=x_2\\ x_6=x_3=x_1}} = \mathscr{M}_-. \tag{8.39}$$

Eventually, we could construct the following tree of wave function reductions:

$$\begin{tikzcd}
 & \Psi\left[\begin{array}{ccc|cc} x_1 & x_5 & x_3 & 0 & \theta_3 \\ x_1 & x_5 & x_2 & 0 & \theta_4 \end{array}\right], \mathscr{M}_1 \arrow[r, "x_3=x_2"] & \begin{array}{c}\Psi\left[\begin{array}{ccc|cc} x_1 & x_5 & x_2 & 0 & 0 \\ x_1 & x_5 & x_2 & 0 & \theta_4 \end{array}\right], \\ \mathscr{M}_+ + p_{n-1}(x_5,x_2)\hat{\theta}_4 \end{array} \\
\Psi\left[\begin{array}{ccc|cc} x_1 & x_5 & x_3 & \theta_1 & \theta_3 \\ x_4 & x_6 & x_2 & \theta_2 & \theta_4 \end{array}\right], \mathscr{M}_{\Gamma_{11}} \arrow[ur, "\substack{x_4=x_1\\ x_6=x_5}"] \arrow[dr, "\substack{x_4=x_5\\ x_6=x_1}"'] & & \\
 & \Psi\left[\begin{array}{ccc|cc} x_1 & x_5 & x_3 & 0 & \theta_3 \\ x_5 & x_1 & x_2 & 0 & \theta_4 \end{array}\right], \mathscr{M}_2 \arrow[r, "\substack{x_5=x_1\\ x_3=x_2}"] \arrow[dr, "\substack{x_5=x_2\\ x_3=x_1}"'] & \Psi\left[\begin{array}{ccc|cc} x_1 & x_1 & x_2 & 0 & 0 \\ x_1 & x_1 & x_2 & 0 & 0 \end{array}\right], \mathscr{M}_+ \\
 & & \Psi\left[\begin{array}{ccc|cc} x_1 & x_2 & x_1 & 0 & 0 \\ x_2 & x_1 & x_2 & 0 & 0 \end{array}\right], \mathscr{M}_-
\end{tikzcd} \tag{8.40}$$

According to Sec. IV C 2 the wave function for MF operator $\mathscr{M}_1$ takes the following form:

$$\Psi_1 = U_1(x_5^k\theta_4\psi_+(x_1,x_2)) + \mathscr{M}_1\Theta_1, \qquad 0 \le k \le n-2, \tag{8.41}$$

where U_1 is an appropriate dressing operator and $\psi_+ \in H^*(\mathscr{M}_+)$.

For MF operator $\mathscr{M}_2$ we find following Sec. IV C 3:

$$\Psi_2 = U_2\left(\frac{1}{2}\left(1 + \frac{x_5 - x_3}{x_1 - x_2}\right)\xi_+(x_1,x_2) + \frac{1}{2}\left(1 + \frac{x_3 - x_5}{x_1 - x_2}\right)\xi_-(x_1,x_2)\right) + \mathscr{M}_2\Theta_2, \tag{8.42}$$

where U_2 is also a dressing operator, and $\xi_\pm \in H^*(\mathscr{M}_\pm)$, $\xi_+(x,x) = \xi_-(x,x)$. Now we should impose a matching point constraint from the first four-valent vertex splitting $\Psi_1|_{x_5=x_1} = \Psi_2|_{x_5=x_1}$. Actually,

$$\mathscr{M}_3 \coloneqq \mathscr{M}_1|_{x_5=x_1} = \mathscr{M}_2|_{x_5=x_1} = \mathscr{M}_G(x_1,x_2,x_3,x_1) + u_1\begin{bmatrix} x_2 & x_1 \\ x_3 & x_1 \end{bmatrix}\hat{\theta}_3 + (x_2-x_3)\hat{\theta}_3^\dagger + u_2\begin{bmatrix} x_2 & x_1 \\ x_3 & x_1 \end{bmatrix}\hat{\theta}_4 + x_1(x_2-x_3)\hat{\theta}_4^\dagger. \tag{8.43}$$

Its cohomology coincides with cohomology of $\mathscr{M}_1$ with substituted $x_5 = x_1$. So we impose the following condition:

$$U_1'(x_1^k\theta_4\psi_+(x_1,x_2)) - U_2'\left(\xi_+(x_1,x_2) + \frac{x_2-x_3}{x_1-x_2}\cdot\frac{\xi_+(x_1,x_2) - \xi_-(x_1,x_2)}{2}\right) = \mathscr{M}_3(\Theta_2' - \Theta_1'), \tag{8.44}$$

where primes denote proper projected functions to $x_5 = x_1$. We should note that dressing operators $U_{1,2}$ do not alter the lowest fermionic degree, so for the functions ψ appearing at the first degree and $\xi_\pm$ appearing at the zeroth degree the only option not to belong to $\mathrm{Im}\,\mathscr{M}_3$ is to be proportional to root $x_1 - x_5$. Requiring this proportionality from wave functions (8.41), (8.42) we will find that the generic cohomological solution reads,

$$\Psi = \frac{1}{2}\left(1 + \frac{x_6 - x_4}{x_1 - x_5}\right) U_1(x_5^{k-1}(x_5 - x_1)\theta_4\psi_+(x_1, x_2)) + \frac{1}{2}\left(1 - \frac{x_6 - x_4}{x_1 - x_5}\right) U_2((x_5 - x_1)\xi_-(x_1, x_2)),$$
$$1 \le k \le n-2, \psi_+ \in H^*(\mathscr{M}_+), \xi_- \in H^*(\mathscr{M}_-), \tag{8.45}$$

the component ξ_+ is a projection to $x_5 = x_1$ so it is annihilated, also mode $x_5^0\theta_4\psi_+$ is annihilated by constraint (8.44). This relation is a wave function manifestation of the MOY move (4.13e).

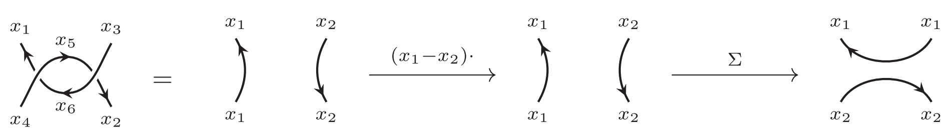
Having constructed wave functions explicitly one could solve for the cohomologies of the respective operator $\mathscr{O}_L$. This equation can be solved partly by separating variables analogously to Sec. VII. We will present a detailed derivation elsewhere. One would find that the wave function splits in three parts,

$$\Psi_{\mathrm{bipt}} = V_1\psi_1(x_1, x_2) + \epsilon_1 V_2\psi_2(x_1, x_2) + \epsilon_1\epsilon_2 V_3\psi_3(x_1, x_2), \tag{8.46}$$

where $\psi_{1,2} \in H^*(\mathscr{M}_+)$ and $\psi_3 \in H^*(\mathscr{M}_-)$, and V_i are invertible dressing factor operators we do not need to know an explicit form of at the moment. The horizontal morphism would reduce to,

$$\mathfrak{D} = \mathfrak{D}_G + \hat{\epsilon}_1 V_2(x_1 - x_2)V_1^{-1} + \hat{\epsilon}_2\hat{\epsilon}_1\hat{\epsilon}_1^\dagger V_3\Sigma V_2^{-1},$$
$$\Sigma = \frac{\mathscr{M}_G(x_1, x_2, x_1, x_2) - \mathscr{M}_G(x_1, x_2, x_2, x_1)}{x_1 - x_2}. \tag{8.47}$$

This statement is equivalent to Krasner's complex [45] for a bipartite vertex,



(8.48)

IX. CONCLUSION

The powerful functional integral technique allows one to calculate correlators of Wilson loops in Yang-Mills theory. In the particular case of the 3D Chern-Simons theory, it reduces to the RT formalism [7,8,63] for the evaluation of topologically invariant HOMFLY-like polynomials, which can also be considered as Euler characteristics of certain complexes. Usually, the Euler characteristic has two levels of description—as an alternated sum of dimensions of spaces (of all forms with different gradings) and as that of their zero modes (holomorphic or harmonic). Once zero modes show up, one can study them separately and not obligatorily only alternatively summed—this leads to a generalization of Euler characteristic to Poincaré polynomial and, perhaps, even further. Instead the integrals of curvatures and their functional-integral generalizations no longer work—and a new formalism is needed. Moreover, the study of zero modes requires a kind of Hamiltonian, which distinguishes them from everything else—and conceptually this adds one extra dimension to the problem. This addition is not important for the Euler characteristic itself, but becomes crucial for Poincaré generalization. In the case of the 3D Chern-Simons theory, this leads from RT to KR calculus, which naturally possesses formulations in 4D and even 5D and 6D terms [12,26].

If we are interested in building an efficient calculational technique, these general concepts should be reduced to more compact and handleable kinds of formalism. In this paper, we substitute a sophisticated machinery of abstract matrix factorization by a straightforward analysis of odd differential operators, associated with link diagrams. The calculus is essentially cohomological, still applicable to open tangles—what is crucially important for knot theory, at least for calculational purposes. As such, this formalism seems very promising for further analysis of KR [1,2] and superpolynomials [10,11], which so far remains restricted to just a few examples—either for very small knots or to knots with special additional structures, like torus [14,64], twist [65], or pretzel [66] knots.

We list here just a few straightforward applications, which hopefully will not take too much time to complete.

(i) Generalizations from fundamental to other representations and algebras—straightforward in the RT approach—imply the change from $w(x) = x^{n+1}$ for the fundamental representation of $\mathfrak{su}_n$ to more sophisticated functions of more sophisticated variables. Does the cabling procedure [67–69] lifted to operators $\mathcal{O}_L$ allow one to produce new superpotentials and vertices?

(ii) Bipartite knots, made entirely from antiparallel lock tangles, possess Kauffman planar decomposition [70] for any n, not just $n = 2$. This makes the theory of fundamental bipartite HOMFLY polynomials nearly trivial [46]—though some problems persist for higher representations, even symmetric [47]. Planar decomposition is immediately seen in our operator formalism, and building a systematic bipartite KR theory, originated in [22,45], seems now within reach.

(iii) A and C polynomials [71,72] can now be lifted to the level of KR polynomials. A mixture of approaches of [73,74] and the present paper seems especially interesting.

Coming back to the main story of this paper, the main challenge is to make it fully algorithmic and computerized—so that operators, their zero modes, and the Poincare polynomial for a given link diagram could be found automatically. In fact, we are already close to this level of understanding. What remains rather abstract and requires more work, are certain steps in computation of the operators $\mathcal{O}_L$, especially for open tangles. As to the zero modes of the Hamiltonian (1.5) on the space of graded polynomials, the procedure is nearly algorithmic. All the graded subspaces of polynomials are finite and the range of gradings, contributing to the zero modes, could be estimated from counting the resolutions (MOY dimensions). Surely, there is a need for optimization to make the procedure practical, and better algorithms are still to be found.

ACKNOWLEDGMENTS

We would like to thank Alexandra Anokhina and Radomir Stepanov for illuminating discussions. The work was partially funded within the state assignment of the Institute for Information Transmission Problems of RAS.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: MF FROM TOPOLOGICAL STRINGS

One of the most popular models for topological strings is a 2D $\mathcal{N} = (2,2)$ supersymmetric Landau-Ginzburg model [49,75]. For simplicity we present its action for a single chiral multiplet with top complex scalar component ϕ and fermionic partners ψ_α. The potential for this model is defined by a holomorphic superpotential function $W(\phi)$. The action of this model reads,

$$\begin{aligned} \mathcal{S}_\chi(\phi, \psi_\alpha, W) = \int dx^0 dx^1 \Big[& |\partial_0\phi|^2 - |\partial_1\phi|^2 \\ & + \mathrm{i}\bar{\psi}_-(\partial_0 + \partial_1)\psi_- + \mathrm{i}\bar{\psi}_+(\partial_0 - \partial_1)\psi_+ \\ & - \frac{1}{4}|W'|^2 - \frac{1}{2}W''\psi_+\psi_- + \frac{1}{2}\bar{W}''\bar{\psi}_+\bar{\psi}_- \Big]. \end{aligned} \tag{A1}$$

It is invariant (if no boundary of the worldsheet is present) with respect to the following SUSY transforms:

$$\begin{aligned} \delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+, \\ \delta\psi_+ &= \mathrm{i}\bar{\epsilon}_-(\partial_0 + \partial_1)\phi - \frac{1}{2}\epsilon_+\bar{W}', \\ \delta\psi_- &= -\mathrm{i}\bar{\epsilon}_+(\partial_0 - \partial_1)\phi - \frac{1}{2}\epsilon_-\bar{W}'. \end{aligned} \tag{A2}$$

Supercharges anticommute to translation symmetries, and if some boundary of the worldsheet is present some part of the supersymmetry should be broken as translation perpendicular to the boundary is no more symmetry of the problem. We choose preserved subsymmetry to be of B type by setting without loss of generality $\epsilon_+ = \epsilon_- = \epsilon$.

Consider a junction of two Landau-Ginzburg (LG) worldsheets located at spatial coordinate $x^1 = 0$ as it is depicted in Fig. 3. To describe the junction defect we put on it a 1D model of the Fermi multiplet [31,49] with fermionic field ρ. It is allowed to interact with the bulk chirals via superapotentials A and $\overline{B}$ that are holomorphic and antiholomorphic functions of boundary values $\boldsymbol{\phi}_i(x^0) \coloneqq \phi_i(x^0, x^1 = 0)$. From the fermionic part of the chiral multiplets it interacts only with combinations $\boldsymbol{\psi}_i(x^0) \coloneqq (\psi_+^{(i)}(x^0, x^1 = 0) - \psi_-^{(i)}(x^0, x^1 = 0))$,

$$\mathcal{S}_{\mathrm{F}}(\rho, A, B) = \int dx^0 \left(\mathrm{i}\bar{\rho}\partial_0\rho - \frac{1}{4}|A|^2 - \frac{1}{4}|B|^2 - \frac{1}{\sqrt{2}}\mathrm{Re}\rho\left(\sum_i \boldsymbol{\psi}^{(i)}\partial_{\boldsymbol{\phi}_i}A - \sum_i \overline{\boldsymbol{\psi}^{(i)}\partial_{\boldsymbol{\phi}_i}B} \right) \right). \tag{A3}$$

SUSY transforms for field ρ read,

$$\delta\rho = \frac{1}{\sqrt{2}}(\epsilon\bar{A} + \bar{\epsilon}B). \tag{A4}$$

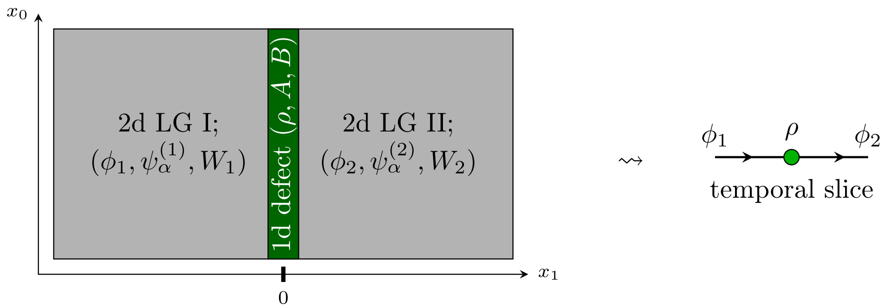


FIG. 3. Junction of two LG worldsheets.

For the total SUSY variation of combined actions we find,

$$\delta(\mathcal{S}_\chi(\phi_1,\psi_\alpha^{(1)},W_1)+\mathcal{S}_\chi(\phi_2,\psi_\alpha^{(2)},W_2)+\mathcal{S}_{\mathrm{F}}(\rho,A,B)) \\ =\int dx^0\mathrm{Re}\left(\bar{\epsilon}\sum_i \boldsymbol{\psi}^{(i)}\partial_{\boldsymbol{\phi}_i}\left(\frac{\mathrm{i}}{2}W_1-\frac{\mathrm{i}}{2}W_2-\frac{1}{4}AB\right)+(-1)^i(-\bar{\epsilon}(\bar{\psi}_+^{(i)}-\bar{\psi}_-^{(i)})\partial_1\phi+\bar{\epsilon}(\bar{\psi}_+^{(i)}+\bar{\psi}_-^{(i)})\partial_0\phi)|_{x^1=0}\right). \tag{A5}$$

For this system to remain supersymmetric we should impose constraints on a relation between bulk and defect superpotentials,

$$AB=2\mathrm{i}(W_1-W_2), \tag{A6}$$

as well for fields ϕ_i and $\bar{\psi}_+^{(i)}-\bar{\psi}_-^{(i)}$ we impose Neumann boundary conditions at the defect boundary, and for fields $\bar{\psi}_+^{(i)}+\bar{\psi}_-^{(i)}$ we impose Dirichlet boundary conditions correspondingly.

Now we consider a horizontal section of the worldsheet and Hamiltonian dynamics on it. Wave functions spanning the Hilbert space of this system are functionals $\Psi[\phi_i(x),\bar{\phi}_i(x),\psi_\alpha^{(i)}(x),\rho]$ of spatial field distributions $\phi_i(x)$, $\psi_\alpha^{(i)}$ and a function of variable ρ. The Hamiltonian of this supersymmetric system can be derived as an anticommutator of supercharge operators $\mathcal{H}=\frac{1}{2}\{\mathfrak{Q},\mathfrak{Q}^\dagger\}$, where [31],

$$\mathfrak{Q}_\Lambda^\dagger=\sum_i\int dx^1\left(-\mathrm{i}(\hat{\psi}_+^{(i)\dagger}-\hat{\psi}_-^{(i)\dagger})\frac{\delta}{\delta\bar{\phi}_i}+e^\Lambda(\hat{\psi}_+^{(i)\dagger}+\hat{\psi}_-^{(i)\dagger})\partial_1\phi_i-e^{-\Lambda}\frac{\mathrm{i}}{2}(\hat{\psi}_+^{(i)}+\hat{\psi}_-^{(i)})W_i\right)+\frac{1}{\sqrt{2}}(A\hat{\rho}+B\hat{\rho}^\dagger), \tag{A7}$$

where we have already introduced rescaling Λ according to localization prescription [35]. All the Hamiltonians obtained from $\mathfrak{Q}_\Lambda^\dagger$ for different Λ correspond to different systems, nevertheless supersymmetric ground states for all of them correspond to cohomologies $H^*(\mathfrak{Q}_\Lambda^\dagger)$ and are isomorphic among each other. It is easy to calculate such cohomologies in the limit $\Lambda\to\infty$ since the dominant term localizes field configurations to constant modes $\partial_1\phi_i=0$. The constant modes coincide with values $\boldsymbol{\phi}_i$ at the boundary. The wave function factorizes as

$$\Psi[\phi_i(x),\bar{\phi}_i(x),\psi_\alpha^{(i)}(x),\rho] \\ =\Psi_0(\boldsymbol{\phi}_i,\bar{\boldsymbol{\phi}}^{(i)},\boldsymbol{\psi}^{(i)},\rho)\times\Psi[\text{heavy modes}], \tag{A8}$$

where Ψ_0 is described as cohomology of the following effective supercharge:

$$\mathfrak{Q}_0^\dagger=-\mathrm{i}\sum_i\boldsymbol{\psi}^{(i)\dagger}\frac{\partial}{\partial\bar{\boldsymbol{\phi}}_i}+\frac{1}{\sqrt{2}}(A\hat{\rho}+B\hat{\rho}^\dagger). \tag{A9}$$

In this supercharge variables split, so we find its cohomologies in the following form:

$$\Psi_0(\boldsymbol{\phi}_i,\bar{\boldsymbol{\phi}}^{(i)},\boldsymbol{\psi}^{(i)},\rho)=\psi(\boldsymbol{\phi}_i,\rho)\times\prod_i\boldsymbol{\psi}^{(i)}, \tag{A10}$$

where ψ are strictly holomorphic in all variables $\boldsymbol{\phi}_i$ and are cohomologies of the following effective supercharge:

$$\mathfrak{Q}^\dagger=\frac{1}{\sqrt{2}}(A\hat{\rho}+B\hat{\rho}^\dagger), \tag{A11}$$

which is analogous to MF operators $\mathscr{M}$ introduced in Sec. II.

It is easy to generalize this story to any number of 2D sheets and 1D junctions. A temporal slice of the string worldsheet tube is encoded literally by a graph Γ whose oriented edges correspond to intersections of 2D worldsheets with the slice, and vertices correspond to intersections with junction defect worldlines (see Fig. 3). Orientation on edges reflects worldsheet orientation. The supersymmetric ground state wave function could be determined as cohomology of an effective supercharge,

$$\mathfrak{Q}_\Gamma^\dagger = \sum_{v\in\text{vertices}} \mathfrak{Q}_v^\dagger, \tag{A12}$$

where local vertex supercharges $\mathfrak{Q}_v^\dagger$ correspond to matrix factorization. We can easily generalize a junction defect to a vertex of any valency. Due to (A6) the respective matrix factorization is constrained,

$$(\mathfrak{Q}_v^\dagger)^2 = 2\mathrm{i}\sum_{a:\ *\to v} W_a - 2\mathrm{i}\sum_{b:\ v\to *} W_b, \tag{A13}$$

where we split sums over edges a flowing inward vertex v and edges b flowing outward vertex v.

APPENDIX B: QUOTIENTS OF POLYNOMIAL RINGS

Working with the framework of Khovanov-Rozansky double complexes one encounters often a necessity to describe a quotient ring. For example, cohomologies of a simplistic nilpotent MF operator $\mathscr{M} = p(x)\hat{\theta}$, where $p(x)$ is a polynomial, acting on a ring $\mathbb{Q}[x]\oplus\mathbb{Q}[x]\theta$ is isomorphic as a vector space to quotient ring $\mathbb{Q}[x]/\langle p(x)\rangle \coloneqq \mathbb{Q}[x]/(p(x)\mathbb{Q}[x])$. This fact has an analogous rephrasing as the short exact sequence of modules,

$$0 \longrightarrow \mathbb{Q}[x] \xrightarrow{p(x)\cdot} \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]/\langle p(x)\rangle \longrightarrow 0 \quad . \tag{B1}$$

So here we gather a few handy quotient relations that might appear useful for the task of a quotient ring construction.

1. Linear quotient

Let us start with quotients modulo linear relations. Consider a polynomial function F in a ring of N pairs of variables z_i and w_i. Apparently, one could work equally well with polynomials over fields of other unconstrained variables. Let us emphasize the dependence of F on z_i only and omit the dependence on w_i. Assume z_i are variables we would like to exclude by relations $z_i = w_i$, $i = 1,\ldots,N$. In terms of rings we consider F in the ring modulo ideals $z_i - w_i$. So we decompose F over a zeroth term with substituted $z_i = w_i$ and a sum of ideal members,

$$F(z_1,\ldots,z_N) = F(w_1,\ldots,w_N) + \sum_{i=1}^N (z_i - w_i)\times G_i,$$
$$G_i = \frac{F(w_1,\ldots,w_{i-1},z_i,z_{i+1},\ldots,z_N) - F(w_1,\ldots,w_{i-1},w_i,z_{i+1},\ldots,z_N)}{z_i - w_i}. \tag{B2}$$

Here all the ratios G_i are polynomial integer quotients indeed as the denominators cancel factors in the numerators having respective roots at $z_i = w_i$.

Apparently, fractions G_i are not defined uniquely, rather they are subjected to the gauge transforms parametrized by arbitrary polynomials H,

$$G_i \to G_i + (z_j - w_j)H, \quad G_j \to G_j - (z_i - w_i)H. \tag{B3}$$

2. Quadratic quotient

Here we would like to discuss a specific quotient ring $\mathbb{Q}[x_,x_2,x_3,x_4]/\langle s_1,s_2\rangle$, where

$$s_1 = x_1 + x_2 - x_3 - x_4, \qquad s_2 = x_1x_2 - x_3x_4. \tag{B4}$$

Here again an attempt to factor out a function by s_i would lead to localization of the function to solutions $s_1 = 0$, $s_2 = 0$. Let us assume that variables x_3 and x_4 are to be excluded, following Viète's theorem we find two possible roots,

$$(x_3 = x_1, x_4 = x_2) \quad \text{and} \quad (x_3 = x_2, x_4 = x_1). \tag{B5}$$

To construct the quotient of an arbitrary function $f(x_1,x_2,x_3,x_4)$ first we extract dependence on combinations $x_3 + x_4$ and x_3x_4 by decomposing f over the symmetric and antisymmetric part,

$$f(x_1,x_2,x_3,x_4) = \frac{1}{2}\underbrace{(f(x_1,x_2,x_3,x_4)+f(x_1,x_2,x_4,x_3))}_{\ell_1(x_1,x_2,x_3+x_4,x_3x_4)} + \frac{1}{2}(x_3-x_4)\underbrace{\frac{f(x_1,x_2,x_3,x_4)-f(x_1,x_2,x_4,x_3)}{x_3-x_4}}_{\ell_2(x_1,x_2,x_3+x_4,x_3x_4)}. \tag{B6}$$

Again for the ratio the denominator is canceled with a root in the numerator, so both functions $\ell_{1,2}$ are polynomials. These polynomials are symmetric with respect to permutation $x_3 \leftrightarrow x_4$, therefore they could be decomposed over elementary symmetric polynomials x_3+x_4 and x_3x_4. Further we construct the linear quotient (B2) of $\ell_{1,2}$ for variables $z_1 = x_3+x_4$, $z_2 = x_3x_4$, $w_1 = x_1+x_2$, $w_2 = x_1x_2$. Eventually we would arrive to the following decomposition of function f,

$$f(x_1,x_2,x_3,x_4) = \frac{1}{2}(1+h)\times f(x_1,x_2,x_1,x_2) + \frac{1}{2}(1-h)\times f(x_1,x_2,x_2,x_1) + s_1v_1(f) + s_2v_2(f), \tag{B7}$$

where $h \coloneqq \frac{x_3-x_4}{x_1-x_2}$, and linear maps $v_{1,2}$ represent integer quotients of f for s_1 and s_2, respectively, according to (B2). They are defined modulo the gauge transform,

$$v_1(f) \to v_1(f) + s_2 g, \qquad v_2(f) \to v_2(f) - s_1 g, \qquad \forall\ g. \tag{B8}$$

Let us note that,

$$h^2 = \left(\frac{x_3-x_4}{x_1-x_2}\right)^2 = 1 - s_1\frac{x_1+x_2+x_3+x_4}{(x_1-x_2)^2} + s_2\frac{4}{(x_1-x_2)^2} = 1 \mod \langle s_1,s_2\rangle. \tag{B9}$$

Thus we see that $P_\pm = \frac{1}{2}(1\pm h)$ are projectors to disjoint roots (B5),

$$P_+^2 = P_+ \mod \langle s_1,s_2\rangle, \qquad P_-^2 = P_- \mod \langle s_1,s_2\rangle, \qquad P_+P_- = P_-P_+ = 0 \mod \langle s_1,s_2\rangle. \tag{B10}$$

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